# Homework 1: Solutions

(Note: all interest rates in this HW are compounded continuously.)

## Problems to turn in individually

1. Ratios: What is the ratio of an investment over 30 years that earns 7% to an investment that earns 6%? What is the ratio of an investment over 30 years that earns 5% to an investment that earns 4%? Compare your two answers. Why is your comparison unsurprising?

   ANS: With continuous compounding the ratio is

   $$\frac{y_1(t)}{y_2(t)} = \frac{Pe^{r_1 t}}{Pe^{r_2 t}} = e^{(r_1 - r_2)t}$$

   which is the same in both cases because $r_1 - r_2 = 1\%$, $t = 30$ in both. Plugging in these values, we get the ratio to be $e^{(0.01)30} = 1.35$.

2. The rule of 72: There is a classic quick and dirty rule for approximating the amount of time needed to double the worth of an investment called the "rule of 72". This means that money approximately doubles in about 9 years if invested at 8% (since $9\times 8 = 72$) or in about 12 years if invested at 6% (since $12\times 6 = 72$), etc. Assuming continuous compounding, determine what the quantity $rt$ must exactly equal for an investment to double. This rule holds rigorously and is only a bit less quick and a bit more dirty.

   ANS: We want to find $rt$ so that $y(t) = Pe^{rt} = 2P$. Cancelling $P$ and solving for $rt$, we get $rt = \ln 2 = 0.69 = 69\%$.

3. Varying interest rates: Let's say $r(t) = 0.05 + 0.02t$, so, for example, over the first two years the rate continuously increases from 5% to 9%. Use separation of variables to determine how much \$1000 in principal will be worth after these two years.

   ANS: The ODE is $dy = r(t)ydt$, where $r(t) \geq 0$ is now a function of $t$. Separating the variables and integrating from 0 to $t$ we get

   $$\int_{y(0)}^{y(t)} \frac{1}{y}\, dy = \int_0^t r(s)\, ds$$

$$\ln |y(t)| - \ln |y(0)| = \int_0^t r(s)\, ds$$
$$\ln \frac{y(t)}{y(0)} = \int_0^t r(s)\, ds$$
$$y(t) = y(0) e^{\int_0^t r(s)\, ds}$$

where we dropped the absolute signs since $y(t) > 0$, which follows from $r(t) \geq 0,\ y(0) > 0$.

With $r(s) = 0.05 + 0.02s$, we have $\int_0^t r(s)\, ds = 0.05t + 0.01t^2$. Upon substituting $y(0) = 1000,\ t = 2$, we get $y(t) = 1000e^{0.14} = 1150.27$.

4. Why you should start saving early for retirement: Consider Person A and Person B. Starting today, Person A will put money into a 401k at a rate of $k$ dollars per year for 10 years. (The value of $k$ actually doesn't matter.) After 10 years, Person A will stop contributing to their 401k, due to deciding that their salary is better diverted to a brand new cocaine habit.

   During the 10 years Person A contributes to their 401k, Person B decides to attend 6 years of graduate school to obtain a Ph.D., followed by 4 years of low paying postdoctoral work. After that Person B will find a job and begin contributing to their 401k at a rate of $k$ dollars per year, just as Person A begins their destructive cocaine binge.

   How many years must Person B contribute $k$ per year to catch up to the worth of Person A's 401k if

   (a) ...their 401k accounts grow at 5%.
   (b) ...their 401k accounts grow at 6%.
   (c) ...their 401k accounts grow at 7%.

   The answer for (c) may surprise you. Explain it. Most 401k companies (e.g., Fidelity, Vanguard,...) suggest assuming 8% in a 401k, which represents a mix of stocks and bonds.

   To do this calculation, you can use either use the differential equations from class or an excel spreadsheet.

   ANS: First let's consider the first 10 years for person A. From equation (1) in the answer to question 6, we replace $-k$ with $k$, since we are adding, not subtracting money, we set $y(0) = 0$, since we start with no money, and we set the time equal to 10 to yield a 401k worth at the end of these 10 years of

$$\frac{k}{r}\left(e^{10r} - 1\right).$$

Now let $t$ represent the time after these 10 years have elapsed, so, for example, $t = 1$ is the 11th year. Person A contributes nothing while $t > 0$, so $k = 0$ and we have that $y_A$, the worth of person A's 401k, is

$$y_A = \left[\frac{k}{r}\left(e^{10r} - 1\right)\right] e^{rt}.$$

For Person B, we again use equation (1) with $k$ instead of $-k$ and $y(0) = 0$ to obtain

$$y_B = \frac{k}{r}\left(e^{rt} - 1\right).$$

We want the time, $t$, where $y_A = y_B$. Setting these equal, we see that we can cancel a factor of $\frac{k}{r}$ (which is why the value of $k$ is irrelevant in the problem), which yields

$$\left(e^{rt} - 1\right) = \left(e^{10r} - 1\right) e^{rt}.$$

Dividing by $e^{rt}$, and then solving for $t$, we have

$$t = -\frac{1}{r}\ln\left(2 - e^{10r}\right).$$

Plugging in $r = 0.05$ yields $t = 20.9$ years. For $r = 0.06$, we have $t = 28.8$ years. At $r = 0.07$, we see that the input into the logarithm is negative. This indicates no solution, which is exactly right. At 7% (or higher), Person B will *never* catch up to Person A. This is because the interest off of Person A's 10 years of savings is bigger than Person B's contribution of $k$ dollars per year.

