Serie n°3 : Signaux et systèmes continus

Exercice 1.

C) signal sinusoidal :

$$x(t) = A\cos(2\pi f_0 (t - \alpha))$$

1) $x(t) = A\cos(2\pi f_0 t - \alpha)$

$$= \frac{A}{2}\left[e^{j(2\pi f_0 t - \alpha)} + e^{-j(2\pi f_0 t - \alpha)}\right]$$

$$= \frac{A}{2}\left[e^{-j\alpha} \cdot e^{j2\pi f_0 t} + e^{j\alpha} \cdot e^{-j2\pi f_0 t}\right]$$

On pose $X_1 = A\frac{e^{-j\alpha}}{2}$

$X_2 = A\frac{e^{j\alpha}}{2}$ (ne varie pas en fonction de t (constante))

$$X(f) = X_1 \delta(f - f_0) + X_2 \delta(f + f_0)$$

$$X(f) = \frac{e^{-j\alpha}}{2}\delta(f - f_0) + \frac{e^{j\alpha}}{2}\delta(f + f_0)$$

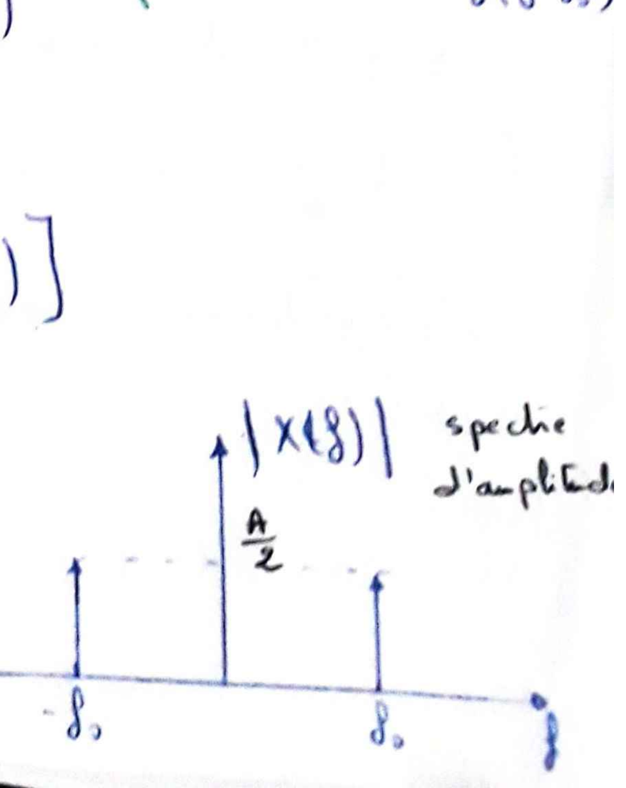
(car $TF(e^{-(2j\pi f_0 t)}) = \delta(f+f_0)$
et $TF(e^{(2j\pi f_0 t)}) = \delta(f - f_0)$)

2) si $\alpha = 0$

$$X(f) = \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

* spectre d'amplitude :

$$|X(f)| = \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

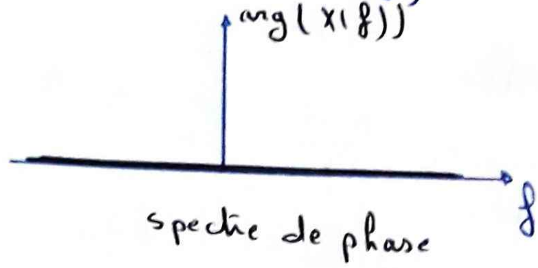


③ Fonction et corrélation :

Rappel :

* spectre de phase :

