M 348 HOMEWORK 1

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1. Chapter 0.1 Ex (Exercises) # 2

Rewrite the following polynomials in nested form and evaluate at $x = -1/2$:

(a) $P(x) = 6x^3 - 2x^2 - 3x + 7$
(b) $P(x) = 8x^5 - x^4 - 3x^3 + x^2 - 3x + 1$
(c) $P(x) = 4x^6 - 2x^4 - 2x + 4$

Solution. Nested Form and when $x = -1/2$:

(a) $P(x) = 7 + x(-3 + x(-2 + x(6))$
$P(-1/2) = 7 + (-\frac{1}{2})(-3 + (-\frac{1}{2})(-2 + (-\frac{1}{2})(6))) = 7.25$

(b) $P(x) = 1 + x(-3 + x(1 + x(-3 + x(-1 + x(8)))))$
$P(-1/2) = 1+(-\frac{1}{2})(-3+(-\frac{1}{2})(1+(-\frac{1}{2})(-3+(-\frac{1}{2})(-1+(-\frac{1}{2})(8))))) = 2.8125 \approx 2.81$

(c) $P(x) = 4 + x(-2 + x^3(-2 + x^2(4)))$
$P(-1/2) = 4 + (-\frac{1}{2})(-2 + (-\frac{1}{2})^3(-2 + (-\frac{1}{2})^2(4))) = 4.9375 \approx 4.94$

2. Chapter 0.1 Ex (Exercises) # 6a

Explain how to evaluate the polynomial for a given input x, using as few operations as possible. How many multiplications and how many additions are required?

(a) $P(x) = a_0 + a_5x^5 + a_{10}x^{10} + a_{15}x^{15}$

Solution. We can apply nest multiplication (Horner's method) to evaluate the polynomial for a given input x, which reducing the number of multiplications and additions.
First, turn this equation to nest form.

$$P(x) = a_0 + x^5(a_5 + x^5(a_{10} + a_{15}x^5))$$

Note that we should first find x^5, that is:

$$x^5 = (x \cdot x)^2 \cdot x$$

This utilizes 3 multiplications.$(x \cdot x, x^2 \cdot x^2, x^4 \cdot x)$
Then, analysis this nest form we can conclude that number of operations required will be:

- **Multiplications:**
  - 3 multiplications to calculate x^5
  - 3 multiplications within the nested form of $P(y)$.
  - Total multiplications = 3 (for x^5) + 3 (in nested form) = 6 multiplications
- **Additions:**
  - 3 additions in the nested form.

Therefore, 6 multiplications and 3 additions are required to evaluate the polynomial.

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3. Chapter 0.4 CP (Computer Problems) # 1
Calculate the expressions that follow in double precision arithmetic (using Matlab, for example) for $x = 10^{-1}, ..., 10^{-14}$. Then, using an alternative form of the expression that doesn't suffer from subtracting nearly equal numbers, repeat the calculation and make a table of results. Report the number of correct digits in the original expression for each x.

(a) $\frac{1-secx}{tan^2x}$
(b) $\frac{1-(1-x)^3}{x}$

Solution. (a) First we try to find an alternative form of the expression that doesn't suffer from subtracting nearly equal numbers. By applying trigonometric identities:

$$secx = \frac{1}{cosx}, \quad tan^2x = \frac{sin^2x}{cos^2x}$$

$$\begin{aligned}
\frac{1-secx}{tan^2x} &= \frac{1-secx}{tan^2x} \cdot \frac{1+secx}{1+secx} \\
&= \frac{1-sec^2x}{tan^2x(1+secx)} \\
&= \frac{1-\frac{1}{cos^2x}}{\frac{sin^2x}{cos^2x}(1+secx)} \\
&= \frac{\frac{cos^2x-1}{cos^2x}}{\frac{sin^2x}{cos^2x}(1+secx)} \\
&= \frac{cos^2x-1}{sin^2x(1+secx)} \\
&= \frac{-sin^2x}{sin^2x(1+secx)} \\
&= \frac{-1}{1+secx}
\end{aligned}$$

