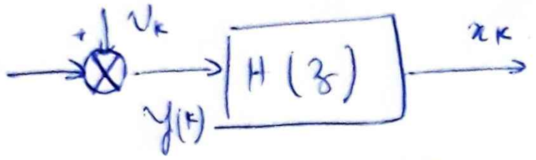
Travaux Dirigés :

Exercice 1 :

$$\begin{cases} x_{k+1} = 0{,}25\, x_k + w_k \\ y_k = x_k + v_k \end{cases}$$

$$S_{xx}(z) = \frac{-3{,}75\, z}{(z-0{,}25)(z-4)} \; ; \; S_{vv}(z) = 1$$



$$H(z) = \frac{1}{S^{+}_{yy}(z)} \cdot \left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{-1})} \right]_{+}$$

$S_{xy}(z) = ?$

$$\phi_{xy}(j) = E\left[x(i+j)\, y(i)\right]$$

$$= E\left[x(i+j)\left(x(i) + v(i)\right)\right]$$

$$= E\left[x(i+j) \cdot x(i)\right] + \underbrace{E\left[x(i+j)\, v(i)\right]}_{0}$$

$$\boxed{\phi_{xy}(j) = \phi_{xx}(j)}$$

TZ, $S_{xy}(z) = S_{xx}(z) = \dfrac{-3{,}75 z}{(z-0{,}25)(z-4)}$.

$S_{yy}(z) = ?$

$$\phi_{yy}(j) = E\left[y(i+j) \cdot y(i)\right] = E\left[x(i+j) + v(i+j)\right]\left[x(i) + v(i)\right]$$

$$= E\left[x(i+j)\, x(i)\right] + \underbrace{E\left[x(i+j)\, v(i)\right]}_{0} + \underbrace{E\left[v(i+j)\, x(i)\right]}_{0} + E\left[v(i+j)\, v(i)\right]$$

$$\phi_{yy}(j) = \phi_{xx}(j) + \phi_{vv}(j).$$

TZ, $S_{yy}(z) = S_{xx}(z) + S_{vv}(z) = \dfrac{-3{,}75 z}{(z-0{,}25)(z-4)} + 1$

③

$$S_{yy}(z) = \frac{-3{,}75z + (z-0{,}25)(z-4)}{(z-0{,}25)(z-4)}.$$

$$S_{yy}(z) = \frac{-3{,}75z + z^2 - 4{,}25z + 1}{(z-0{,}25)(z-4)} = \frac{z^2 - 8z + 1}{(z-0{,}25)(z-4)}.$$

Soit $P(z) = z^2 - 8z + 1$.

$\Delta = 8^2 - 4 = 60$. $\Rightarrow$ $z_1 = \frac{8+\sqrt{60}}{2} = 4 + \sqrt{15} = 7{,}87$.

$\sqrt{\Delta} = \sqrt{60}$ $\qquad z_2 = \frac{8-\sqrt{60}}{2} = 4 - \sqrt{15} = 0{,}127$.

$$S_{yy}(z) = \frac{(z-7{,}87)(z-0{,}127)}{(z-0{,}25)(z-4)}$$

$$S^{+}_{yy}(z) = \frac{(z-0{,}127)}{(z-0{,}25)} \qquad S^{-}_{yy}(z) = \frac{z-7{,}87}{z-4} = S^{+}_{yy}(z^{+})$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{+})}\right]_{+} = \frac{-3{,}75z}{(z-0{,}25)(z-4)} \times \frac{(z-4)}{(z-7{,}87)} = \frac{-3{,}75z}{(z-0{,}25)(z-7{,}87)}$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{*})}\right] = \frac{-3{,}75z}{(z-0{,}25)(z-7{,}87)z} = \frac{\alpha}{(z-0{,}25)} + \frac{\beta}{(z-7{,}87)}$$

$$\Rightarrow \alpha = \frac{-3{,}75 \times 0{,}25}{0{,}25 - 7{,}87} = 0{,}12 \qquad \beta = \frac{-3{,}75 \times 7{,}87}{7{,}87 - 0{,}25} = -3{,}87.$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z)}\right] = \frac{0{,}12}{(z-0{,}25)} - \frac{3{,}87}{z-7{,}87}$$

$$\Rightarrow \left[\frac{S_{xy}(z)}{S^{+}_{yy}(z)}\right]_{+} = \frac{0{,}12}{z-0{,}25}$$

Donc $$H(z) = \frac{(z-0{,}25) \times 0{,}12}{(z-0{,}127)(z-0{,}25)}$$

$$\Rightarrow \quad H(z) = \frac{0,12}{z - 0,127}$$

Exercice 1 :

