Homework 10: Solutions

Problems to turn in individually

Problem 1

Follow the derivation done in class and convince yourself that we get the same Black-Scholes PDE when $\mu(S,t)$ and $\sigma(S,t)$ are deterministic functions of S and t.

Problem 2

We have the Black-Scholes PDE

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0. \tag{1}$$

Given the final condition $V(S,T) = S$, let us guess that the solution to (1) is $V(S,t) = S$. Then $V_t = 0$, $V_S = 1$, $V_{SS} = 0$, which (along with $V = S$) satisfy eq. (1). Thus, we have verified that the price of this derivative instrument should *at all times* be equal to the price of the underlying stock.

Here is why the above solution is enforced by the "no-arbitrage" principle. Suppose $V(S,t) \neq S$ at some time $t < T$; then we can construct an arbitrage strategy by

- ☼ selling the costlier asset and buying the cheaper one at t,
- ☼ closing out the positions at T.

We spent no money out of pocket at time t, and made an immediate profit. Moreover, since the two assets are worth the same at T, there is no chance of a loss. Such riskless profits are ruled out by the assumption of "no-arbitrage". Hence $V(S,t) \neq S$ cannot be true. In other words, two assets that are worth the same at some time T must be worth the same at all earlier times $t < T$. This is the *Law of One Price*.

(a) Now the final condition is $V(S,T) = S^2(T)$. Following the hint, guess $V(S,t) = S^2 h$. Plugging $V_t = S^2 h'$, $V_S = 2Sh$, $V_{SS} = 2h$ into eq. (1) gives

$$h' = -(r+\sigma^2)h \quad \Longrightarrow \quad h(t) = ke^{-(r+\sigma^2)t}.$$

Imposing $h(T) = 1$ gives $k = e^{(r+\sigma^2)T}$, thus the value of the derivative is

$$V(S,t) = S^2 h(t) = e^{(r+\sigma^2)(T-t)} S^2.$$

Problems to turn in as a group

Problem 1

Stock paying continuous dividends: As usual, consider the self-financing portfolio

$$\Pi = V(S,t) + nS + c.$$

As time changes,

$$\begin{aligned}
d\Pi &= dV(S,t) + d(nS) + dc \\
&= dV(S,t) + ndS + Sdn + dndS + dc_{\text{buy/sell}} + dc_{\text{int}} + dc_{\text{div}} && \text{(product rule)} \\
&= dV(S,t) + ndS + \cancel{Sdn} + \cancel{dndS} + \cancel{dc_{\text{buy/sell}}} + dc_{\text{int}} + dc_{\text{div}} && \text{(self-financing)} \\
&= V_t dt + V_S dS + \tfrac{1}{2}V_{SS}(dS)^2 + ndS + rcdt + nSDdt && \text{(chain rule)} \\
&= V_t dt + \cancel{V_S dS} + \tfrac{1}{2}V_{SS}(\sigma^2 S^2 dt) + \cancel{ndS} + rcdt + nSDdt && (\text{choose } n = -V_S) \\
&= \left(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rc - V_S SD\right) dt.
\end{aligned}$$

This is risk-free, because it has no dB term. By "no-arbitrage" principle, we must have

$$d\Pi = r\Pi dt = r(V + nS + c)dt = r(V - V_S S + c)dt.$$

Equating the two, we get $r(V - SV_S + \cancel{c})dt = \left(V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + \cancel{rc} - SV_S D\right) dt$. Thus we get the following PDE that holds for all S and t:

$$V_t + (r-D)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0.$$

Remark: The self-financing condition follows by considering what happens at the end of a Δt time interval, and the "conservation" equation

$$(n + \Delta n)(S + \Delta S) + \Delta c_{\text{buy/sell}} = n(S + \Delta S) \tag{2}$$

