Donc :

$$I_4 = 2\,\beta\left(\frac{1-x}{2}, \frac{1+x}{2}\right)$$

$$= 2 . \frac{\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{1+x}{2}\right)}{\Gamma(1)}$$

$$= 2\,\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{1+x}{2}\right)$$

$$= 2\,\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1 - \frac{1-x}{2}\right)$$

$$= 2 . \frac{\pi}{\sin\left(\pi\left(\frac{1-x}{2}\right)\right)}$$

car $\begin{cases} -1 < x < 1 \\ -1 < -x < 1 \\ 0 < -x+1 < 2 \\ 0 < \frac{-x+1}{2} < 1 \end{cases}$

<u>Ex3</u> ,

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \,, \quad a>0, b>0$$

1) a) Cvs.

b) $\beta(a,b) = \beta(b,a)$ (Cvg).

c) $a>0, b>0$ mq :

$$\beta(a+1,b) + \beta(a,b+1)$$

$$= \int_0^1 t^{a}(1-t)^{b-1}\,dt + \int_0^1 t^{a-1}(1-t)^{b}\,dt$$

$$= \int_0^1 t^{a-1}(1-t)^{b-1}\left[t + 1 - t\right]dt$$

$$= \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

$$= \beta(a,b)$$

d) $\beta\left(\frac{3}{2}, \frac{3}{2}\right) = \int_0^1 t^{\frac{3}{2}-1}(1-t)^{\frac{3}{2}-1}\,dt$

$$= \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt$$

$t = \cos^2\theta$.

$dt = -2\cos\theta\sin\theta\,d\theta$.

$$= \int_{\frac{\pi}{2}}^{0} \cos\theta . \sin\theta . (-2)\cos\theta\sin\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta\,\sin^2\theta\,d\theta .$$

$$\cancel{= \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,\frac{1-\cos(2\theta)}{2}\,d\theta}$$

$$\cancel{= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))(1-\cos(2\theta))\,d\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos\theta\sin\theta)^2\,d\theta$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin(2\theta)}{2}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^2\,d\theta$$

$$\boxed{\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}}$$

$$= \frac{1}{2} . \int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos(4\theta))\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{8} .$$

(c) $n\in\mathbb{N}^*$, ma:

$$\beta(1,n) = \int_0^1 t^{1-1}(1-t)^{n-1}\,dt$$
$$= \int_0^1 (1-t)^{n-1}\,dt$$
$$= \left[-\frac{1}{n}(1-t)^n\right]_0^1$$
$$= \frac{1}{n} \Rightarrow \beta(1,n)=\frac{1}{n},\ n\in\mathbb{N}^*$$

(2) (a)

$$\beta(a+1,b) = \int_0^1 t^a(1-t)^{b-1}\,dt$$

$$\text{IPP} \begin{cases} t^a \longrightarrow a t^{a-1} \\ (1-t)^{b-1} \longrightarrow -\frac{1}{b}(1-t)^b \end{cases}$$

$$\beta(a+1,b) = \left[-\frac{t^a}{b}(1-t)^b\right]_0^1 + \frac{a}{b}\int_0^1 t^{a-1}(1-t)^b$$
$$= 0 + \frac{a}{b}\int_0^1 t^{a-1}(1-t)^b\,dt$$
$$= \frac{a}{b}\int_0^1 t^{a-1}(1-t)^{b-1}(1-t)\,dt$$
$$= \frac{a}{b}\int_0^1 t^{a-1}(1-t)^{b-1} - t^a(1-t)^{b-1}\,dt$$
$$= \frac{a}{b}\left(\beta(a,b) - \beta(a+1,b)\right)$$
$$\beta(a+1,b)\left[1+\frac{a}{b}\right] = \frac{a}{b}\beta(a,b)$$
$$\Rightarrow \boxed{\beta(a+1,b) = \left(\frac{a}{a+b}\right)\beta(a,b)}$$

(b) $n,p\in\mathbb{N}^*$, ma?

