

# Homework 4

## Problems to turn in individually

1. Let  $M, N$ , and  $O$  be  $n \times n$  matrices and  $v$  be an  $n$ -vector. Rewrite

$$\sum_{i,j,k} M_{ik} O_{kj} N_{jl} v_i$$

in matrix-vector notation ending with the vector  $v$  as in class. Now write the expression in matrix-vector notation but starting, as opposed to ending, with the vector  $v$ . Notice that when you start with a vector, it must be the transpose of the vector for the multiplication to make sense. Compare your two expressions, which are the transpose of each other. This comparison should make clear how to take the transpose of a product of matrices and vectors. Use this to rewrite  $(w^T B C^T A^T u)^T$  in terms of the product of the matrices  $B, C$ , and  $A$  and their transposes and the vectors  $w$  and  $u$  and their transposes. This last question should take you under 30 seconds to do if you understood how to generalize the comparison correctly.

2. Let  $A$  be a matrix of constants and  $x$  be a vector. Determine the gradient of  $x^T A x$  with respect to  $x$ . (That is determine the derivative of  $x^T A x$  with respect to each component in the  $x$  vector.) If  $A$  is symmetric, how can you simplify your results? (You should be able to combine the two terms in your answer when  $A$  is not symmetric into a single term.)
3. In this question, you will continue to use Python with the data you extracted from HW2. (Or you can use different data, it's up to you.) Recall that you had a time series of stock returns for your favorite stock. Now do the following:
  - (a) Sort the returns and find the 99th percentile value.
  - (b) Find the 1st percentile value.
  - (c) What is the mean of all returns greater than 5%?
  - (d) How many sigma away from the mean is the lowest return?

## Problems to turn in as a group

1. Let  $X$  be a random variable. Let's say that we take  $n$  samples  $x_i$ , where  $i = 1, 2, \dots, n$ , from the distribution of  $X$ . Similarly, take  $n$  samples  $y_i$ , where  $i = 1, 2, \dots, n$ , from the distribution of a random variable  $Y$ . These sample data values can be used to create *point estimators* (i.e., best approximations) to various parameters for the random variables. For example, the point estimator (i.e., the best approximation) of the mean of  $X$  (i.e.,  $E[X]$ ) from the sampled data is called the *sample mean*  $\bar{x}$ :

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}.$$

The point estimator of  $V[X]$ , the variance of  $X$ , is the *sample variance*:

$$\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1} = \frac{\sum_{i=1}^n (x_i^2 - 2x_i\bar{x} + \bar{x}^2)}{n - 1} = \frac{\sum_{i=1}^n (x_i^2) - n\bar{x}^2}{n - 1}.$$

The point estimator of  $Cov[X, Y]$ , the covariance of  $X$  and  $Y$ , is the *sample covariance*:

$$\frac{\sum_{i=1}^n [(x_i - \bar{x})(y_i - \bar{y})]}{n - 1} = \frac{\sum_{i=1}^n (x_i y_i) - n\bar{x}\bar{y}}{n - 1}.$$

If we now consider the  $n$  data points  $(x_i, y_i)$  and try to best fit these data points with the line

$$y = a + bx,$$

show that the value of  $b$  that comes from the regression calculation happens to equal the sample covariance of the  $x_i$  and  $y_i$  divided by the sample variance of the  $x_i$ .

2. Let's say the monthly returns over the past seven months for stock in Amalgamated Dental Floss, Inc. are the following: .00852, .01045, .00573, .01578, -.01223, .00682, .00452. In general, we do not expect that these returns are correlated with time, but given statistical variation and the small number of data points, we expect there to be some slope to the line that best fits these points. (Although the likelihood of the slope being positive or negative is the same.)

- (a) Determine the line  $a + bt$  (where  $t$  is in months) that minimizes the sum of the square differences between the returns and the line.
- (b) Determine the constant  $a$  that minimizes the sum of the square differences between the returns and the constant. Compare this to the average value of the returns. Do you expect the minimizing constant and the average value to be close? the same? Explain.
- (c) Determine the parabola  $a + bt + ct^2$  that minimizes the sum of the square differences between the returns and the parabola. (Hint: think of  $t^2$  as  $x_2$  from class.)