Homework 8: Solutions

Math Finance

Preamble

In this homework, we learn to "RAIL" (Relentlessly Apply Itō's Lemma). First let us recap the multidimensional version of Itō's Lemma:

Multidimensional Itō chain rule. *Consider an m-dimensional Itō process $\underline{X}$. Let*

$$d\underline{X} = \underline{\mu}dt + \underline{\underline{\sigma}}d\underline{B},$$

with the i^{th} component given by

$$\begin{aligned} dX_i &= \mu_i dt + \sigma_{i1}dB_1 + \sigma_{i2}dB_2 + \cdots + \sigma_{in}dB_n \\ &= \mu_i dt + \sum_{j=1}^{n} \sigma_{ij}dB_j \end{aligned}$$

*where the B_j are BMs, and the processes** $\underline{\mu}$, $\underline{\underline{\sigma}}$ *satisfy appropriate conditions.*

If $f(t, X_1, X_2, \ldots, X_m)$ is a smooth function, then it is an Itō process and

$$\boxed{df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_k}dX_i\, dX_k}$$

with the usual "multiplication" rules

$$(dt)^2 = 0, \quad dt\, dB_i = 0, \quad (dB_i)^2 = dt$$

and, if the B_j are independent BMs, $dB_i\, dB_k = 0$, $i \neq k$. □

In particular, we have

Case $m = 1$. Then $f(t, X)$ satisfies

$$df = f_t dt + f_X dX + \frac{1}{2}f_{XX}(dX)^2. \tag{1}$$

Case $m = 2$. Then $f(t, X, Y)$ satisfies

$$df = f_t dt + f_X dX + f_Y dY + \frac{1}{2}f_{XX}(dX)^2 + \frac{1}{2}f_{YY}(dY)^2 + f_{XY}dX\, dY. \tag{2}$$

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*Note that μ_i and σ_{ij} need not be constants, or even deterministic. This is not GBM!

Observe that applying eq. (2) to the function $f(x, y) = xy$ gives the **Itō product rule**:

$$d(XY) = X\,dY + Y\,dX + (dX)(dY). \tag{3}$$

Problems to turn in individually

Problem 1

We are told that $X(t)$ is a GBM with drift μ and volatility σ:

$$dX = \mu X dt + \sigma X dB. \tag{4}$$

Here μ and σ are constants. We are asked to find the SDE for $\boxed{Y = X^a}$ where $a \in \mathbb{R}$. It is clear we must apply Itō's Lemma to $f(x) = x^a$ since $Y = f(X)$. Now, $f'(x) = ax^{a-1}$ and $f''(x) = a(a-1)x^{a-2}$, so

$$\begin{aligned} dY &= aX^{a-1}dX + \frac{1}{2}a(a-1)X^{a-2}(dX)^2 \\ &= aX^{a-1}(\mu X dt + \sigma X dB) + \frac{1}{2}a(a-1)X^{a-2}\sigma^2 X^2 dt \\ &= a\mu X^a dt + a\sigma X^a dB + \frac{1}{2}a(a-1)\sigma^2 X^a dt, \end{aligned}$$

where we invoked the "multiplication" rules to write $(dX)^2 = \sigma^2 X^2 (dB)^2 = \sigma^2 X^2 dt$. Therefore

$$dY = a\left[\mu + \tfrac{1}{2}(a-1)\sigma^2\right] Y dt + a\sigma Y dB \tag{5}$$

which shows that Y is also a GBM.

Application: (Note that Sreedhar gets a little ahead of the material here. This application will make more sense after we do Black-Scholes later in class.) Where might this be used? Suppose we are asked to price a contingent claim that pays

$$(S^a(T) - K)^+ = \max(S^a(T) - K,\, 0) = \begin{cases} S^a(T) - K & \text{when } S^a(T) > K, \\ 0 & \text{else.} \end{cases}$$

at time T. This is known as a *power call option*. The analysis we did in this problem would be the first step in deriving a Black-Scholes type expression, $C(S(t), t)$, for the time-t (arbitrage-free) price of the power call option.

Problem 2

We are asked to find the SDE for fg using the Itō product rule (see eq. (3))

$$d(fg) = f\,dg + g\,df + (df)(dg). \tag{6}$$

(a) Applying Itō's Lemma to $f(t, B)$ and $g(t, B)$ we get

$$\begin{aligned} df &= f_t dt + f_B dB + \frac{1}{2} f_{BB}(dB)^2 = \big[f_t + \frac{1}{2} f_{BB}\big] dt + f_B dB \\ dg &= g_t dt + g_B dB + \frac{1}{2} g_{BB}(dB)^2 = \big[g_t + \frac{1}{2} g_{BB}\big] dt + g_B dB. \end{aligned}$$

Using the "multiplication" rules we get $df\,dg = f_B g_B (dB)^2 = f_B g_B dt$, and therefore eq. (6) becomes

$$d(fg) = \big[f g_t + \frac{1}{2} f g_{BB} + g f_t + \frac{1}{2} g f_{BB} + f_B g_B\big] dt + \big[f g_B + g f_B\big] dB.$$