Given that 8% is realistic in 401k plans, the moral of this problem is: START YOUR 401K AS SOON AS YOU START WORKING. DON'T WAIT.

## Problems to turn in as a group

1. Constant removal/addition of money in an interest bearing account: As mentioned in class, if you remove money from your account at a constant rate of $k$ dollars/year, you have the differential equation

$$\frac{dy}{dt} = ry - k.$$

   (a) If you originally invest \$1,000,000 and earn 7% interest while removing money at a constant rate of \$100,000 per year, determine $y(t)$ for future times and also determine the time, $t_{broke}$, at which you go broke. (Note that normally in applications you can use $\int \frac{1}{x} dx = ln(x) + C$ since $x$ is positive, but here you may need to use that the actual formula involves the absolute value: $\int \frac{1}{x} dx = ln|x| + C$.)

   (b) If you originally invest \$1,000,000 and earn 7% interest, determine the constant rate $k$ at which you can remove money while keeping the amount of money in the account fixed at \$1,000,000

   (c) If you originally invest \$1,000,000 and earn 7% interest while continuing to **add** money at a constant rate of \$100,000 per year, determine $y(t)$ for future times.

ANS: Separating the variables and integrating, we get

$$\int \frac{1}{ry-k}\, dy = \int dt$$
$$\frac{1}{r} \ln |ry(t) - k| = t + c$$
$$|ry(t) - k| = e^{rt+rc} = \tilde{c}e^{rt}$$

where $c$ is the constant of integration, and $\tilde{c} = e^{rc}$. Note that, whether $ry(t) - k$ is positive or negative, we can express the solution as

$$y(t) = \frac{k}{r} + c_0 e^{rt}$$

for a suitable constant $c_0$. We can determine $c_0$ by substituting $t = 0$, so clearly $c_0 = y(0) - \frac{k}{r}$. Thus

$$y(t) = \left(y(0) - \frac{k}{r}\right) e^{rt} + \frac{k}{r}. \tag{1}$$

(a) Substituting $y(0) = 10^6$, $r = 7\%$, $k = 10^5$ in our previous equation and setting $y(t) = 0$, we get

$$t = \frac{1}{r} \ln\left(\frac{k}{k - ry(0)}\right) = 17.2.$$

(b) The question meant that $y(t)$ is kept fixed *for all* time, not just brought back to the initial value $y(0)$ at the start of each year. If $y(t)$ is constant (always equal to $y(0)$), then

$$\frac{dy}{dt} = 0 \quad \Rightarrow \quad ry - k = 0 \quad \Rightarrow \quad k = ry(0) = 0.07 \times 10^6 = 70,000.$$

(c) Now $k = -10^5$ (we are *adding* money). Using $y(0) = 10^6$, $r = 7\%$, $k = -10^5$ gives

$$y(t) = \left(10^6 + \frac{10^5}{0.07}\right) e^{0.07t} - \frac{10^5}{0.07} = 2,428,571.43 e^{0.07t} - 1,428,571.43$$

2. Continuous active investing in a taxable account: Many actively invested mutual funds and hedge funds have very high turnover rates. Rates of over 100% are not at all uncommon (and for day traders it's pretty much guaranteed!) In terms of taxes, this is very well approximated by continuously selling and rebuying, which, in a taxable account, leads to the differential equation

$$\frac{dy}{dt} = (1 - T)ry$$

where $T$ is the tax rate. Of course, when you remove the funds at the end, you owe no additional taxes since you've been paying capital gains taxes all along. Let's say you start with \$100,000 principal and invest it for 40 years at a rate $r = 0.10$.

(a) Determine how much money you would have if the money were in a Roth IRA and so you owed no taxes at any time during the 40 years or at the end of them. (In other words, if $T = 0$.)

(b) Determine how much money you would have if you were charged the tax rate $T = 0.24$ This is actually the federal + state long term capital gains rate as opposed to $T = 0.37$, which is the more likely relevant short term capital gains rate for stocks held less than a year. Just to get an idea of what a day trader makes (since everything for them is short term), determine how much money you would have if you were charged the tax rate $T = 0.37$. Compare it to your answer for $T = 0.24$. Yes, that is a big difference!

