P7486 [Stoi2031] 彩虹-莫比乌斯反演

题目

$$给定n \in N^*, 询问t组, 每组1 \le l \le r \le n, 求\prod_{i=l}^{r}\prod_{j=l}^{r} lcm(i,j)^{lcm(i,j)} \bmod 32465177$$

$$1 \le n \le 10^6, 1 \le t \le 10$$

分析

$$不难想到用前缀积\prod_{i=1}^{n}\prod_{j=1}^{m} lcm(i,j)^{lcm(i,j)}来求区间[l,r]的答案$$

$$\begin{aligned}
\prod_{i=l}^{r}\prod_{j=l}^{r} lcm(i,j)^{lcm(i,j)} &= \prod_{i=1}^{r}\frac{\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)}}{\prod_{j=1}^{l-1} lcm(i,j)^{lcm(i,j)}}\\
&= \frac{\prod_{i=l}^{r}\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)}}{\prod_{i=l}^{r}\prod_{j=1}^{l-1} lcm(i,j)^{lcm(i,j)}}\\
&= \frac{\frac{\prod_{i=1}^{r}\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)}}{\prod_{i=1}^{l-1}\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)}}}{\frac{\prod_{i=1}^{r}\prod_{j=1}^{l-1} lcm(i,j)^{lcm(i,j)}}{\prod_{i=1}^{l-1}\prod_{j=1}^{l-1} lcm(i,j)^{lcm(i,j)}}}\\
&= \frac{\prod_{i=1}^{r}\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)} * \prod_{i=1}^{l-1}\prod_{j=1}^{l-1} lcm(i,j)^{lcm(i,j)}}{\left[\prod_{i=1}^{l-1}\prod_{j=1}^{r} lcm(i,j)^{lcm(i,j)}\right]^2}
\end{aligned}$$

$$下面来计算\prod_{i=1}^{n}\prod_{j=1}^{m} lcm(i,j)^{lcm(i,j)}:$$

$$\begin{aligned}
\prod_{i=1}^{n}\prod_{j=1}^{m} lcm(i,j)^{lcm(i,j)} &= \left(\frac{i*j}{gcd(i,j)}\right)^{\frac{i*j}{gcd(i,j)}}\\
&= \prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{d=1}^{min(n,m)}\left(\frac{i*j}{d}\right)^{\frac{i*j}{d}*[gcd(i,j)=d]}\\
&= \prod_{d=1}^{min(n,m)}\prod_{i=1}^{n}\prod_{j=1}^{m}\left(d*\frac{i}{d}*\frac{j}{d}\right)^{d*\frac{i}{d}*\frac{j}{d}*[gcd(\frac{i}{d},\frac{j}{d})=1]}\\
&= \prod_{d=1}^{min(n,m)}\prod_{i=1}^{\lfloor\frac{n}{d}\rfloor}\prod_{j=1}^{\lfloor\frac{m}{d}\rfloor}\left(d*i*j\right)^{d*i*j*[gcd(i,j)=1]}\\
&= \prod_{d=1}^{min(n,m)} d^{d*\sum_{i=1}^{\lfloor\frac{n}{d}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{d}\rfloor} i*j*[gcd(i,j)=1]}\prod_{i=1}^{\lfloor\frac{n}{d}\rfloor}\prod_{j=1}^{\lfloor\frac{m}{d}\rfloor}\left(i*j\right)^{d*i*j*[gcd(i,j)=1]}
\end{aligned}$$

$$\begin{aligned}
&记f_1(n) = \sum_{i=1}^{n} i,\\
&f_2(n) = \prod_{i=1}^{n} i^i,\\
&S(n,m) = \sum_{i=1}^{n}\sum_{j=1}^{m} i*j*[gcd(i,j)=1],\\
&G(n,m) = \prod_{i=1}^{n}\prod_{j=1}^{m} (i*j)^{i*j*[gcd(i,j)=1]},\\
&则:\\
&上式 = \prod_{d=1}^{min(n,m)} d^{d*S(\lfloor\frac{n}{d}\rfloor,\lfloor\frac{m}{d}\rfloor)} * \left[G(\lfloor\frac{n}{d}\rfloor,\lfloor\frac{m}{d}\rfloor)\right]^d
\end{aligned}$$

$$\begin{aligned}
S(n,m) &= \sum_{i=1}^{n}\sum_{j=1}^{m} i*j*[gcd(i,j)=1]\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} i*j*\sum_{d|gcd(i,j)}\mu(d)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d|min(n,m)} \sum_{d|i} \sum_{d|j} \mu(d) * d^2 * \frac{i}{d} * \frac{j}{d} \\
&= \sum_{d=1}^{min(n,m)} \mu(d) * d^2 \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} i \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} j \\
&= \sum_{d=1}^{min(n,m)} \mu(d) * d^2 * f_1(\lfloor \frac{n}{d} \rfloor) * f_1(\lfloor \frac{m}{d} \rfloor) \\
&= \sum_{l=1, r=\lfloor \frac{min(n,m)}{\lfloor \frac{min(n,m)}{l} \rfloor} \rfloor}^{l \le min(n,m)} f_1(\lfloor \frac{n}{d} \rfloor) * f_1(\lfloor \frac{m}{d} \rfloor) * \left(\sum_{i=l}^{r} \mu(d) * d^2 \right) --\text{数论数块}
\end{aligned}
$$