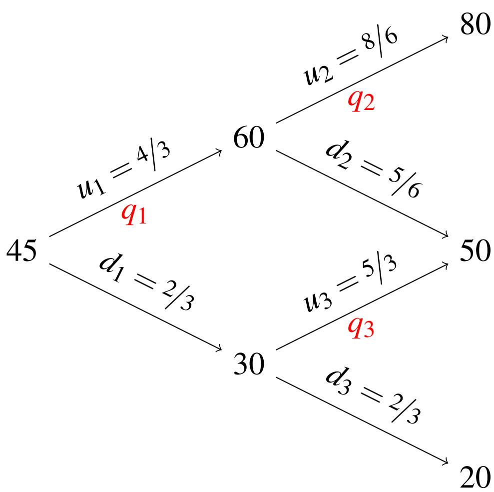
where the lhs (respectively, rhs) represents the situation after (before) rebalancing. Since no money enters or exits the overall portfolio, any change in the value of the stock position due to rebalancing must be accounted for by an equal and opposite change in the cash position. Note that, after a simple cancellation, eq. (2) becomes

$$\Delta n(S + \Delta S) + \Delta c_{\text{buy/sell}} = 0$$

which shows clearly that Δn and $\Delta c_{\text{buy/sell}}$ have opposite signs.

Problem 2

We are given $R = 1 + r = 1.02$. It is easy to calculate the "up" move and "down" move factors for each of the 3 nodes for which risk-neutral probabilities need to be computed; see fig. We apply the usual formula to find the risk-neutral up-move probabilities, i.e.,



$$q_1 = \frac{R - d_1}{u_1 - d_1} = \frac{1.02 - 2/3}{4/3 - 2/3} = 0.53$$
$$q_2 = \frac{R - d_2}{u_2 - d_2} = \frac{1.02 - 5/6}{8/6 - 5/6} = 0.3733$$
$$q_3 = \frac{R - d_3}{u_3 - d_3} = \frac{1.02 - 2/3}{5/3 - 2/3} = 0.3533.$$

All the q's are different, which contrasts with the standard binomial tree we have been used to, where they are the same across the tree. Here, since the up and down moves are different at each node, we also obtain different q's. There is nothing wrong with this, the principle of *replication* we studied for the one-step binomial tree works just as well for each single branching node. Hence there is no error here, nor does this situation admit arbitrage possibilities. (It can be proven that, for a one-step binomial tree, the no-arbitrage condition is equivalent to $d < R < u$, which is the same as $0 < q < 1$.)

Problem 3

Part (a) is left as an exercise (but see Table 1). For part (b), we are given $S = 100$, $K = 100$, $u = 1.10$, $d = 0.90$, $R = 1.00$. Thus the risk-neutral probability is

$$q = \frac{R - d}{u - d} = \frac{1}{2}.$$

A one-period put pays $P_u = 0$ in state u and $P_d = 10$ in state d. Its initial value is

$$P(1) = \frac{1}{R}\left[qP_u + (1 - q)P_d\right] = 5.$$

A two-period put pays $P_{uu} = 0$ in the state u^2S, $P_{ud} = 1$ in the state udS, and $P_{dd} = 19$ in the state d^2S. Its initial value is

$$P(2) = \frac{1}{R^2}\left[q^2 P_{uu} + 2q(1-q)P_{ud} + (1-q)^2 P_{dd}\right] = 5.25. \tag{3}$$

Remarks: In contrast to part (a), the value of the two-period put here is more than that of the one-period put. In both parts (a) and (b), as an alternative to eq. (3), we can compute the value of the two-period put by working backwards recursively in the tree, one period at a time.

Stock price tree		
100	110	121
	90	99
		81
One-period put		
3.922	0	
	10	
Two-period put		
3.383	0.392	0
	8.039	1
		19

Table 1: "High" R $(= 1.02)$

Stock price tree		
100	110	121
	90	99
		81
One-period put		
5	0	
	10	
Two-period put		
5.25	0.5	0
	10	1
		19

Table 2: "Low" R $(= 1)$