# Homework 6: Solutions

(Note: This the first of a number of homework solutions that were prepared, in large part, by Sreedhar Jayaraman, who worked tirelessly to help the students in this course the first time we taught it in 2008. You'll note the tremendous detail he incorporates into these solutions, which are characterized by fancier fonts, problem headers, etc. This is a shout out to the tremendous work he has done from which you now benefit!)

### Problems to turn in individually

## *Problem 1*

Markowitz problem: Straightforward. Following the derivation done in class, we get the same optimal portfolio weights $\underline{w}$. Our intuition says that this must indeed be the case—the function to be minimized has merely been multiplied by a constant factor of 2, therefore the minimum would be achieved at the same point $\underline{w}$.

Note that when we solve for the optimal $\underline{w}$ in terms of the Lagrange multipliers ($\lambda_1$ and $\lambda_2$), there is an extra factor of 0.5 in front. However, the Lagrange multipliers here are two times those found in class; therefore we end up with the same optimal $\underline{w}$.

To be precise, the optimal $\underline{w}$ is given by

$$2\underline{\underline{\Sigma}}\,\underline{w} = \lambda_1\underline{\mu} + \lambda_2\underline{1} \qquad \Longrightarrow \qquad \underline{w} = 0.5\underline{\underline{\Sigma}}^{-1}(\lambda_1\underline{\mu} + \lambda_2\underline{1})$$

where $\lambda_1$ and $\lambda_2$ satisfy the simultaneous linear equations

$$\begin{aligned} B\lambda_1 + A\lambda_2 &= 2E_\Pi \\ A\lambda_1 + C\lambda_2 &= 2 \end{aligned} \qquad \Longrightarrow \qquad \begin{bmatrix} B & A \\ A & C \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2E_\Pi \\ 2 \end{bmatrix}. \tag{1}$$

The constants $B, A$ and $C$ are the same as in class. Expressed in vector notation, eq. (1) makes it clear why $\lambda_1$ and $\lambda_2$ are twice those found in class. The right hand side vector $\left[\begin{smallmatrix} 2E_\Pi \\ 2 \end{smallmatrix}\right]$ is twice as big here; therefore, by linearity, so is the solution vector $\left[\begin{smallmatrix} \lambda_1 \\ \lambda_2 \end{smallmatrix}\right]$.

### Problems to turn in as a group

# *Problem 1*

$$\min_{\underline{w}} \quad \tilde{L} := \tfrac{1}{2}\underline{w}'\underline{\underline{\Sigma}}\,\underline{w} + \lambda\left[E(r_p) - \underline{w}'\underline{\mu} - (1 - \underline{w}'\underline{1})r_f\right]$$

From class, we know that the *first order necessary conditions* (FONC) are

$$\begin{aligned}
\frac{\partial \tilde{L}}{\partial \underline{w}} &= \underline{\underline{\Sigma}}\,\underline{w} - \lambda\underline{\mu} + \lambda\,\underline{1}\,r_f \qquad \text{(since } \underline{\underline{\Sigma}} \text{ is symmetric)} \\
&= \underline{0}, \\
\text{and} \quad \frac{\partial \tilde{L}}{\partial \lambda} &= E(r_p) - \underline{w}'\underline{\mu} - (1 - \underline{w}'\underline{1})\,r_f \\
&= 0.
\end{aligned}$$

Upon rearranging, the above equations become

$$\underline{\underline{\Sigma}}\,\underline{w} = \lambda(\underline{\mu} - \underline{1}r_f) \tag{2}$$

$$E(r_p) - r_f = \underline{w}'(\underline{\mu} - \underline{1}r_f). \tag{3}$$

Note that (2) is a vector equation while (3) is a scalar equation. (In this problem there is only one Lagrange multiplier.) At this point, one might be tempted to argue as follows:

> We set out to find $\underline{w}$. Equation (3) has “just” $\underline{w}$ as the unknown, so let us solve it and get the answer.

This won’t work because $\underline{w}$ is a vector (with $n$ components, which are the unknowns). We cannot find $n$ unknowns (uniquely) using a single equation. In fact, eq. (3) is merely the original constraint in a rearranged form! Had we been able to find the optimal $\underline{w}$ from the original constraint itself, we wouldn’t have had to define the Lagrange multiplier $\lambda$, etc.

The correct steps are

(a) Use eq. (2) to express the optimal $\underline{w}$ in terms of $\lambda$. This means multiplying both sides of eq. (2) by $\underline{\underline{\Sigma}}^{-1}$, which is possible since $\underline{\underline{\Sigma}}$ is invertible.

Now we are almost done, except that we don’t know $\lambda$. (Recall that $\lambda$ is a scalar in this problem.)

The optimal $\underline{w}$ must satisfy these conditions.

(b) Substitute in eq. (3) and solve for $\lambda$. One equation, one (scalar) unknown.

(c) Now that we know $\lambda$, substitute it in the expression for $\underline{w}$ obtained in step (a) to arrive at the final answer.

Following the above steps diligently, we get

$$\underline{w} = \frac{E(r_p) - r_f}{H} \underline{\underline{\Sigma}}^{-1}(\underline{\mu} - \underline{1} r_f)$$

where

$$H := (\underline{\mu} - r_f \underline{1})' \underline{\underline{\Sigma}}^{-1} (\underline{\mu} - r_f \underline{1}).$$