Donc :

$$I_4 = 2\,\beta\left(\frac{1-x}{2}, \frac{1+x}{2}\right)$$

$$= 2 \cdot \frac{\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{1+x}{2}\right)}{\Gamma(1)}$$

$$= 2\,\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{1+x}{2}\right)$$

$$= 2\,\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(1 - \frac{1-x}{2}\right)$$

$$= 2 \cdot \frac{\pi}{\sin\left(\pi\left(\frac{1-x}{2}\right)\right)}$$

car
$$\begin{cases} -1 < x < 1 \\ -1 < -x < 1 \\ 0 < -x+1 < 2 \\ 0 < \frac{-x+1}{2} < 1 \end{cases}$$

Ex3,

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \quad a>0, b>0$$

(a) cvs. (cvrg).

(b) $\beta(a,b) = \beta(b,a)$

(c) $a>0, b>0$ ma :

$$\beta(a+1,b) + \beta(a,b+1)$$

$$= \int_0^1 t^{a}(1-t)^{b-1}\,dt + \int_0^1 t^{a-1}(1-t)^{b}\,dt$$

$$= \int_0^1 t^{a-1}(1-t)^{b-1}\left[t + 1 - t\right]dt$$

$$= \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$$

$$= \beta(a,b)$$

(d) $$\beta\left(\tfrac{3}{2}, \tfrac{3}{2}\right) = \int_0^1 t^{\frac{3}{2}-1}(1-t)^{\frac{3}{2}-1}\,dt$$

$$= \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt$$

$t = \cos^2\theta$.

$dt = -2\cos\theta\sin\theta\,d\theta$.

$$= \int_{\frac{\pi}{2}}^{0} \cos\theta \cdot \sin\theta \cdot (-2)\cos\theta\sin\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2\theta\,\sin^2\theta\,d\theta.$$

$$\cancel{= \int_0^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,\frac{1-\cos(2\theta)}{2}\,d\theta}$$

$$\cancel{= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))(1-\cos(2\theta))\,d\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos\theta\sin\theta)^2\,d\theta$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin(2\theta)}{2}\right)^2 d\theta$$

$$= \frac{2}{4}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^2\,d\theta$$

$$\boxed{\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (1-\cos(4\theta))\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{8}.$$

(c) $n \in \mathbb{N}^*$, ma :

$$\beta(1,n) = \int_0^1 t^{1-1}(1-t)^{n-1}\,dt$$
$$= \int_0^1 (1-t)^{n-1}\,dt$$
$$= \left[-\frac{1}{n}(1-t)^n\right]_0^1$$
$$= \frac{1}{n} \Rightarrow \beta(1,n) = \frac{1}{n},\ n \in \mathbb{N}^*$$

(2) (a)

$$\beta(a+1,b) = \int_0^1 t^a (1-t)^{b-1}\,dt$$

IPP $\begin{cases} t^a \longrightarrow a t^{a-1} \\ (1-t)^{b-1} \longrightarrow -\frac{1}{b}(1-t)^b \end{cases}$

$$\beta(a+1,b) = \left[-\frac{t^a}{b}(1-t)^b\right]_0^1 + \frac{a}{b}\int_0^1 t^{a-1}(1-t)^b$$
$$= 0 + \frac{a}{b}\int_0^1 t^{a-1}(1-t)^b\,dt$$
$$= \frac{a}{b}\int_0^1 t^{a-1}(1-t)^{b-1}(1-t)\,dt$$
$$= \frac{a}{b}\int_0^1 t^{a-1}(1-t)^{b-1} - t^a(1-t)^{b-1}\,dt$$
$$= \frac{a}{b}\left(\beta(a,b) - \beta(a+1,b)\right)$$
$$\beta(a+1,b)\left[1 + \frac{a}{b}\right] = \frac{a}{b}\beta(a,b)$$
$$\Rightarrow \boxed{\beta(a+1,b) = \left(\frac{a}{a+b}\right)\beta(a,b)}$$

(b) $n, p \in \mathbb{N}^*$, ma ?

$\beta(n,p) = ?$

$$\beta(n,p) = \beta((n-1)+1, p)$$
$$= \frac{(n-1)}{(n-1)+p}\beta(n-1,p)$$
$$= \frac{n-1}{n-1+p}\,\frac{n-2}{n-2+p}\beta(n-2,p)$$
$$= \frac{(n-1)}{n-1+p}\,\frac{(n-2)}{n-2+p}\beta(n-2,p)$$
$$= \frac{n-1}{n-1+p}\,\frac{n-2}{n-2+p}\,\frac{n-3}{n-3+p}\beta(n-3,p)$$
$$= \frac{(n-1)}{n-1+p}\,\frac{(n-2)}{n-2+p}\,\frac{n-3}{n-3+p}\cdots\cdots\frac{1}{p+1}\beta(n-(n-1),p)$$
$$= \frac{(n-1)(n-2)(n-3)\cdots 1}{(n+p-1)(n+p-2)\cdots(p+1)}\beta(1,p)$$
$$= \frac{(n-1)!}{(n+p-1)(n+p-2)\cdots(p+1)}\,\frac{1}{p}$$
$$= \frac{(n-1)!\ (p-1)\cdots 1}{(n+p-1)\cdots p\,(p-1)\cdots 1}$$
$$= \frac{(n-1)!\,(p-1)!}{(n+p-1)!}$$