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C:\Users\13464\AppData\Local\Programs\Python\Python310\python.exe C:\Users\13464\Desktop\M348\10.py
Results for Expression 1:
                  x     Original Expr1  Alternative Expr1
0   0.10000000000000 -0.498747913711413 -0.498747913711429
1   0.01000000000000 -0.499987499790956 -0.499987499791664
2   0.00100000000000 -0.499999875014289 -0.499999874999979
3   0.00010000000000 -0.499999993627931 -0.499999998750000
4   0.00001000000000 -0.500000041336852 -0.499999999987500
5   0.00000100000000 -0.500044450290837 -0.499999999999875
6   0.00000010000000 -0.510702591327569 -0.499999999999999
7   0.00000001000000  0.000000000000000 -0.500000000000000
8   0.00000000100000  0.000000000000000 -0.500000000000000
9   0.00000000010000  0.000000000000000 -0.500000000000000
10  0.00000000001000  0.000000000000000 -0.500000000000000
11  0.00000000000100  0.000000000000000 -0.500000000000000
12  0.00000000000010  0.000000000000000 -0.500000000000000
13  0.00000000000001  0.000000000000000 -0.500000000000000

Process finished with exit code 0
```

x	original expression	alternative expression	correct digits' number
10^{-1}	-0.498747913711413	-0.498747913711429	14
10^{-2}	-0.499987499790956	-0.499987499791664	12
10^{-3}	-0.499999875014289	-0.499999874999979	9
10^{-4}	-0.499999993627931	-0.499999998750000	9
10^{-5}	-0.500000041336852	-0.499999999987500	1
10^{-6}	-0.500044450290837	-0.499999999999875	1
10^{-7}	-0.510702591327569	-0.499999999999999	1
10^{-8}	0.000000000000000	-0.500000000000000	1
10^{-9}	0.000000000000000	-0.500000000000000	1
10^{-10}	0.000000000000000	-0.500000000000000	1
10^{-11}	0.000000000000000	-0.500000000000000	1
10^{-12}	0.000000000000000	-0.500000000000000	1
10^{-13}	0.000000000000000	-0.500000000000000	1
10^{-14}	0.000000000000000	-0.500000000000000	1

(b) First we try to find an alternative form of the expression that doesn't suffer from subtracting nearly equal numbers.

$$\begin{aligned}\frac{1-(1-x)^3}{x} &= \frac{1-(1-3x+3x^2-x^3)}{x}\\ &= \frac{3x-3x^2+x^3}{x}\\ &= 3-3x+x^2\end{aligned}$$

```
C:\Users\13464\AppData\Local\Programs\Python\Python310\python.exe C:\Users\13464\Desktop\M348\11.py
Results for Expression 2:
                   x     Original Expr2  Alternative Expr2
0   0.10000000000000  2.709999999999999   2.71000000000000
1   0.01000000000000  2.970099999999998   2.97010000000000
2   0.00100000000000  2.997000999999999   2.99700100000000
3   0.00010000000000  2.999700010000161   2.99970001000000
4   0.00001000000000  2.999970000083784   2.99997000010000
5   0.00000100000000  2.999997000041610   2.99999700000100
6   0.00000010000000  2.999999698660716   2.99999970000001
7   0.00000001000000  2.999999981767587   2.99999997000000
8   0.00000000100000  2.999999915154206   2.99999999700000
9   0.00000000010000  3.000000248221113   2.99999999970000
10  0.00000000001000  3.000000248221113   2.99999999997000
11  0.00000000000100  2.999933634839635   2.99999999999700
12  0.00000000000010  3.000932835561798   2.99999999999970
13  0.00000000000001  2.997602166487923   2.99999999999997

