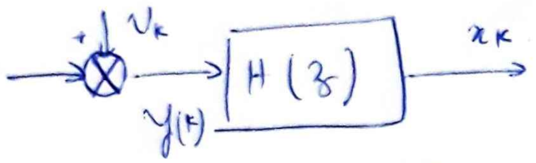
Travaux Dirigés :

Exercice (1) :

$$\begin{cases} x_{k+1} = 0{,}25\, x_k + w_k \\ y_k = x_k + v_k \end{cases}$$

$$S_{xx}(z) = \frac{-3{,}75\, z}{(z-0{,}25)(z-4)} \quad ; \quad S_{vv}(z) = 1$$



$$H(z) = \frac{1}{S^{+}_{yy}(z)} \left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{-1})} \right]_{+}$$

$S_{xy}(z) = ?$

$$\phi_{xy}(j) = E\left[x(i+j)\, y(i)\right]$$

$$= E\left[x(i+j)\left(x(i) + v(i)\right)\right]$$

$$= E\left[x(i+j) \cdot x(i)\right] + \underbrace{E\left[x(i+j)\, v(i)\right]}_{0}$$

$$\boxed{\phi_{xy}(j) = \phi_{xx}(j)}$$

TZ $\Big($

$$S_{xy}(z) = S_{xx}(z) = \frac{-3{,}75\, z}{(z-0{,}25)(z-4)}$$

$S_{yy}(z) = ?$

$$\phi_{yy}(j) = E\left[y(i+j) \cdot y(i)\right] = E\left[x(i+j) + v(i+j)\right]\left[x(i) + v(i)\right]$$

$$= E\left[x(i+j)\, x(i)\right] + \underbrace{E\left[x(i+j)\, v(i)\right]}_{0} + \underbrace{E\left[v(i+j)\, x(i)\right]}_{0} + E\left[v(i+j)\, v(i)\right]$$

$$\phi_{yy}(j) = \phi_{xx}(j) + \phi_{vv}(j).$$

TZ $\Big($,

$$S_{yy}(z) = S_{xx}(z) + S_{vv}(z) = \frac{-3{,}75\, z}{(z-0{,}25)(z-4)} + 1$$

$$S_{yy}(z) = \frac{-3{,}75z + (z-0{,}25)(z-4)}{(z-0{,}25)(z-4)}$$

$$S_{yy}(z) = \frac{-3{,}75z + z^2 - 4{,}25z + 1}{(z-0{,}25)(z-4)} = \frac{z^2 - 8z + 1}{(z-0{,}25)(z-4)}$$

Soit $P(z) = z^2 - 8z + 1$

$\Delta = 8^2 - 4 = 60$ $\Rightarrow$ $z_1 = \dfrac{8+\sqrt{60}}{2} = 4 + \sqrt{15} = 7{,}87$

$\sqrt{\Delta} = \sqrt{60}$ $\quad z_2 = \dfrac{8-\sqrt{60}}{2} = 4 - \sqrt{15} = 0{,}127$

$$S_{yy}(z) = \frac{(z-7{,}87)(z-0{,}127)}{(z-0{,}25)(z-4)}$$

$$S^{+}_{yy}(z) = \frac{(z-0{,}127)}{(z-0{,}25)} \qquad S^{-}_{yy}(z) = \frac{z-7{,}87}{z-4} = S^{+}_{yy}(z^{+})$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{+})}\right]_{+} = \frac{-3{,}75z}{(z-0{,}25)(z-4)} \times \frac{(z-4)}{(z-7{,}87)} = \frac{-3{,}75z}{(z-0{,}25)(z-7{,}87)}$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z^{*})}\right] = \frac{-3{,}75z}{(z-0{,}25)(z-7{,}87)z} = \frac{\alpha}{(z-0{,}25)} + \frac{\beta}{(z-7{,}87)}$$

