6. En el caso $\mathbb{R}^3$, si $\{e_i\}$ define un sistema de coordenadas [...] demuestre que:

a) $$e^i = \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)}$$

$e_i e^i = \delta_i^j$ | como la base es dextrógira, ; ese triple producto es | positivo.
$e^i \downarrow$ base dual

Denotamos:
$V := e_1 \cdot (e_2 \times e_3)$

Ahora sup un $e_m \in \mathbb{R}^3$

$$e_m \cdot e^i = e_m \cdot \frac{e_j \times e_k}{e_i \cdot (e_j \times e_k)} \Rightarrow \frac{e_m \cdot (e_j \times e_k)}{e_i \cdot (e_j \times e_k)}$$ | si $m = i$ $\frac{e_i \cdot (e_j \times e_k)}{e_i \cdot (e_j \times e_k)} = 1$

si $m = j$ $e_j \times e_k = v_a$ a ser $e_j \times e_k \perp e_j \wedge$
$$\frac{e_j \cdot (e_j \times e_k)}{e_i \cdot (e_j \times e_k)} = 0$$
$e_j \times e_k \perp e_k$
$\Rightarrow e_j \cdot (e_j \times e_k) = 0$
$e_k \cdot (e_j \times e_k) = 0$

si $m = k$
$$\frac{e_k \cdot (e_j \times e_k)}{e_i \cdot (e_j \times e_k)} = 0$$

Entonces $$e_m \cdot e_i = \begin{cases} 1, & m = i \\ 0, & m \neq i \end{cases} = \delta_m^i$$ esto prueba que la formula de la base es recíproca.

b) $V = e_1 \cdot (e_2 \times e_3)$ $\quad V \cdot \tilde{V} = 1$
$\tilde{V} = e^1 \cdot (e^2 \times e^3)$

$E = e_1 | e_2 | e_3$. el tripe producto de $V = \det(E)$
$F = e^1 | e^2 | e^3 = (E^T)^{-1} \Rightarrow \tilde{V} = \det[(E^T)^{-1}]$ | $\det(A) = \det(A^T)$
$\det[(E^T)^{-1}] = \det(E^{-1}) = \frac{1}{\det(E)}$
$\Rightarrow V \cdot \tilde{V} = \cancel{\det(E)} \cdot \frac{1}{\cancel{\det(E)}} = 1$

c) ¿Qué vector satisface $a \cdot e^i = 1$? Demuestre que a es unico
$a \cdot e^i = a \cdot e^1 + a \cdot e^2 + a \cdot e^3 = 1$ $\quad a \cdot e^i = \frac{1}{3}$
$a \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + a \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + a \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = 1$
$a = e_1 + e_2 + e_3$
$a = A^j e_j \Rightarrow a \cdot e^i = A^j (e_j \cdot e^i) = A^j \delta_j^i = A^i$
$a \cdot e^i = 1$ da $A^i = 1$ para cada i. $\Rightarrow a = e_1 + e_2 + e_3$

d) Dado $w_1 = (4, 2, 1)$; $w_2 = (3, 3, 0)$, $w_3 = (0, 0, 2)$ $\quad a = (1, 2, 3)$
$$w^1 = \frac{w_2 \times w_3}{w_1 \cdot (w_2 \times w_3)} \quad w^2 = \frac{w_3 \times w_1}{w_1 \cdot (w_2 \times w_3)}, \quad w^3 = \frac{w_1 \times w_2}{w_1 (w_2 \times w_3)}$$
$w_2 \times w_3 = (3, 3, 0) \times (0, 0, 2) = (6, -6, 0)$
$w_1 \cdot (w_2 \times w_3) = (4, 2, 1) \cdot (6, -6, 0) = 12$
$w_3 \times w_1 = (0, 0, 2) \times (4, 2, 1) = (-2, 8, 0)$
$w_1 \times w_2 = (4, 2, 1) \times (3, 3, 0) = (-1, 1, 2)$
$w^1 = \frac{1}{12}(6, -6, 0)$; $w^2 = \frac{1}{12}(-2, 8, 0)$; $w^3 = \frac{1}{12}(-1, 1, 2)$
$w^1 = (\frac{1}{2}, -\frac{1}{2}, 0)$; $w^2 = (-\frac{1}{3}, \frac{2}{3}, 0)$; $w^3 = (-\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

$a^1 = w^1 \cdot a = -\frac{1}{2}$, $a^2 = w^2 \cdot a = 1$, $a^3 = w^3 \cdot a = \frac{7}{4}$
$a^i = (-0.5, 1, 1.75) \rightarrow$ contravariantes
Covariantes se obtiene proyectando en la base original
$a_i = w_i \cdot a \Rightarrow a_i = (11, 9, 6)$

D) Considere el espacio vectorial con producto interno interno
$\langle a | b \rangle = Tr(A^\dagger B)$
$\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ hallar $\{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$
$\sigma_0 \cdot \sigma^0 = 1$; ... ; $\sigma_3 \cdot \sigma^3 = 1$ por $\sigma_i \cdot \sigma^j = \delta_i^j$
$Tr(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$
la base dual $\{\tau^\mu\}$ está definida por $\tau^\mu(\sigma_\nu) = \delta_\nu^\mu$
como $\langle A | \cdot \rangle = Tr(A \cdot) \Rightarrow Tr(\tau^\mu \sigma_\nu) = \delta_\nu^\mu$
$Tr(\sigma_\mu \cdot \sigma_\nu) = 2\delta_{\mu\nu}$ resulta inmediatamente
$\tau^\mu(\cdot) = \frac{1}{2} Tr(\sigma_\mu \cdot)$, $\tau^\mu = \frac{1}{2}\sigma_\mu$
la base dual será
$\{\frac{1}{2}\sigma_\mu\}$

Un hermítico genérico
$X = \begin{pmatrix} a & c - id \\ c + id & b \end{pmatrix}$ $\quad a, b, c, d \in \mathbb{R}$
$e^i = \sigma^i \cdot e \Rightarrow e^i = \frac{1}{2} Tr(\sigma_i e)$ $\quad e^0 = \frac{a+b}{2}$ $\quad e^1 = c$, $e^2 = d$, $e^3 = \frac{a-b}{2}$
$w_e(\cdot) := \langle e | \cdot \rangle = Tr(e \cdot)$

en la base dual $\{\tau^\mu\}$
$w_e = (w_e)_\mu \tau^\mu$, $(w_e)_\mu$
En la base dual $\{\tau^\mu\}$
$w_e = (w_e)_\mu \tau^\mu$, $(w_e)_\mu = w_e(\sigma_\mu) = Tr(e\sigma_\mu) = 2e^\mu$.
$(w_e)_0 = a + b$ $\quad (w_e)_1 = 2c$ $\quad (w_e)_2 = 2d$, $(w_e)_3 = a - b$.