# Ejercicio clase 3

Sunday, August 17, 2025 4:00 PM

Sección 2.1.5 ejercicio 10

10. Sea $\mathcal{P}_n$ el conjunto de todos los polinomios de grado $n$, en $x$, con coeficientes reales:

$$|p_n\rangle = p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = \sum_{i=0}^{n-1} a_i x^i .$$

(a). Demostrar que $\mathcal{P}_n$ es un espacio vectorial respecto a la suma de polinomios y a la multiplicación de polinomios por un número (número real).

(b). Si los coeficientes $a_i$ son enteros ¿$\mathcal{P}_n$ será un espacio vectorial? ¿Por qué?

(c). ¿Cuál de los siguientes subconjuntos de $\mathcal{P}_n$ es un subespacio vectorial?

I. El polinomio cero y todos los polinomios de grado $n-1$.

II. El polinomio cero y todos los polinomios de grado par.

III. Todos los polinomios que tienen a $x$ como un factor (grado $n > 1$).

IV. Todos los polinomios que tienen a $x-1$ como un factor.

$$|P_n\rangle \doteq P(x) = \sum_{i=0}^{n-1} a_i x^i$$

a) Sup $|P_n\rangle = \sum_{i=0}^{n-1} a_i x^i \wedge |q_n\rangle = \sum_{i=0}^{n-1} b_i x^i$

$$|P_n + q_n\rangle = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} b_i x^i$$

$$= \sum_{i=0}^{n-1} a_i x^i + b_i x^i$$

$$= \sum_{i=0}^{n-1} (a_i + b_i) x^i \Rightarrow \in \mathbb{V}$$

$$|\alpha P_x\rangle = \alpha |P_x\rangle \quad \alpha \in \mathbb{R}$$

$$= \sum_{i=0}^{n-1} \alpha a_i x^i = \alpha \sum_{i=0}^{n-1} a_i x^i$$

b) Si los coeficientes $a_i$ son $\mathbb{Z}$

· El problema está en si definimos un $\alpha \in \mathbb{R}$ ej: $\alpha = \frac{1}{2}$ $P(x) = x \Rightarrow \alpha P(x) = \frac{1}{2}x \notin$ espacio vectorial por tanto ya no sería espacio vectorial.

C.I) El polinomio 0 y los polinomios hasta $n-1$ para pertenecer a $\omega$ (el subespacio), el grado del polinomio tiene que ser $n-1$, ahora sup:

$$|P_x\rangle = x^{n-1} \in S \wedge |q_x\rangle = -x^{n-1} + 1$$

$$|P_x\rangle + |q_x\rangle = |P_x + q_x\rangle = \cancel{x^{n-1}} + (-\cancel{x^{n-1}}) + 1$$

$$|P_x + q_x\rangle = 1 \notin \omega$$

C.II) $|P(x)\rangle = -x^4 \wedge |q(x)\rangle = x^4 + x^3 \in S$

$$|P(x)\rangle + |q(x)\rangle = -\cancel{x^4} + \cancel{x^4} + x^3$$

$$|P(x) + q(x)\rangle \notin S$$

C III) $|P_x\rangle = x \cdot q(x) \wedge |r_x\rangle = x \cdot s(x)$

$$|P_x + r_x\rangle = x \cdot q(x) + x \cdot s(x)$$

$$|P_x + r_x\rangle = x(q(x) + s(x))$$

$$|\alpha P_x\rangle = x \alpha q(x) = \alpha x q(x)$$

Si algún $a_i = 0 \Rightarrow 0_S \in 0_{\mathbb{V}}$

C.IV) $|P_x\rangle = (x-1) \cdot q(x)$ $|r_x\rangle = (x-1) \cdot s(x)$

$$|P_x + r_s\rangle = (x-1) \cdot q(x) + (x-1) \cdot s(x)$$

$$= (x-1)(q(x) + s(x)) \in S$$

$P(1) = 0 \wedge r(1) = 0$

$(P + r)(1) = P(1) + r(1) = 0 + 0 = 0$ $0_S \in 0_{\mathbb{V}}$

$$|\alpha P_x\rangle = \alpha |P_x\rangle = (x-1)\alpha q(x) = \alpha (x-1) \cdot q(x)$$

