Math221 – Mathematics for Computer Science

S1-2024 Assignment 1 - Sample Answer

(Each question carries 10 marks)

1. Let Z be the set of integers, ℝ be the set of real numbers, N be the set of natural numbers, $A = \{x \in \text{R}: -5 < x \leq 5\}$ and $\text{B} = \{x \in \text{N}: -5 < x \leq 5\}$. Find the following and specify them either directly or using set builder notations:

   (i) (Z – R)

   (ii) (N – R)

   (iii) (N – Z).

   (iv) $(A - B)$

   (v) $\{9, 3, 8\} \cap (\{-9, 9\} \cup \{1, 10\})$

Answer:

(i) $\varnothing$

(ii) $\varnothing$

(iii) $\varnothing$

(iv) $\{x \in \text{R}: (x \notin \text{N}) \wedge (-5 < x < 5)\}$

Or $\{x \in \text{R}: (-5 < x < 1) \vee (1 < x < 2) \vee (2 < x < 3) \vee (3 < x < 4) \vee (3 < x < 4)\}$

(v) $\{9\}$

2. Use element argument method to prove that for any two sets A and B, if A ⊆ B, then P(A) ⊆ P(B), where P(A) and P(B) are power sets of A and B respectively. You must state your reasons clearly for every statement in your proof.

Proof.

Let A and B be sets, such that A ⊆ B.
Let X ∈ P(A).
From definition of power set, X ⊆ A.
Let y ∈ X.

Since y ∈ X and X ⊆ A, from the definition of subset, y ∈ A.
Since A ⊆ B, from the definition of subset, y ∈ B.
Thus, from definition of subset, we have X ⊆ B.
Hence, from the definition of power set, X ∈ P(B).
Therefore, from the definition of subset, we have P(A) ⊆ P(B).

3. Prove using mathematical induction that if a and r are real numbers and r ≠ 1,

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

for all n ≥ 0.

Proof:

Basis Step:

For n = 0,

$$\text{LHS} = \sum_{i=0}^{0} ar^i = a \times r^0 = a$$

$$\text{RHS} = \frac{a(r^{n+1}-1)}{r-1} = \frac{a(r^1-1)}{r-1} = a$$

Hence, LHS = RHS

Hence, the formula holds.

Inductive Step:

Assume that the equality holds when n =k, where k ≥ 0, that is,

$$\sum_{i=0}^{k} ar^i = \frac{a(r^{k+1} - 1)}{r - 1}$$

We shall prove that the equality still holds when n = k+1, that is:

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2} - 1)}{r - 1}$$

When n=k+1,

$$\sum_{i=0}^{k+1} ar^i = \sum_{i=0}^{k} ar^i + ar^{k+1}$$

$$= \frac{a(r^{k+1}-1)}{r-1} + ar^{k+1} \qquad \text{from the assumption}$$

$$= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1}$$

$$= \frac{a(r^{k+2} - 1)}{r-1}$$

Hence, the formula holds when n = k+1.

Therefore, by the principle of mathematical induction, the given formula holds for all $n \geq 0$.

4. Let b $\in$ R. Prove that if $b^2 + 7b + 777$ is an irrational number, then b irrational numbers.

Proof:

We shall use Proof by Contradiction.

Let b be a rational number.

Since b is a rational number, from definition of rational number:

$$b = \frac{p}{q} \quad \text{where } p, q \in Z \text{ and } q \neq 0$$

Then,

$$b^2 + 7b + 777 = \left(\frac{p}{q}\right)^2 + 7\left(\frac{p}{q}\right) + 777$$

$$= \frac{p^2}{q^2} + \frac{7p}{q} + 777$$

$$= \frac{p^2 + 7pq + 777q^2}{q^2}$$

$= \frac{m}{n}$, where $m = p^2 + 7pq + 777q^2$, $n = q^2$, m, n $\in \mathbb{Z}$ $n \neq 0$

This is a Contradiction to the given assumption.

Therefore, if $b^2 + 7b + 777$ is an irrational number, b be an irrational number.

5. Using substitution method to prove the following equivalence:

$$(\sim x \Rightarrow y \vee z) \equiv (\sim x \wedge \sim y \Rightarrow z)$$

Proof:

$(\sim x \Rightarrow y \vee z)$	$\equiv \sim(\sim x) \vee (y \vee z)$	implication law
	$\equiv (\sim(\sim x) \vee y) \vee z$	associative law
	$\equiv (\sim\sim x \vee \sim \sim y) \vee z$	double negation law
	$\equiv \sim(\sim x \wedge \sim y) \vee z$	DeMozgan's law
	$\equiv (\sim x \wedge \sim y \Rightarrow z)$	implication law

Or

Proof:

$(\sim x \Rightarrow y \vee z) \equiv \sim(\sim x) \vee (y \vee z)$

$\equiv (\sim(\sim x) \vee y) \vee z$

$\equiv (\sim\sim x \vee \sim \sim y) \vee z$

$\equiv (x \vee y) \vee z$

$(\sim x \wedge \sim y \Rightarrow z) \equiv$……… $(x \vee y) \vee z$

………..

$\equiv (x \vee y) \vee z$

Therefore, $(\sim x \Rightarrow y \vee z) \equiv (\sim x \wedge \sim y \Rightarrow z)$

6. Let P, Q and R be simplest statements. Determine whether the following statement is a tautology, contradiction or contingent statement:

$$(Q \wedge P \Rightarrow R) \Rightarrow Q \wedge R$$

Answer:

P	Q	R	$Q \wedge P$	$Q \wedge P \Rightarrow R$	$Q \wedge R$	$(Q \wedge P \Rightarrow R) \Rightarrow Q \wedge R$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	F	T	T	T
F	T	F	F	T	F	F
F	F	T	F	T	F	F
F	F	F	F	T	F	F

Since $(Q \wedge P \Rightarrow R) \Rightarrow Q \wedge R$ is true only for 3 rows, the given statement is a contingent statement.

