Homework 3: Solutions

Problems to turn in individually

1. Later in class (when we begin to learn Ito Calculus) we're going to need $Var\left[(\Delta B)^2\right]$, where ΔB is the change in a Brownian motion from t to $t+\Delta t$. In this problem, you will compute this by using integration by parts.

   Note that because we're talking about a finite number Δt, as opposed to an infinitesimally small dt, we have from class that $\Delta B = \sqrt{\Delta t}Z$. So, $E\left[\Delta B\right] = E\left[\sqrt{\Delta t}Z\right] = \sqrt{\Delta t}E\left[Z\right] = 0$. Since we know from class that $Var[\Delta B] = \Delta t$, and we know that $Var\left[\Delta B\right] = E\left[(\Delta B - E[\Delta B])^2\right] = E\left[(\Delta B)^2\right] = E\left[(\Delta t)\, Z^2\right] = \Delta t E\left[Z^2\right]$, we see that $E\left[Z^2\right] = 1$. This can also be shown via integration by parts:

   (a) $E\left[Z^2\right] = \int_{-\infty}^{\infty} z^2 f_Z(z)dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}}dz$. Apply integration by parts to this integral using $u = z$ and $dv = ze^{-\frac{z^2}{2}}dz$. Now use that $E[1] = \int_{-\infty}^{\infty} f_Z(z)dz = 1$ to finish showing that $E\left[Z^2\right] = 1$.
   ANS:

$$
\begin{aligned}
E[1] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}dz = 1 \\
E[Z^2] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\underbrace{z}}_{u}\,\underbrace{\underbrace{ze^{\frac{-z^2}{2}}}}_{dv}dz \\
&\qquad\qquad du = dz \quad v = -e^{\frac{-z^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}\left(\underbrace{-ze^{\frac{-z^2}{2}}\Big|_{-\infty}^{\infty}}_{0} + \underbrace{\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}dz}_{1}\right) = \boxed{1}
\end{aligned}
$$

   The choice of $u = z$ and $dv = ze^{\frac{-z^2}{2}}$ allows us to integrate dv.
   The fact that the two terms "evaluated at $\pm\infty$" equal 0 can be seen using L'Hôpital's Rule.

(b) Now determine $E\left[Z^4\right]$. You'll want to apply integration by parts using $u = z^3$ and $dv = ze^{-\frac{z^2}{2}}dz$ and then use your previous result for $E\left[Z^2\right]$.

ANS:

$$
\begin{aligned}
E[Z^4] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\underbrace{z^3}_{u}}_{du = 3z^2} \underbrace{\underbrace{ze^{\frac{-z^2}{2}}}_{dv}}_{v = -e^{\frac{-z^2}{2}}} dz \\
&= \frac{1}{\sqrt{2\pi}} \left(\underbrace{-z^3 e^{\frac{-z^2}{2}} \Big|_{-\infty}^{\infty}}_{0} + 3 \int_{-\infty}^{\infty} z^2 e^{\frac{-z^2}{2}} dz \right) = 3E[Z^2] = \boxed{3}
\end{aligned}
$$

(c) Finally, you're ready to get what we'll need later! Determine $Var\left[(\Delta B)^2\right]$ using your results for $E\left[Z^4\right]$, $E\left[Z^2\right]$ and $E[1]$, and also noting from above that $E\left[(\Delta B)^2\right] = \Delta t$.

ANS:

$$
\begin{aligned}
\Delta B &= \sqrt{\Delta t} * Z \\
Var\left[(\Delta B)^2\right] &= E\left[\left((\Delta B)^2 - E[(\Delta B)^2]\right)^2\right] \\
&= E\left[\left((\Delta t Z^2 - E[\Delta t Z^2]\right)^2\right] \\
&= (\Delta t)^2 E\left[\left(Z^2 - E[Z^2]\right)^2\right] \\
&= (\Delta t)^2 E\left[\left(Z^2 - 1\right)^2\right] \\
&= (\Delta t)^2 E\left[Z^4 - 2Z^2 + 1\right] \\
& \quad (\Delta t)^2 E\left[3 - 2*1 + 1\right] \\
&= \boxed{2(\Delta t)^2}
\end{aligned}
$$

2. How good a model is geometric Brownian motion?

The geometric Brownian motion (GBM) model

$$dS = \mu S dt + \sigma S dB$$

works quite well, especially given its simplicity, but it does have weak points that can be improved:

(a) GBM underestimates the tails of observed stock price distributions. We saw in class that $B(t)$ has a normal distribution. In reality the distribution of stocks have bigger tails. That is large gains and large losses happen more often than GBM predicts. And investors can get particularly upset when large losses happen more often than predicted! There are a couple of ways to compensate for this. One, which we will cover at the end of this course (time permitting), is to add Poisson distributed jumps to GBM. This has the advantage of also being able to add observed skew into the price distribution as well. A second way to compensate for the small tails is to use a probability distribution that has larger tails than the standard normal distribution. What distribution do you know from statistics that could be used because it looks like a standard normal distribution but with larger tails?

ANS: t-distributions (and the degrees of freedom can be reduced to make the tails bigger).

(b) GBM is only affected by the current state of the stock, not the past behavior. Therefore, it cannot take into account reversion to the mean, which we discussed on the first day of class. Here's a model for the stock price $S = S(t)$ that does allow for reversion to the mean:

$$dS = \left[\mu \langle \pm_1 \rangle \eta \left(S - S(0)e^{\mu t}\right)\right] S dt \langle \pm_2 \rangle \sigma S dB.$$

Here η is a positive constant, $S(0)$ is the initial stock price, and $\langle \pm_i \rangle$ may mean a plus sign or a minus sign. For any reasonable model:

i. Determine if the symbol $\langle \pm_1 \rangle$ should be a plus sign, a minus sign, or it doesn't matter.
ANS: A minus sign since $S > S(0)e^{\mu t}$ means the stock is outperforming its mean. Therefore, there should be a progressively negative push to get S back to its mean value, $S(0)e^{\mu t}$.

ii. Determine if the symbol $\langle \pm_2 \rangle$ should be a plus sign, a minus sign, or it doesn't matter.