Système : $\begin{cases} x(k+1) = F.X(k) + G.U(k) \\ y(k) = H.X(k) \end{cases}$

observateur $\begin{cases} \hat{x}(k+1) = F\hat{x}(k) + GU(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = H.\hat{x}(k) \end{cases}$

1) $\delta(k) = x(k) - \hat{x}(k)$

$\delta(k+1) = x(k+1) - \hat{x}(k+1)$

1) $\delta(k+1) = x(k+1) - \hat{x}(k+1)$

$= F.X(k) + GU(k) - F\hat{X}(k) - GU(k) - L(y(k) - \hat{y}(k))$

$FX(k) + GU(k) + F\hat{X}(k) - GU(k) - L(H(X(k) - \hat{X}(k)))$

$= (F - LH)\,\delta(k)$

pour que l'erreur soit stable, il faut que $(F-LH)$ possède des valeurs propres stables.

2) $F = \begin{bmatrix} 2 & -1 \\ 1 & 0,5 \end{bmatrix}$ $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Etape 1

$\Theta = \begin{pmatrix} H \\ HF \end{pmatrix}$; $HF = (1\ 0)\begin{pmatrix} 2 & -1 \\ 1 & 0,5 \end{pmatrix} = (2\ \ -1)$

$\Theta = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$; $\det \neq 0 \rightarrow$ Système observable.

Etape 2 : $\det(zI - F) = \det\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & 0,5 \end{pmatrix}\right]$

$$= \det \begin{pmatrix} z-2 & 1 \\ -1 & z-0{,}5 \end{pmatrix} = (z-2)(z-0{,}5)+1$$

$$= z^2 - 2{,}5z + 2$$

$$\begin{cases} a_0 = 2 \\ a_1 = -2{,}5 \end{cases}$$

étape ③ :

$$K = \begin{bmatrix} -2{,}5 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1} = K.O = \begin{bmatrix} -2{,}5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -0{,}5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$P = \frac{1}{\det(P^{-1})} \cdot com\,(P^{-1})^t = \begin{bmatrix} 0 & 1 \\ -1 & -0{,}5 \end{bmatrix}$$

$$\det(P^{-1}) = 1$$

étape ④ :

$$(z-0{,}2)(z-0{,}6) = z^2 - 0{,}2z + 0{,}12 - 0{,}6z$$

$$= z^2 - 0{,}8z + 0{,}12 \Rightarrow \alpha_0 = 0{,}12, \; \alpha_1 = -0{,}8$$

étape ⑤ :

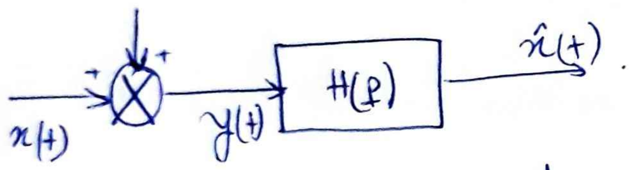
$$L = P. \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0{,}5 \end{bmatrix} \begin{bmatrix} -1{,}88 \\ 1{,}7 \end{bmatrix} = \begin{bmatrix} 1{,}7 \\ 2{,}73 \end{bmatrix}$$

Exercice ② :

$$\begin{cases} \dot{x}(t) = -4x(t) + w(t). \\ y(t) = x(t) + v(t). \end{cases}$$

$$S_{xx}(p) = \frac{1}{16 - p^2} \quad ; \quad S_{vv}(p) = 1.$$

1) filtre de Winner.



hypothèse : $x(t)$ et $v(t)$ sont indépendants

$$H(p) = \frac{1}{S^{*}_{yy}(p)} \cdot \left[\frac{S_{xy}(p)}{S^{*}_{yy}(-p)} \right]$$

$S_{xy}(p) = ?$ $\phi_{xy}(\tau) = E\left[x(t+\tau)\, y(t)\right].$

$$= E\left[x(t+\tau)\left(x(t) + v(t)\right)\right] = E\left[x(t+\tau) . x(t)\right] + \underbrace{E\left[x(t+\tau)\, v(t)\right]}_{0}$$

$$\phi_{xy}(\tau) = \phi_{xx}(\tau).$$

TL

$$S_{xy}(p) = S_{xx}(p) = \frac{1}{16 - p^2}.$$

$S_{yy}(p) = ?$ $\phi_{yy}(\tau) = E\left[y(t+\tau) . y(t)\right].$

$$= E\left[\left(x(t+\tau) + v(t+\tau)\right) . \left(x(t) + v(t)\right)\right].$$

