Creating a jump formula for a platformer game

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When developing independant video games, a customary way of setting up game physics is by tweaking and testing constants into a formula until desired behavior seem to be attained. While it may have been successful for many games, this process can take much more time than intended and isn't open to modifications. Magic values obtained from tweaking and testing does not contain any information regarding the criterias used to find them and will cause problems to the next developer working on it if not very well documented.

Hence, a solution to this problem would be to find those game physics constants from those criterias automatically. With algebra and differential equations, it is totally possible to express desired constants from inputs criterias like timings and distance.

While this process of expressing game physics constants as relations of game physics criterias can be done for any type of game physics, this paper focus on the jump formula of platformer games.

The following sections go through the whole process of finding specific values from criterias. Then the last section describes an implementation of the formula in a game.

Velocity function

Let $V(t)$ represent the vertical velocity of the moving object at any time t.

$$V(t) = V_0 + s \cdot R - G \cdot t + j \cdot F \cdot t$$

$V_0 \geq 0$ is the initial velocity

$G > 0$ is vertical velocity resulting from

$F \geq 0$ is the *floating* constant. It provides a height bonus based on how long the jump button is held after the start of the jump.

$j = \{0, 1\}$ is a boolean representing the status of the jump button, $j = 1$ if pressed, otherwise $j = 0$

$R \geq 0$ is the *running* constant. It provides a height bonus based on the objet's horizontal speed.

$s \in [0, 1]$ indicates how fast the object is going from 0 to 1 in relation to its maximum speed.

Height function

In order to impose height constants as inputs, we will need to define a height function.

Let $y(t)$ represent the height of the object at any time t. The derivative of y is its velocity.

$$\begin{aligned}
\frac{dy}{dt} = V(t) \Rightarrow \frac{dy}{dt} &= V_0 + s \cdot R - G \cdot t + j \cdot F \cdot t \\
dy &= (V_0 + s \cdot R - G \cdot t + j \cdot F \cdot t)dt \\
\int dy &= V_0 \int dt + s \cdot R \int dt - G \int t\,dt + j \cdot F \int t\,dt \\
y &= V_0 \cdot t + s \cdot R \cdot t - \frac{G \cdot t^2}{2} + \frac{j \cdot F \cdot t^2}{2} + C
\end{aligned}$$

$C = 0$ is the initial vertical position (and can safely be ignored)

Input constants

The input constants are criterias imposed by the developer to adjust the jump making it follow a specific pattern.

The input constants are

$Y_1 \geq 0$ is the maximum height of a jump with no bonus for horizontal velocity or button holding.

$Y_2 \geq Y_1$ is the maximum height of a jump with the button held all the time but no horizontal velocity.

$Y_3 \geq Y_2$ is the maximum height of a jump with both bonuses.

$L > 0$ is the time needed to complete the jump if it lands on the same height it started. The jump of length L reaches height Y_2 meaning it's done with the button held all the time but without horizontal velocity.

These conditions assume that

If $j = 0$ and $s = 0$ then $y_{max} = Y_1$

If $j = 1$ and $s = 0$ then $y_{max} = Y_2$ and $t_2 - t_1 = L$ where $y(t_1)y(t_2) = C$ and $t_1 \neq t_2$

If $j = 1$ and $s = 1$ then $y_{max} = Y_3$

Finding when the object is at its max height

Let's find t for y_{max}. We know at the extremums of a function the value of the derivative is 0.

$$y_{max} \Rightarrow V(t) = 0$$

$$\begin{aligned} V(t) = 0 \Rightarrow & V_0 + s \cdot R - G \cdot t + j \cdot F \cdot t = 0 \\ & V_0 + s \cdot R = G \cdot t - j \cdot F \cdot t \\ & V_0 + s \cdot R = (G - j \cdot F)t \\ & t = \frac{V_0 + s \cdot R}{G - j \cdot F} \end{aligned}$$

Then we can find y_{max}:

