Homework 12: Solutions

Problems to turn in individually

Problem 1

Call option gamma: From eq. (4), the gamma for a call option is

$$\Gamma := \frac{\partial^2 C}{\partial S^2} = \Phi'(d_1)\frac{\partial d_1}{\partial S} = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \cdot \frac{1}{S\sigma\sqrt{T-t}}$$

where we used eqs. (3a) and (3b) at the last step. As an aside, differentiating the *put-call parity* equation twice, we get the gamma for a put option:

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0 \quad \Longrightarrow \quad \frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 C}{\partial S^2}.$$

The Taylor series associated with C about the current stock price S is

$$\Delta C = C(S+\Delta S) - C(S) = \underbrace{\frac{\partial C}{\partial S}}_{\text{delta}}\Delta S + \frac{1}{2}\underbrace{\frac{\partial^2 C}{\partial S^2}}_{\text{gamma}}(\Delta S)^2 + \cdots \tag{1}$$

which can be used to approximate the change in the call's value when (only) the stock price changes. Given $\sigma = 0.3$, $r = 0.04$, $K = 79$, $T - t = 3/4$ and $S = 60$, we calculate

$$d_1 = -0.8135,\ d_2 = -1.0733,\ \frac{\partial C}{\partial S} = 0.20797,\ \frac{\partial^2 C}{\partial S^2} = 0.018382.$$

We are told that $\Delta S = +4$ (the stock price increases to 64). We can obtain two formulas to estimate ΔC by truncating the Taylor series in eq. (1) after one term and two terms, respectively:

$$\Delta C \approx \frac{\partial C}{\partial S}\Delta S = 0.20797 \times 4 = 0.83186.$$
$$\Delta C \approx \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 = 0.20797 \times 4 + \frac{1}{2} \times 0.018382 \times 4^2 = 0.97892.$$

For comparison the actual change, computed using eq. (2), is

$$\Delta C = C(64) - C(60) = 2.6099 - 1.6247 = 0.98518.$$

Problem 2

Delta hedging: Suppose we hold one call option and x shares of ABW. We can choose x such that the *delta of our portfolio* becomes zero:

$$0 = \frac{\partial}{\partial S}(C + xS) = \frac{\partial C}{\partial S} + x\frac{\partial S}{\partial S} = \Delta + x \cdot 1 \quad \Longrightarrow \quad x = -\Delta = -0.20797.$$

Thus we should be short 0.20797 shares of ABW in order to "delta hedge" our portfolio.

Problem 3

(a) The security is more likely to be a put and not a call, because the value of the security increases as j declines, i.e. as the stock price falls.

(b) In the explicit model, we can solve for the values of the trinomial probabilities, as follows: ($v = 0.3$, $r = 0.1$)

$$\begin{aligned} a &= v^2 h/(2k^2) + h(r - 0.5v^2)/(2k) = 0.398 \\ c &= v^2 h/(2k^2) - h(r - 0.5v^2)/(2k) = 0.352 \\ b &= 1 - a - c = 0.25 \end{aligned}$$

Using these values we can solve for

$$\begin{aligned} V(i,j) &= \frac{a\,V(i+1, j+1) + b\,V(i+1, j) + c\,V(i+1, j-1)}{1 + rh} \\ &= \frac{0.398(4.5) + 0.25(5.2) + 0.352(6.3)}{1 + 0.1/12} \\ &= 5.2647 \end{aligned}$$

Problems to turn in as a group

Problem 1

Call option delta: The Black-Scholes formula for a call option is

$$C = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2}$$

$$\text{where} \qquad \Phi(d_1) = \int_{-\infty}^{d_1} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$
$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Differentiating eq. (2) with respect to S gives

$$\frac{\partial C}{\partial S} = 1\cdot\Phi(d_1) + S\cdot\Phi'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}\cdot\Phi'(d_2)\frac{\partial d_2}{\partial S}.$$

We can show that the second and third terms cancel by using

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1/S}{\sigma\sqrt{T-t}} \tag{3a}$$

$$\Phi'(d_1) = \tfrac{1}{\sqrt{2\pi}}e^{-d_1^2/2}, \quad \Phi'(d_2) = \tfrac{1}{\sqrt{2\pi}}e^{-d_2^2/2} \tag{3b}$$

$$d_1^2 - d_2^2 = (d_1-d_2)(d_1+d_2) = \cancel{\sigma\sqrt{T-t}}\cdot\frac{2\left[\ln(S/K)+r(T-t)\right]}{\cancel{\sigma\sqrt{T-t}}} \quad\Longrightarrow\quad e^{\frac{d_1^2-d_2^2}{2}} = \frac{S}{K}e^{r(T-t)}.$$

Thus the *delta for a call option* is

$$\boxed{\Delta := \frac{\partial C}{\partial S} = \Phi(d_1).} \tag{4}$$

An easy way to remember this result is to pretend that no term except S in the rhs of eq. (2) depends on S. ☺

Note that, once we have (4), we can find the delta for a *put* option easily via call-put parity:

$$C - P = S - Ke^{-r(T-t)} \quad\Longrightarrow\quad \frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1.$$
$$\therefore\ \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = \Phi(d_1) - 1 = -\Phi(-d_1).$$