$$= E\left[x(t+\tau) . x(t)\right] + E\left[x(t+\tau) . v(t)\right] + E\left[v(t+\tau) . x(t)\right] +$$

$$\left[\frac{S_{xy}(p)}{S^*_{yy}(p)}\right] = \frac{\frac{1}{\sqrt{17}+4}}{4+p} = \frac{1}{(\sqrt{17}+4)(4+p)}$$

$$H(p) = \frac{1}{S^*_{yy}(p)} \cdot \left[\frac{S_{xy}(p)}{S^*_{yy}(p)}\right]$$

$$= \frac{4+p}{\sqrt{17}+p} \cdot \frac{1}{(\sqrt{17}+4)(4+p)}$$

$$\boxed{H(p) = \frac{1}{(\sqrt{17}+p)(\sqrt{17}+4)}}$$

2) $A=-4$, $B=0$, $G=1$, $C=1$, $R=1$, $Q=1$.

- filtre de Kalmam (continu).

$$\begin{cases} \dot{\hat{x}}(t) = A.\hat{x}(t) + B.w(t) + K(t).\left[y(t) - C.\hat{x}(t)\right] \\ K(t) = P(t).C^T.R^{-1} \\ \dot{P}(t) = G\,Q.G^T + A.P(t) + P(t).A^T - P(t).C^T.R^{-1}C\,P(t) \end{cases}$$

$$\begin{cases} \dot{\hat{x}}(t) = -4\hat{x}(t) + K(t)\left[y(t) - \hat{x}(t)\right] \\ K(t) = p(t) \\ \dot{P}(t) = 1 - 4p(t) - 4p(t) - p^2(t) \\ \quad = 1 - 8p(t) - p^2 \end{cases}$$

En régime permanent: $P = \text{cte}$, $\dot{P}(t) = 0$

$$1 - 8p - p^2 = 0.$$

$\Delta = b^2 - 4ac = 68$.

$$\begin{cases} p_1 = \dfrac{-b-\sqrt{\Delta}}{2a} = \dfrac{8-2\sqrt{17}}{-2} = -4+\sqrt{17}. \\[2ex] p_2 = \dfrac{-b+\sqrt{\Delta}}{2a} = \dfrac{8+2\sqrt{17}}{-2} = -4-\sqrt{17}. \end{cases}$$

$$\boxed{p = -4+\sqrt{17}}$$

$$\Rightarrow \boxed{K = p = -4+\sqrt{17}}$$

$$\dot{\hat{x}}(t) = -4\hat{x}(t) + K(t)\left[y(t) - \hat{x}(t)\right].$$

$$= -4\hat{x}(t) - (4-\sqrt{17})(y(t) - \hat{x}(t)).$$

$$= -4\hat{x}(t) - (4-\sqrt{17})\,y(t) + 4\hat{x}(t) - \sqrt{17}\,\hat{x}(t).$$

$$\dot{\hat{x}}(t) = \sqrt{17}\,\hat{x}(t) + (-4+\sqrt{17})\,y(t).$$

$$\mathcal{L}\quad p\cdot\hat{X}(p) = -\sqrt{17}\,\hat{X}(p) + (-4+\sqrt{17})\,Y(p).$$

$$H_{Kalman}(p) = \frac{\hat{X}(p)}{Y(p)} = \frac{-4+\sqrt{17}}{p+\sqrt{17}}.$$

$$\text{or}\quad H_{Wiener}(p) = \frac{1}{(\sqrt{17}+p)(\sqrt{17}+4)} \cdot \frac{\sqrt{17}-4}{\sqrt{17}-4}.$$

$\Rightarrow$ Les deux filtres sont identique.

Suite exercice 4 :

2) filtre de Kalman.

$$\begin{cases} x_{k+1} = 0{,}25\, x_k + w_k \\ y_k = x_k + v_k \end{cases}$$

$$\begin{cases} x_{k+1} = A_k x_k + B_k U_k + G_k w_k \\ y_k = C_k . x_k + v_k \end{cases}$$

$A_k = 0$; $B_k = 0$; $G_k = 1$, $C_k = 1$, $R_k = 1$, $Q_k = 1$.