$$\beta(n,p) = ?$$

$$\beta(n,p) = \beta((n-1)+1,p)$$
$$= \frac{(n-1)}{(n-1)+p}\beta(n-1,p)$$
$$= \frac{n-1}{n-1+p}\,\frac{n-2}{n-2+p}\,\beta(n-2,p)$$
$$= \frac{(n-1)}{n-1+p}\,\frac{(n-2)}{n-2+p}\,\beta(n-2,p)$$
$$= \frac{n-1}{n-1+p}\,\frac{n-2}{n-2+p}\,\frac{n-3}{n-3+p}\,\beta(n-3,p)$$
$$= \frac{(n-1)}{n-1+p}\,\frac{(n-2)}{n-2+p}\,\frac{n-3}{n-3+p}\cdots$$
$$\cdots\ \frac{1}{p+1}\beta(n-(n-1),p)$$
$$= \frac{(n-1)(n-2)(n-3)\cdots 1}{(n+p-1)(n+p-2)\cdots(p+1)}\beta(1,p)$$
$$= \frac{(n-1)!}{(n+p-1)(n+p-2)\cdots(p+1)}\,\frac{1}{p}$$
$$= \frac{(n-1)!\quad (p-1)\cdots 1}{(n+p-1)\cdots p(p-1)\cdots 1}$$
$$= \frac{(n-1)!\,.\,(p-1)!}{(n+p-1)!}$$

(c) $n,p\in\mathbb{N}^*$ ma?

$$\beta\left(n+\frac{1}{2},\ p+\frac{1}{2}\right)$$
$$\beta\left(\left(n-1+\frac{1}{2}\right)+1,\ p+\frac{1}{2}\right)$$
$$= \frac{(n-\frac{1}{2})}{(n-\frac{1}{2}+p+\frac{1}{2})}\,\beta\left(\left(p-1+\frac{1}{2}\right)+1,\ n-\frac{1}{2}\right)$$

$(n-1+\frac{1}{2})$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n-\frac{1}{2}+p+\frac{1}{2})(p-\frac{1}{2}+n-\frac{1}{2})}\,\beta\left(n-\frac{1}{2},p-\frac{1}{2}\right)$$
$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)}\,\beta\left(n-\frac{1}{2},p-\frac{1}{2}\right)$$

$(n-2+$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)}\,\frac{(n-\frac{3}{2})(p-\frac{3}{2})}{(n-\frac{3}{2}+p-\frac{1}{2})(p-\frac{3}{2}+n-\frac{3}{2})}\,\beta\left(n-\frac{3}{2},p-\frac{3}{2}\right)$$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)} \frac{(n-\frac{3}{2})(p-\frac{3}{2})}{(n+p-2)(n+p-3)}$$

$$\beta\left(\cancel{n-1}\ (n-1)-\tfrac{1}{2},\ p-(p-1)+\tfrac{1}{2}\right)$$

$$\frac{(n-\frac{5}{2})\quad(p-\frac{5}{2})}{\underbrace{(n-\frac{5}{2}+p-\frac{3}{2})}_{n+p-4}\ \underbrace{(n-\frac{5}{2}+p-\frac{5}{2})}_{n+p-5}} \beta\left(n-\tfrac{5}{2}, p-\tfrac{5}{2}\right)$$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)} \cdot \frac{(n-\frac{3}{2})(p-\frac{3}{2})}{(n+p-2)(n+p-3)} \frac{(n-\frac{5}{2})(p-\frac{5}{2})}{(n+p-4)(n+p-5)} \cdots$$

$$\frac{\frac{3}{2}\qquad\frac{3}{2}}{(\frac{3}{2}+\frac{2+1}{2})\ (\frac{3}{2}+\frac{3}{2})} \beta\left(\overbrace{n-(n-1)+\tfrac{1}{2}}^{\frac{3}{2}}, \overbrace{p-(p-1)+\tfrac{1}{2}}^{\frac{3}{2}}\right)$$

$$= \frac{(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}\times 4}{(n+p)(n+p-1)(n+p-2)(n+p-3)\cdots 4.3} \beta\left(\tfrac{3}{2},\tfrac{3}{2}\right) \quad \frac{\pi}{8}$$

$$\cancel{(2n-1)(2n-3)(2n-5)}$$

$$= \frac{(\frac{1}{2})^n (2n-1)(2n-3)\cdots 1\ (\frac{1}{2})^p (2p-1)(2p-3)\cdots 1\ \cancel{4}}{(n+p)(n+p-1)(n+p-2)\cdots 4.3} \cdot \frac{\pi}{8}\quad 8=4\times 2$$

