整除分块

$$\forall n \in N^*, \exists \{x | x \in [1, n], x_{i+1} = x_i + 1\}, \forall x_i, x_j \in \{x\}, \lfloor \frac{n}{x_i} \rfloor = \lfloor \frac{n}{x_j} \rfloor$$

证明：

令$n = q * x + r, (r \in [1, x-1], q \le r)$, 则：

$$\lfloor \frac{n}{x} \rfloor = \lfloor \frac{q * x + r}{x} \rfloor = q$$

$$\lfloor \frac{n}{x+m} \rfloor = \lfloor \frac{q * x + r}{x+m} \rfloor = \lfloor \frac{q * (x+m) + (r - q * m)}{x+m} \rfloor = q + \lfloor \frac{r - q * m}{x+m} \rfloor = q, \text{当} m \le \lfloor \frac{r}{q} \rfloor \text{时成立}$$

此时$x, x+m$在同一个块$\{q\}$中, 不难知道$\forall v \in [x, x+m], v \in \{q\}$

进一步的, 假设$x+m$是$\{q\}$的边缘, 那么对于$x+m+1$来说：

$$\lfloor \frac{n}{x+m+1} \rfloor = \lfloor \frac{q * x + r}{x+m+1} \rfloor = \lfloor \frac{(q-1) * (x+m+1) + [r + x - (q-1) * (m+1)]}{x+m+1} \rfloor$$

$$= q - 1 + \lfloor \frac{r + x - (q-1) * (m+1)}{x+m+1} \rfloor = q - 1 + \lfloor \frac{(r - q * m) \ge 0 + (x + m + 1 - q) \ge 0}{x+m+1} \rfloor$$

$$= q - 1$$

故：块也是连续的

下面来确定一下每个块边界$[l, r]$的方法:

根据上面论证的对于x为起点所在的最大块为$[x, x+m]$, 此时$m = \lfloor \frac{r}{q} \rfloor, (n = q * x + r)$, 即：

$$l = x, r = x + \lfloor \frac{r}{q} \rfloor = x + \lfloor \frac{n\%x}{\lfloor \frac{n}{x} \rfloor} \rfloor$$

$$\because \lfloor \frac{a}{b} \rfloor = \frac{a - a\%b}{b}$$

$$\therefore \text{上式} = x + \frac{n\%x - n\%x\%\lfloor \frac{n}{x} \rfloor}{\lfloor \frac{n}{x} \rfloor}$$

$$= \frac{x * \lfloor \frac{n}{x} \rfloor + n\%x - (\lfloor \frac{n}{x} \rfloor * x + n\%x)\%\lfloor \frac{n}{x} \rfloor}{\lfloor \frac{n}{x} \rfloor}$$

$$= \frac{x * \frac{n - n\%x}{x} + n\%x - n\%\lfloor \frac{n}{x} \rfloor}{\lfloor \frac{n}{x} \rfloor}$$

$$= \frac{n - n\%\lfloor \frac{n}{x} \rfloor}{\lfloor \frac{n}{x} \rfloor}$$

$$= \lfloor \frac{n}{\lfloor \frac{n}{x} \rfloor} \rfloor$$

故：对于每个块$[l, r] = \left[l, \lfloor \frac{n}{\lfloor \frac{n}{l} \rfloor} \rfloor \right]$

结论

$$\sum_{i=1}^{n} f(i) * g(\lfloor \frac{n}{i} \rfloor) = \sum_{l=1, r=\lfloor \frac{n}{\lfloor \frac{n}{l} \rfloor} \rfloor}^{l \le n} g(\lfloor \frac{n}{l} \rfloor) * \left[\sum_{i=l}^{i \le r} f(i)\right], \text{预处理} \sum_{i=1}^{n} f(i)$$

$$\prod_{i=1}^{n} f(i)^{g(\lfloor \frac{n}{i} \rfloor)} = \prod_{l=1, r=\lfloor \frac{n}{\lfloor \frac{n}{l} \rfloor} \rfloor}^{l \le n} \left[\prod_{i=l}^{r} f(i)\right]^{g(\lfloor \frac{n}{l} \rfloor)}, \text{预处理} \prod_{i=1}^{n} f(i)$$

$$\prod_{i=1}^{n} g(\lfloor \frac{n}{i} \rfloor)^{f(i)} = \prod_{l=1, r=\lfloor \frac{n}{\lfloor \frac{n}{l} \rfloor} \rfloor}^{l \le n} \left[g(\lfloor \frac{n}{l} \rfloor)\right]^{\sum_{i=l}^{r} f(i)}, \text{预处理} \sum_{i=1}^{n} f(i)$$

推论

多重分块：

当式子中有$|m = \{n_i\}|$个$\lfloor \frac{n_i}{d} \rfloor$函数, 不难知道对于$min(n_i)$的分块成立时, 所有$n_i$的块也成立, 故按$min(n_i)$进行分块

$$\sum_{i=1}^{n} \left[f(i) * \prod_{j=1}^{m} g_j(\lfloor \frac{n_j}{i} \rfloor)\right] = \sum_{l=1, r=\lfloor \frac{min(n_j)}{\lfloor \frac{min(n_j)}{l} \rfloor} \rfloor}^{l \le min(n_j)} \left[\prod_{j=1}^{m} g_j(\lfloor \frac{n_j}{l} \rfloor)\right] * \left[\sum_{i=l}^{i \le r} f(i)\right]$$