(c) Let's say you instead invested your \$100,000 in an index fund for the entire stock market and assume that there is no turnover in this fund (although in reality there is about a 4% annual turnover but let's ignore that and, while we're at it, we'll also ignore the even bigger issue of dividends.) So $T = 0$ for the 40 years, but then you need to pay 24% of your capital gains at the end of the 40 years. What $r$ will give you the same amount of money at the end of the 40 years as you had at the end of the previous example where $r = 0.10$ but $T = 0.24$ throughout?

ANS:

(a) With $T = 0$, the ODE is $\frac{dy}{dt} = (1-0)ry = ry$, which, after solving, yields $y(t) = Pe^{rt}$. Substituting $P = 10^5$ and $r = 0.10$, we get $y(t) = 10^5 e^{0.10t}$. Thus $y(40) = 5,459,815$.

(b) With $T = 0.24$, the ODE is $\frac{dy}{dt} = (1 - 0.24)ry = 0.76ry$. Thus $y(t) = Pe^{0.76rt} = Pe^{0.076t}$, and $y(40) = 10^5 \exp[0.076 \times 40]$ =2,090,524.

This is much smaller than the answer to (a).

With $T = 0.37$ we have $y(40) = 10^5 \exp[0.063 \times 40]$ =1,242,860.
Short term capital gains really hurt.

(c) We want to find $r$ such that

$$Pe^{0.076\times 40} = P\left(e^{r40}0.76 + 0.24\right)$$

Solving, we get $e^{r40} = (e^{3.04} - 0.24)/0.76$, or $r = 8.257\%$.

3. Change in the capital gains tax rate: Congress likes to mess with the capital gains rate every once in a while just to keep things interesting. If you're an investor and the rate is set to go up, what should you do? Let's say at some time, call it $t = 0$, the capital gains rate is set to increase from 24% to 29%. Let's say that just before the rate is set to

switch you have an account that started with a cost basis of $B$ dollars and has now accrued capital gains worth $C$ dollars. Let's also say that you know the account will grow at a constant rate $r$ and that you will pull everything out of the fund at time $t = t_{final}$ after the rate switch. On one hand you know that if you sell your account now and then immediately rebuy it, you get to pay the lower tax rate and only have to pay the higher rate on the new gains accrued, but you also know that by paying the gains on $C$ early, you lose money early which can't be used to grow your account. If $r = 0.07$, determine the time $t = t_0$ after the switch in rates, where for any $t_{final} < t_0$ you should immediately sell and then rebuy, but for any $t_{final} > t_0$, you should not sell and rebuy.

Hints: It turns out that $t_0$ depends on neither $B$ nor $C$ so you should see $B$ and $C$ cancel as you work through this problem. Also, you should solve for $rt_0$ as a combined value. Your final step will be to set $r = 0.07$ and divide by 0.07 to get $t_0$. (Note: In reality you can't sell and then immediately rebuy stock without waiting a month in between. If you wait less than a month, it's called a "wash sale" which means from the IRS's point of view, it's as if you never sold or rebought the stock and so there is no immediate capital gains consequence.)

ANS: The capital gains rate is set to increase at $t = 0$, and we are sitting on a total wealth of $B + C$, of which $C$ is capital gains.

*Strategy 1: Sit tight.* Sell everything at the final time $t > 0$, paying taxes (only) then. In that case, we would end up with

$$(B + C)e^{rt} - \Big[(B + C)e^{rt} - B\Big]\, 0.29 = (B + C)e^{rt} 0.71 + B0.29.$$

*Strategy 2: Sell and re-buy.* Selling now means we must pay capital gains tax on $C$ (at the lower rate). This reduces our wealth to $B + C - 0.24C = B + 0.76C$, which then grows for $t > 0$ years. At the final time $t$, when we cash out, we must pay taxes again. Our final wealth would be given by the formula derived in class,

$$Pe^{rt} - \Big[Pe^{rt} - P\Big]\, 0.29 = P\left[e^{rt} 0.71 + 0.29\right]$$

with $P = B + 0.76C$.

Equating the two strategies, we get

$$(B + C)e^{rt} 0.71 + B0.29 = (B + 0.76C)\left[e^{rt} 0.71 + 0.29\right]$$

for $t = t_0$. After some algebra, both $B$ and $C$ disappear (!) and we get

$$e^{rt_0} = \frac{(.76)(.29)}{(.24)(.71)} = 1.29343$$

hence $t_0 = (\ln 1.29343)/r = 3.676$. Strategy 1 wins when $t > t_0$, and strategy 2 wins when $t < t_0$.

4. Bond prices are very vulnerable to interest rate changes, especially when the term of the bond, $T$, is long. In this problem we'll see the exact relationship between interest rates, the term of the bond, and the value of the bond.