# Ejercicios clase 4

Sunday, August 17, 2025 4:02 PM

*ejercicio 6 sección 2.2.4*

$\langle v_i|v_j\rangle \equiv \|\|v_i\rangle\|^2 \in$

$\langle v_i|v_j\rangle = \langle v_j|v_i\rangle^*$

$\langle v_i|(|v_j\rangle + |v_k\rangle) = \langle v_i|v_j\rangle + \langle v_i|v_k\rangle$

$\langle v_i|(\alpha|v_j\rangle + \beta|v_k\rangle) = \alpha\langle v_i|v_j\rangle + \beta\langle v_i|v_k\rangle$

$\langle \alpha v_i|v_j\rangle = \alpha^*\langle v_i|v_j\rangle$

$\langle v_i|\beta v_j\rangle = \beta\langle v_i|v_j\rangle$

$\langle v_i|v_i\rangle = 0$

$d(|v_i\rangle, |v_j\rangle) = \|\|v_i\rangle - |v_j\rangle\| = \sqrt{\langle v_i - v_j|v_i - v_j\rangle}$

6. $\langle e^1|e_3\rangle$

| | $\hat{i}$ | $\hat{j}$ | $\hat{k}$ |
|---|---|---|---|
| $\hat{i}$ | 1 | 0 | 0 |
| $\hat{j}$ | 0 | 1 | 0 |
| $\hat{k}$ | 0 | 0 | 1 |

$\langle e^1|e_1\rangle + \langle e^1|e_2\rangle + \langle e^1|e_3\rangle$

$\langle e^3|e_3\rangle$

$|a\rangle = a^k|g_k\rangle = a^0 + a^1|g_1\rangle = a_0 + a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

con $k = 0, 1, 2, 3$

$= a_0 + \underbrace{a^1|g_1\rangle + a^2|g_2\rangle + a^3|g_3\rangle}_{\text{parte compleja}}$

$|b\rangle^* = b^0|g_0\rangle - b^1|g_1\rangle$

$|a\rangle = a^k|g_k\rangle$
$|b\rangle = b^k|g_k\rangle$ $\Big\}$ $= |c\rangle = c^k|g_k\rangle$

$\Rightarrow |a\rangle + |b\rangle = (a^k + b^k)|g_k\rangle$

$\Rightarrow c^k = (a^k + b^k)$

$\alpha c^k|g_k\rangle$

a) Si los cuaterniones forman un espacio vectorial.

$|a\rangle + |b\rangle \in \mathbb{V}$

$|a\rangle + |b\rangle = |b\rangle + |a\rangle$

$|a\rangle + (-|a\rangle) = 0 \mathbb{V}$

$|\alpha a\rangle \in \mathbb{V}$

$(\alpha + \beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle$

$0 \cdot |a\rangle = 0$

$|a\rangle + |b\rangle = a_0 + a^1|g_1\rangle + a^2|g_2\rangle + a^3|g_3\rangle + b_0 + b^1|g_1\rangle + b^2|g_2\rangle + b^3|g_3\rangle$

$= (a_0 + b_0) + (a^1 + b^1)|g_1\rangle + (a^2 + b^2)|g_2\rangle + (a^3 + b^3)|g_3\rangle$

$|\alpha a\rangle = \alpha a_0 + \alpha a^1|g_1\rangle + \alpha a^2|g_2\rangle + \alpha a^3|g_3\rangle = \alpha(c_0 + a^1|g_1\rangle + a^2|g_2\rangle + a^3|g_3\rangle)$

$= \alpha(|a\rangle)$

$|0\rangle = 0 + 0|g_1\rangle + 0|g_2\rangle + 0|g_3\rangle = 0$

$|a\rangle + (-|a\rangle) = 0$

$|a\rangle + (-|a\rangle) = a_0 + a^1\cancel{|g_1\rangle} + a^2\cancel{|g_2\rangle} + a^3\cancel{|g_3\rangle} + (-a_0 - a^1\cancel{|g_1\rangle} - a^2\cancel{|g_2\rangle} - a^3\cancel{|g_3\rangle})$

