Ejercicios clase 1

Saturday, August 9, 2025 4:23 PM

Taller

ejercicios 3 y 4 de la sección 1.4.3

3. Considere lo expuesto en la sección 1.4.3 y demuestre que:

$$A^i_j\,\tilde{A}^j_k = \delta^i_k$$

y además, como un caso especial, establezca la relación con los cosenos directores que satisfacen:

$$\cos(\alpha)^2 + \cos(\beta)^2 + \cos(\gamma)^2 = 1\,.$$

4. Considere el radio vector posición $\mathbf{r} = x^i\mathbf{e}_i \equiv x\hat{\imath} + y\hat{\jmath}$ en 2 dimensiones. Dado el conjunto de transformaciones que se indican a continuación, demuestre en cuales casos las componentes de $\mathbf{r}$ transforman como verdaderas componentes de vectores.

$(x,y) \to (-y,x)$, $(x,y) \to (x,-y)$, $(x,y) \to (x-y,x+y)$, $(x,y) \to (x+y,x-y)$.

$\delta^j_k = 1$ si $j = k$ $\wedge$ $\delta^j_k = 0$ si $j \neq k$

Podemos representar como

$$x^{i'} = A^{i'}_j x^j \qquad A^{i'}_k = \frac{\partial x^{i'}}{\partial x^k}$$

$$x^j = \tilde{A}^j_{i'} x^{i'} \; ; \; \tilde{A}^j_{i'} = \frac{\partial x^j}{\partial x^{i'}} \qquad \tilde{A}^j_{i'} = \frac{\partial x^j}{\partial x^{i'}}$$

$$\delta^j_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$j \neq k$

$$A^{i'}_j = \frac{\partial x^{i'}}{\partial x^j} \qquad \frac{\partial x^{i'}}{\partial x^k} \cdot \frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^k}$$

$$A^{i'}_k \cdot \tilde{A}^j_{i'}$$

$$x^j = x^j(x^{i'}(x^k))$$

$j = k$

$$\frac{\partial x^{i'}}{\partial x^k} \cdot \frac{\partial x^k}{\partial x^{i'}} \qquad \frac{\partial x^j}{\partial x^k} = \frac{\partial x^j}{\partial x^{i'}} \cdot \frac{\partial x^{i'}}{\partial x^k}$$

$$\frac{\partial x^{i'}}{\partial x^{i'}} = 1 \qquad \frac{\partial x^j}{\partial x^k} = A^j_{i'} \cdot A^{i'}_k$$

en particular:

fijemos $i = 1$ y consideremos la primera base nueva sus componentes A_{1j}. Por definición entre $\hat{e}_1$ y $\hat{e}_j$.

$\cos\alpha = \hat{e}_1' \cdot \hat{e}_1 = A_{11}$, $\cos\beta = \hat{e}_1' \cdot \hat{e}_2 = A_{12}$, $\cos\gamma = \hat{e}_1' \cdot \hat{e}_3 = A_{13}$

la condición ortogonal $A_{1j} \cdot A_{1j} = 1$

$A_{11}^2 + A_{12}^2 + A_{13}^2 = 1$,

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

4. Considere...

$r = x^i \hat{u}_i$

$x^{i'} = A^{i'}_j x^j$

A debe ser ortogonal, $A^T \cdot A = I$

a) $(x^1, x^2) \to (-x^2, x^1)$

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = I$$

✓ transforma y ortonormal

b) $(x^1, x^2) \to (x^1, -x^2)$

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

transforma y ortonormal

$$A^T \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

✓ transforma y ortonormal

c) $(x^1, x^2) \to (x^1 - x^2, x^1 + x^2)$

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$\det(A) \neq 0$ transforma

d) $(x^1, x^2) \to (x^1 + x^2, x^1 - x^2)$

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$\det(A) \neq 0$ transforma

$r = x^i \hat{u}_j \equiv x\hat{\imath} + y\hat{\jmath}$

$(x,y) \to (-y,x)$, $(x,y) \to (x,-y)$

$(x,y) \to (x-y, x+y)$, $(x,y) \to (x+y, x-y)$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$u = \begin{bmatrix} a \\ b \end{bmatrix} ; \; v = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} + \begin{pmatrix} -d \\ c \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} -b-d \\ a+c \end{pmatrix}$$