(b) Applying Itō's Lemma to $f(t, B_1)$ and $g(t, B_2)$ we get

$$\begin{aligned} df &= f_t dt + f_{B_1} dB_1 + \frac{1}{2} f_{B_1 B_1}(dB_1)^2 = \big[f_t + \frac{1}{2} f_{B_1 B_1}\big] dt + f_{B_1} dB_1 \\ dg &= g_t dt + g_{B_2} dB_2 + \frac{1}{2} g_{B_2 B_2}(dB_2)^2 = \big[g_t + \frac{1}{2} g_{B_2 B_2}\big] dt + g_{B_2} dB_2. \end{aligned}$$

Since B_1 and B_2 are **independent** BM, $df\,dg = f_{B_1} g_{B_2} dB_1 dB_2 = 0$, and eq. (6) becomes

$$d(fg) = \big[f g_t + \frac{1}{2} f g_{B_2 B_2} + g f_t + \frac{1}{2} g f_{B_1 B_1}\big] dt + g f_{B_1} dB_1 + f g_{B_2} dB_2.$$

Thus, in this case (independent BM), eq. (6) reduces to the usual product rule of calculus, viz., $d(fg) = f\,dg + g\,df$.

Problems to turn in as a group

Problem 1

For each of $Z = e^{B^2}$, and $Z = (B + t)e^{-(B+\frac{t}{2})}$, we are asked to find

$$dZ = \underline{\qquad}\, dt + \underline{\qquad}\, dB$$

via two different methods.[‡]

(a) First method: Let $X = B^2$. If we consider the function g defined by $g(y) = y^2$, we have $X = g(B)$. Since $g'(y) = 2y$ and $g''(y) = 2$, Itō says

$$\begin{aligned} dX &= g'(B)dB + \frac{1}{2}g''(B)(dB)^2 = 2B\,dB + \frac{1}{\cancel{2}}\cancel{2}\,(dB)^2 = 2B\,dB + dt \\ (dX)^2 &= (2B\,dB + dt)^2 = 4B^2(dB)^2 = 4B^2dt. \end{aligned}$$

Similarly, if we consider the function $f(x) = e^x$, then $f'(x) = f''(x) = e^x$, and Itō's Lemma applied to $Z = f(X) = e^X$ gives

$$\begin{aligned} dZ &= Z_X\,dX + \frac{1}{2}Z_{XX}(dX)^2 \\ &= e^X(2B\,dB + dt) + \frac{1}{2}e^X(4B^2dt) \\ &= e^{B^2}(1 + 2B^2)dt + e^{B^2}2B\,dB. \end{aligned}$$

Second method: This is straightforward. We can let $Y = B$ and apply Itō's Lemma to $Z = f(Y) = f(B)$, where the function f is defined by $f(y) = e^{y^2}$. Then

$$\begin{aligned} dZ &= Z_B\,dB + \frac{1}{2}Z_{BB}(dB)^2 \\ &= 2Be^{B^2}dB + \frac{1}{\cancel{2}}\cancel{2}(e^{B^2} + 2B^2e^{B^2})dt \\ &= e^{B^2}(1 + 2B^2)dt + 2Be^{B^2}dB. \end{aligned}$$

(b) First method: This is straightforward. Let $X = B$, then[§] $dX = dB$. Hence

$$(dX)^2 = (dB)^2 = dt.$$

Since $Z = (X + t)e^{-\left(X+\frac{t}{2}\right)}$ we have, by Itō's Lemma,

$$\begin{aligned} dZ &= Z_t\,dt + Z_X\,dX + \frac{1}{2}Z_{XX}(dX)^2 \\ &= \left[\cancel{e^{-\left(X+\frac{t}{2}\right)}} - \frac{1}{2}\cancel{(X+t)}e^{-\left(X+\frac{t}{2}\right)}\right]dt + \left[e^{-\left(X+\frac{t}{2}\right)} - (X+t)e^{-\left(X+\frac{t}{2}\right)}\right]dB \\ &\qquad + \frac{1}{2}\left[-\cancel{e^{-\left(X+\frac{t}{2}\right)}} - \cancel{e^{-\left(X+\frac{t}{2}\right)}} + \cancel{(X+t)}e^{-\left(X+\frac{t}{2}\right)}\right]dt \\ &= e^{-\left(B+\frac{t}{2}\right)}\left[1 - (B+t)\right]dB. \end{aligned}$$

[‡] We should double-check our answer through an alternative method whenever possible. Besides catching silly mistakes this will deepen our understanding of the subject.

[§] Convince yourself using Itō's Lemma.

