Homework 7: Solutions

Problems to turn in individually

1. SGI (Soylent Green Industries) will make an annual profit of $P(x, y)$ (in millions of dollars), where x is the cost (in dollars) per pound of obtaining the raw materials to make their product and y is the average salary (in thousands of dollars) paid to employees. This year the value of x is 100 and y is 50. Next year the company expects x to go up to 103 and they are thinking of reducing y to 48.

   (a) SGI scientists have determined that, near $(x, y) = (100, 50)$,

   $$P(x, y) \approx ye^{-\frac{xy}{10,000}}.$$

   Determine two approximations of $P(103, 48)$ by using a first order Taylor polynomial approximation and then a second order Taylor polynomial approximation centered at $(x, y) = (100, 50)$. (Note, a first order approximation uses up to first derivatives (that is, no second, third, etc. derivatives). A second order approximation uses up to second derivatives.)

   ANS: For notational ease, define $E = e^{-\frac{xy}{10,000}}$, so $P(x, y) = yE$. Taking derivatives, we find that

   $$\begin{aligned}
   P_x &= -\frac{y^2}{10,000}E \\
   P_y &= \left(1 - \frac{xy}{10,000}\right)E \\
   P_{xx} &= \frac{y^3}{(10,000)^2}E \\
   P_{xy} &= \left(-\frac{y}{5000} + \frac{xy^2}{(10,000)^2}\right)E \\
   P_{yy} &= \left(-\frac{x}{5000} + \frac{x^2 y}{(10,000)^2}\right)E
   \end{aligned}$$

For the first order approximation, we have

$$\begin{aligned}
P(103,48) &\approx P(100,50) + P_x(100,50)\Delta x + P_y(100,50)\Delta y \\
&= \left(50 - \frac{1}{4} * 3 + \frac{1}{2} * (-2)\right) e^{-\frac{1}{2}} \\
&= 29.265
\end{aligned}$$

For the second order approximation, we have

$$\begin{aligned}
P(103,48) &\approx 29.265 + \frac{1}{2} P_{xx}(100,50)(\Delta x)^2 \\
&\quad + P_{xy}(100,50)\Delta x \Delta y + \frac{1}{2} P_{yy}(100,50)(\Delta y)^2 \\
&= 29.265 + \left(\frac{1}{2}(0.00125)(3)^2 \right. \\
&\quad \left. -0.005(3)(-2) - \frac{1}{2}(0.015)(-2)^2\right) e^{-\frac{1}{2}} \\
&= 29.268
\end{aligned}$$

(b) If the SGI scientists' formula ended up being valid at $(x, y) = (103, 48)$, as well as at $(x, y) = (100, 50)$, how close were the approximations in part (a) to the actual value for $P(103, 48)$?

ANS: $P(103, 48) = 48e^{-\frac{103*48}{10,000}} = 29.277$ Note that $P(100, 50) = 50e^{-\frac{1}{2}} = 30.33$ is the zeroth order approximation. The first order approximation, 29.265, is *much* better. In this case the second order approximation, 29.268, is better, but not by that much.

2. You enter into a short crude oil futures contract at \$43 per barrel. The initial margin is \$3,375 and the maintenence margin is \$2,500. One contract is for 1,000 barrels of oil. By how much do oil prices have to change before you receive a margin call?

   ANS: If the margin account falls to a value of \$2500 then a call will occur. Therefore, the loss on the position must be equal to \$3375-\$2500=\$875 for a margin call. Solving the following equation

   $$1000\,(P - 43) = 875$$

   gives $P = 43.875$, which is the price at which a margin call will take place.

3. The forward price of wheat for delivery in three months is \$3.90 per bushel, while the spot price is \$3.60. The three-month interest rate in continuously compounded terms is 8% per annum. Is there an arbitrage opportunity in this market if wheat may be stored costlessly?

   ANS: The spot price of wheat is \$3.60. Since there are no storage costs, we compute the theoretical forward price of wheat as $3.60 \exp(0.08 \times 3/12) = 3.6727$ which is less than the forward price. Hence, there is an arbitrage opportunity.

   In order to arbitrage this situation, we would undertake the following strategy:

   - Sell wheat forward at 3.90.
   - Buy wheat spot at 3.60.
   - Borrow 3.60 for three months .

   At inception, the net cash-flow is zero. At maturity, we deliver the wheat we own and receive the forward price of \$3.90. We return the borrowed amount with interest for a cash outflow of $3.60 \exp(0.08 \times 0.25) = 3.6727$.This results in a net cash inflow of 0.2273. The following table summarizes:

	Cash Flows	
Source	Initial	Final
Short Forward	-	$+3.9000$
Long spot	-3.6000	-
Borrowing	$+3.6000$	-3.6727
Net	-	$+0.2273$

   Note that it makes no difference if the contract is cash-settled instead of settled by physical delivery. If it is cash-settled, letting W_T denote the spot price of wheat at date T, we receive $3.90 - W_T$ on the forward contract, sell the spot wheat we own for W_T, and repay the borrowing, for exactly the same final cash flow.