7. Prove that for any integer $m \in Z$, $m^2 + 3m + 9$ is odd.

Proof:

We shall use proof by cases: Case 1 – m is odd; Case 2- m is even.

Case 1 – m is odd

We shall use direct proof for proving this case.

Let m be an odd integer.

Then from definition, $m = 2p + 1$, where $p \in Z$.

Hence,

$$\begin{aligned} m^2 + 3m + 9 &= (2p + 1)^2 + 3(2p + 1) + 9 \\ &= (4p^2 + 4p + 1) + (6p + 3) + 9 \\ &= 4p^2 + 10p + 13 \\ &= 4p^2 + 10p + 12 + 1 \\ &= 2(2p^2 + 5p + 6) + 1 \\ &= 2x + 1 \quad \text{where } x = (2p^2 + 5 + 6),\ x \in Z. \end{aligned}$$

Hence, if m is odd, $m^2 + 3m + 9$ is odd.

Case 2 – m is even

We shall use direct proof for proving this case.

Let m be an even integer.

Then from definition, $m = 2q$, where $q \in Z$.

Hence,

$$\begin{aligned} m^2 + 3m + 9 &= (2q)^2 + 3(2q) + 9 \\ &= 4q^2 + 6q + 9 \\ &= 4q^2 + 6q + 8 + 1 \\ &= 2(2q^2 + 3q + 4) + 1 \\ &= 2y + 1 \quad \text{where } y = (2q^2 + 3q + 4),\ y \in Z. \end{aligned}$$

Hence, if m is even, $m^2 + 3m + 9$ is odd.

Therefore, from the proofs for the two cases, we conclude that for any integer $m \in Z$, $m^2 + 3m + 9$ is odd.

8. Specify the following statement in symbolic form:

(i) The sum of an odd integer and an even integer is odd.

(ii) Some integers are not natural numbers.

(iii) There are computes that do not use any Microsoft product.

Answer:

(i) The sum of an odd integer and an even integer is odd.

$\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}$, (m is odd)$\wedge$(n is even) $\Rightarrow$ (m + n) is odd

(ii) Some integers are not natural numbers.

$\exists\, m \in \mathbb{Z}, m \notin N$

(iii) There are computes that do not use any Microsoft product.

$\exists$ computer C, $\forall$ Microsoft product S, C does not use S

9. Determine whether the following statements are True or False.

(i) $\{a, b\} \not\subset \{a, b\}$;

True

(ii) $\varnothing \subseteq P(\{a, b\})$;

True

(iii) $\varnothing \in P(\{a, b\})$;

True

(iv) $\varnothing \not\subseteq P(\{a, b\})$;

False

(v) $\varnothing \in \{\{\varnothing\}\}$;

False

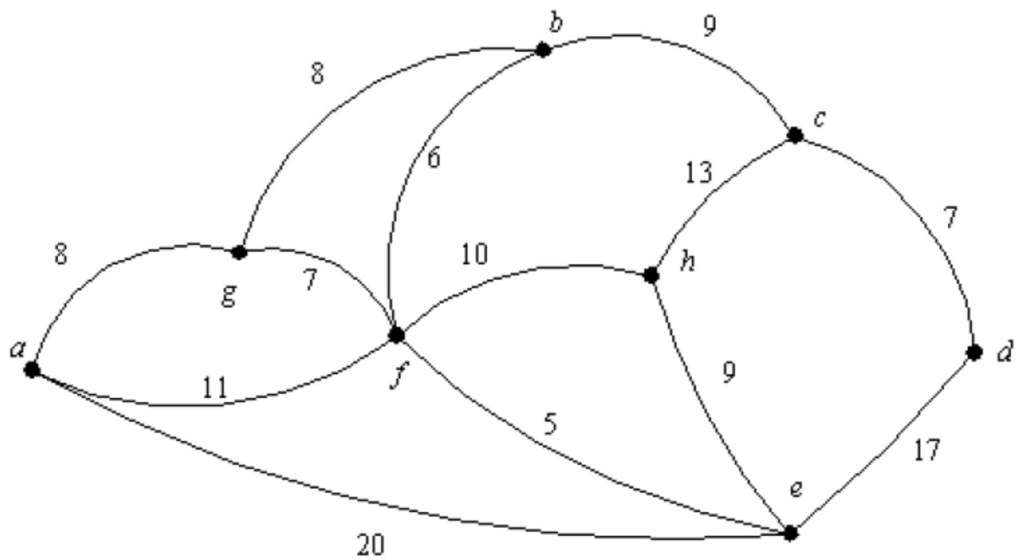
(vi) $\varnothing \not\subseteq \{\varnothing, a, b\}$;

False.

(vii) $\{\emptyset\} \in P(\{a, b\})$;

False

10. Start from vertex e, find a minimum spanning tree (MST) of the following graph using *Prim*'s algorithm: Show the minimum weight and the sequence of edges in the MST according to the order added.



Answer:

A possible minimum spanning tree can be formed by selecting the following edges:

(e, f), (f, b), (f, g), (g, a), (e, h), (b, c), (c, d)

The minimum total weight = 5 + 6 + 7 + 8 + 9 + 9 + 7 = 51