$\cdot f' \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$

$f'(0^+) = -1$ et $f'(0^-) = 1$

$T_f'' = (T_{f'})' = T_{f''} + (-1-1)\delta$

$= T_{f''} - 2\delta$ avec $f''(x) = \begin{cases} e^{-x} & \text{si } x \geq 0 \\ e^{x} & \text{si } x < 0 \end{cases}$

$= \cancel{\int_{-\infty}^{0} e^{x} \varphi(x)\,dx + \int_{0}^{+\infty} e^{-x} \varphi(x)\,dx - 2\delta}$

$$\boxed{\Rightarrow \mathcal{F}(T_f'') = \mathcal{F}(T_f - 2\delta)}$$

$(2i\pi x)^2 \hat{T}_f = \hat{T}_f - 2\hat{\delta} = \hat{T}_f - 2$

$\Longleftrightarrow \hat{T}_f (1 + 4\pi^2 x^2) = 2$

$\hat{T}_f = \hat{T}_{e^{-|x|}} = \widehat{e^{-|x|}} = \dfrac{2}{1 + 4\pi^2 x^2}$

(5)

$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\xi,t))(x) = \mathcal{F}^{-1}(\hat{\varphi}) * \mathcal{F}^{-1}(\cos(2\pi c \xi t))$

$+ \mathcal{F}^{-1}(\hat{\psi}) * \mathcal{F}^{-1}\left(\dfrac{\sin(2\pi c \xi t)}{2\pi c \xi}\right)(x) =$

$= \left[\varphi(x) * \frac{1}{2}(\delta_{ct} + \delta_{-ct}) \right.$ (5) $\left. + \psi(x) * \frac{1}{2c} \mathbb{1}_{[-ct, ct]} \right]$

Exercice 4 :

$-1-\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & * \\ u(x,0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x) \end{cases}$

T.F. partielle

$\hat{u}(\xi,t) = \mathcal{F}(\underbrace{u(x,t)}_{H(x)})(\xi) = \int_{-\infty}^{+\infty} u(x,t) e^{-2i\pi \xi x}$

- Appliquons la transformation de Fourier variable d'intég. sur $*$:

$\mathcal{F}\left(\dfrac{\partial^2 u}{\partial t^2}\right) - c^2 \mathcal{F}\left(\dfrac{\partial^2 u}{\partial x^2}\right) = 0$

$* \mathcal{F}\left(\dfrac{\partial^2 u}{\partial t^2}(x,t)\right)(\xi) = \dfrac{\partial^2}{\partial t^2} \hat{u}(\xi,t)$

$* \mathcal{F}\left(\dfrac{\partial^2 \overbrace{u(x,t)}^{H(x)}}{\partial x^2}\right)(\xi) = (2i\pi\xi)^2 \hat{u}(\xi,t)$

$\mathcal{F}(f''(t))(x) = (2i\pi x)^2 \hat{f}(x)$

$\Rightarrow \dfrac{\partial^2}{\partial t^2} \hat{u}(\xi,t) - c^2 (2i\pi\xi)^2 \hat{u}(\xi,t) = 0$

$$\boxed{\frac{\partial^2}{\partial t^2}\hat{U}(\xi,t)+(2\pi c\xi)^2\,\hat{U}(\xi,t)=0}$$

Eq diff d'ordre 2 homogène

-2- (ça-d resoudre cette équation)

Ec : $r^2+(2\pi c\xi)^2 = 0$

$r^2 = -(2\pi\xi c)^2$

$\Rightarrow r = 0 \pm i 2\pi c\xi$

la forme d'eq

$\hat{U}(\xi,t) = [A\cos(2\pi c\xi t)+B\sin(2\pi c\xi t)]\, e^0$

partie réel

[illegible] cos ξ

condition initial

$* \; U(x,0) = \varphi(x)$

$\Rightarrow \mathcal{F}(U(x,0))(\xi) = \hat{\varphi}(\xi)$

$\Rightarrow \boxed{\hat{U}(\xi,0) = \hat{\varphi}(\xi)}$ 1ère cond

or $\hat{U}(\xi,0) = A \Rightarrow \boxed{A = \hat{\varphi}(\xi)}$

$\frac{\partial U(x,0)}{\partial t} = \psi(x)$

appliquons Fourier

$\frac{\partial}{\partial t}\hat{U}(\xi,0) = \hat{\psi}(\xi)$

or $\frac{\partial \hat{U}(\xi,t)}{\partial t} = -A\,2\pi c\xi\sin(2\pi c\xi t)+B\,2\pi c\xi\cos(2\pi c\xi t)$

