Homework 9: Solutions

Problems to turn in individually

1. Microsoft is currently trading at \$26. You expect that prices will increase but not rise above \$28 per share. Options on Microsoft with strikes of \$22.50, \$25, \$27.50 and \$30 are available. What options portfolio would you construct from these options to incorporate your views?

   ANS: You create a bullish vertical spread by selling the call at a strike of 27.50 and buying the call at a strike of 25. This way you do not pay for the appreciation above \$30. Different combinations may be tried as well, for example, the sale of the 30 strike call might be a more expensive approach but then you would pay for the range from 28 to 30, which you do not believe will occur. But it builds in some error in the upper bound of the view on the upside.

2. There are call and put options on a stock with strike 40, 50, 55. Which of the following inequalities must hold?

   (a) $0.5C(40) + 0.5C(55) > C(50)$
   (b) $(1/3)C(40) + (2/3)C(55) > C(50)$
   (c) $(2/3)C(40) + (1/3)C(55) > C(50)$
   (d) $0.5P(40) + 0.5P(55) > P(50)$
   (e) $(1/3)P(40) + (2/3)P(55) > P(50)$
   (f) $(2/3)P(40) + (1/3)P(55) > P(50)$

   ANS: This question makes use of the general form of convexity in the strike for options:

   $$wC(K_1) + (1-w)C(K_3) \geq C(K_2)$$

   and

   $$wP(K_1) + (1-w)P(K_3) \geq P(K_2)$$

where

$$w = \frac{K_3 - K_2}{K_3 - K_1}$$

(a) Because options are convex in their strike prices, we know that

$$0.5C(40) + 0.5C(55) > C(47.5) > C(50).$$

Hence this inequality is valid.

(b) $(1/3)C(40)+(2/3)C(55) > C(1/3\times 40+2/3\times 55) = C(50)$. Therefore, this also holds.

(c) Compared to (b) above, since this expression overweights the more expensive option, the LHS is even more valuable than in (b); thus, the expression is valid.

(d) This is exactly the same as (a); put options are also convex in their strike prices, and hence this may be approached in the same way, i.e.

$$0.5P(40) + 0.5P(55) > P(47.5) < P(50)$$

Therefore, this may not be valid.

(e) From convexity, we see that $(1/3)P(40) + (2/3)P(55) > P(50)$, hence valid.

(f) This may not be valid as it overweights the cheaper option compared to (e) above.

$$(2/3)P(40) + (1/3)P(55) > P(45)$$

but this may not be greater than $P(50)$.

3. The current price of ABC stock is \$50. The term structure of interest rates (continuously compounded) is flat at 10%. What is the 6-month forward price of the stock? Denote this F. The 6-month call price at strike F is equal to \$8. The 6-month put price at strike F is equal to \$7. Construct a risk free arbitrage given these prices.

   ANS: First, we compute the forward price, which is

$$F = Se^{rT} = 50e^{0.10\times 0.5} = 52.564.$$

   We now demonstrate the arbitrage as follows:

Set up the following portfolio: Short the call and go long the put option. This generates a cashflow of \$1. Buy the stock and borrow the required money at 10%.

At maturity, two scenarios are considered:

(a) $S_T < F$: here the call expires worthless, but the put pays us $52.564 - S_T$. Also, we sell the stock and get cashflow S_T, and repay the borrowing with interest i.e. -52.564. Hence, the net cashflow is zero.

(b) $S_T > F$: In this case, we pay out $-(S_T - 52.564)$ on the call, sell the stock and get cashflow S_T, and repay the borrowing with interest i.e. -52.564. Hence, the net cashflow is zero.

Thus, at maturity, the cashflow is always zero, but we pick up a free \$1 at inception, resulting in an arbitrage.

4. Using internet or library search, find out what is meant by a "butterfly spread". What is your trading view if you sell a butterfly spread?

   ANS: A butterfly spread is

$$C(K_1) - 2C(K_2) + C(K_3), \quad K_! < K_2 < K_3$$

   The butterfly spread pays off for the buyer when the stock at maturity is in the range (K_1, K_3). You sell it when you expect the stock volatility to be high and the stock will trade well out of its current range, so that you will keep the premium but not have to pay anything to the option buyer.

Problems to turn in as a group

1. Recall from class that the solution to the PDE

$$f_t = \frac{1}{2} f_{xx} \,, \qquad f(x,0) = f_0(x) \tag{1}$$

   is

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f_0(y) e^{-\frac{(x-y)^2}{2t}} dy. \tag{2}$$

(a) Determine the partial of equation (2) with respect to t, and the second partial of equation (2) with respect to x. Show that these satisfy $f_t = \frac{1}{2} f_{xx}$.