$$T\left(\alpha \begin{pmatrix} a \\ b \end{pmatrix}\right) = \alpha T\begin{pmatrix} a \\ b \end{pmatrix}$$

$$T\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \begin{pmatrix} -\alpha b \\ \alpha a \end{pmatrix} = \alpha \begin{pmatrix} -b \\ a \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} a+c+b+d \\ a+c-b-d \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ a-b+c-d \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix}$$

$$T\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \begin{pmatrix} \alpha a + \alpha b \\ \alpha a - \alpha b \end{pmatrix} = \begin{pmatrix} \alpha(a+b) \\ \alpha(a-b) \end{pmatrix}$$

$$\alpha T\begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} a+c \\ -b-d \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} c \\ -d \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix}$$

$$T\left(\alpha \begin{pmatrix} a \\ b \end{pmatrix}\right) = T\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \begin{pmatrix} \alpha a \\ -\alpha b \end{pmatrix} = \alpha \begin{pmatrix} a \\ -b \end{pmatrix} = \alpha T\begin{pmatrix} a \\ b \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} a+c-(b+d) \\ a+c+b+d \end{pmatrix} = \begin{pmatrix} a+c-b-d \\ a+c+b+d \end{pmatrix} \neq \begin{pmatrix} a-b+c-d \\ a+b+c+d \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix}$$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a-b \\ a+b \end{pmatrix}; \; T\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c-d \\ c+d \end{pmatrix}$$

$$T\begin{pmatrix} a+c \\ b+d \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} + T\begin{pmatrix} c \\ d \end{pmatrix}$$

$$T\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \begin{pmatrix} \alpha a - \alpha b \\ \alpha a + \alpha b \end{pmatrix} = \begin{pmatrix} \alpha(a-b) \\ \alpha(a+b) \end{pmatrix} = \alpha \begin{pmatrix} a-b \\ a+b \end{pmatrix} = \alpha T\begin{pmatrix} a \\ b \end{pmatrix}$$

Ejercicios clase 2

Sunday, August 10, 2025 3:46 PM

De la 1.6.6 resuelva 2, 5 y 6.

$\cos(3\theta) + i\,\mathrm{sen}(3\theta)$ $\qquad \cos(3\theta) = \cos^3(\theta) - 3\cos\theta\,\mathrm{sen}^2\theta$

$= [\cos(\theta) + i\,\mathrm{sen}(\theta)]^3$

$= \cos^3\theta + 3\cos^2\theta\,(i\,\mathrm{sen}\theta) + 3\cos\theta\,(i\,\mathrm{sen}\theta)^2 + (i\,\mathrm{sen}\theta)^3$ $\qquad i^3 = i^2 \cdot i = -i$

$= \cos^3\theta + 3i\cos^2\theta \cdot \mathrm{sen}\,\theta - 3\cos\theta\,\mathrm{sen}^2\theta - i\,\mathrm{sen}^3\theta$

$= \cos^3\theta - 3\cos\theta\,\mathrm{sen}^2\theta + i(3\cos^2\theta \cdot \mathrm{sen}\,\theta - \mathrm{sen}^3\theta)$ $\qquad 3(1-\mathrm{sen}^2\theta)\cdot\mathrm{sen}\theta - \mathrm{sen}^3\theta$

$= \cos^3\theta - 3\cos\theta\,\mathrm{sen}^2\theta + i(3\,\mathrm{sen}\theta - 4\,\mathrm{sen}^3\theta)$ $\qquad 3(\mathrm{sen}\theta - \mathrm{sen}^3\theta) - \mathrm{sen}^3\theta$

$\Rightarrow \cos(3\theta) = \cos^3\theta - 3\cos\theta\,\mathrm{sen}^2\theta$ $\qquad 3\,\mathrm{sen}\theta - 3\,\mathrm{sen}^3\theta - \mathrm{sen}^3\theta$

$\mathrm{sen}(3\theta) = 3\,\mathrm{sen}\theta - 4\,\mathrm{sen}^3\theta$ $\qquad 3\,\mathrm{sen}\theta - 4\,\mathrm{sen}^3\theta$