ANS: It doesn't matter, since dB has a symmetrical distribution centered at 0. That is, the distribution for dB and $-dB$ are identical.

(c) The assumption that the volatility is constant is highly questionable. For example during the great recession and most other times when we've had a bear market, the volatility of stocks has increased. To account for the fact that volatility is not constant, we have so called "vol-vol" models to account for the volatility of σ. One of the best known vol-vol models is the Heston model from 1993. This model uses GBM

$$dS = \mu S dt + \sigma(t) S dB,$$

where the volatility, $\sigma(t)$, is given by

$$d\sigma(t) = -\kappa(\sigma(t) - \bar{\sigma})dt + \alpha\sqrt{\sigma(t)}d\tilde{B},$$

where α and κ are positive constants, $\bar{\sigma}$ is a constant that is approximately equal to the average volatility, and $\tilde{B}(t)$ is a different Brownian motion than $B(t)$. Are $B(t)$ and $\tilde{B}(t)$ positively correlated, negatively correlated, or not correlated?

ANS: Negatively correlated because when $B(t)$ is more negative (bear market), we know that volatility tends to be higher, meaning $\tilde{B}(t)$ is more positive.

Problems to turn in as a group

1. Recall from class that $B(t)$ is a Brownian motion if it satisfies the following four properties:

1 $B(0) = 0$

2 $B(t)$ is continuous.

3 If $0 \leq t_1 < t_2 \leq t_3 < t_4$, then $[B(t_4) - B(t_3)]$ is independent of $[B(t_2) - B(t_1)]$.

4 If $0 \leq t_1 < t_2$, then $[B(t_2) - B(t_1)]$ has mean zero and variance $(t_2 - t_1)$.

Given this, if $B(t)$ is a Brownian motion, determine whether or not

(a) $F(t) = B(t+s) - B(s)$ is a Brownian motion, where $s > 0$ is a fixed number.

ANS: Yes, $F(t)$ is a Brownian motion.

1 $F(0) = 0$ because $F(0) = B(s) - B(s) = 0$.

2 $F(t)$ is continuous because a continuous function, $B(t+s)$, minus a constant, $B(s)$, is a continuous function.

3 Since $B(t)$ is a Brownian motion, we know that if $0 \leq t_1 + s < t_2 + s \leq t_3 + s < t_4 + s$, then $[B(t_4+s) - B(t_3+s)] = [F(t_4) - F(t_3)]$ is independent of $[B(t_2+s) - B(t_1+s)] = [F(t_2) - F(t_1)]$. Therefore, if $0 \leq t_1 < t_2 \leq t_3 < t_4$, $[F(t_4) - F(t_3)]$ is independent of $[F(t_2) - F(t_1)]$.

4 Similarly, since $B(t)$ is a Brownian motion, we know that if $0 \leq t_1+s < t_2+s$, then $[B(t_2+s) - B(t_1+s)] = [F(t_2) - F(t_1)]$ has a mean of zero and a variance of $(t_2+s) - (t_1+s) = t_2 - t_1$. Since this holds when $0 \leq t_1 + s < t_2 + s$, it certainly hold for the subset where $0 \leq t_1 < t_2$.

(b) $G(t) = cB\left(\frac{t}{c^2}\right)$ is a Brownian motion, where $c > 0$ is a fixed number.

ANS: Yes, $G(t)$ is a Brownian motion.

1 $G(0) = cB(0) = 0$

2 Since $B(t)$ is continuous, $B\left(\frac{t}{c^2}\right)$ must also be continuous and so must $cB\left(\frac{t}{c^2}\right)$ be. Therefore, $G(t)$ is continuous.

3 If $0 \leq t_1 < t_2 \leq t_3 < t_4$, then $0 \leq \frac{t_1}{c} < \frac{t_2}{c} \leq \frac{t_3}{c} < \frac{t_4}{c}$. Therefore, $\left[B\left(\frac{t_4}{c^2}\right) - B\left(\frac{t_3}{c^2}\right)\right] = \frac{1}{c}[G(t_4) - G(t_3)]$ is independent of $\left[B\left(\frac{t_2}{c^2}\right) - B\left(\frac{t_1}{c^2}\right)\right] = \frac{1}{c}[G(t_2) - G(t_1)]$. Since the $\frac{1}{c}$ factor doesn't affect independence, $[G(t_4) - G(t_3)]$ is independent of $[G(t_2) - G(t_1)]$.

4 If $0 \leq t_1 < t_2$, then $0 \leq \frac{t_1}{c} < \frac{t_2}{c}$. Therefore, $\left[B\left(\frac{t_2}{c^2}\right) - B\left(\frac{t_1}{c^2}\right)\right] = \frac{1}{c}[G(t_2) - G(t_1)]$ has mean 0 and variance $\frac{t_2}{c^2} - \frac{t_1}{c^2} = \frac{t_2 - t_1}{c^2}$. Since $E\left[\frac{1}{c}[G(t_2) - G(t_1)]\right] = \frac{1}{c}E\left[G(t_2) - G(t_1)\right]$ and

$Var\left[\frac{1}{c}[G(t_2) - G(t_1)]\right] = \left(\frac{1}{c}\right)^2 Var\,[G(t_2) - G(t_1)]$, we therefore have that $[G(t_2) - G(t_1)]$ has mean 0 and variance $t_2 - t_1$.