(c) $n, p \in \mathbb{N}^*$ ma ?

$$\beta\left(n+\frac{1}{2}, p+\frac{1}{2}\right)$$
$$\beta\left(\left(n-1+\frac{1}{2}\right)+1,\ p+\frac{1}{2}\right)$$
$$= \frac{\left(n-\frac{1}{2}\right)}{\left(n-\frac{1}{2}+p+\frac{1}{2}\right)}\beta\left(\left(p-1+\frac{1}{2}\right)+1,\ n-\frac{1}{2}\right)$$

$\left(n-1+\frac{1}{2}\right)$

$$= \frac{\left(n-\frac{1}{2}\right)\left(p-\frac{1}{2}\right)}{\left(n-\frac{1}{2}+p+\frac{1}{2}\right)\left(p-\frac{1}{2}+n-\frac{1}{2}\right)}\beta\left(n-\frac{1}{2}, p-\frac{1}{2}\right)$$
$$= \frac{\left(n-\frac{1}{2}\right)\left(p-\frac{1}{2}\right)}{(n+p)(n+p-1)}\beta\left(n-\frac{1}{2}, p-\frac{1}{2}\right)$$

$(n-2+$

$$= \frac{\left(n-\frac{1}{2}\right)\left(p-\frac{1}{2}\right)}{(n+p)(n+p-1)}\,\frac{\left(n-\frac{3}{2}\right)\left(p-\frac{3}{2}\right)\beta\left(n-\frac{3}{2}, p-\frac{3}{2}\right)}{\left(n-\frac{3}{2}+p-\frac{1}{2}\right)\left(p-\frac{3}{2}+n-\frac{3}{2}\right)}$$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)} \; \frac{(n-\frac{3}{2})(p-\frac{3}{2})}{(n+p-2)(n+p-3)}$$

$$\cancel{\beta(n-(n-1)+\tfrac{1}{2}, p-(p-1)+\tfrac{1}{2})}$$

$$\frac{(n-\frac{5}{2})\quad(p-\frac{5}{2})}{\underbrace{(n-\frac{5}{2}+p-\frac{3}{2})}_{n+p-4}\;\underbrace{(n-\frac{5}{2}+p-\frac{5}{2})}_{n+p-5}} \beta(n-\tfrac{5}{2}, p-\tfrac{5}{2})$$

$$= \frac{(n-\frac{1}{2})(p-\frac{1}{2})}{(n+p)(n+p-1)} \cdot \frac{(n-\frac{3}{2})(p-\frac{3}{2})}{(n+p-2)(n+p-3)} \; \frac{(n-\frac{5}{2})(p-\frac{5}{2})}{(n+p-4)(n+p-5)} \cdots$$

$$\frac{\frac{3}{2}\qquad\frac{3}{2}}{(\frac{3}{2}+2+\frac{1}{2})\;(\frac{3}{2}+\frac{3}{2})} \beta(\overbrace{n-(n-1)+\tfrac{1}{2}}^{\frac{3}{2}}, \overbrace{p-(p-1)+\tfrac{1}{2}}^{\frac{3}{2}})$$

$$= \frac{(n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})\cdots\frac{3}{2}\frac{1}{2}\times 4}{(n+p)(n+p-1)(n+p-2)(n+p-3)\cdots 4.3} \beta(\tfrac{3}{2},\tfrac{3}{2}) \quad \frac{\pi}{8}$$

$$\cancel{(2n-1)(2n-3)(2n-5)}$$

$$= \frac{(\frac{1}{2})^n (2n-1)(2n-3)\cdots 1 \; (\frac{1}{2})^p (2p-1)(2p-3)\cdots 1 \; \cancel{4}}{(n+p)(n+p-1)(n+p-2)\cdots\; 4.3} \cdot \frac{\pi}{8} \quad 8=4\times 2$$

$$\cancel{= \frac{(\frac{1}{2})^{n+p}(2n-1)!\,(2p-1)!\,\pi}{(n+p)!}} = \left(\frac{1}{2}\right)^{n+p} \frac{(2n)!\,(2p)!}{2n(2n-2)\cdots 2 \;\; 2p(2p-2)\cdots 2\,(n+p)!}\,\pi$$

$$\cancel{= \pi \left(\frac{1}{2}\right)^{n+p} \frac{(2n-1)!\,(2p-1)!}{(n+p)!}} = \left(\frac{1}{2}\right)^{n+p} \frac{(2n)!\,(2p)!\,\pi}{2^n\, n!\; 2^p\, p!\;(n+p)!}$$

$$= \frac{(2n)!\,(2p)!}{n!\,p!\;2^{2(n+p)}}\,\pi.$$