$e^{i\theta} = \cos\theta + i\,\mathrm{sen}\theta$

$e^{in\theta} = \cos n\theta + i\,\mathrm{sen}\,n\theta$

$z = re^{i\theta} = r(\cos\theta + i\,\mathrm{sen}\theta)$

$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\,\mathrm{sen}(n\theta))$

$z^{1/n} = r^{1/n} e^{i\frac{\theta}{n}} = r^{\frac{1}{n}}(\cos(\frac{\theta}{n}) + i\,\mathrm{sen}(\frac{\theta}{n}))$

5. a) $\sqrt{2i} = (2i)^{1/2} = 2^{1/2} \cdot i^{1/2}$

$|z| = r = \sqrt{2^2} = 2$

$(2i)^{1/2} = 2^{1/2} e^{i\frac{\theta}{2}}$

$2^{1/2} e^{i\frac{\theta}{2}} = 2^{1/2}\cos\left(\frac{\theta}{2}\right) + i\,\mathrm{sen}\left(\frac{\theta}{2}\right)$

$\mathrm{Arg}(z) = \tan(\theta) = \frac{b}{a} = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\theta = \tan^{-1}\left(\frac{2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$ (primer cuadrante)

$\sqrt{2i} = 2^{1/2}\cos\left(\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}\right)$

$= 2^{1/2}(2^{1/2}/2 + i\,2^{1/2}/2)$

$= 1 + i$

b) $\sqrt{1-\sqrt{3}\,i}$ → cuarto cuadrante

$r = \sqrt{1^2 + (-\sqrt{3})^2} = r = \sqrt{4} = 2$

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

$(1-\sqrt{3}\,i)^{1/2} = 2^{1/2}\left(\cos\left(-\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{6}\right)\right)$

$2^{1/2}\left(\cos\left(\frac{\pi}{6}\right) - i\,\mathrm{sen}\left(\frac{\pi}{6}\right)\right)$

$2^{1/2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$

c) $i^{1/3}$

$r = 1 \quad \theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$i^{1/3} = 1^{1/3}\cos\left(\frac{\pi}{6}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6}\right)$

$i^{1/3} = \frac{\sqrt{3}}{2} + \frac{i}{2}$

d) $8^{1/6} = 1.414$

e) $\sqrt[4]{-8-8\sqrt{3}\,i}$

$r = \sqrt{8^2 + (8\sqrt{3})^2} = 16$

$\theta = \tan^{-1}\left(\frac{-8\sqrt{3}}{-8}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

$(-8-8\sqrt{3}\,i)^{1/4} = 16^{1/4}\cos\left(\frac{4\pi}{3\cdot 4}\right) + i\,\mathrm{sen}\left(\frac{4\pi}{3\cdot 4}\right)$

$= 16^{1/4}\cos\left(\frac{\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{3}\right)$

$= (16^{1/2})^{1/2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 1 + i\sqrt{3}$

6) Demuestre que:

a) $\mathrm{Log}(-ie) = 1 - \frac{\pi}{2}i$. $\qquad \mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$

$|z| = \sqrt{(-e)^2}$ $\qquad -\pi < \mathrm{Arg}(z) \leq \pi$

$|z| = e$ $\qquad \mathrm{Arg}(z) = \tan^{-1}\left(-\frac{e}{0}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$

$\log(-ie) = \ln(e) - i\frac{\pi}{2}$

$= 1 - i\frac{\pi}{2}$

b) $\log(1-i) = \frac{1}{2}\ln(2) - \frac{\pi}{4}i$ $\qquad |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\ln(2^{1/2}) = \frac{1}{2}\ln(2)$

$\mathrm{Log}(1-i) = \ln|1-i| + i\,\mathrm{Arg}(1-i)$

$\mathrm{Log}(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}$ $\qquad \mathrm{Arg}(z) = \tan^{-1}\left(-\frac{1}{1}\right) = -\tan^{-1}(1) = -\frac{\pi}{4}$

c) $\mathrm{Log}(e)$ $\qquad \mathrm{Log}(z) = \ln|z| + i\,(\mathrm{Arg}\,z + 2\pi n)$