$$\Rightarrow \alpha = \frac{-3{,}75 \times 0{,}25}{0{,}25 - 7{,}87} = 0{,}12 \qquad \beta = \frac{-3{,}75 \times 7{,}87}{7{,}87 - 0{,}25} = -3{,}87$$

$$\left[\frac{S_{xy}(z)}{S^{+}_{yy}(z)}\right] = \frac{0{,}12}{(z-0{,}25)} - \frac{3{,}87}{z-7{,}87}$$

$$\Rightarrow \left[\frac{S_{xy}(z)}{S^{+}_{yy}(z)}\right]_{+} = \frac{0{,}12}{z-0{,}25}$$

Donc $\quad H(z) = \dfrac{(z-0{,}25) \times 0{,}12}{(z-0{,}127)(z-0{,}25)}$

$$\Rightarrow H(z) = \frac{0,12}{z - 0,127}$$

Exercice 1 :

Système : $\begin{cases} x(k+1) = F.X(k) + G.U(k). \\ y(k) = H.X(k) \end{cases}$

observateur $\begin{cases} \hat{x}(k+1) = F\hat{x}(k) + GU(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) = H.\hat{x}(k) \end{cases}$

1) $\delta(k) = x(k) - \hat{x}(k)$.

$\delta(k+1) = x(k+1) - \hat{x}(k+1)$.

1) $\delta(k+1) = x(k+1) - \hat{x}(k+1)$.

$= F.X(k) + GU(k) - F\hat{X}(k) - GU(k) - L(y(k) - \hat{y}(k))$.

$FX(k) + GU(k) + F\hat{X}(k) - GU(k) - L(H(X(k) - \hat{X}(k)))$.

$= (F - LH)\,\delta(k)$.

pour que l'erreur soit stable, il faut que $(F - LH)$ possède des valeurs propres stables.

2) $F = \begin{bmatrix} 2 & -1 \\ 1 & 0,5 \end{bmatrix}$ $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Étape 1

$\theta = \begin{pmatrix} H \\ HF \end{pmatrix}$; $HF = (1\ 0)\begin{pmatrix} 2 & -1 \\ 1 & 0,5 \end{pmatrix} = (2\ \ -1)$

$\theta = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$; $\det \neq 0 \rightarrow$ Système observable.

Étape 2 : $\det(zI - F) = \det\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & 0,5 \end{pmatrix}\right]$

$$= \det\begin{pmatrix} z-2 & 1 \\ -1 & z-0{,}5 \end{pmatrix} = (z-2)(z-0{,}5)+1$$

$$= z^2 - 2{,}5z + 2$$

$$\begin{cases} a_0 = 2 \\ a_1 = -2{,}5 \end{cases}$$

étape ③ :

$$K = \begin{bmatrix} -2{,}5 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^{-1} = K.Q = \begin{bmatrix} -2{,}5 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -0{,}5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$P = \frac{1}{\det(P^{-1})} \cdot com(P^{-1})^t = \begin{bmatrix} 0 & 1 \\ -1 & -0{,}5 \end{bmatrix}$$

$$\det(P^{-1}) = 1$$

étape ④

$$(z-0{,}2)(z-0{,}8) = z^2 - 0{,}2z + 0{,}12 - 0{,}6z$$

$$= z^2 - 0{,}8z + 0{,}12 \Rightarrow \alpha_0 = 0{,}12,\ \alpha_1 = -0{,}8$$

étape ⑤

$$L = P.\begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0{,}5 \end{bmatrix}\begin{bmatrix} -1{,}88 \\ 1{,}7 \end{bmatrix} = \begin{bmatrix} 1{,}7 \\ 2{,}73 \end{bmatrix}$$

Exercice ② :