$\Rightarrow \frac{\partial}{\partial t}\hat{U}(\xi,0) = B\,2\pi c\xi = \hat{\psi}(\xi)$

$$\boxed{B = \frac{1}{2\pi c\xi}\hat{\psi}(\xi)}$$

$$\boxed{\hat{U}(\xi,t) = \hat{\varphi}(\xi)\cos(2\pi c\xi t)+\frac{1}{2\pi c\xi}\hat{\psi}(\xi)\sin(2\pi c\xi t)}$$

$\mathcal{F}(?)$

$U(x,t) = \mathcal{F}^{-1}(\hat{U}(\xi,t))(x)$

d'après exo 2 : $\hat{\delta}_a = e^{-2i\pi a x} \Rightarrow \hat{\delta}_{-a} = e^{2i\pi a x}$

donc $\mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{\delta_a+\delta_{-a}}{2}\right)\right) = \mathcal{F}^{-1}\left(\frac{e^{-2i\pi a x}+e^{2i\pi a x}}{2}\right) = \mathcal{F}^{-1}(\cos(2\pi a x))$

$$\boxed{\mathcal{F}^{-1}(\cos(2\pi a x)) = \frac{\delta_a+\delta_{-a}}{2}}$$ cas général

variable de fourier

$$\boxed{\mathcal{F}^{-1}(\cos(2\pi c\xi t)) = \frac{\delta_{ct}+\delta_{-ct}}{2}}$$

$\mathcal{F}^{-1}(\mathcal{F}(1\!\!1_{[-a,a]})(x)) = \mathcal{F}^{-1}\left(\frac{\sin(2\pi a x)}{\pi x}\right)$

$\mathcal{F}^{-1}\left(\frac{\sin(2\pi c\xi t)}{2\pi c\xi}\right) = \mathcal{F}^{-1}\left(\frac{1}{2c}\frac{\sin(2\pi c\xi t)}{\pi\xi}\right) = \frac{1}{2c}\mathcal{F}^{-1}\left(\frac{\sin(2\pi c\xi t)}{\pi\xi}\right)$

$= \boxed{\frac{1}{2c}1\!\!1_{[-ct,ct]}}$

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Exercice 3 :

$\varphi \in S(\mathbb{R})$

-1- a) $\langle \hat{\delta}_a, \varphi \rangle = \langle \delta_a, \hat{\varphi}(x) \rangle$

définition de Dirac $= \hat{\varphi}(a) = \int_{-\infty}^{+\infty} \varphi(x)\, e^{-2i\pi x a}\, dx$

$= \langle T_{e^{-2i\pi a x}}, \varphi(x) \rangle$

$\Rightarrow \hat{\delta}_a = T_{e^{-2i\pi a x}} = e^{-2i\pi a x}$

-b- $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi}(x) \rangle$

fct est appliqué à chapeau

$= \int_{-\infty}^{+\infty} c\, \hat{\varphi}(x)\, dx = c \int_{-\infty}^{+\infty} \hat{\varphi}(x)\, dx$

e^{0} $\quad 2i\pi x 0 = 0$

. FIF
. φ continue en 0

$= c\, \varphi(0)$

$= c \langle \delta, \varphi \rangle$

$= \langle c\delta, \varphi \rangle$

$$\Rightarrow \boxed{\hat{f} = c\,\delta}$$

formule d'inversion de fourier

FIF

$f(x) = \int \hat{f}(t)\, e^{2i\pi x t}\, dt$

lorsque f continue en x

sinon limite à droite et à gauche en x

. la fct φ continue en 0.