$\mathrm{Log}(e) = 1 + i(0 + 2\pi n)$ $\qquad \ln|z| = \ln(e) = 1$

$\mathrm{Arg}(z) = \tan^{-1}(0) = 0$

$\log(e) = 1 + i\,2\pi n$

d) $\log(i) = (2n + \frac{1}{2})\pi i$ $\qquad \ln|i| = \ln|1| = 0$

$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

$\log(i) = i\left(\frac{\pi}{2} + 2\pi n\right)$

$\log(i) = i\pi\left(\frac{1}{2} + 2n\right)$

• De la sección 1.5.7 2a, 2d, 2f

2a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$\phi = \phi(x,y,z)$; $\psi(x,y,z)$

$\nabla(\phi\psi) = \left(\frac{\partial(\phi\psi)}{\partial x}, \frac{\partial(\phi\psi)}{\partial y}, \frac{\partial(\phi\psi)}{\partial z}\right)$ $\qquad \left(\overset{\triangle}{\frac{\partial\phi}{\partial x}}\psi + \overset{\nabla}{\frac{\partial\psi}{\partial x}}\phi, \overset{\triangle}{\frac{\partial\phi}{\partial y}}\psi + \overset{\nabla}{\frac{\partial\psi}{\partial y}}\phi, \overset{\triangle}{\frac{\partial\phi}{\partial z}}\psi + \overset{\nabla}{\frac{\partial\psi}{\partial z}}\phi\right)$

$\frac{\partial(\phi\psi)}{\partial x} = \frac{\partial\phi}{\partial x}\psi + \frac{\partial\psi}{\partial x}\phi$

$\left(\psi\frac{\partial\phi}{\partial x}, \psi\frac{\partial\phi}{\partial y}, \psi\frac{\partial\phi}{\partial z}\right) + \left(\phi\frac{\partial\psi}{\partial x}, \phi\frac{\partial\psi}{\partial y}, \phi\frac{\partial\psi}{\partial z}\right)$

$\frac{\partial(\phi\psi)}{\partial y} = \frac{\partial\phi}{\partial y}\cdot\psi + \frac{\partial\psi}{\partial y}\cdot\phi$

$\psi\nabla(\phi) + \phi\nabla(\psi)$ ∴

$\frac{\partial(\phi\psi)}{\partial z} = \frac{\partial\phi}{\partial z}\cdot\psi + \frac{\partial\psi}{\partial z}\cdot\phi$

$\nabla(\phi) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

$\nabla(\psi) = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right)$

2d) $\nabla\cdot(\nabla\times a)$ ¿Qué puede decir de $\nabla\times(\nabla\cdot a)$?

$\nabla\cdot(\nabla\times a) = \partial^i(\nabla\times a)_i$

$= \partial^i(\epsilon_{ijk}\partial^j a^k)$

$= -\epsilon_{jik}\partial^j(\partial^i a^k)$

$= \epsilon_{ijk}\partial^j(\partial^i a^k) = -\epsilon_{ijk}\partial^j(\partial^i a^k)$

$= \epsilon_{ijk}\partial^j(\partial^i a^k) = 0$

$= \nabla\cdot(\nabla\times a) = 0$

$\nabla\times(\nabla\cdot a)$ no está definido si a es un escalar, esta se define para campos vectoriales.

f) $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

Sup un $a(x)$ (campo vectorial)

$[\nabla\times(\nabla\times A)]_i = \epsilon_{ijk}\partial_j(\nabla\times A)_k = \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m)$,

donde $\partial_j = \frac{\partial}{\partial x^j}$

ahora, $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$,

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m$. $\qquad (\delta_{im}\delta_{jl})\partial_j\partial_l A_m = \partial_j\partial_i A_j$

$= (\delta_{il}\delta_{jm})\partial_j\partial_l A_m = \partial_j\partial_i A_j$. $\qquad = \nabla^2 A_i$

$= \partial_i(\partial_j A_j) - \nabla^2 A_i$ $\qquad \partial_j A_j = \nabla\cdot A$

$= \nabla(\nabla\cdot A) - \nabla^2 A_i$