Second method: Let $Z = XY$, where $X = B + t$, and $Y = e^{-\left(B+\frac{t}{2}\right)}$. First we find dX and dY by Itō's Lemma:

$$\begin{aligned} dX &= X_t\,dt + X_B\,dB + \frac{1}{2}\cancel{X_{BB}(dB)^2} \\ &= dt + dB, \end{aligned} \qquad \text{and} \qquad \begin{aligned} dY &= Y_t\,dt + Y_B\,dB + \frac{1}{2}Y_{BB}(dB)^2 \\ &= \cancel{-\tfrac{1}{2}Y\,dt} - Y\,dB + \cancel{\tfrac{1}{2}Y\,dt}. \end{aligned}$$

Then, by the Itō product rule,

$$\begin{aligned} dZ &= X\,dY + Y\,dX + dX\,dY \\ &= (B+t)(-Y\,dB) + Y(dt + dB) + (dt + dB)(-Y\,dB) \\ &= -(B+t)Y\,dB + \cancel{Y\,dt} + Y\,dB - \cancel{Y(dB)^2} \\ &= \big[-(B+t) + 1\big]Y\,dB \\ &= \big[1 - (B+t)\big]e^{-\left(B+\frac{t}{2}\right)}dB. \end{aligned}$$

Problem 2

We are asked to find the SDE for $d(\ln S_i)$ given that

$$\frac{dS_i}{S_i} = \mu_i\,dt + \sum_{j=1}^{n} \sigma_{ij}\,dB_j.$$

By Itō's Lemma*,

$$\begin{aligned} d(\ln S_i) &= \frac{1}{S_i}dS_i + \frac{1}{2}\left(-\frac{1}{S_i^2}\right)(dS_i)^2 = \frac{dS_i}{S_i} - \frac{1}{2}\left(\frac{dS_i}{S_i}\right)^2 \\ &= \left(\mu_i dt + \textstyle\sum_{j=1}^n \sigma_{ij} dB_j\right) - \frac{1}{2}\left(\mu_i dt + \textstyle\sum_{j=1}^n \sigma_{ij} dB_j\right)\left(\mu_i dt + \textstyle\sum_{k=1}^n \sigma_{ik} dB_k\right) \\ &= \left(\mu_i dt + \textstyle\sum_{j=1}^n \sigma_{ij} dB_j\right) - \frac{1}{2}\textstyle\sum_{j=1}^n \sum_{k=1}^n \sigma_{ij}\sigma_{ik} I_{jk} dt \\ &= \mu_i dt + \sum_{j=1}^n \sigma_{ij} dB_j - \frac{1}{2}\sum_{j=1}^n \sigma_{ij}\sigma_{ij} dt \\ &= \left(\mu_i - \frac{1}{2}\sum_{j=1}^n \sigma_{ij}^2\right)dt + \sum_{j=1}^n \sigma_{ij} dB_j. \end{aligned}$$

*What else?! ☺

Problem 3

This is straightforward but involves some algebra. We must use eq. (6), so let us first determine df and dg. By Itō's Lemma,

$$\begin{aligned} df &= f_t dt + f_Y dY + \frac{1}{2} f_{YY} (dY)^2 \\ dg &= g_t dt + g_Z dZ + \frac{1}{2} g_{ZZ} (dZ)^2. \end{aligned} \tag{7}$$

We are given

$$\begin{aligned} dY &= \alpha_{11} dB_1 + \alpha_{12} dB_2 \\ dZ &= \alpha_{21} dB_1 + \alpha_{22} dB_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} (dY)^2 &= \alpha_{11}^2 (dB_1)^2 + \alpha_{12}^2 (dB_2)^2 = \big(\alpha_{11}^2 + \alpha_{12}^2\big) dt \\ (dZ)^2 &= \alpha_{21}^2 (dB_1)^2 + \alpha_{22}^2 (dB_2)^2 = \big(\alpha_{21}^2 + \alpha_{22}^2\big) dt. \end{aligned}$$

Upon collecting terms, eq. (7) becomes

$$\begin{aligned} df &= \big[f_t + \tfrac{1}{2} f_{YY} \big(\alpha_{11}^2 + \alpha_{12}^2\big)\big] dt + f_Y \alpha_{11} dB_1 + f_Y \alpha_{12} dB_2 \\ dg &= \big[g_t + \tfrac{1}{2} g_{ZZ} \big(\alpha_{21}^2 + \alpha_{22}^2\big)\big] dt + g_Z \alpha_{21} dB_1 + g_Z \alpha_{22} dB_2 \end{aligned} \tag{8}$$

which means that

$$(df)(dg) = f_Y \alpha_{11} g_Z \alpha_{21} (dB_1)^2 + f_Y \alpha_{12} g_Z \alpha_{22} (dB_2)^2 = f_Y g_Z (\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22}) dt. \tag{9}$$

Substituting (8) and (9) in eq. (6) and collecting terms, we get

$$\begin{aligned} d(fg) &= f\, dg + g\, df + df\, dg \\ &= \Big\{ f\big[g_t + \tfrac{1}{2} g_{ZZ} \big(\alpha_{21}^2 + \alpha_{22}^2\big)\big] + g\big[f_t + \tfrac{1}{2} f_{YY} \big(\alpha_{11}^2 + \alpha_{12}^2\big)\big] + f_Y g_Z (\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22}) \Big\} dt \\ &\quad + \big[f g_Z \alpha_{21} + g f_Y \alpha_{11}\big] dB_1 + \big[f g_Z \alpha_{22} + g f_Y \alpha_{12}\big] dB_2. \end{aligned}$$