(b) Let's say we have a financial instrument worth $X(t)$ millions of dollars. Let's say that, at $t = 0$, X has a uniform probability distribution between 0 and 1 million dollars and $X(t)$ evolves in time by Brownian motion.

   i. Use R to approximate the solution to equation (1) and, from your results, plot $f_{X(1)}$, the density function for X after one year. You will solve the PDE over the region $-5 \leq x \leq 5$ and $0 \leq t \leq 1$. Use $h = \frac{1}{252}$ and the stability condition $k = 2\sqrt{h}$. Make sure to check that your solution stays near zero near $x = \pm 5$ as is required at all times for this method to be valid.

   ii. Use equation (2) and R's "pnorm" function to determine $f(2,1)$. Compare this with your PDE approximation for $f(2,1)$. As you make h smaller, you should see your PDE approximation for $f(2,1)$ get closer and closer to the correct solution for $f(2,1)$ determined from your "pnorm" method. Use five or so values of h to show that this is the case.

ANS:

(a) Show $f_t = \frac{1}{2} f_{xx}$:

$$\begin{aligned}
f(x,t) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f_0(y) e^{\frac{-(x-y)^2}{2t}} dy \\
f_t &= -\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{t^{\frac{3}{2}}} \int_{-\infty}^{\infty} f_0(y) e^{\frac{-(x-y)^2}{2t}} dy \\
&\quad + \frac{1}{\sqrt{2\pi}}\frac{1}{t^{\frac{1}{2}}} \int_{-\infty}^{\infty} f_0(y) e^{\frac{-(x-y)^2}{2t}} \frac{(x-y)^2}{2t^2} dy \\
f_x &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f_0(y) e^{\frac{-(x-y)^2}{2t}} \left(\frac{-(x-y)}{t}\right) dy \\
f_{xx} &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f_0(y) e^{\frac{-(x-y)^2}{2t}} \left(\frac{(x-y)^2}{t^2} - \frac{1}{t}\right) dy \\
\text{so } f_t &= \frac{1}{2} f_{xx}
\end{aligned}$$

(b) Since $\frac{h}{k^2} = \frac{1}{4}$,

$$h = \frac{1}{252}, \quad k = 2\sqrt{h} = .126 \Rightarrow \frac{5}{k} = 39.68, \text{ round up to } 40$$

i.

$$a = \frac{1}{2}\frac{h}{k^2} = \frac{1}{8}, \quad b = 1 - \frac{h}{k^2} = \frac{3}{4}, \quad c = \frac{1}{2}\frac{h}{k^2} = \frac{1}{8}$$

$$-5 \leq x \leq 5 \Rightarrow \text{use } f^n = \begin{bmatrix} f^n_{40} \\ f^n_{39} \\ . \\ . \\ f^n_0 \\ . \\ . \\ f^n_{-40} \end{bmatrix} \quad \text{so } f^n_i \approx (ik, nh)$$

Since the initial condition equals 1 for x between 0 and 1 and equals 0 for all other x, we have that

$$f^0_i = \begin{cases} 1 & \text{if } 0 \leq i \leq 7 \\ 0 & \text{otherwise} \end{cases}, \qquad f^0 = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \\ . \\ . \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \longleftarrow f^0_{40} \\ \\ \\ \\ \longleftarrow f^0_7 \\ \\ \\ \longleftarrow f^0_0 \\ \\ \\ \\ \\ \longleftarrow f^0_{-40} \end{array}$$

So let $Q = \begin{bmatrix} \frac{3}{4} & \frac{1}{8} & & & & & & \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & & & & & \\ & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ & & & & & & & \frac{1}{8} & \frac{3}{4} \end{bmatrix}$ be an 81×81 matrix.

We approximate f at time 1 by f^{252} and so compute

$$f^{\overbrace{252}^{\text{Superscript}}} = Q^{\overbrace{252}^{\text{Power}}} f^0$$

ii.

$$\begin{aligned} f(2,1) &= \frac{1}{\sqrt{2\pi \cdot 1}} \int_0^1 e^{\frac{-(y-2)^2}{2}} dy \\ \text{Let } z &= (y-2), \text{ then } dz = dy, \text{ so} \\ f(2,1) &= \frac{1}{\sqrt{2\pi}} \int_{-2}^{-1} e^{-\frac{z^2}{2}} dz \\ &= \mathbf{pnorm}(-1) - \mathbf{pnorm}(-2) \\ &= \boxed{.136} \end{aligned}$$

Your approximations will vary depending on h.