Process finished with exit code 0
```

x	original expression	alternative expression	correct digits' number
10^{-1}	2.709999999999999	2.71000000000000	2
10^{-2}	2.970099999999998	2.97010000000000	4
10^{-3}	2.997000999999999	2.99700100000000	6
10^{-4}	2.999700010000161	2.99970001000000	13
10^{-5}	2.999970000083784	2.99997000010000	10
10^{-6}	2.999997000041610	2.99999700000100	11
10^{-7}	2.999999698660716	2.99999970000001	7
10^{-8}	2.999999981767587	2.99999997000000	8
10^{-9}	2.999999915154206	2.99999999700000	8
10^{-10}	3.000000248221113	2.99999999970000	0
10^{-11}	3.000000248221113	2.99999999997000	0
10^{-12}	2.999933634839635	2.99999999999700	5
10^{-13}	3.000932835561798	2.99999999999970	0
10^{-14}	2.997602166487923	2.99999999999997	3

4. Additional Problem 1

Find both roots of the equation $3x^2 - 8^{14}x + 100 = 0$ with three-digit accuracy.

Solution. We want to find the roots of the equation

$$3x^2 - 8^{14}x + 100 = 0.$$

We use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 3$, $b = -8^{14}$, and $c = 100$. Substituting these values, we get

$$x = \frac{-(-8^{14}) \pm \sqrt{(-8^{14})^2 - 4 \cdot 3 \cdot 100}}{2 \cdot 3}.$$

Simplifying further, we can find the roots of the equation, which are:

$$x_1 = 2.27 \times 10^{-11}, \quad x_2 = 1.47 \times 10^{12}$$

5. Additional Problem 2

The function $f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^{x} - 1.99$ is given. Use three-digit rounding arithmetic, the assumption that $e^{1.53} = 4.62$, and the fact that $e^{nx} = (e^x)^n$ to evaluate $f(1.53)$ with the following methods:

(a) naive (adding left to right);
(b) naive (adding right to left);
(c) Show that the polynomial nesting technique described in class can also be applied to the evaluation of f(x). Evaluate $f(1.53)$ by first nesting the calculations.
(d) Compare the approximations in parts (a), (b) and (c) to the true three-digit result $f(1.53) = -7.61$ by calculating the absolute and the relative errors.

Solution.

1. Naive Evaluation (Left to Right).

$$\begin{aligned}
f(1.53) &= 1.01(4.62)(4.62)(4.62)(4.62) - 4.62(4.62)(4.62)(4.62) - 3.11(4.62)(4.62) + 12.2 \times 4.62 - 1.99 \\
&= 4.67(4.62)(4.62)(4.62) - 21.3(4.62)(4.62) - 14.4(4.62) + 56.4 - 1.99 \\
&= 21.6(4.62)(4.62) - 98.4(4.62) - 66.5 + 56.4 - 1.99 \\
&= 99.8(4.62) - 455 - 66.5 + 56.4 - 1.99 \\
&= 461 - 455 - 66.5 + 56.4 - 1.99 \\
&\approx -6.09.
\end{aligned}$$

2. Naive Evaluation (Right to Left).

$$\begin{aligned}
f(1.53) &= -1.99 + 12.2 \times 4.62 - 3.11(4.62)(4.62) - 4.62(4.62)(4.62)(4.62) + 1.01(4.62)(4.62)(4.62)(4.62) \\
&= -1.99 + 12.2 \times 4.62 - 3.11 \times 21.3 - 4.62 \times 98.6 + 1.01 \times 455 \\
&= -1.99 + 56.4 - 66.2 - 455 + 460 \\
&\approx -7.00.
\end{aligned}$$

3. Polynomial Nesting Technique. Rewrite the function in nested form

$$f(x) = (((1.01e^x - 4.62)\, e^x - 3.11)\, e^x + 12.2)e^x - 1.99$$

$$
\begin{aligned}
f(1.53) &= (((1.01 \times 4.62 - 4.62) \times 4.62 - 3.11) \times 4.62 + 12.2) \times 4.62 - 1.99 \\
&= ((4.67 - 4.62) \times 4.62 - 3.11) \times 4.62 + 12.2) \times 4.62 - 1.99 \\
&= ((0.05 \times 4.62 - 3.11) \times 4.62 + 12.2) \times 4.62 - 1.99 \\
&= ((0.23 - 3.11) \times 4.62 + 12.2) \times 4.62 - 1.99 \\
&= (-2.88 \times 4.62 + 12.2) \times 4.62 - 1.99 \\
&= (-13.3 + 12.2) \times 4.62 - 1.99 \\
&= -1.10 \times 4.62 - 1.99 \\
&= -5.08 - 1.99 \\
&\approx -7.07.
\end{aligned}
$$

4. Comparison with True Value. The true value at $x = 1.53$ is -7.61. According to the proposition:

$$
AbsoluteError = |X_c - X|, \quad RelativeError = \frac{|X_c - X|}{|X|}
$$

- Method 1: Absolute Error $= |-6.09-(-7.61)| = 1.52$, Relative Error $= \frac{1.52}{7.61} \approx 0.2000$ (20.0%).
- Method 2: Absolute Error $= |-7.00-(-7.61)| = 0.61$, Relative Error $= \frac{0.61}{7.61} \approx 0.0802$ (8.02%).
- Method 3: Absolute Error $= |-7.07-(-7.61)| = 0.54$, Relative Error $= \frac{0.54}{7.61} \approx 0.0710$ (7.10%).