$$\cancel{= \frac{(\frac{1}{2})^{n+p}(2n-1)!\,(2p-1)!\,\pi}{(n+p)!}} = \left(\frac{1}{2}\right)^{n+p} \frac{(2n)!\ (2p)!}{2n(2n-2)\cdots 2\ \ 2p(2p-2)\cdots 2\ (n+p)!}\,\pi$$

$$\cancel{= \pi\left(\frac{1}{2}\right)^{n+p} \frac{(2n-1)!\,(2p-1)!}{(n+p)!}} = \left(\frac{1}{2}\right)^{n+p} \frac{(2n)!\ (2p)!\ \pi}{2^n\, n!\ 2^p\, p!\ (n+p)!}$$

$$= \frac{(2n)!\ (2p)!}{n!\ p!\ 2^{2(n+p)}}\ \pi.$$

Ex $x > 0$

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt.$$

① $y = \frac{t-x}{\sqrt{x}} \Rightarrow y\sqrt{x} = t - x.$

$dy = \frac{1}{\sqrt{x}} dt \quad t = (y\sqrt{x} + x)$

$$\Gamma(x+1) = \int_{-\sqrt{x}}^{+\infty} (y\sqrt{x}+x)^{x}\, e^{-(y\sqrt{x}+x)} \sqrt{x}\, dy$$

$$= \int_{-\sqrt{x}}^{+\infty} x^x \left(1 + \frac{y}{\sqrt{x}}\right)^x e^{-x} e^{-y\sqrt{x}} \sqrt{x}\, dy$$

$$= \int_{-\sqrt{x}}^{+\infty} \left(\frac{x}{e}\right)^x \sqrt{x} \left(1+\frac{y}{\sqrt{x}}\right)^x e^{-y\sqrt{x}}\, dy.$$

$$= \left(\frac{x}{e}\right)^x \sqrt{x} \int_{-\sqrt{x}}^{+\infty} e^{x\log(1+\frac{y}{\sqrt{x}})} e^{-y\sqrt{x}}\, dy$$

$$= \left(\frac{x}{e}\right)^x \sqrt{x} \int_{-\sqrt{x}}^{+\infty} e^{x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x}}\, dy$$

ⓑ pour $x \geq 1$, $y \geq 0$ on a :

$$x\log\left(1+\frac{y}{\sqrt{x}}\right) - y\sqrt{x} \leq (1+y)e^{-y}$$

• $e^{x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x}} \xrightarrow[x\to+\infty]{CS}$?

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4!} + o(x^4).$

$\log\left(1+\frac{y}{\sqrt{x}}\right) = \frac{y}{\sqrt{x}} - \frac{y^2}{2x} + o\left(\frac{1}{x^2}\right).$

donc.

$x\left(\frac{y}{\sqrt{x}} - \frac{y^2}{2x}\right) - y\sqrt{x} =$

$= \cancel{y\sqrt{x}} - \frac{y^2}{2} - \cancel{y\sqrt{x}} = -\frac{y^2}{2}$

$x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x} \sim x\left(\frac{y}{\sqrt{x}} - \frac{y^2}{2x}\right)^{+o(\frac{1}{x^2})} - y\sqrt{x}$

$\sim -\frac{y^2}{2} + o\left(\frac{1}{x}\right).$

donc,

$$e^{x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x}} \xrightarrow[x\to+\infty]{CS} e^{-\frac{y^2}{2}}$$

• on a :

$$e^{x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x}} \leq (1+y)e^{-y}$$

$\forall x \geq 1, y \geq 0$

et $\int_0^{+\infty} (1+y)e^{-y}\, dy = \left[-(1+y)e^{-y}\right]_0^{+\infty} + \left[-e^{-y}\right]_0^{+\infty}$

$1+y \to 1$

$e^{-y} \to -e^{-y}$

$= 1 + 1 = 2 < +\infty$

TCD :

$$\lim_{x\to+\infty} \int_0^{+\infty} e^{x\log(1+\frac{y}{\sqrt{x}}) - y\sqrt{x}}\, dy$$

$$= \int_0^{+\infty} \lim_{x\to+\infty} (-)\, dy$$

$$= \int_0^{+\infty} e^{-\frac{y^2}{2}}\, dy.$$

ⓒ pour $x > 0$, $y \in]-\sqrt{x}, 0]$ on a :

$$x\log\left(1+\frac{y}{\sqrt{x}}\right) - \sqrt{x}y \leq -\frac{y^2}{2}.$$

• on a $e^{x\log(1+\frac{y}{\sqrt{x}}) - \sqrt{x}y} \leq e^{-\frac{y^2}{2}}$,

$\int_{\mathbb{R}} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$ avec $\int_{-\sqrt{x}}^{0} e^{-\frac{y^2}{2}}\, dy < +\infty.$

