$$a_1 = \frac{i}{2}\left(\frac{1}{a^3} - \frac{i}{2a}\right) \text{ donc } a_3 = -a_1 = \frac{-i}{2}\left(\frac{1}{a^3} - \frac{1}{2a}\right)$$

$$g(p) = -\frac{i}{2}\left(\frac{1}{a^3} - \frac{1}{2a}\right)\left[\frac{1}{p+ia} - \frac{1}{p-ia}\right] + \frac{1}{4a^2}\left[\frac{1}{(p+ia)^2} + \frac{1}{(p-ia)^2}\right]$$

$$g(p) = \frac{1}{p^2+a^2}\,\frac{1}{p^2+a^2} = \frac{1}{a^2}\mathcal{L}(H(t)\sin(at))(p)\,\mathcal{L}(H(t)\sin(at))(p)$$

$$\mathcal{L}(f*g)(p) = \mathcal{L}(f)(p)\,\mathcal{L}(g)(p) = \frac{1}{a^2}\mathcal{L}(H(\cdot)\sin(a\,\cdot) * H(\cdot)\sin(a\,\cdot))(t))(p)$$

$$\Rightarrow \mathcal{L}^{-1}(g)(t) = (H(\cdot)\sin(a\,\cdot)) * (H(\cdot)\sin(a\,\cdot))(t) = \int_{\mathbb{R}} H(x)\sin(ax)H(t-x)\sin(a(t-x))\,dx$$

or $H(x) = \begin{cases} 1 & ; \text{si } x \geq 0 \\ 0 & ; \text{sinon} \end{cases}$ $\quad H(t-x) = \begin{cases} 1 & ; t-x \geq 0 \Rightarrow x \leq t \\ 0 & t-x < 0 \end{cases}$

$$\Rightarrow \mathcal{L}^{-1} g(t) = \int_0^{+\infty} \sin(ax)H(t-x)\sin(a(t-x))\,dx = \int_0^t \sin(ax)\sin(a(t-x))\,dx$$

$\cos^2(x) + \sin^2(x) = 1$ | $\sin(a+b) = \sin a\cos b - \sin b\cos a$ | $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(2x) = \cos^2 x - \sin^2 x$ | $\sin(a-b) = \sin a\cos b + \sin b\cos a$ | $\cos(a-b) = \sin a\sin b + \cos a\cos b$

$$Ff(x) = \frac{2}{\pi x}\left\{\left[\frac{-t}{2\pi x}\cos(2\pi x t)\right]_{t=0}^{t=1} + \frac{1}{2\pi x}\int_0^1 \cos(2\pi x t)\,dt\right\}$$

$$= \frac{-1}{(\pi x)^2}\cos(2\pi x) + \frac{1}{(\pi x)^2}\,\frac{\sin(2\pi x)}{2\pi x}$$

$$\Longrightarrow Ff(x) = -\frac{\cos(2\pi x)}{(\pi x)^2} + \frac{1}{2(\pi x)^3}\sin(2\pi x)$$

2 _ soit $I = \int_0^{+\infty} \frac{x\cos(x) - \sin(x)}{x^3}\cos\left(\frac{x}{2}\right)dx$

on a $g(x) = Ff(x) = -\frac{\cos(2\pi x)}{(\pi x)^2} + \frac{1}{2(\pi x)^3}\sin(2\pi x)$ est paire donc

① $F(g)(x) = F_c\, g(x) = 2\int_0^{+\infty} g(t)\cos(2\pi x t)\,dt$ d'une part, et d'autre part d'après la formule d'inversion : $Fg(x) = F(Ff)(x) = \tilde{f}(x) = f(-x)$

$= f(x)$ car f est paire

on a $Fg(x) = 2\int_0^{+\infty}\left\{-\frac{\cos(2\pi t)}{(\pi t)^2} + \frac{1}{2(\pi t)^3}\sin(2\pi t)\right\}\cos(2\pi x t)\,dt$

soit le CV $y = 2\pi t \Rightarrow dy = 2\pi\,dt$; $t = \frac{y}{2\pi}$

$$\Longrightarrow Fg(x) = -2\int_0^{+\infty}\left\{\frac{\cos(y)}{\left(\frac{y}{2}\right)^2} - \frac{1}{2\left(\frac{y}{2}\right)^3}\sin(y)\right\}\cos(yx)\frac{dy}{2\pi}$$

$$= \frac{-4}{\pi}\int_0^{+\infty}\frac{y\cos(y) - \sin(y)}{y^3}\cos(xy)\,dy$$

En particulier pour $x = \frac{1}{2}$; $\Longrightarrow Fg\left(\frac{1}{2}\right) = -\frac{4}{\pi}I = f\left(\frac{1}{2}\right) = \frac{3}{4} \Longrightarrow I = -\frac{3\pi}{16}$