4. A stock will pay a dividend of \$1 in one month and \$2 in four months. The risk-free rate of interest for all maturities is 12%. The current price of the stock is \$90.

(a) Calculate the arbitrage-free price of (i) a three-month forward contract on the stock and (ii) a six-month forward contract on the stock.

(b) Suppose the six-month forward contract is quoted at 100. Identify the arbitrage opportunities, if any, that exist, and explain how to exploit them.

ANS: We are given: $S = 90$; $r = 0.12$ for all maturities; and that dividends of \$1 and \$2 will be paid in one and four months, respectively.

(a) First, consider the case of a three-month horizon. There is only one dividend to be considered, viz. the payment of \$1 in one month. The present value of this dividend is

$$\exp\{-(0.12)(\frac{1}{12})\} \times 1 \;=\; 0.99.$$

Since the dividend represents a cash inflow, we have $M = -0.99$. Therefore, the arbitrage-free forward price is

$$F \;=\; (S+M)e^{rT} \;=\; (90-0.99)e^{(0.12)(0.25)} \;=\; 91.72.$$

Now, consider the six-month horizon. There are two dividend payments that occur. The present value of the first dividend is 0.99, as we have seen above. The present value of the second dividend is

$$\exp\{-(0.12)(\frac{1}{3})\} \times 2 \;=\; 1.92.$$

Therefore, the present value of the dividends combined is $0.99 + 1.92 = 2.91$. Since the dividends represent a cash inflow, we must have $M = -2.91$.

It follows that the arbitrage-free forward price for a six-month horizon is

$$F \;=\; (S+M)e^{rT} \;=\; (90-2.91)e^{(0.12)(0.50)} \;=\; 92.475.$$

(b) The six-month forward is quoted at 100, so it is *overvalued relative to spot*. To make an arbitrage profit, one should sell forward, buy spot, and borrow. Specifically:

i. At time 0: Buy one unit of spot; borrow 87.09 for repayment in six months; borrow 1.92 for repayment in four months; and borrow 0.99 for repayment in one month.
Net cash flow: 0.

ii. In one month: receive dividend of \$1; use this to repay the one-month loan.
Net cash flow: 0.

iii. In four months: receive dividend of \$2; use this to repay the four-month loan.
Net cash flow: 0.

iv. In six months: Use the unit of spot to settle the short forward position; receive 100 from the forward position; use 92.475 of this to repay the six-month loan.
Net cash flow: $100 - 92.475 = 7.525$.

5. A three-month forward contract on a non-dividend-paying asset is trading at 90, while the spot price is 84.

   (a) Calculate the implied repo rate.

   (b) Suppose it is possible for you to borrow at 8% for three months. Does this give rise to any arbitrage opportunities? Why or why not?

   ANS: The implied repo rate is

$$r = \frac{1}{T}[\ln F - \ln S] = \frac{1}{0.25}[\ln 90 - \ln 84] = 0.27957,$$

   or 27.98%. Since we can borrow at 8% for three months, there is a clear arbitrage opportunity:

   - Sell the forward at 90.
   - Borrow 84 at 8%.
   - Buy spot at 84.

   The net cash-flow at inception is zero. The net cash-flow at maturity is

$$(90 - S_T) - 84\exp(0.08 \times 3/12) + S_T = 4.3031$$

which is the difference between the repo rate and market borrowing rate on a base price of 84. To see this, note that

$$84[\exp(r \times 3/12) - \exp(0.08 \times 3/12)] = 4.3031.$$

Problems to turn in as a group

1. You have a position in 200 shares of a technology stock with an annualized standard deviation of changes in the price of the stock being 30. Say that you want to hedge this position over a one-year horizon with a technology stock index. Suppose that the index value has an annual standard deviation of 20. The correlation between the two annual changes is 0.8. How many units of the index should you hold to have the best hedge?

   ANS: In the notation of the chapter, we are given that $\sigma(\Delta_S) = 30$, $\sigma(\Delta_F) = 20$, and $\rho = 0.8$. So the minimum-variance hedge ratio is

   $$h^* \ = \ \rho \frac{\sigma(\Delta_S)}{\sigma(\Delta_F)} = 0.8(30/20) = 1.20$$

   Hence, you need to short $1.2 \times 200 = 240$ units of the index to set up the hedge.

2. You manage a portfolio of GM bonds and run a regression of your bond's price changes on the changes in the S&P 500 index futures and changes in the ten-year Treasury note futures. The regression result is as follows:

   $$\delta_P = 0.02 - 0.2\,\delta_{S\&P} + 0.5\,\delta_{TRY}, \quad R^2 = 0.7$$

   where the regression above is in changes in index values for all the right-hand side variables. What positions in the two index futures will you take? What proportion of the risk remains unhedged? What implicit assumption might you be making in this case?