$|a\rangle + (-|a\rangle) = 0$

b) $|b\rangle \equiv (b^0, b)$ $|r\rangle = (r^0, r)$

$|d\rangle = |b\rangle \otimes |r\rangle$

$|d\rangle = |b\rangle \otimes |r\rangle \Leftrightarrow (b^0, b) = (b^0 r^0 - b \cdot r, r^0 b + b \times r)$

$B = b^0 + b$, $R = r^0 + r$

DONDE $(b^0 + b)(r^0 + r)$

$= b^0 \cdot r^0 + b^0 \cdot r + b \cdot r^0 + b \cdot r$

$b \cdot r = b^1|g_1\rangle + b^2|g_2\rangle + b^3|g_3\rangle \cdot (r^1|g_1\rangle + r^2|g_2\rangle + r^3|g_3\rangle)$

$b^1|g_1\rangle \cdot r^1|g_1\rangle + b^1|g_1\rangle \cdot r^2|g_2\rangle + b^1|g_1\rangle \cdot r^3|g_3\rangle$

$+ b^2|g_2\rangle \cdot r^1|g_1\rangle + b^2|g_2\rangle \cdot r^2|g_2\rangle + b^2|g_2\rangle \cdot r^3|g_3\rangle$

$+ b^3|g_3\rangle \cdot r^1|g_1\rangle + b^3|g_3\rangle \cdot r^2|g_2\rangle + b^3|g_3\rangle \cdot r^3|g_3\rangle$

$= -b^1 \cdot r^1 + b^1(r^2|g_3\rangle - b^1 \cdot r^3|g_2\rangle$

$- b^2 \cdot r^1|g_3\rangle - b^2 \cdot r^2 + b^2 \cdot r^3|g_1\rangle$

$+ b^3 \cdot r^1|g_2\rangle - b^3 \cdot r^2|g_1\rangle - b^3 \cdot r^3$

$= -b^1 \cdot r^1 - b^1 \cdot r^1 - b^3 \cdot r^3 + b^2 \cdot r^3|g_1\rangle - b^3 \cdot r^2|g_1\rangle$

$+ b^3 \cdot r^1|g_2\rangle - b^1 \cdot r^3|g_2\rangle + b^1 \cdot r^2|g_3\rangle - b^2 \cdot r^1|g_3\rangle$

$-b \cdot r = -(b^1 \cdot r^1 + b^2 \cdot r^2 \cdot b^3 \cdot r^3)$

$b \times r = \begin{vmatrix} i & j & k \\ b^1 & b^2 & b^3 \\ r^1 & r^2 & r^3 \end{vmatrix}$

$= (b^2 r^3 - b^3 \cdot r^2)|g_1\rangle$

$+ (b^3 \cdot r^1 - r^1 \cdot b^3)|g_2\rangle$

$+ (b^1 \cdot r^2 - b^2 \cdot r^1)|g_3\rangle$

$= -b \cdot r + b \times r$

$= -b \cdot r + b \times r$

$= -b \cdot r - b \cdot r + b \times r$

$b^0 \cdot r^0 + b^0 \cdot r + r^0 \cdot b - b \cdot r + b \times r$

$b^0 \cdot r^0 - b \cdot r + b^0 \cdot r + r^0 \cdot b + b \times r$

$|d\rangle = (b^0 \cdot r^0 - b \cdot r, r^0 b + b^0 \cdot r + b \times r)$

c) $|b\rangle = b^k|g_k\rangle$ $|r\rangle = r^k|g_k\rangle$

$|d\rangle = |b\rangle \otimes |r\rangle = a|g_0\rangle + j^{ik} \delta_k^0|g_i\rangle$