Exercice :

soit $g(p) = \frac{1}{(p^2+a^2)^2}$; chercher l'origine de g par la transformation de Laplace

$= \frac{1}{(p+ia)^2(p-ia)^2}$; on a $g(p) = \frac{a_1}{p+ia} + \frac{a_2}{(p+ia)^2} + \frac{a_3}{p-ia} + \frac{a_4}{(p-ia)^2}$

$a_2 = (p+ia)^2 g(p)\Big|_{p=-ia} = -\frac{1}{4a^2}$; $a_4 = (p-ia)^2 g(p)\Big|_{p=ia} = -\frac{1}{4a^2}$; $\lim_{p\to+\infty} p\,g(p) = 0 = a_1 + a_3 \Rightarrow a_1 = -a_3$

et $g(0) = \frac{1}{a^4} = \frac{a_1}{ia} + \frac{a_2}{ia^2} - \frac{a_3}{ia} - \frac{a_4}{a^2} = \frac{2a_1}{ia} + \frac{1}{2a^2} = \frac{1}{a^4} \Rightarrow a_1 = \left(\frac{1}{a^4} - \frac{1}{2a^2}\right)\frac{ia}{2}$

Exercice :

-1- Calculer la transformation de Fourier $f(x) = \begin{cases} 1-x^2 ; & |x|<1 \\ 0 ; & \text{sinon} \end{cases}$

-2- Evaluez l'intégrale $\int_0^{+\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

- $\mathcal{F}f(x) = \int_{\mathbb{R}} f(t)\, e^{-2i\pi x t}\, dt$; • $\mathcal{F}(\tau_a f)(u) = \mathcal{F}(f(t-a))(x) = e^{-2i\pi \dots}$
- $\mathcal{F}(f(at))(x) = \frac{1}{|a|} \mathcal{F} f\left(\frac{x}{a}\right)$; • $\mathcal{F}(f^{(k)})(x) = (2i\pi x)^k \mathcal{F} f(x)$

$\frac{d^k}{dx^k}(\mathcal{F} f)(x) = (\mathcal{F} f)^{(k)}(x) = (-2i\pi)^k \mathcal{F}(x \dots$

- $f * g(x) = \int_{\mathbb{R}} f(t)\, g(x-t)\, dt$; • $\mathcal{F}(f*g)(x) = \mathcal{F}f(x)\, \mathcal{F}g(x)$
- $\int_{\mathbb{R}} \mathcal{F}f(x)\, \overline{\mathcal{F}g(x)}\, dx = \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, dx$; • $\mathcal{F}^{-1} f(x) = \mathcal{F}f(-x) = \int_{\mathbb{R}} f(t)\, e^{2i\pi \dots}$
- $\mathcal{F} \circ \mathcal{F} f(x) = \tilde{f}(x) = f(-x)$; • $\mathcal{F} \circ \mathcal{F}^{-1} f(x) = f(x)$
- $\|\mathcal{F} f\|_2 = \|f\|_2$; • $\int_{\mathbb{R}} |\mathcal{F} f(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx$

Réponse :

on peut voir que $f \in L^1$ car $\int |f| < \infty$ et que f est paire donc

$$\mathcal{F} f(x) = \mathcal{F}_c f(x) = 2\int_0^1 \cos(2\pi x t)(1-t^2)\, dt \quad ; \begin{cases} u = 1-t^2 \rightarrow \dots \\ v' = \cos(2\pi x t) \end{cases}$$

$$\mathcal{F} f(x) = 2\Big[\underbrace{\frac{1-t^2}{2\pi x} \sin(2\pi x t)\Big]_0^1}_{=0} + \frac{2}{2\pi x}\int_0^1 t \sin(2\pi x t)\, dt \quad ; \quad v = \frac{1}{2\pi x}\sin(2\pi x t \dots$$

IPP ; $u = t \longrightarrow u' = 1$

$v' = \sin(2\pi x t) \longrightarrow v = -\frac{1}{2\pi x}\cos(2\pi x t)$