Ejercicios clase 5

Sunday, August 24, 2025 9:02 PM

5. Considere

$A \leftrightarrow \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = A^\dagger = \begin{pmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{pmatrix}$ es decir $\begin{cases} z_1^* = z_1 & \text{real} \\ z_4^* = z_4 & \text{real}, \\ z_2^* = z_3 & \text{complejos} \end{cases}$ $z_2 = z_3^*$

muestre que las matrices de Pauli $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ forman una base para este espacio vectorial

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} z_1 & z_2 \\ z_2^* & z_4 \end{pmatrix} = \begin{pmatrix} \alpha & z \\ z^* & \beta \end{pmatrix} = \begin{pmatrix} \alpha & x+iy \\ x-iy & \beta \end{pmatrix} = A$

$A = a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3 = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix}$

$\begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} = \begin{pmatrix} \alpha & x+iy \\ x-iy & \beta \end{pmatrix}$

$\alpha = a+d \quad a = \alpha - d$

$a-d = \beta \quad \alpha - d - d = \beta \rightarrow \alpha - 2d = \beta$

$b - ic = x+iy$ $\quad -2d = \beta - \alpha$

$b = x$ $\quad a - \frac{\alpha-\beta}{2} = \beta \quad d = \frac{\alpha-\beta}{2}$

$-c = y \quad c = -y$ $\quad a = \beta + \frac{\alpha-\beta}{2}$

$a = \frac{\alpha+\beta}{2} \quad b = x$ $\quad a = \frac{2\beta}{2} + \frac{\alpha-\beta}{2} = \frac{\alpha+\beta}{2}$

$c = -y$

$b = \frac{\alpha-\beta}{2}$ $\hookrightarrow$ Podemos escribir cualquier hermítico como una combinación de las matrices de pauli.

b) ortogonal con $\langle a|b\rangle = Tr(A^\dagger B)$

$\langle \sigma_0|\sigma_1\rangle = Tr(\sigma_0^\dagger \sigma_1)$

$\sigma_0^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Tr(\sigma_0^\dagger \sigma_1) = 0$

$\langle \sigma_i|\sigma_j\rangle = Tr(\sigma_i^\dagger \sigma_j) = Tr(\sigma_i \sigma_j)$

$\sigma_0 \sigma_j = \sigma_j \rightarrow Tr(\sigma_0 \sigma_j) = Tr(\sigma_j)$

Pero $Tr(\sigma_i) = 0$; para $i = 1, 2, 3$

$Tr(\sigma_0 \sigma_0) = 2$

$\sigma_i^2 = I$

$= \delta_{ij} I \rightarrow \sigma_1 \cdot \sigma_1, \sigma_2 \cdot \sigma_2, \sigma_3 \cdot \sigma_3$

$$\left.\begin{aligned} \sigma_1 \cdot \sigma_2 &= i\sigma_3 \\ \sigma_2 \cdot \sigma_3 &= i\sigma_1 \\ \sigma_3 \cdot \sigma_2 &= -i\sigma_1 \\ \sigma_3 \cdot \sigma_1 &= i\sigma_2 \\ \sigma_1 \cdot \sigma_3 &= -i\sigma_2 \end{aligned}\right\} i\varepsilon_{ij}^k \sigma_k$$

$\sigma_i \cdot \sigma_j = \delta_{ij} I + i\varepsilon_{ij}^k \sigma_k$

$Tr(\sigma_i \cdot \sigma_j) = \delta_{ij} Tr(I) + i\varepsilon_{ij}^k Tr(\sigma_k)$

$Tr(\sigma_i \cdot \sigma_j) = 2\delta_{ij} + i\varepsilon_{ij}^k Tr(\sigma_k)$

con $k = 1, 2, 3$

$i\varepsilon_{ij}^k Tr(\sigma_k) = 0$

$\Rightarrow Tr(\sigma_i \cdot \sigma_j) = 2\delta_{ij}$

cuando $i \neq j$ $Tr(\sigma_i \cdot \sigma_j) = 0$

c) $V_{real} = \{A \in \mathbb{H}_{2\times2} : A$ tiene todas las entradas reales$\}$

$\begin{pmatrix} \alpha & x \\ x & \beta \end{pmatrix} = A, \quad \alpha, x, \beta \in \mathbb{R}$

$A = a\sigma_0 + b\sigma_1 + c\sigma_3$

$A = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix}$

$\begin{pmatrix} \alpha & x \\ x & \beta \end{pmatrix} = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix}$

$\alpha = a+c \quad x = b \quad \beta = a-c$