$+ A^{ijk} b_j \varepsilon_k|g_i\rangle$

$A^{ijk} = j^{ik} \rightarrow j^{ik} = -A^{ijk} \rightarrow (A^{ijk} b_j \varepsilon_k - A^{ijk} b_j \varepsilon_k)|g_i\rangle$

una forma de expresar la tabla de multiplicación

Sup $|b\rangle = b^k|g_k\rangle$ $|r\rangle = r^k|g_k\rangle$

Ahora

$|g_i\rangle|g_j\rangle = -\delta_{ij}|g_0\rangle + \varepsilon^k_{ij}|g_k\rangle$ → cuando $\delta_{ij}$ es igual a $i = j$ es un producto punto entre $|g_i\rangle|g_j\rangle$

$|g_j\rangle|g_k\rangle = |g_i\rangle|g_0\rangle = |g_k\rangle$

$\delta_{ij} = \delta_{ij}$

$\varepsilon^k_{ij} = -\varepsilon^k_{ji}$

$|b\rangle = b^0|g_0\rangle + b^i|g_i\rangle$

$|r\rangle = r^0|g_0\rangle + r^i|g_i\rangle$

$|b\rangle \otimes |r\rangle = (b^0|g_0\rangle + b^i|g_i\rangle) \cdot (r^0|g_0\rangle + r^i|g_i\rangle)$

$= b^0|g_0\rangle \cdot r^0|g_0\rangle + b^0|g_0\rangle \cdot r^i|g_i\rangle + b^i|g_i\rangle \cdot r^0|g_0\rangle + b^i|g_i\rangle \cdot r^i|g_i\rangle$

$= b^0 \cdot r^0 + b^0 \cdot r^i|g_i\rangle + r^0 \cdot b^i|g_i\rangle + b^i|g_i\rangle \cdot r^i|g_i\rangle$

$= b^0 \cdot r^0 + b^0 \cdot r^i|g_i\rangle + r^0 \cdot b^i|g_i\rangle + b^i|g_i\rangle \cdot r^i|g_i\rangle$

$= b^0 \cdot r^0 - b^0 \cdot r^i \delta_{ij}|g_0\rangle + \varepsilon^k_{ij}|g_k\rangle$

$b^0 \cdot r^0$ → Producto punto

$b^0 \cdot r^i \delta_{ij}$

$= b^0 \cdot r^i(-\delta_{ij}|g_0\rangle + \varepsilon^k_{ij}|g_k\rangle)$

$b^0 \cdot r^i|g_i\rangle + r^0 \cdot b^i|g_i\rangle$

$= -b^0 \cdot r^i \delta_{ij}|g_0\rangle + b^i \cdot r^i \varepsilon^k_{ij}|g_k\rangle$

$a = b^0 \cdot r^0 - b^0 \cdot r^i \delta_{ij}$

$b^0 \cdot r^i|g_i\rangle + r^0 \cdot b^i|g_i\rangle$

$= (b^0 \cdot r^i + r^0 \cdot b^i)|g_i\rangle$

$b^0 \cdot r^0 = b^0 \cdot r^0 + b^0 \cdot r^i \delta_{ij}|g_0\rangle = b^0 \cdot r^0 + b^0 \cdot r^i \delta_{ij}$

$b^0 \cdot r^i \varepsilon^k_{ij}|g_k\rangle = b^i \cdot r^k \varepsilon^k_{ij}|g_k\rangle$

$i = j \Rightarrow k = i$

$a|g_0\rangle + b^0 \cdot r^i|g_i\rangle + r^0 \cdot b^i|g_i\rangle + b^0 \cdot r^i \varepsilon^k_{ij}|g_k\rangle$

$a|g_0\rangle + (b^0 \cdot r^0 + r^0 \cdot b^0)|g_i\rangle + b^i \cdot r^i \varepsilon^k_{ii}|g_i\rangle$

$\delta^{ki}\delta^0_k = \delta^{0i} = b^0 r^i + r^0 b^i$

$|d\rangle = a|g_0\rangle + j^{ik}\delta^0_k|g_i\rangle + b_j \cdot r_k A^{ijk\varepsilon k}|g_i\rangle$

d) $a$, $S^{(ij)}$ y $A^{ijk\varepsilon k}$

$a = b^0 \cdot r^0 - b^i \cdot r^i \delta_{ij}$

$S^{0i} = b^0 r^i + r^0 b^i$, $S^{ij} = b^0 r^j + r^0 b^i$.