$$\arg(X(f)) = 0 \quad (\text{car } X(f) \text{ réel positif})$$

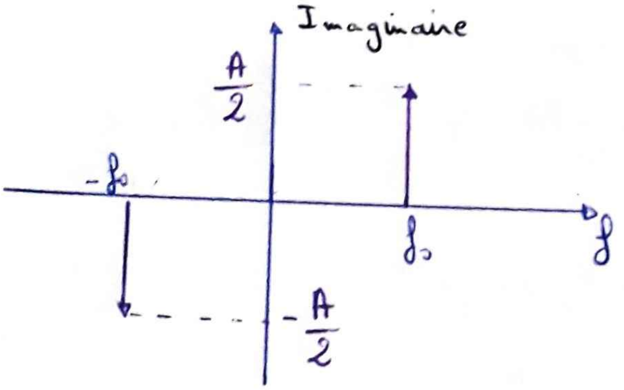


spectre de phase

* si $\alpha = \frac{\pi}{2}$

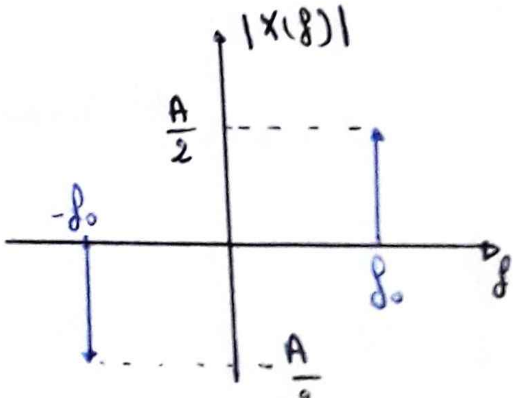
$$X(f) = \frac{A e^{-j\frac{\pi}{2}}}{2} \delta(f-f_0) + \frac{e^{j\frac{\pi}{2}}}{2} \delta(f+f_0)$$

$$= \frac{-Aj}{2} \delta(f-f_0) + \frac{jA}{2} \delta(f+f_0)$$



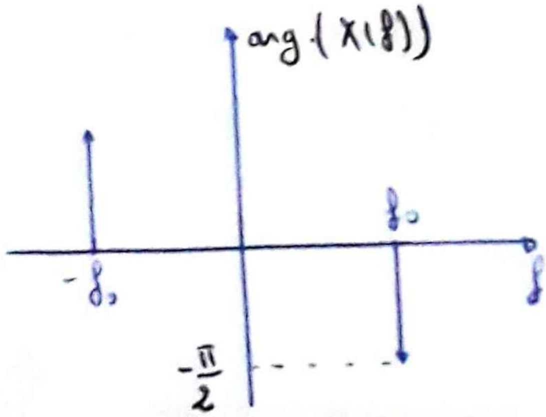
* spectre d'amplitude :

$$|X(f)| = -\frac{A}{2} \delta(f-f_0) + \frac{A}{2} \delta(f-f_0)$$



* spectre de phase :

$$\arg X(f) = \begin{cases} -\frac{\pi}{2} & \text{si } f = f_0 \\ \frac{\pi}{2} & \text{si } f = -f_0 \\ 0 & \text{sinon} \end{cases}$$



③ Fonction de corrélation :

Rappel :

la fonction corrélation d'un signal periodique est :

$$R_{xx}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x^{*}(t-\tau)\, dt$$

avec T est la periode du signal

On a :

$x(t) = A\cos(2\pi f_0 t - \alpha)$ de periode $T_0 = \frac{1}{f_0}$

$$R_{xx}(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t - \alpha) . A\cos(2\pi f_0 (t-\tau) - \alpha)\, dt$$

On a $\cos a . \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}$

Donc

$$R_{xx}(\tau) = \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{\cos(2\pi f_0 \tau) + \cos(4\pi f_0 t - 2\pi f_0 \tau - 2\alpha)}{2}\, dt$$

$$= \frac{A^2}{2T_0}\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos 2\pi f_0 \tau\, dt + \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(4\pi f_0 t - 2\pi f_0 \tau - 2\alpha)\, dt\right]$$

$$= \frac{A^2}{2T_0}\left[T_0\cos(2\pi f_0 \tau) + \left(\frac{1}{4\pi f_0}\left[\sin(4\pi f_0 t - 2\pi f_0 \tau - 2\alpha\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\right)\right]$$

$$= \frac{A^2}{2T_0}\left[T_0\cos(2\pi f_0 \tau) + \left(\frac{1}{4\pi f_0}\underbrace{\left[\sin(2\pi - \cancel{2\pi f_0 \tau - 2\alpha}) - \sin(\cancel{-2\pi - 2\pi f_0 \tau - 2\alpha})\right.}_{0}\right)\right.$$

$$\boxed{R_{xx}(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)}$$

4)
$$S_{xx}(f) = TF[R_{xx}(\tau)]$$
$$= \frac{A^2}{4}[\delta(f-f_0) + \delta(f+f_0)]$$

5) D'après la relation de Parseval

$$\boxed{P_x = R_{xx}(0) = \frac{A^2}{2}} \quad (\text{pour } \tau = 0)$$

et

$$P_x = \int_{-\infty}^{+\infty} S_{xx}(f)\,df = \frac{A^2}{4}\Big[\underbrace{\int_{-\infty}^{+\infty}\delta(f-f_0)\,df}_{1} + \underbrace{\int_{-\infty}^{+\infty}\delta(f+f_0)\,df}_{1}\Big]$$