显然上式可以预处理$\sum_{i=1}^{n} \mu(i) * i^2$, 然后数论分块

$$
\begin{aligned}
G(n,m) &= \prod_{i=1}^{n} \prod_{j=1}^{m} (i*j)^{i*j*[gcd(i,j)=1]} \\
&= \prod_{i=1}^{n} \prod_{j=1}^{m} (i*j)^{i*j*\sum_{d|gcd(i,j)} \mu(d)} \\
&= \prod_{d=1}^{min(n,m)} \prod_{d|i}^{n} \prod_{j|d}^{m} (d^2 * \frac{i}{d} * \frac{j}{d})^{\mu(d)*d^2*\frac{i}{d}*\frac{j}{d}} \\
&= \prod_{d=1}^{min(n,m)} \prod_{i=1}^{\lfloor \frac{n}{d} \rfloor} \prod_{j=1}^{\lfloor \frac{m}{d} \rfloor} (d^2 * i * j)^{\mu(d)*d^2*i*j} \\
&= \prod_{d=1}^{min(n,m)} (d^2)^{\mu(d)*d^2*\sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} i \sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} j} * \prod_{i=1}^{\lfloor \frac{n}{d} \rfloor} \prod_{j=1}^{\lfloor \frac{m}{d} \rfloor} (i*j)^{\mu(d)*d^2*i*j} \\
&= \prod_{d=1}^{min(n,m)} (d^2)^{\mu(d)*d^2*f_1(\lfloor \frac{n}{d} \rfloor)*f_1(\lfloor \frac{m}{d} \rfloor)} * \left[(\prod_{i=1}^{\lfloor \frac{n}{d} \rfloor} i^i)^{\sum_{j=1}^{\lfloor \frac{m}{d} \rfloor} j} * (\prod_{j=1}^{\lfloor \frac{m}{d} \rfloor} j^j)^{\sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} i} \right]^{\mu(d)*d^2} \\
&= \prod_{d=1}^{min(n,m)} \left[(d^2)^{\mu(d)*d^2} \right]^{f_1(\lfloor \frac{n}{d} \rfloor)*f_1(\lfloor \frac{m}{d} \rfloor)} * \left[f_2(\lfloor \frac{n}{d} \rfloor)^{f_1(\lfloor \frac{m}{d} \rfloor)} * f_2(\lfloor \frac{m}{d} \rfloor)^{f_1(\lfloor \frac{n}{d} \rfloor)} \right]^{\mu(d)*d^2} \\
&= \prod_{l=1, r=\lfloor \frac{min(n,m)}{\lfloor \frac{min(n,m)}{l} \rfloor} \rfloor}^{l \le min(n,m)} \left[\prod_{i=l}^{r} (i^2)^{\mu(i)*i^2} \right]^{f_1(\lfloor \frac{n}{d} \rfloor)*f_1(\lfloor \frac{m}{d} \rfloor)} * \left[f_2(\lfloor \frac{n}{d} \rfloor)^{f_1(\lfloor \frac{m}{d} \rfloor)} * f_2(\lfloor \frac{m}{d} \rfloor)^{f_1(\lfloor \frac{n}{d} \rfloor)} \right]^{\sum_{i=l}^{r} \mu(i)*i^2} --\text{数论分块}
\end{aligned}
$$

预处理$\prod_{i=1}^{n} (i^2)^{\mu(i)*i^2}, \sum_{i=1}^{n} \mu(i) * i^2$, 然后数论分块

$$
\begin{aligned}
\text{看回最终式} &= \prod_{d=1}^{min(n,m)} d^{d*S(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} * \left[G(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor) \right]^d \\
&= \prod_{l=1, r=\lfloor \frac{min(n,m)}{\lfloor \frac{min(n,m)}{l} \rfloor} \rfloor}^{l \le min(n,m)} (\prod_{i=l}^{r} d^d)^{S(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} * \left[G(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor) \right]^{\sum_{i=l}^{r} i} \\
&= \prod_{l=1, r=\lfloor \frac{min(n,m)}{\lfloor \frac{min(n,m)}{l} \rfloor} \rfloor}^{l \le min(n,m)} \left[f_2(r) - f_2(l-1) \right]^{S(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor)} * \left[G(\lfloor \frac{n}{d} \rfloor, \lfloor \frac{m}{d} \rfloor) \right]^{f_1(r)-f_1(l-1)} --\text{数论分块}
\end{aligned}
$$