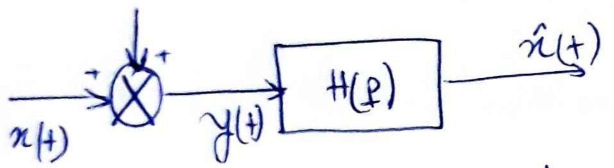
A G B : coefficient de u(t)

$$\begin{cases} \dot{x}(t) = -4x(t) + w(t). \\ y(t) = x(t) + v(t). \end{cases}$$

C

$$S_{xx}(p) = \frac{1}{16-p^2} \quad ; \quad S_{vv}(p) = 1.$$

1) filtre de Wimner.



hypothèse : $x(t)$ et $v(t)$ sont indépendants

$$H(p) = \frac{1}{S^*_{yy}(p)} \cdot \left[\frac{S_{xy}(p)}{S^*_{yy}(-p)} \right]$$

$S_{xy}(p) = ?$ $\quad \phi_{xy}(\tau) = E\left[x(t+\tau)\, y(t)\right].$

$$= E\left[x(t+\tau)\left(x(t)+v(t)\right)\right] = E\left[x(t+\tau).\,x(t)\right] + E\left[\cancel{x(t+\tau)\,v(t)}\right]$$

$\to 0$

$$\phi_{xy}(\tau) = \phi_{xx}(\tau).$$

Tb

$$S_{xy}(p) = S_{xx}(p) = \frac{1}{16-p^2}.$$

$S_{yy}(p) = ?$ $\quad \phi_{yy}(\tau) = E\left[y(t+\tau).\,y(t)\right].$

$$= E\left[\left(x(t+\tau)+v(t+\tau)\right).\left(x(t)+v(t)\right)\right].$$

$$= E\left[x(t+\tau).\,x(t)\right] + E\left[x(t+\tau).\,v(t)\right] + E\left[v(t+\tau).\,x(t)\right] +$$

$$\left[\frac{S_{xy}(p)}{S^{*}_{yy}(p)}\right] = \frac{\frac{1}{\sqrt{17}+4}}{4+p} = \frac{1}{(\sqrt{17}+4)(4+p)}.$$

$$H(p) = \frac{1}{S^{*}_{yy}(p)} \cdot \left[\frac{S_{xy}(p)}{S^{*}_{yy}(p)}\right].$$

$$= \frac{4+p}{\sqrt{17}+p} \cdot \frac{1}{(\sqrt{17}+4)(4+p)}.$$

$$\boxed{H(p) = \frac{1}{(\sqrt{17}+p)(\sqrt{17}+4)}}.$$

2) $A = -4,\ B = 0,\ G = 1,\ C = 1,\ R = 1,\ Q = 1.$

- filtre de Kalman (continu).

$$\begin{cases} \dot{\hat{x}}(t) = A.\hat{x}(t) + B.w(t) + K(t).\left[y(t) - C.\hat{x}(t)\right]. \\ K(t) = P(t).C^T.R^{-1} \\ \dot{P}(t) = G\,Q.G^T + A.P(t) + P(t).A^T - P(t).C^T.R^{-1}C\,P(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = -4\hat{x}(t) + K(t)\left[y(t) - \hat{x}(t)\right]. \\ K(t) = p(t). \\ \dot{P}(t) = 1 - 4p(t) - 4p(t) - p^2(t). \\ \quad = 1 - 8p(t) - p^2. \end{cases}$$

En régime permanent: $P = \underset{?}{cte}$, $\dot{P}(t) = 0$

$$1 - 8p - p^2 = 0.$$

$\Delta = b^2 - 4ac = 68$.

