Exercices du cours : 3.6.1 :

4/ $u \in \mathcal{C}^2$ E_7 : $\begin{cases} \partial^2_{tt} u(t,x) - 3\partial^2_{tx} u(t,x) - 4\partial^2_{xx} u(t,x) = 0 \\ u(0,x) = x^2 \\ \partial_t u(0,x) = 0. \end{cases}$

a/ Soit $u : \mathbb{R}^2 \to \mathbb{R}$ de classe $\mathcal{C}^2$

on a : $(\partial_t - 4\partial_x)((\partial_t + \partial_x)u)(t,x) = \partial^2_{tt} u(t,x) + \partial^2_{tx} u(t,x) - 4\partial^2_{xt} u(t,x)$
$-4\partial^2_{xx} u(t,x)$.

Comme u est de classe $\mathcal{C}^2$ alors : $\partial^2_{tx} = \partial^2_{xt}$.

d'où : $(\partial_t - 4\partial_x)((\partial_t + \partial_x)u)(t,x) = \partial^2_{tt} u(t,x) - 3\partial^2_{tx} u(t,x) - 4\partial^2_{xx} u(t,x)$.

b/ (1) : $\partial_t v(t,x) - 4\partial_x v(t,x) = 0$.

par la méthode de changements des variables : $a = 1$. $b = 4$.

on pose : $t' = -4t - x$
$x' = t - 4x$.
$w(t',x') = v(t,x)$

l'éq vérifiée par w est $\partial_2 w(t',x') = 0$

donc $w(t',x') = g(t') \Leftrightarrow v(t,x) = g(-4t-x)$ avec g est fction de classe $\mathcal{C}^1$.

c/ (2) : $\partial_t u(t,x) + \partial_x u(t,x) = f(4t+x)$.

on pose : $t' = -t + x$.
$x' = 4t + x$. $w(t',x') = u(t,x)$.

c1/ on a : $\partial_t u(t,x) = \partial_t w(\underbrace{-t+x}_{t'}, \underbrace{4t+x}_{x'})$
$= -\partial_1 w(t',x') + 4\partial_2 w(t',x')$.

$\partial_x u(t,x) = \partial_x w(-t+x, 4t+x)$
$= \partial_1 w(t',x') + \partial_2 w(t',x')$.

(2) $\Leftrightarrow$ $-\partial_1 w(t',x') + 4\partial_2 w(t',x') + \partial_1 w(t',x') + \partial_2 w(t',x') = f(4t+x)$

$\Leftrightarrow 5\,\partial_2 w(t',x') = f(\overbrace{4t+x}^{x'})$.

c_2/ on a : $5\,\partial_2 w(t',x') = f(4t+x) = f(x')$

$\Leftrightarrow w(t',x') = \frac{1}{5}\int f(x')\,dx'$.

c.à.d : $u(t,x) = \frac{1}{5}\int f(4t+x)\,dx$

$x' = 4t+x$

$\frac{dx'}{dx} = 1$

f de classe C^1 donc : $dx' = dx$

on a : (1) : $(\partial_t - 4\partial_x)\,v(t,x) = 0$

si on pose : $v(t,x) = \partial_t u(t,x) + \partial_x u(t,x)$

on aura l'éq : $(\partial_t - 4\partial_x)(\partial_t u(t,x) + \partial_x u(t,x)) = 0$.

d'après la résolution on a : $\partial_t u(t,x) + \partial_x u(t,x) = f(-(4t+x))$.

d'où $u(t,x) = -\frac{1}{5}\int f(-(4t+x))\,dx$.