$$\begin{aligned} y(\frac{V_0 + s \cdot R}{G - j \cdot F}) &= V_0 \cdot \frac{V_0 + s \cdot R}{G - j \cdot F} + s \cdot R \cdot \frac{V_0 + s \cdot R}{G - j \cdot F} - \frac{G \cdot (\frac{V_0 + s \cdot R}{G - j \cdot F})^2}{2} + \frac{j \cdot F \cdot (\frac{V_0 + s \cdot R}{G - j \cdot F})^2}{2} \\ &= \frac{V_0(V_0 + s \cdot R)}{G - j \cdot F} + \frac{s \cdot R(V_0 + s \cdot R)}{G - j \cdot F} - \frac{G(V_0 + s \cdot R)^2}{2(G - j \cdot F)^2} + \frac{j \cdot F(V_0 + s \cdot R)^2}{2(G - j \cdot F)^2} \\ &= \frac{V_0(V_0 + s \cdot R) + s \cdot R(V_0 + s \cdot R)}{G - j \cdot F} - \frac{G(V_0 + s \cdot R)^2 + j \cdot F(V_0 + s \cdot R)^2}{2(G - j \cdot F)^2} \\ &= \frac{(V_0 + s \cdot R)^2}{G - j \cdot F} - \frac{(G + j \cdot F)(V_0 + s \cdot R)^2}{2(G - j \cdot F)^2} \\ &= (V_0 + s \cdot R)^2 \left(\frac{1}{G - j \cdot F} - \frac{G + j \cdot F}{2(G - j \cdot F)^2} \right) \\ &= (V_0 + s \cdot R)^2 \left(\frac{2(G - j \cdot F) - G + j \cdot F}{2(G - j \cdot F)^2} \right) \\ &= (V_0 + s \cdot R)^2 \left(\frac{2G - 2 \cdot j \cdot F - G + j \cdot F}{2(G - j \cdot F)^2} \right) \\ &= (V_0 + s \cdot R)^2 \left(\frac{G - j \cdot F}{2(G - j \cdot F)^2} \right) = \frac{(V_0 + s \cdot R)^2}{2(G - j \cdot F)} \end{aligned}$$

Using inputs to find relations

Finding L as a relation of other constants

Since $L = t_2 - t_1$, let's find t_1 and t_2

$$y(t_1) = y(t_2) = C \Rightarrow V_0 \cdot t + s \cdot R \cdot t - \frac{G \cdot t^2}{2} + \frac{j \cdot F \cdot t^2}{2} + C = C$$

$$V_0 \cdot t + 0 \cdot R \cdot t - \frac{G \cdot t^2}{2} + \frac{1 \cdot F \cdot t^2}{2} = 0$$

$$t^2 \frac{F - G}{2} + t \cdot V_0 = 0$$

$$t \left(t \frac{F - G}{2} + V_0 \right) = 0$$

$$t_1 = 0$$

$$t_2 \frac{F - G}{2} + V_0 = 0$$

$$t_2 = \frac{2V_0}{G - F}$$

So

$$L = \frac{2V_0}{G - F} - 0 = \frac{2V_0}{G - F}$$

V_0 in relation with G

If $j = 0$ and $s = 0$ then $y_{max} = Y_1$, so

$$Y_1 = \frac{(V_0 + s \cdot R)^2}{2(G - j \cdot F)}$$

$$Y_1 = \frac{V_0^2}{2G}$$

$$2G \cdot Y_1 = V_0^2$$

$$V_0 = \sqrt{2G \cdot Y_1}$$

F in relation with G

If $j = 1$ and $s = 0$ then $y_{max} = Y_2$

$$Y_2 = \frac{(V_0 + s \cdot R)^2}{2(G - j \cdot F)}$$

$$Y_2 = \frac{(\sqrt{2G \cdot Y_1})^2}{2(G - F)}$$

$$Y_2 = \frac{G \cdot Y_1}{G - F}$$

$$G - F = \frac{G \cdot Y_1}{Y_2}$$

$$F = G - \frac{G \cdot Y_1}{Y_2}$$

R in relation with G

If $j = 1$ and $s = 1$ then $y_{max} = Y_3$

$$Y_3 = \frac{(V_0 + s \cdot R)^2}{2(G - j \cdot F)}$$

$$Y_3 = \frac{(\sqrt{2G \cdot Y_1} + R)^2}{2(G - (-\frac{G \cdot Y_1}{Y_2} + G))}$$

$$Y_3 = \frac{(\sqrt{2G \cdot Y_1} + R)^2}{2(\frac{G \cdot Y_1}{Y_2})}$$

$$Y_3 = \frac{Y_2(\sqrt{2G \cdot Y_1} + R)^2}{2(G \cdot Y_1)}$$

$$Y_3 = \frac{Y_2(2G \cdot Y_1 + 2R\sqrt{2G \cdot Y_1} + R^2)}{2(G \cdot Y_1)}$$

$$\frac{Y_2(2G \cdot Y_1) + Y_2 \cdot 2R\sqrt{2G \cdot Y_1} + Y_2 \cdot R^2}{2(G \cdot Y_1)} - Y_3 = 0$$

$$Y_2 + \frac{Y_2 \cdot 2R}{\sqrt{2G \cdot Y_1}} + \frac{Y_2 \cdot R^2}{2(G \cdot Y_1)} - Y_3 = 0$$

$$\frac{Y_2}{2(G \cdot Y_1)}R^2 + \frac{2Y_2}{\sqrt{2G \cdot Y_1}}R + (Y_2 - Y_3) = 0$$

