Homework 8

Problems to turn in individually

1. Suppose that X satisfies the stochastic differential equation (SDE)

$$dX = \mu X dt + \sigma X dB.$$

Clearly X is governed by geometric Brownian motion. Now define $Y = X^a$, where a is a constant. In this problem, you are to establish that Y is also governed by geometric Brownian motion. To see this, compute the SDE for dY, which means express dY in the form of a term (called the drift term) times dt plus another term (called the diffusion term) times dB. That is,

$$dY = (\text{drift term})\, dt + (\text{diffusion term})\, dB.$$

To show that Y is a geometric Brownian motion, you just need to show that both the drift and diffusion terms are of the form of a constant times Y. (For X, clearly the constant in the drift term is μ and the constant for the diffusion term is σ.)

2. *Ito Product Rule.* Just as there's an Ito chain rule, there's also an Ito product rule. Recall the derivation of the product rule from your Calculus class; you first show that

$$d(fg) = f\, dg + g\, df + (df)(dg),$$

and then you quickly realize that the $(df)(dg)$ term is negligible so you forget about its existence. But it's no longer negligible here, so we have to keep it in! Then we can use the Taylor series expressions from class for df and dg to express $d(fg)$ as an SDE.

   (a) Let's say that both f and g are functions of t and the Brownian motion $B(t)$. Determine the SDE for $d(fg)$.

   (b) Let's say that f is a function of t and $B_1(t)$, while g is a function of t and $B_2(t)$, where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Determine the SDE for $d(fg)$. Note that your SDE will have two diffusion terms corresponding to the two Brownian motions.

Problems to turn in as a group

1. In part (a) and part (b) of this problem, find the SDE for dZ via each of two given methods.

   (a) $Z = e^{(B(t))^2}$

      i. First method: Let $X = \big(B(t)\big)^2$ and apply the Ito chain rule to $f(x) = e^x$.

      ii. Second method: Let $Y = B(t)$ and apply the Ito chain rule to $f(y) = e^{y^2}$.

   (b) $Z = \big(B(t) + t\big)e^{-\left(B(t)+\frac{t}{2}\right)}$

      i. First method: Let $X = B(t)$ and apply the Ito chain rule to $f(x,t) = (x+t)e^{-\left(x+\frac{t}{2}\right)}$.

      ii. Second method: Let $X = \big(B(t) + t\big)$ and $Y = e^{-\left(B(t)+\frac{t}{2}\right)}$ and use the Ito product rule to $f(x,y) = xy$.

2. Consider the price S_i of stock i which is chosen from a collection of n stocks. Derive the SDE given in class (which now has n diffusion terms) for the differential change in the natural logarithm of S_i:

$$d\big(\ln(S_i)\big) = \left(\mu_i - \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}\sigma_{ji}^T\right) dt + \sum_{j=1}^{n} \sigma_{ij}\, dB_j.$$

   To do this, use the Ito calculus methods shown in class, combined with the multidimensional geometric Brownian motion formula

$$\frac{dS_i}{S_i} = \mu_i\, dt + \sum_{j=1}^{n} \sigma_{ij}\, dB_j,$$

   introduced a few classes ago. Note that $\big(\sum_{j=1}^{n} \sigma_{ij}\, dB_j\big)^2$ is a function solely of i, and is best written out in the form

$$\Big(\sum_{j=1}^{n} \sigma_{ij}\, dB_j\Big)\Big(\sum_{k=1}^{n} \sigma_{ik}\, dB_k\Big),$$

   where the summation variables j and k are made different so that the two sums may be combined without confusion.

3. Let's say that f is a function of t and $Y(t)$, while g is a function of t and $Z(t)$, where

$$dY(t) = \alpha_{11}\, dB_1(t) + \alpha_{12}\, dB_2(t),$$
$$dZ(t) = \alpha_{21}\, dB_1(t) + \alpha_{22}\, dB_2(t).$$

If the four α's are known constants, and $B_1(t)$ and $B_2(t)$ are independent Brownian motions, determine the SDE for $d(fg)$.