$A^{ijk\varepsilon k} = \varepsilon^k_{ji}$

el resultado $|d\rangle$ no es un vector pero si un pseudovector puro:

es un cuaternión cómo parte escalar y parte imaginaria.

e)

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Puede representar la base de los cuaternios $\{|g_1\rangle, |g_2\rangle, |g_3\rangle, |g_0\rangle\}$

Para comprobar que sea base $\alpha\, \beta\, \delta\, \theta$

$\begin{matrix} 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -1 & 1 \end{matrix} \Big| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$ $\begin{matrix} 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{matrix} \Big| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$

$f_3 : f_2 + f_3$

$f_4 : f_4 + f_1$

$\alpha - i\beta = 0$ | $\delta + \theta = 0$

$-i\beta = 0$ | $\theta = 0$

$2\theta = 0$ $2\alpha = 0$

$\theta = 0$ $\alpha = 0$

forman una base

$|b\rangle \longleftrightarrow \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$ $z = x + iy$ $w = a + ib$

$|g\rangle = b^0 g_0 + b^1 g_1 + b^2 g_2 + b^3 g_3$

$M(g) = b^0 I + b^1(-i\sigma_1) + b^2(-i\sigma_2) + b^3(-i\sigma_3)$

$M(g) = \begin{pmatrix} b^0 - ib^3 & -b^2 - ib^1 \\ b^2 - ib^1 & b^0 + ib^3 \end{pmatrix}$

$z = b^0 - ib^3$, $w = -b^2 - ib^1$.

$M(g) = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}$

c)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\cdot$ | 1 | -1 | $\|g_1\rangle$ | $-\|g_1\rangle$ | $\|g_2\rangle$ | $-\|g_2\rangle$ | $\|g_3\rangle$ | $-\|g_3\rangle$ |
| 1 | 1 | -1 | $\|g_1\rangle$ | $-\|g_1\rangle$ | $\|g_2\rangle$ | $-\|g_2\rangle$ | $\|g_3\rangle$ | $-\|g_3\rangle$ |
| -1 | -1 | 1 | $-\|g_1\rangle$ | $\|g_1\rangle$ | $-\|g_2\rangle$ | $\|g_2\rangle$ | $-\|g_3\rangle$ | $\|g_3\rangle$ |
| $\|g_1\rangle$ | $\|g_1\rangle$ | $-\|g_1\rangle$ | -1 | 1 | $\|g_3\rangle$ | $-\|g_3\rangle$ | $-\|g_2\rangle$ | $\|g_2\rangle$ |
| $-\|g_1\rangle$ | $-\|g_1\rangle$ | $\|g_1\rangle$ | 1 | -1 | $-\|g_3\rangle$ | $\|g_3\rangle$ | $\|g_2\rangle$ | $-\|g_2\rangle$ |
| $\|g_2\rangle$ | $\|g_2\rangle$ | $-\|g_2\rangle$ | $-\|g_3\rangle$ | $\|g_3\rangle$ | -1 | 1 | $\|g_1\rangle$ | $-\|g_1\rangle$ |
| $-\|g_2\rangle$ | $-\|g_2\rangle$ | $\|g_2\rangle$ | $\|g_3\rangle$ | $-\|g_3\rangle$ | 1 | -1 | $-\|g_1\rangle$ | $\|g_1\rangle$ |
| $\|g_3\rangle$ | $\|g_3\rangle$ | $-\|g_3\rangle$ | $\|g_2\rangle$ | $-\|g_2\rangle$ | $-\|g_1\rangle$ | $\|g_1\rangle$ | -1 | 1 |
| $-\|g_3\rangle$ | $-\|g_3\rangle$ | $\|g_3\rangle$ | $-\|g_2\rangle$ | $\|g_2\rangle$ | $\|g_1\rangle$ | $-\|g_1\rangle$ | 1 | -1 |