$$a = \frac{Y_2}{2(G \cdot Y_1)},\ b = \frac{2Y_2}{\sqrt{2G \cdot Y_1}},\ c = Y_2 - Y_3$$

$$\Delta = b^2 - 4ac \Rightarrow \Delta = \left(\frac{2Y_2}{\sqrt{2G \cdot Y_1}}\right)^2 - 4\frac{Y_2}{2(G \cdot Y_1)}(Y_2 - Y_3)$$

$$\Delta = \frac{4Y_2^2}{2G \cdot Y_1} - \frac{4Y_2(Y_2 - Y_3)}{2(G \cdot Y_1)}$$

$$\Delta = \frac{2Y_2^2 - 2Y_2(Y_2 - Y_3)}{G \cdot Y_1}$$

$$\Delta = \frac{2Y_2^2 - 2Y_2^2 + 2Y_2 \cdot Y_3}{G \cdot Y_1}$$

$$\Delta = \frac{2Y_2 \cdot Y_3}{G \cdot Y_1}$$

$$R = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow R = \frac{-\frac{2Y_2}{\sqrt{2G \cdot Y_1}} \pm \sqrt{\frac{2Y_2 \cdot Y_3}{G \cdot Y_1}}}{2\frac{Y_2}{2(G \cdot Y_1)}}$$

$$R = \frac{\frac{-2Y_2}{\sqrt{2}} \pm \sqrt{2Y_2 \cdot Y_3}}{\frac{Y_2\sqrt{G \cdot Y_1}}{G \cdot Y_1}}$$

$$R = \sqrt{2G \cdot Y_1}\left(-1 \pm \sqrt{\frac{Y_3}{Y_2}}\right)$$

Since $R \geq 0$,

$$\sqrt{2G \cdot Y_1}\left(-1 \pm \sqrt{\frac{Y_3}{Y_2}}\right) > 0$$

Since $G \geq 0$ and $Y_1 > 0$,

$$-1 \pm \sqrt{\frac{Y_3}{Y_2}} > 0$$

$$-1-\sqrt{\frac{Y_3}{Y_2}} \ngtr 0$$

So

$$R = \sqrt{2G \cdot Y_1}\left(-1+\sqrt{\frac{Y_3}{Y_2}}\right)$$

$$R = \sqrt{2G \cdot Y_1}\left(\sqrt{\frac{Y_3}{Y_2}}-1\right)$$

Finding constants in relation to input

We have

$$L = \frac{2V_0}{G - F}$$

$$V_0 = \sqrt{2G \cdot Y_1}$$

$$F = G - \frac{G \cdot Y_1}{Y_2}$$

$$R = \sqrt{2G \cdot Y_1}\left(\sqrt{\frac{Y_3}{Y_2}} - 1\right)$$

So

$$G - F = \frac{G \cdot Y_1}{Y_2}$$

$$L = \frac{2V_0}{G - F} \Rightarrow L = \frac{2V_0 \cdot Y_2}{G \cdot Y_1}$$

$$G = \frac{2V_0 \cdot Y_2}{L \cdot Y_1}$$

Then we can find V_0

$$V_0 = \sqrt{2G \cdot Y_1} \Rightarrow V_0 = \sqrt{2\frac{2V_0 \cdot Y_2}{L \cdot Y_1} \cdot Y_1}$$

$$V_0 = \sqrt{\frac{4V_0 \cdot Y_2}{L}}$$

$$V_0^2 = \frac{4V_0 \cdot Y_2}{L}$$

$$V_0 = \frac{4Y_2}{L}$$

We can find G

$$G = \frac{2V_0 \cdot Y_2}{L \cdot Y_1}$$

$$G = \frac{2\frac{4Y_2}{L} \cdot Y_2}{L \cdot Y_1}$$

$$G = \frac{8Y_2^2}{L^2 \cdot Y_1}$$

We can find F

$$F = G - \frac{G \cdot Y_1}{Y_2}$$

$$F = \frac{8Y_2^2}{L^2 \cdot Y_1} - \frac{\frac{8Y_2^2}{L^2 \cdot Y_1} \cdot Y_1}{Y_2}$$

$$F = \frac{8Y_2^2}{L^2 \cdot Y_1} - \frac{8Y_2}{L^2}$$

$$F = \frac{8Y_2^2 - 8Y_2 \cdot Y_1}{L^2 \cdot Y_1}$$

$$F = \frac{8Y_2(Y_2 - Y_1)}{L^2 \cdot Y_1}$$

Finally we can find R

$$R = \sqrt{2\frac{8Y_2^2}{L^2 \cdot Y_1} \cdot Y_1}\left(\sqrt{\frac{Y_3}{Y_2}} - 1\right)$$

$$R = \frac{4Y_2}{L}\left(\sqrt{\frac{Y_3}{Y_2}} - 1\right)$$

Generating the specific formula from inputs

Now that all constants has been found as relations of inputs, all left to do is to simply insert in values and find the specific formula.