Car $\int_{-\infty}^{+\infty}\delta(f)\,df = 1$

donc $\boxed{P_x = \frac{A^2}{2}}$

D) Autocorrélation et intercorrélation

$x(t) = A\sin(2\pi f_0 t)$ et $y(t) = \frac{A}{2}\cos(2\pi f_0 t + \frac{\pi}{4})$ et $f_0 = \frac{1}{T_0}$

$x(t)$ et $y(t)$ deux signaux periodique de même periode

1) Rappel :

La fonction d'intercorrélation de deux signaux de même periode

$$R_{xy}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot y^*(t-\tau)\,dt = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\cdot y^*(t-\tau)\,dt$$

$$R_{xy}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\sin(2\pi f_0 t)\cdot\frac{A}{2}\cos(2\pi f_0(t-\tau) + \frac{\pi}{4})\,dt$$

$$= \frac{A^2}{2T_0} \int_{-T_0/2}^{T_0/2} \sin(2\pi f_0 t) \cdot \cos\left(2\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{4}\right) dt$$

on a :

$$\sin a \cdot \cos b = \frac{\sin(a-b) + \sin(a+b)}{2}$$

Donc :

$$R_{xy} = \frac{A^2}{4T_0} \int_{-T_0/2}^{T_0/2} \sin\left(2\pi f_0 \tau - \frac{\pi}{4}\right) + \sin\left(4\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{4}\right) dt$$

$$= \frac{A^2}{4T_0}\left[\int_{-T_0/2}^{T_0/2} \sin\left(2\pi f_0 \tau - \frac{\pi}{4}\right) dt + \int_{-T_0/2}^{T_0/2} \sin\left(4\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{4}\right) dt\right]$$

$$= \frac{A^2}{4T_0}\left[T_0 \cdot \sin\left(2\pi f_0 \tau - \frac{\pi}{4}\right) + \left(\frac{1}{4\pi f_0} \sin\left(4\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{4}\right)\right)\right]$$

$$= \frac{A^2}{4} \sin\left(2\pi f_0 \tau - \frac{\pi}{4}\right) + \frac{A^2}{4T_0} \frac{1}{4\pi f_0} \underbrace{\left[\sin\left(2\pi - 2\pi f_0 \tau + \frac{\pi}{4}\right) - \sin\left(-2\pi - 2\pi f_0 \tau + \frac{\pi}{4}\right)\right]}_{0}$$

$$\boxed{R_{xy} = \frac{A^2}{4} \sin\left(2\pi f_0 \tau - \frac{\pi}{4}\right)}$$

2) $Z(t) = x(t) + y(t)$

$$R_{zz} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} Z(t) \cdot Z^*(t-\tau) \, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left[A \sin(2\pi f_0 t) + \frac{A}{2} \cos\left(2\pi f_0 t + \frac{\pi}{4}\right)\right]\left[A \sin(2\pi f_0 (t-\tau)) + \frac{A}{2} \cos\left(2\pi f_0 (t - \tau) + \frac{\pi}{4}\right)\right]$$

$$x(\tau) = \; = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\Big[A\sin(2\pi f_0 t) \cdot A\sin(2\pi f_0(t-\tau)) + \frac{A}{2}\cos(2\pi f_0 t + \frac{\pi}{4})\, A\sin(2\pi f_0(t-\tau)$$

$$+ A\sin(2\pi f_0 t) \cdot \frac{A}{2}\cos(2\pi f_0(t-\tau) + \frac{\pi}{4})$$

$$+ \frac{A}{2}\cos(2\pi f_0 t + \frac{\pi}{4}) \cdot \cos(2\pi f_0(t-\tau) + \frac{\pi}{4})\Big]\, dt$$

a : si

$$(\tau) = \; = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

$= \frac{1}{2}$

$= \frac{A^2}{2T_0}$ c

$\frac{A^2}{2}$ co

$\tau) =$

$= \frac{1}{T_0}$

$\frac{A^2}{4T_0}$

^{2} co

$\frac{A^2}{8}$ c

on

$$R_{xx}(\tau) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2 \sin(2\pi f_0 t) \sin(2\pi f_0 (t-\tau))\, dt$$