   Let $p$ be the par value of a bond, that is, the amount of money collected at time $T$, the end of the bond term. Let $c$ be the continuous coupon rate for the bond, paid between time 0 and time $T$. Let $r$ be the current (assumed constant) interest rate over the term of the bond.

   From class, we know that the value of the bond at the present time, time 0, must be the present value of the entire coupon payment stream generated between time 0 and time $T$ plus the present value of the par value at time $T$:

$$\int_0^T pce^{-rt}dt + pe^{-rT}.$$

   (a) Show that if you buy the bond at par, that is, if you pay $p$ at time 0, then $c$ must equal $r$.

   ANS: Since the present value of the worth of the bond is $p$, we have

$$p = \int_0^T pce^{-rt}dt + pe^{-rT}.$$

   And therefore,

$$\begin{aligned} p &= p\left(-\frac{c}{r}\int_0^T -re^{-rt}dt + e^{-rT}\right) \\ 1 &= -\frac{c}{r}\left(e^{-rT}-1\right) + e^{-rT} \\ \frac{c}{r}\left(e^{-rT}-1\right) &= \left(e^{-rT}-1\right) \\ c &= r. \end{aligned}$$

(b) Let's say you bought a bond with a par value $p$ when the interest rate was $r_{old}$. It is now time 0 and the bond will expire at time $T$. The current interest rate is $r_{new}$. The fraction of the value of your bond that is lost or gained going forward due to the interest rate shift can be calculated from computing

$$\frac{PV(\text{your bond}) - PV(\text{new bond with par value } p)}{PV(\text{new bond with par value } p)},$$

which is equal to

$$\frac{\left(\int_0^T pr_{old}e^{-r_{new}t}dt + pe^{-r_{new}T}\right) - \left(\int_0^T pr_{new}e^{-r_{new}t}dt + pe^{-r_{new}T}\right)}{\int_0^T pr_{new}e^{-r_{new}t}dt + pe^{-r_{new}T}}.$$

Canceling the par values in the numerator and recognizing the denominator must equal the par value $p$ given our calculation in part (a), we have

$$\frac{p\left(\int_0^T r_{old}e^{-r_{new}t}dt - \int_0^T r_{new}e^{-r_{new}t}dt\right)}{p}.$$

After canceling the $p$, calculate the resulting integral to obtain an expression for the fractional loss or gain in the bond value. When we have a loss, this fraction, expressed as a percentage, is called the bond *discount*. When we have a gain, this fraction, expressed as a percentage, is called the bond *premium*.

ANS:

$$\begin{aligned}
\int_0^T r_{old}e^{-r_{new}t}dt - \int_0^T r_{new}e^{-r_{new}t}dt &= \int_0^T (r_{old} - r_{new})e^{-r_{new}t}dt \\
&= \frac{(r_{old} - r_{new})}{-r_{new}} \int_0^T -r_{new}e^{-r_{new}t}dt \\
&= \frac{(r_{old} - r_{new})}{-r_{new}} \left(e^{-r_{new}T} - 1\right) \\
&= \left(\frac{r_{old}}{r_{new}} - 1\right)\left(1 - e^{-r_{new}T}\right)
\end{aligned}$$

i. Consider a bond bought at $r = 0.05$ (i.e., 5%). Use your formula to determine the discount or premium created by a 1, 2, and 4 percentage point gain or loss in $r$. These six values should be determined for a short term bond with $T = 3$ years, for an intermediate bond with $T = 8$ years, and for a long term bond with $T = 20$ years.

ANS: Using the formula above, we can produce the following chart using, say, Excel:

| $r_{new}$ | $T = 3$ | $T = 8$ | $T = 20$ |
|---|---|---|---|
| 0.01 | 11.8% | 30.7% | 72.5% |
| 0.03 | 5.74% | 14.2% | 30.1% |
| 0.04 | 2.83% | 6.85% | 13.8% |
| 0.06 | -2.75% | -6.35% | -11.6% |
| 0.07 | -5.41% | -12.3% | -21.5% |
| 0.09 | -10.5% | -22.8% | -37.1% |

Note how much more strongly interest rates affect the worth of the bond when $T$ is big!

5. Download the data on daily stock prices for your favorite stock or index for 3 years: 2006, 2007, 2008. Keep each year's data separate.

   (a) Compute the series of daily returns from the data. The return is nothing but $\frac{S_t - S_{t-1}}{S_{t-1}}$.

   (b) Compute the autocorrelation for each year's data for lags 1 through 6. Comment on what you find.

   (c) Variance ratios. For each year compute the variance of the returns. Call this $V_1$.

   (d) Then compute the 2-day interval returns, and compute the variances, call these $V_2$.

   (e) Compute the variance ratios, i.e. $VR = V_2/(2V_1)$. Comment on what you find for each year.

   ANS: Answers will vary based on data.