复杂度

$$
\text{预处理}\mu(n), f_1(n), f_2(n), \sum_{i=1}^{n} \mu(i) * i^2, \prod_{i=1}^{n} (i^2)^{\mu(i)*i^2} --O(4*n + n*log_2(n)) = O(n*log_2(n))
$$
$$
\text{求一次}S(n,m) --O(\sqrt{n}), \text{求一次}G(n,m) --O(\sqrt{n}*log_2(n))
$$
$$
\text{处理一次询问} --O(\sqrt{n}*(log_2(n) + \sqrt{n} + \sqrt{n}*log_2(n))) = O(n*log_2(n))
$$
$$
\text{总复杂度} --O(n*log_2(n))
$$

拓欧优化

因为$32465177 \in P$, 所以根据费马小定理$a^{p-1} \equiv 1$, 所以对于$a^x \equiv a^{\lambda*(p-1)+x\%(p-1)} \equiv 1 * a^{x\%(p-1)} \mod 32465177$

代码

```cpp
#include<bits/stdc++.h>
using namespace std;
#define int long long
int qpow(int a,int b,int p){
    int res = 1;
    for(a %= p,b=(b % (p - 1) + p - 1) % (p - 1);b;b >>= 1,a = a * a % p){
        if(b & 1) res = res * a % p;
    }
    return res;
}
const int mod = 32465177;
const int N = 5 + 1e6;
bool vis[N];
int tot,pri[N];
int mu[N],f2[N],f3[N],f5[N];
void seive(int n){
    f2[0] = 1;
    for(int i = 1 ; i <= n; i ++){
        f2[i] = f2[i-1] * qpow(i,i,mod);
        f2[i] %= mod;
    }
    mu[1] = 1,f3[1] = 1;
    for(int i = 2;i <= n;i ++){
        if(!vis[i]){
            vis[i] = 1;
            pri[++tot] = i;
            mu[i] = -1;
        }
        for(int j = 1;j <= tot && i * pri[j] <= n; j++){
            int v = i * pri[j];
            vis[v] = 1;
            if(i % pri[j] == 0){
                mu[v] = 0;
                break;
            }else{
                mu[v] = - mu[i];
            }
        }
    }
    f5[0] = 1;
    for(int i = 1;i <= n;i ++){
        f3[i] = f3[i-1] + mu[i] * i * i % (mod - 1);
        f3[i] %= mod - 1;
        f5[i] = f5[i-1] * qpow(i * i,mu[i] * i * i,mod);
        f5[i] %= mod;
    }
}
int f1(int n,int p = mod){
    return n * (n+1) / 2 % p;
}
int f6(int n,int m,int p = mod){
    return qpow(f2[n],f1(m,p - 1),p);
}
unordered_map<int,int> s;
int S(int n,int m){
    if(s[n*N+m]) return s[n*N+m];
    int res=0;
    for(int l = 1,r;l <= min(n,m);l = r + 1){
        r = min(n / (n / l),m / (m / l));
        res += f1(n / l,mod - 1) * f1(m / l,mod - 1) % (mod - 1) * (f3[r] - f3[l-1]) % (mod - 1);
        res %= mod - 1;
    }
    return s[n*N+m]=res;
}
unordered_map<int,int> g;
int G(int n,int m){
    if(g[n*N+m]) return g[n*N+m];
    int res = 1;
    for(int l = 1,r;l <= min(n,m);l = r + 1){
        r = min(n / (n / l),m / (m / l));
        res *= qpow(f5[r] * qpow(f5[l-1],mod - 2,mod),f1(n / l,mod - 1) * f1(m / l,mod - 1),mod);
        res %= mod;
```

```
        res *= qpow(f6(n / l,m / l,mod) * f6(m / l,n / l,mod) % mod,f3[r] - f3[l-1],mod);
        res %= mod;
    }
    return g[n*N+m]=res;
}
unordered_map<int,int> sol;
int solve(int n,int m){
    if(sol[n*N+m]) return sol[n*N+m];
    int res = 1;
    for(int l = 1,r;l <= min(n,m);l = r+1){
        r=min(n / (n / l),m / (m / l));
        res *= qpow(f2[r] * qpow(f2[l-1],mod - 2,mod),S(n / l,m / l),mod);
        res %= mod;
        res *= qpow(G(n / l,m / l),f1(r,mod - 1) - f1(l - 1,mod - 1),mod);
        res %= mod;
    }
    return sol[n*N+m]=res;
}
signed main(){
    ios::sync_with_stdio(false),cin.tie(0),cout.tie(0);
    int t,n; cin>>t>>n;
    seive(n);
    while(t--){
        int l,r; cin>>l>>r;
        int ans = solve(r,r) * solve(l - 1,l - 1) % mod * qpow(solve(l - 1,r),mod - 1 - 2,mod) %
mod;
        cout<<(ans + mod) % mod<<endl;
    }
}
```