on a : $\sin a \sin b = \dfrac{\cos(a-b) - \cos(a+b)}{2}$

$$R_{xx}(\tau) = \frac{A^2}{2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left(\cos(2\pi f_0 \tau) - \cos(4\pi f_0 t - 2\pi f_0 \tau)\right) dt$$

$$= \frac{A^2}{2T_0}\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 \tau)\, dt - \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(4\pi f_0 t - 2\pi f_0 \tau)\, dt \right]$$

$$= \frac{A^2}{2T_0} \cos(2\pi f_0 \tau) . T_0 - \frac{A^2}{2T_0} \left[\frac{1}{4\pi f_0} \sin(4\pi f - 2\pi f_0 \tau) \right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \frac{A^2}{2} \cos(2\pi f_0 \tau) - \frac{A^2}{2T_0} \left[\frac{1}{4\pi f_0} \left(\underbrace{\sin(2\pi - 2\pi f_0 \tau) - \sin(-2\pi - 2\pi f_0 \tau)}_{0} \right) \right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$\boxed{R_{xx}(\tau) = \frac{A^2}{2} \cos(2\pi f_0 \tau)}$$

$$R_{yy}(\tau) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{A}{2} \cos\left(2\pi f_0 t + \frac{\pi}{4}\right) . \frac{A}{2} \cos\left(2\pi f (t-\tau) + \frac{\pi}{4}\right) dt$$

$$= \frac{A^2}{4T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{\cos(2\pi f_0 \tau) + \cos(4\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{2})}{2}\, dt$$

$$= \frac{A^2}{8} \cos 2\pi f \tau$$

$$\boxed{R_{yy}(\tau) = \frac{A^2}{8} \cos(2\pi f_0 \tau)}$$

$$R_{yx} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{A}{2} \cos\left(2\pi f_0 t + \frac{\pi}{4}\right) . A \sin\left(2\pi f_0 (t-\tau)\right) dt$$

$$= \frac{A^2}{2T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos\left(2\pi f_0 t + \frac{\pi}{4}\right) \sin\left(2\pi f_0 (t-\tau)\right) dt$$

on a :

$$\boxed{\cos a . \sin b = \frac{\sin(a+b) - \sin(a-b)}{2}}$$

$$\Rightarrow \quad R_{yx}(\tau) = \frac{A^2}{4T_0}\left[\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin\left(4\pi f_0 t - 2\pi f_0 \tau + \frac{\pi}{4}\right) dt}_{0} - \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin\left(2\pi f_0 \tau + \frac{\pi}{4}\right) dt}_{T_0 \sin\left(2\pi f_0 \tau + \frac{\pi}{4}\right)} \right]$$

$$\boxed{R_{yx}(\tau) = \frac{A^2}{4} \sin\left(2\pi f_0 \tau + \frac{\pi}{4}\right)}$$

donc

$$\boxed{R_{zz}(\tau) = \frac{A^2}{2} \cos\left(2\pi f_0 \tau\right) + \frac{A^2}{8} \cos\left(2\pi f_0 \tau\right) + \frac{A^2}{4} \sin\left(2\pi f_0 \tau + \frac{\pi}{4}\right)}$$

3)- $S_{zz}(f) = TF\left(R_{zz}(\tau)\right)$

$$TF\left(\frac{A^2}{4} \sin\left(2\pi f_0 \tau + \frac{\pi}{4}\right)\right) = \frac{A^2}{8j} TF\left(e^{+j\left(2\pi f_0 \tau + \frac{\pi}{4}\right)} - e^{-j\left(2\pi f_0 \tau + \frac{\pi}{4}\right)} \right)$$

$$= \frac{A^2}{8j}\left[e^{j\frac{\pi}{4}} . \delta(f-f_0) - e^{-j\frac{\pi}{4}} \delta(f+f_0) \right]$$

$$S_{zz} = \frac{A}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right] + \frac{A^2}{8}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$+ \frac{A^2}{8j}\left[e^{j\frac{\pi}{4}} \delta(f-f_0) - e^{-j\frac{\pi}{4}} \delta(f+f_0) \right]$$

4) Par application de la formule de Parseval :

$$P_z = R_{zz}(0) = \int_{-\infty}^{+\infty} S_{xx}(f) \cdot df$$

$$R_{zz}(0) = \frac{A^2}{2} + \frac{A^2}{8} + \frac{A^2}{4} \sin\frac{\pi}{4} = \frac{A^2}{2} + \frac{A^2}{8} + \frac{A^2}{4\sqrt{2}}$$