Homework 10

Problems to turn in individually

1. In class we derived the Black-Scholes PDE for constant μ and σ. What if, instead, we had $\mu(S, t)$ and $\sigma(S, t)$? That is, if μ and σ are known functions of S and t, determine how, if at all, the Black-Scholes PDE would change by going step by step through the derivation.

2. Let's say that you had a derivative instrument that, at expiration time T, will be worth the price of its underlying stock; i.e., $V(S, T) = S(T)$. What should be the worth of this derivative at earlier times? You can use "no-arbitrage" arguments to determine this. Show that your guess satisfies the Black-Scholes PDE and the final condition $V(S, T) = S$. If you feel more comfortable with the math than the "no-arbitrage" arguments, you can solve the Black-Scholes PDE by guessing the solution — it's not a complicated solution — and once you've verified that it's a solution, find a "no-arbitrage" argument for why it had to be the solution.

   (a) Now consider a derivative instrument where $V(S, T) = S^2$. It's not so easy to find a "no-arbitrage" argument for the value of this derivative at earlier times; guessing the solution to the Black-Scholes PDE is easier. Try guessing a solution of the form $V(S, t) = g(S)$ for a reasonable function $g(S)$. You should have a good guess for what to try from the $V(S, T) = S$ case. While this won't work, you won't be too far off from a solution. Keep your $g(S)$ but multiply it by a function of t so $V(S, t) = g(S)h(t)$. Determine the functions g and h that work and report $V(S, t)$, the value of this derivative.

Problems to turn in as a group

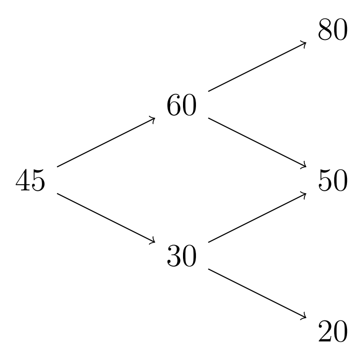
1. Continuous dividends: If we consider a stock that pays continuous dividends at a rate of D dollars per year per dollar's worth of stock, we alter two of our equations. First, we alter the stock differential to

$$dS = (\mu - D)Sdt + \sigma SdB$$

since the payout of dividends reduces the worth of the stock. Second, the cash differential dc now has three parts: $dc_{\text{buy/sell}}$ due to buying and selling, dc_{int} due to the cash gaining interest at the risk-free rate, and now we add $dc_{\text{div}} = nSDdt$ due to the dividend payout. Show, by going step by step through the derivation, that the resulting Black-Scholes PDE for the value V of any derivative instrument which has this dividend paying stock as its underlying equity has the form

$$V_t + (r - D)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0.$$

2. You are given the following binomial tree of stock prices. In addition, the rate of interest per period is constant at 2%. Find the risk-neutral probabilities of the stock movements from each node on the tree. Are these probabilities the same? If not, explain whether the tree is a valid one.



3. In the absence of dividends, the holder of a European call always benefits from an increase in maturity since the insurance value and time value of the call both increase. However, for the holder of a European put in this case, insurance value increases but time value decreases, so the put value could increase or decrease. In general, for a given level

of volatility, if interest rates are "high," the time value effect will outweigh the insurance value effect, so European put values will *decrease* as maturity increases; but if interest rates are "low," the insurance value effect will dominate, so the put value will increase. This question illustrates these arguments.

Consider a binomial model with parameters $S = 100$, $u = 1.10$, and $d = 0.90$, and a European put with a strike of $K = 100$.

(a) First, consider a "high" interest rate environment where $R = 1.02$ (one plus the interest rate). Show that with these parameter values, a one-period put is worth 3.92, but a two-period European put is worth only 3.38. The increase in maturity hurts the put holder because the insurance value effect is outweighed by the time value effect.

(b) Now consider a "low" interest rate environment where $R = 1.00$. Show that in this case, the one-period put is worth less than the two-period put.