   ANS: As noted in Section 5.8, these regression numbers mean that for every unit of the bond held in the portfolio, the minimum-variance hedge ratio requires a long position of 0.2 units of the S&P index and a short position in 0.5 units of the Treasury index. The percentage of the risk that is eliminated is given by the R^2 of the regression, and hence, the residual risk is 30%. In implementing a hedge in this fashion, we are making all the usual assumptions required by regression analysis. The hedge ratio effectiveness is analogous to the fit of a regression model, i.e., residual risk is 30%.

3. You are asked to hedge price changes in a security S over a maturity T. The correlations of price changes in S, and price changes in futures contracts F_1, F_2 are given by the following correlation matrix:

	S	F_1	F_2
S	1.00000	0.98757	0.82923
F_1	0.98757	1.00000	0.84939
F_2	0.82923	0.84939	1.00000

If the standard deviations of the price changes on the three assets are given by

$$\begin{aligned} \sigma(\Delta_S) &= 0.30 \\ \sigma(\Delta_{F_1}) &= 0.25 \\ \sigma(\Delta_{F_2}) &= 0.15 \end{aligned}$$

then, find the minimum-variance hedge for S using *both* futures contracts F_1 and F_2. Express your solution in terms of the number of units in the futures contracts F_1 and F_2 to hedge a 1 unit position in S. What can you say about the solution(s) you have arrived at?

ANS: When we are using a *single* futures contract to hedge, we may calculate the minimum-variance hedge ratio from knowledge of (a) the standard deviation of spot price changes, (b) the standard deviation of futures price changes, and (c) the correlation between the two. This question requires us to derive the corresponding result when we are using *two* futures contracts for hedging. The procedure is effectively the same: for any given hedge ratios h_1 and h_2, we compute the variance of the hedged portfolio and then choose h_1 and h_2 to minimize this variance. We explain the general procedure here even as we apply it to this example.

As a first step, we use the given correlation matrix and the individual standard deviations to calculate the variance-covariance matrix Σ of price changes. For notational simplicity, we write

- σ_S^2 for $\sigma^2(\Delta_S)$, etc.
- σ_{SF_1} for Covariance(Δ_S, Δ_{F_1}) = Covariance(Δ_{F_1}, Δ_S), etc.

These individual entries are easily computed. For example, σ_S is given to be 0.30, so $\sigma_S^2 = 0.09$. Similarly, we have

$$\sigma_{SF_1} \;=\; \rho_{SF_1} \times \sigma_S \times \sigma_{F_1} \;=\; 0.074068$$

Completing the matrix in this fashion, we have

$$\Sigma \;=\; \begin{bmatrix} \sigma_S^2 & \sigma_{SF_1} & \sigma_{SF_2} \\ \sigma_{SF_1} & \sigma_{F_1}^2 & \sigma_{F_1F_2} \\ \sigma_{SF_2} & \sigma_{F_1F_2} & \sigma_{F_2}^2 \end{bmatrix} \;=\; \begin{bmatrix} 0.090000 & 0.074068 & 0.037315 \\ 0.074068 & 0.062500 & 0.031852 \\ 0.037315 & 0.031852 & 0.022500 \end{bmatrix}$$

Now, let w denote our holdings of the spot asset and the futures contracts in the hedged portfolio:

$$w \;=\; \begin{bmatrix} 1 \\ h_1 \\ h_2 \end{bmatrix}$$

That is, we are long one unit of the underlying security, and h_1 and h_2 represent, respectively, the hedge ratios used for futures contracts 1 and 2, respectively. The variance of hedged portfolio cash flows, in matrix notation, is now given by

$$w'\Sigma w$$

or, in expanded notation, by

$$\sigma_S^2 + h_1^2\sigma_{F_1}^2 + h_2^2\sigma_{F_2}^2 + 2h_1\sigma_{SF_1} + 2h_2\sigma_{SF_2} + 2h_1h_2\sigma_{F_1F_2}$$

To minimize this hedged portfolio variance, we take the derivatives of this expression with respect to h_1 and h_2 and set them equal to zero. Doing so and substituting for the covariances and standard deviations results in

$$\begin{aligned} 0 &= 0.148136 + 0.125h_1 + 0.063704h_2 \\ 0 &= 0.074630 + 0.063704h_1 + 0.045h_2 \end{aligned}$$

This yields the minimum variance hedge ratios as

$$\begin{aligned} h_1^* &= -1.22025 \\ h_2^* &= 0.0689978 \end{aligned}$$

In words, if we are currently long the spot asset and are looking to hedge its price changes over the given horizon, then the optimal (minimum-variance) hedge involves

- a short position in 1.22 units in the first futures contract, and
- a long position in 0.069 units of the second futures contract.