$$\begin{cases} P_1 = \dfrac{-b-\sqrt{\Delta}}{2a} = \dfrac{8-2\sqrt{17}}{-2} = -4+\sqrt{17}. \\ P_2 = \dfrac{-b+\sqrt{\Delta}}{2a} = \dfrac{8+2\sqrt{17}}{-2} = -4-\sqrt{17}. \end{cases}$$

$$\boxed{P = -4+\sqrt{17}}$$

$$\Rightarrow \boxed{K = P = -4+\sqrt{17}}$$

$$\dot{\hat{x}}(t) = -4\hat{x}(t) + K(t)\left[y(t) - \hat{x}(t)\right].$$

$$= -4\hat{x}(t) - (4-\sqrt{17})(y(t) - \hat{x}(t)).$$

$$= -4\hat{x}(t) - (4-\sqrt{17})\,y(t) + 4\hat{x}(t) - \sqrt{17}\,\hat{x}(t).$$

$$\dot{\hat{x}}(t) = \sqrt{17}\,\hat{x}(t) + (-4+\sqrt{17})\,y(t).$$

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$$P.\hat{X}(P) = -\sqrt{17}\,\hat{X}(P) + (-4+\sqrt{17})\,Y(P).$$

$$H_{Kalman}(P) = \frac{\hat{X}(P)}{Y(P)} = \frac{-4+\sqrt{17}}{P+\sqrt{17}}.$$

or $$H_{Wimer}(P) = \frac{1}{(\sqrt{17}+P)(\sqrt{17}+4)} \cdot \frac{\sqrt{17}-4}{\sqrt{17}-4}$$

$\Rightarrow$ Les deux filtres sont identique.

Suite exercice 4 :

2) filtre de Kalman.

$$\begin{cases} x_{k+1} = 0{,}25\, x_k + w_k \\ y_k = x_k + v_k \end{cases}$$

$$\begin{cases} x_{k+1} = A_k x_k + B_k U_k + G_k w_k \\ y_k = C_k \cdot x_k + v_k \end{cases}$$

$A_k = 0$; $B_k = 0$; $G_k = 1$, $C_k = 1$, $R_k = 1$, $Q_k = 1$.

$$\begin{cases} \hat{x}_{k+1/k} = A_k \cdot \left[I - K_k C_k \right] \hat{x}_{k/k-1} + B_k \cdot U_k + A_k \cdot K_k y_k \\ P_{k+1/k} = A_k \left[I - K_k \cdot C_k \right] \cdot P_{k/k-1} \cdot A_k^T + G_k Q_k G_k^T \\ K_k = P_{k/k-1} C_k^T \left(R_k + C_k \cdot P_{k/k-1} C_k^T \right)^{-1} \end{cases}$$

$$\begin{cases} \hat{x}_{k+1/k} = 0{,}25 \left(1 - K_k \right) \cdot \hat{x}_{k/k-1} + 0{,}25\, K_k \cdot y_k \\ P_{k+1/k} = 0{,}25 \left(1 - K_k \right) \cdot P_{k/k-1} \cdot 0{,}25 + 1 \\ K_k = P_{k/k-1} \left(1 + P_{k/k-1} \right)^{-1} \end{cases}$$

en régime permanent : $P = cte \rightsquigarrow P_{k+1/k} = P_{k/k-1} = P$.

$$P = 0{,}0625 \left(1 - K_k \right) P + 1$$

or $K_k = P \left(1 + P \right)^{-1} = \dfrac{P}{1+P}$

$$\Rightarrow P = 0{,}0625 \left(1 - \frac{P}{1+P} \right) P + 1$$

$$P = 0{,}0625 \left(\frac{1}{1+P} \right) P + 1$$

$$P = \frac{0{,}0625\,P}{P+1} + 1$$

$$P = \frac{0{,}0625\,P + P + 1}{P+1}$$

$$P(P+1) = 0{,}0625\,P + P + 1$$

$$P\,(P + 1 - 0{,}0625 - 1) = 1$$

$$(P^2 - 0{,}0625\,P - 1) = 0$$

$$\Delta = (0{,}0625)^2 + 4 = 4{,}0039$$

$$P_1 = \frac{0{,}0625 - \sqrt{4{,}0039}}{2} = -0{,}969$$

$$P_2 = \frac{0{,}0625 + \sqrt{4{,}0039}}{2} = 1{,}03$$

$$\Rightarrow P = 1{,}03$$

$$K_k = \frac{P}{P+1} = \frac{1{,}03}{1+1{,}03} = 0{,}503$$

$$\hat{x}_{k+1/k} = 0{,}25\,(1 - 0{,}507)\,\hat{x}_{k|k-1} + 0{,}25 \times 0{,}503\,y_k$$