f) $\sigma_1, \sigma_2, \sigma_3$

$\sigma_i^2 = -I_4$ ; $\sigma_i^2 = -g_0$

$\sigma_1 \sigma_2 = \sigma_3$

$\sigma_2 \sigma_3 = \sigma_1$

$\sigma_3 \sigma_1 = \sigma_2$

$I_4, \sigma_1, \sigma_2, \sigma_3$ son L.I $\alpha|g_1\rangle - \beta|g_2\rangle - \gamma|g_3\rangle - \delta|g_0\rangle$

por tanto forman una base para los cuaternionos.

d) $\langle a|b\rangle = |a\rangle^* \circ b$

$(\langle a|b\rangle)^* \langle a|a\rangle^* \circ |b\rangle)^* = |b\rangle^* \circ |a\rangle = \widetilde{\langle a|b\rangle}$

$\widetilde{\langle a|b\rangle} \circ g = (|a\rangle^* \circ |b\rangle) \circ g = \langle a|b\rangle \circ g$

$\langle a \cdot g|b\rangle = (|a \cdot g\rangle)^* \circ |a\rangle^* \circ \langle a|b\rangle$

$\langle a|a\rangle = |a\rangle^* \circ |a\rangle$ para $g = b^0 + b_i g_i$

$g \cdot g = (b^0)^2 + \sum_i (b^i)^2$ cuyo escalar real positivo

si $a = 0$ $\widetilde{\langle a|a\rangle} = 0$

h)

$\langle a|b\rangle = \frac{1}{2}\left[\langle a|b\rangle - |g_1\rangle \circ \langle a|b\rangle \circ |g_1\rangle\right]$

$\phi(x) = \frac{1}{2}(x - g_1 x g_1)$.

$\langle a|b\rangle = \phi(|a\rangle^* \circ |b\rangle)$

$(\langle a|b\rangle \circ |b\rangle) \circ |a\rangle) = \langle b|a\rangle$

$\langle a|b\rangle = \phi(|a\rangle^* \circ |b\rangle) = \phi(|a\rangle^* g_1 g_1) = |a\rangle^* g_1$ → Real y positivo

$\langle a|b\rangle = \langle a|b\rangle \cdot s$

si $s = g_1$ la igualdad falla, no cumple la propiedad de linealidad a la derecha que escalares cuaternionos esto bivector.

i) $\alpha(|b\rangle) = \| |a\rangle \| = \sqrt{\langle a|a\rangle} = \sqrt{|a|^2 \circ |a\rangle}$

Sup $a = a_0 + a^1|g_1\rangle + a^2|g_2\rangle + a^3|g_3\rangle$

$a^* = a_0 - a_1|g_1\rangle - a_2|g_2\rangle - a_3|g_3\rangle$

$= \sqrt{a_0^2 - a_1|g_1\rangle - a_2|g_2\rangle - a_3|g_3\rangle \cdot a_0 + a_1|g_1\rangle + a_2|g_2\rangle + a_3|g_3\rangle}$

$= \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$

$|a| \geq 0$

$|\lambda a| = |\lambda| |a| = \lambda \cdot \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$

j) $|a\rangle = \frac{|a\rangle^*}{\|a\|^2}$

$|a\rangle \circ |a\rangle = \frac{|a\rangle \circ |a\rangle^*}{\|a\|^2} = \frac{\| |a\rangle \|^2}{\| |a\rangle \|^2} = 1$

l) $v = v^1|g_1\rangle + v^2|g_2\rangle + v^3|g_3\rangle$ $\bar{v} = -v$ $\|v\|^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$

$v^* = \bar{v} \circ v$

$\|v\|^2 = \| |v\rangle \circ v \| = \| |v\rangle \| \cdot \| |v\rangle \| = \| |v\rangle \|^2 \cdot \| |v\rangle \|$

$\| |v\rangle \|^2 = \| |v\rangle \|^2 \cdot \| |v\rangle \|^2$