$$\begin{cases} \hat{x}_{k+1/k} = A_k . \left[I - K_k C_k \right] \hat{x}_{k/k-1} + B_k . U_k + A_k . K_k y_k \\ P_{k+1/k} = A_k \left[I - K_k . C_k \right] . P_{k/k-1} . A_k^T + G_k Q_k G_k^T \\ K_k = P_{k/k-1} C_k^T \left(R_k + C_k . P_{k/k-1} C_k^T \right)^{-1} \end{cases}$$

$$\begin{cases} \hat{x}_{k+1/k} = 0{,}25 \left(1 - K_k \right) . \hat{x}_{k/k-1} + 0{,}25\, K_k . y_k \\ P_{k+1/k} = 0{,}25 \left(1 - K_k \right) . P_{k/k-1} . 0{,}25 + 1 \\ K_k = P_{k/k-1} \left(1 + P_{k/k-1} \right)^{-1} \end{cases}$$

en régime permanent : P = cte $\leadsto$ $P_{k+1/k} = P_{k/k-1} = P$.

$$P = 0{,}0625 \left(1 - K_k \right) P + 1$$

or $K_k = P(1+P)^{-1} = \dfrac{P}{1+P}$

$$\to P = 0{,}0625 \left(1 - \frac{P}{1+P} \right) P + 1$$

$$P = 0{,}0625 \left(\frac{1}{1+P} \right) P + 1$$

$$P = \frac{0,0625\,P}{P+1} + 1$$

$$P = \frac{0,0625\,P + P + 1}{P+1}$$

$$P(P+1) = 0,0625\,P + P + 1$$

$$P\left(P + 1 - 0,0625 - 1\right) = 1$$

$$\left(P^2 - 0,0625\,P - 1\right) = 0$$

$$\Delta = (0,0625)^2 + 4 = 4,0039$$

$$P_1 = \frac{0,0625 - \sqrt{4,0039}}{2} = -0,969$$

$$P_2 = \frac{0,0625 + \sqrt{4,0039}}{2} = 1,03$$

$$\Rightarrow P = 1,03$$

$$K_k = \frac{P}{P+1} = \frac{1,03}{1+1,03} = 0,503$$

$$\hat{x}_{k+1/k} = 0,25\left(1 - 0,507\right)\hat{x}_{k|k-1} + 0,25 \times 0,503\, y_k$$

$$\hat{x}_{k+1/k} = 0,123\,\hat{x}_{k/k-1} + 0,126\, y_k$$

$$\begin{cases} U(k+m) = z^m . U(k) \\ U(k-m) = z^{-m} . U(k) \end{cases}$$

$$H_{kalman}(z) = \frac{\hat{X}_k}{Y_k} = \frac{0,126}{z - 0,123}$$

$$z . \hat{X}_k = 0,123\,\hat{X}_k + 0,126\, Y_k$$

$$\left(z - 0,123\right)\hat{X}(k) = 0,126\, Y_k$$

$$H_{wiener}(z) = \frac{0,12}{z - 0,127}$$

Exercice 3 :

1) mvt à une vitesse constante.

$$\begin{cases} x(t) = v.t + y_0 \\ y(t) = x(t) + e(t). \end{cases}$$

$$y(t) = v.t + y_0 + e(t).$$

$$\hookrightarrow \quad y_k = v.k + y_0 + e_k.$$

2) objectif Estimation des paramètres:

$$\theta = \begin{bmatrix} y_0 & v \end{bmatrix}^T = cte.$$

$$\theta_{k+1} = \theta_k.$$

or $y_k = \begin{bmatrix} 1 & k \end{bmatrix} \begin{bmatrix} y_0 \\ v \end{bmatrix} + e_k.$

$y_k = C_k . \theta_k + e_k$ avec $C_k = \begin{bmatrix} 1 & k \end{bmatrix}.$

$$\begin{cases} \theta_{k+1} = \theta_k. \\ y_k = C_k . \theta_k + e_k \quad \text{avec } C_k = \begin{bmatrix} 1 & k \end{bmatrix}. \end{cases}$$

3)

$$\begin{cases} \hat{\theta}_k = \hat{\theta}_{k-1} + K_k \left[y_k - C_k \hat{\theta}_{k-1} \right]. \\ K_k = P_{k-1} C_k^T \left(R_k + C_k . P_{k-1} . C_k^T \right)^{-1} \\ P_k = P_{k-1} - K_k . C_k . P_{k-1} = (I - K_k C_k) P_{k-1}. \end{cases}$$

$$\begin{cases} \hat{\theta}_1 = \hat{\theta}_0 + K_1\left(y_1 - C_1 \hat{\theta}_0\right) \\ K_1 = P_0\, C_1^T \left(R + C_1 P_0\, C_1^T\right)^{-1} \\ P_1 = P_0 - K_1 C_1 \cdot P_0 \end{cases}$$