• $e^{x\log(1+\frac{y}{\sqrt{x}}) - \sqrt{x}y} \xrightarrow{CV} e^{-\frac{y^2}{2}}.$

TCD $\lim_{+\infty} \int_{-\sqrt{x}}^{0} (-)\, dy = \int_{-\infty}^{0} e^{-\frac{y^2}{2}}\, dy.$

(d) $\underline{\text{On a}}$:

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{x} \int_{-\sqrt{x}}^{+\infty} e^{x\log\left(1+\frac{y}{\sqrt{x}}\right) - y\sqrt{x}}\, dy$$

$$= \left(\frac{x}{e}\right)^x \sqrt{x} \left[\int_{-\sqrt{x}}^{0} (\ )\,dy + \int_{0}^{+\infty} (\ )\,dy \right]$$

$$\lim_{x\to+\infty} \Gamma(x+1) =$$

$$\lim_{x\to+\infty} \left(\int_{-\sqrt{x}}^{0} (\ )\,dy + \int_{0}^{+\infty} (\ )\,dy \right)$$

$$= \lim \int_{-\infty}^{0} e^{-\frac{y^2}{2}}\,dy + \int_{0}^{+\infty} e^{-y^2/2}\,dy$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\,dy = \sqrt{\frac{\pi}{\frac{1}{2}}} = \sqrt{2\pi}.$$

$\underline{\text{alors}}$:

$$\Gamma(x+1) \underset{+\infty}{\sim} \sqrt{2\pi} \cdot \left(\frac{x}{e}\right)^x \cdot \sqrt{x}.$$

(e) pour $x = n \in \mathbb{N}$ $\underline{\text{on a}}$:

$$\underbrace{\Gamma(n+1)}_{n!} \underset{+\infty}{\sim} \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n.$$

$$\Leftrightarrow n! \underset{+\infty}{\sim} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

$\underline{\text{App}}$:

(1) $\lim_{n\to+\infty} \frac{(n!)^{\frac{1}{n}}}{n} = ?$

$$n! \underset{+\infty}{\sim} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

$$(n!)^{\frac{1}{n}} \underset{+\infty}{\sim} (2\pi n)^{\frac{1}{2n}} \cdot \left(\frac{n}{e}\right).$$

$$\Rightarrow \frac{(n!)^{\frac{1}{n}}}{n} \underset{+\infty}{\sim} \frac{1}{e} (2\pi n)^{\frac{1}{2n}}.$$

$$\lim_{+\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \lim_{+\infty} \frac{1}{e} (2\pi n)^{\frac{1}{2n}}.$$

$$= \lim_{+\infty} \frac{1}{e} e^{\frac{1}{2n}\log(2\pi n)}.$$

$$= \frac{1}{e}.$$

$\lim_{+\infty} \frac{\log x}{x} = 0.$

$= \lim_{+\infty} \frac{\log x}{x^n}.$

(2) $\lim_{n\to+\infty} \left(\frac{\log n!}{n} - \log n \right)$

$$n! \underset{+\infty}{\sim} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

$$\log(n!) \underset{+\infty}{\sim} \log\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)$$

$$\frac{\log(n!)}{n} - \log(n) \underset{+\infty}{\sim} \frac{\log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)}{n} - \log n$$

avec $\frac{\log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)}{n} - \log(n)$

$$= \frac{\log(\sqrt{2\pi n}) + \log\left(\frac{n}{e}\right)^n}{n} - \log n.$$

$$= \frac{\log(\sqrt{2\pi n}) + n\log\left(\frac{n}{e}\right)}{n} - \log(n)$$

$$= \frac{\frac{1}{2}\log(2\pi n)}{n} + \cancel{\log(n)} - \log(e) - \cancel{\log(n)} \xrightarrow[n\to+\infty]{} -\log(e) = -1$$