$$V_0 = \frac{4Y_2}{L}$$

$$G = \frac{8Y_2^2}{L^2 \cdot Y_1}$$

$$F = \frac{8Y_2(Y_2 - Y_1)}{L^2 \cdot Y_1}$$

$$R = \frac{4Y_2}{L}\left(\sqrt{\frac{Y_3}{Y_2}} - 1\right)$$

As an example, let's take $L = \frac{3}{4}$, $Y_1 = 2$, $Y_2 = 4$ and $Y_3 = 5$

$$V_0 = \frac{4 \cdot 4}{\frac{3}{4}} = \frac{64}{3} = 21.\overline{3}$$

$$G = \frac{8 \cdot 4^2}{(\frac{3}{4})^2 \cdot 2} = \frac{1024}{9} = 113.\overline{7}$$

$$F = \frac{8 \cdot 4(4-2)}{(\frac{3}{4})^2 \cdot 2} = \frac{512}{9} = 56.\overline{8}$$

$$R = \frac{4 \cdot 4}{\frac{3}{4}}\left(\sqrt{\frac{5}{4}} - 1\right) = \frac{32\sqrt{5} - 64}{3} \approx 2.5181$$

The formula can then be written as

$$V(t) = \frac{64}{3} + s \cdot \frac{32\sqrt{5} - 64}{3} - \frac{1024}{9} \cdot t + j \cdot \frac{512}{9} \cdot t$$

Applying the formula into a game

The formula to be used into the game physics is the velocity formula. Even though the position of the moving object is very relevant when talking about game physics, the jump of the object may not be the only thing affecting the object's position and movement. Thus the moving object should hold a position vector and a velocity vector. The velocity is continuously added to the object's position. In a video game program, this continous action is done every frame, typically 60 times per seconds.

The following example is a class definition with the minimum requirements to implement the formula.

```
class JumpingObject
{
   Vector position, velocity;

   void update(float delta) {
      //velocity formula gets applied here

      position += velocity * delta;
   }
}
```

The formula's implementation then goes as follows:

```
void update(float delta)
{
   if(buttonPressed)
   {
      if(onGround)
         velocity.y = V0 + s * R;
      velocity.y += F * delta;
   }

   velocity.y -= G * delta;

   position += velocity * delta;
}
```

`V0`, `R`, `F` and `G` are constants and should be assigned with the values found earlier. Notice how `j` got implemented from the if statement. However, it is not possible to do that for `s` since jump bonus from horizontal velocity is only applied once. `s` must be implemented as the ratio of speed over maximum speed. If the object has no maximum speed, you can select a value at which the object must jump max height and then ensure the ratio is not bigger than 1. If the ratio becomes bigger than 1, the object will reach heights higher than `Y3`.

A simple 2-dimensional implementation of that horizontal velocity ratio can be written as

```
void update(float delta)
{
   if(buttonPressed)
   {
      if(onGround)
         velocity.y += V0 + min(abs(velocity.x) / maxSpeed, 1) * R;
      velocity.y += F * delta;
   }

   velocity.y -= G * delta;

   position += velocity * delta;
}
```

The `abs` function prevents the ratio from becoming negative and the `min` function ensures the ratio never exceed 1.

Finally, horizontal movement must be implemented. This example presents an algorithm using 2 buttons as inputs to move the object horizontally. It is 2-dimensional but can be easily expanded to 3 dimensions.

```
//Constants
float acceleration, deceleration, maxSpeed;
float V0, G, R, F;

void update(float delta)
{
   int prevDir = signum(velocity.x);
    int newDir = 0;

    if(leftPressed != rightPressed) //if only one of them is pressed
        newDir = leftPressed ? -1 : 1;

    velocity.x -= prevDir * deceleration * delta;

    if(signum(player.velX) != prevDir)
        velocity.x = 0;

    if(abs(velocity.x) < maxSpeed || prevDir != newDir)
    {
        velocity.x += newDir * acceleration * delta;

        if(abs(player.velX) > maxSpeed && prevDir == newDir)
            velocity.x = maxSpeed * prevDir;
    }

   if(jumpPressed)
   {
      if(onGround)
         velocity.y += V0 + min(abs(velocity.x) / maxSpeed, 1) * R;
      velocity.y += F * delta;
   }

   velocity.y -= G * delta;
   position += velocity * delta;
}
```