$$K_1 = P_0\, C_1^T \left(R + C_1 \cdot P_0 C_1^T\right)^{-1}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(0{,}01 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 10 \\ 10 \end{bmatrix} \cdot \left(0{,}01 + 20\right)^{-1} = \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix}.$$

$$P_1 = P_0 - K_1 C_1 P_0$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 0{,}4 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - 0{,}96 \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

$$= \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix}.$$

$$\hat{\theta}_1 = \theta_0 + K_1\left(y_1 - C_1 \cdot \hat{\theta}_0\right).$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix} \cdot \left(9 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right).$$

$$= 9 \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix} = \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix}.$$

$$\begin{cases} \hat{\theta}_2 = \hat{\theta}_1 + K_2\left(y_2 - C_2\hat{\theta}_1\right) \\ K_2 = P_1 \cdot C_2^T \left(R + C_2 P_1 C_2^T\right)^{-1} \\ P_2 = P_1 - K_2 C_2 \cdot P_1 \end{cases}$$

$$K_2 = P_1 \cdot C_2^T \left(R + C_2 P_1 C_2^T\right)^{-1}$$

$$= \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(0{,}01 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} 0{,}4 \\ 0{,}8 \end{bmatrix} \left(0{,}01 + 2\right)^{-1}$$

$$= \begin{bmatrix} 0{,}19 \\ 0{,}39 \end{bmatrix}$$

$$\hat{\theta}_2 = \hat{\theta}_1 + K_2\left(y_2 - C_2\hat{\theta}_1\right)$$

$$= \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix} + \begin{bmatrix} 0{,}19 \\ 0{,}39 \end{bmatrix} \left(10{,}8 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix}\right)$$

⚠ approx de Euler (discret → continue).

$\dot{x}(t) = \frac{x_{k+1} - x_k}{T_e}$

$\frac{x_{k+1} - x_k}{T_e} = A x_k$

$x_{k+1} = T_e A x_k$

$\begin{cases} x_{k+1} = (T_e A + I) x_k \\ y_k = [1 \ 0] x_k + v_k \end{cases}$

A.U. : 2021.2022

Section : GEA2

Enseignant : Anis Messaoud

Travaux Dirigés : Filtrage Optimal

⚠ [illegible]

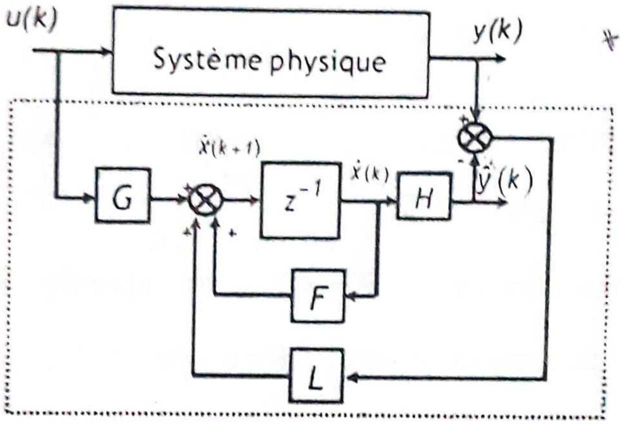
Variance = R

Exercice 1

Soit un système défini par la représentation d'état suivante :

$$\begin{cases} X(k+1) = F.X(k) + G.u(k) \\ y(k) = H.X(k) \end{cases}$$

Pour estimer les variables d'état de ce système, on a conçu un observateur comme indiqué sur la figure suivante.



1/ Établir les équations des erreurs entre les trajectoires d'état et de sortie du système et de l'observateur.

2/ On donne : $F = \begin{bmatrix} 2 & -1 \\ 1 & 0.5 \end{bmatrix}$; $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $H = [1 \quad 0]$

Synthétiser un observateur caractérisé par les valeurs propres $\lambda_1 = 0.2$ et $\lambda_2 = 0.6$.

Exercice 2

Considérons le système stochastique continu suivant :

$$\begin{cases} \dot{x}(t) = -4x(t) + w(t) \\ y(t) = x(t) + v(t) \end{cases}$$

avec : $w(t)$ et $v(t)$ sont deux bruits blancs gaussiens de variance respectivement $q = 1$ et $r = 1$. Le spectre de covariance des signaux $v(t)$ est $S_{vv}(p)$ et le spectre de covariance du signal $x(t)$ est $S_{xx}(p)$. On donne :

$$S_{xx}(p) = \frac{1}{16 - p^2} \quad ; \quad S_{vv}(p) = 1$$