$\hat{\;}(\xi\ ...) = A 2\pi \xi \sin(2\pi \xi ...) + A 2\pi \xi \cos ...$

$\boxed{\cos x}$

Transformé de fourier est une app linéaire donc

$\widehat{\cos x} = \frac{1}{2} \left(\widehat{e^{ix}} + \widehat{e^{-ix}} \right)$

$F(e^{2i\pi a x}) = ?$

$\langle \hat{T}_{e^{2i\pi a x}}, \varphi \rangle = \langle T_{e^{2i\pi a x}}, \hat{\varphi}(x) \rangle$

$= \int e^{2i\pi a x}\, \hat{\varphi}(x)\, dx$

FIF
φ est continue en a

$= \varphi(a) = \langle \delta_a, \varphi \rangle$

$\Rightarrow \widehat{e^{2i\pi a x}} = \delta_a$

$a = \frac{1}{2\pi}$; $a = \frac{-1}{2\pi} \Rightarrow \widehat{\cos x} = \frac{1}{2} \left(\delta_{1/2\pi} + \delta_{-1/2\pi} \right)$

-2- $f(x) = e^{-|x|} = \begin{cases} e^{-x} & \text{si } x \geq 0 \\ e^{x} & \text{si } x < 0 \end{cases}$

$\boxed{\text{Formule de Saut}}$

continue en 0 $\Downarrow$ p.p.l.o.

$* \; f' = T_f' = T_{f'}$ avec $f'(x) = \begin{cases} -e^{-x} & \text{si } x > 0 \\ e^{x} & \text{si } x < 0 \end{cases}$

dérivable alors T_f'' existe

$* \; T_f'' = (T_f')' = (T_{f'})'$

par formule de saut pour f' car il y a discontinuité

donc on va ...

$\bullet\ f' \in \mathcal{C}^1(\mathbb{R}\setminus\{0\})$

$f'(0^+) = -1$ et $f'(0^-) = 1$

$T_f'' = (T_{f'})' = T_{f''} + (-1-1)\delta$

$= T_{f''} - 2\delta$ avec $f''(x) = \begin{cases} e^{-x} & \text{si } x \geq 0 \\ e^{x} & \text{si } x < 0 \end{cases}$

$= \cancel{\int_{-\infty}^{0} e^{x}\varphi(x)\,dx + \int_{0}^{+\infty} e^{-x}\varphi(x)\,dx - 2\delta}$

$$\boxed{\Rightarrow \mathcal{F}(T_f'') = \mathcal{F}(T_f - 2\delta)}$$

$(2i\pi x)^2 \hat{T}_f = \hat{T}_f - 2\hat{\delta} = \hat{T}_f - 2$

$\Longleftrightarrow \hat{T}_f(1 + 4\pi^2 x^2) = 2$

$$\hat{T}_f = \hat{T}_{e^{-|x|}} = \widehat{e^{-|x|}} = \frac{2}{1 + 4\pi^2 x^2}$$

(3)

$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\xi,t))(x) = \mathcal{F}^{-1}(\hat{\varphi}) * \mathcal{F}^{-1}(\cos(2\pi c\xi t))(x)$

$+ \mathcal{F}^{-1}(\hat{\psi}) * \mathcal{F}^{-1}\left(\frac{\sin(2\pi c\xi t)}{2\pi c\xi}\right)(x) =$

$= \varphi(x) * \frac{1}{2}(\delta_{ct} + \delta_{-ct})$ (5) $+ \psi(x) * \frac{1}{2c}\, 1_{[-ct,ct]}$

Exercice 4 :

$-1-\ \begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & * \\ u(x,0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \psi(x) \end{cases}$

T.F. partielle

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$\circ$ Appliquons la transformation de Fourier variable d'intégration sur $*$:

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$* \ \mathcal{F}\left(\dfrac{\partial^2 \overbrace{u(x,t)}^{H(x)}}{\partial x^2}\right)(\xi) = (2i\pi\xi)^2 \hat{u}(\xi,t)$

$\mathcal{F}(f''(t))(x) = (2i\pi x)^2 \hat{f}(x)$

$\Rightarrow \dfrac{\partial^2}{\partial t^2}\hat{u}(\xi,t) - c^2 (2i\pi\xi)^2 \hat{u}(\xi,t) = 0$

$$\boxed{\frac{\partial^2}{\partial t^2}\hat{U}(\xi,t)+(2\pi c\xi)^2\,\hat{U}(\xi,t)=0}$$

Eq diff d'ordre 2 homogène

-2- (ça de resoudre cette équation)

$E_c : r^2+(2\pi c\xi)^2=0$

$r^2=-(2\pi\xi c)^2$

$\Rightarrow r=0\pm i2\pi c\xi$

la forme d'eq ?

$\hat{U}(\xi,t)=[A\cos(2\pi c\xi t)+B\sin(2\pi c\xi t)]\,e^{0}$

partie réel

condition initial

$* \; U(x,0)=\varphi(x)$

$\Rightarrow F(U(x,0))(\xi)=\hat{\varphi}(\xi)$

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or $\hat{U}(\xi,0)=A \Rightarrow \boxed{A=\hat{\varphi}(\xi)}$

$\frac{\partial U}{\partial t}(x,0)=\psi(x)$

appliquons Fourier

$\frac{\partial}{\partial t}\hat{U}(\xi,0)=\hat{\psi}(\xi)$

or $\frac{\partial \hat{U}(\xi,t)}{\partial t}=-A2\pi c\xi\sin(2\pi c\xi t)+B2\pi c\xi\cos(2\pi c\xi t)$

$\Rightarrow \frac{\partial\hat{U}(\xi,0)}{\partial t}=B2\pi c\xi=\hat{\psi}(\xi)$

$$\boxed{B=\frac{1}{2\pi c\xi}\hat{\psi}(\xi)}$$

$$\boxed{\hat{U}(\xi,t)=\hat{\varphi}(\xi)\underbrace{\cos(2\pi c\xi t)}_{F(?)}+\frac{1}{2\pi c\xi}\hat{\psi}(\xi)\sin(2\pi c\xi t)}$$

$U(x,t)=F^{-1}(\hat{U}(\xi,t))(x)$

d'après exo 2 : $\hat{\delta}_a=e^{-2i\pi ax}\Rightarrow\hat{\delta}_{-a}=e^{2i\pi ax}$

donc $F^{-1}\left(\frac{\delta_a+\delta_{-a}}{2}\right)=\frac{e^{-2i\pi ax}+e^{2i\pi ax}}{2}=F^{-1}(\cos(2\pi ax))$

$$\boxed{F^{-1}(\cos(2\pi a x))=\frac{\delta_a+\delta_{-a}}{2}}$$ cas général

variable de fourier

$$\boxed{F^{-1}(\cos(2\pi c\xi t))=\frac{\delta_{ct}+\delta_{-ct}}{2}}$$

$F^{-1}(F(1\!\!1_{[-a,a]})(x))=\frac{\sin(2\pi ax)}{\pi x}$

$F^{-1}\left(\frac{\sin(2\pi c\xi t)}{2\pi c\xi}\right)=F^{-1}\left(\frac{1}{2c}\frac{\sin(2\pi c\xi t)}{\pi\xi}\right)=\frac{1}{2c}F^{-1}\left(\frac{\sin(2\pi c\xi t)}{\pi\xi}\right)$

$=\frac{1}{2c}1\!\!1_{[-ct,ct]}$

8

$$\frac{1}{2}\left[\varphi(x) * (\delta_{ct} + \delta_{-ct})\right]$$

$$= \frac{1}{2}\left[\varphi * \delta_{ct} + \varphi * \delta_{-ct}\right]$$

$\boxed{\varphi * \delta_a ?}$

$$\langle T * \delta_a, \varphi \rangle =$$

$$= \langle T_x, \langle \delta_{a,y}, \varphi(x+y) \rangle \rangle$$

$$= \langle T_x, \varphi(x+a) \rangle$$

$$= \langle T_x, \tau_a \varphi(x) \rangle$$

$$= \langle \tau_a T, \varphi \rangle$$

نحيوها كيف نبدلو ب variable واحد

$$T * \delta_a = \delta_a * T = \tau_a T$$

$$= T(x-a)$$

$$= \frac{1}{2}\left[\tau_{ct}\varphi(x) + \tau_{-ct}\varphi(x)\right]$$

$$= \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right]$$

$$(f * g)(x) = \int f(y)\, g(x-y)\, dy$$

$$* \ \psi * \frac{1}{2c}\mathbb{1}_{[-ct,ct]}$$

$$= \frac{1}{2c}\int_{-ct}^{ct} \psi(x-y)\, \mathbb{1}(y)\, dy = \frac{1}{2c}\int_{-ct}^{ct} \psi(x-y)\, dy$$

$-ct < y < ct$

$\mathbb{1}(y) = 1$