$$\hat{x}_{k+1/k} = 0{,}123\,\hat{x}_{k/k-1} + 0{,}126\,y_k$$

$$\begin{cases} U(k+m) = z^m \cdot U(k) \\ U(k-m) = z^{-m} \cdot U(k) \end{cases}$$

$$z \cdot \hat{x}_k = 0{,}123\,\hat{x}_k + 0{,}126\,y_k$$

$$(z - 0{,}123)\,\hat{x}(k) = 0{,}126\,y_k$$

$$H_{kalman}(z) = \frac{\hat{x}_k}{y_k} = \frac{0{,}126}{z - 0{,}123}$$

$$H_{wiener}(z) = \frac{0{,}12}{z - 0{,}127}$$

Exercice (3) :

1) mvt à une vitesse constante.

$$\begin{cases} x(t) = v.t + y_0 \\ y(t) = x(t) + e(t). \end{cases}$$

$$y(t) = v.t + y_0 + e(t).$$

$$\hookrightarrow y_k = v.k + y_0 + e_k.$$

2) objectif Estimation des paramètres:

$$\theta = \begin{bmatrix} y_0 & v \end{bmatrix}^T = cte.$$

$$\theta_{k+1} = \theta_k.$$

or $y_k = \begin{bmatrix} 1 & k \end{bmatrix} \begin{bmatrix} y_0 \\ v \end{bmatrix} + e_k.$

$y_k = C_k . \theta_k + e_k$ avec $C_k = \begin{bmatrix} 1 & k \end{bmatrix}.$

$$\begin{cases} \theta_{k+1} = \theta_k. \\ y_k = C_k . \theta_k + e_k \quad \text{avec } C_k = \begin{bmatrix} 1 & k \end{bmatrix}. \end{cases}$$

3/

$$\begin{cases} \hat{\theta}_k = \hat{\theta}_{k-1} + K_k \left[y_k - C_k \hat{\theta}_{k-1} \right]. \\ K_k = P_{k-1} C_k^T \left(R_k + C_k . P_{k-1} . C_k^T \right)^{-1} \\ P_k = P_{k-1} - K_k . C_k . P_{k-1} = (I - K_k C_k) P_{k-1}. \end{cases}$$

$$\begin{cases} \hat{\theta}_1 = \hat{\theta}_0 + K_1 (y_1 - C_1 \theta_0^1) \\ K_1 = P_0 C_1^T (R + C_1 P_0 C_1^T)^{-1} \\ P_1 = P_0 - K_1 C_1 . P_0 \end{cases}$$

$$K_1 = P_0 C_1^T (R + C_1 . P_0 C_1^T)^{-1}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} . \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(0{,}01 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} . (0{,}01 + 20)^{-1} = \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix}.$$

$$P_1 = P_0 - K_1 C_1 P_0$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 0{,}4 \\ 0{,}4 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - 0{,}96 \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}.$$

$$= \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix}.$$

$$\hat{\theta}_1 = \theta_0 + K_1 (y_1 - C_1 . \hat{\theta}_0).$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix} . \left(9 - \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).$$

$$= 9 \begin{bmatrix} 0{,}49 \\ 0{,}49 \end{bmatrix} = \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix}.$$

$$\begin{cases} \hat{\theta}_2 = \hat{\theta}_1 + K_2 \left(y_2 - C_2 \hat{\theta}_1 \right). \\ K_2 = P_1 \cdot C_2^T \left(R + C_2 P_1 C_2^T \right)^{-1}. \\ P_2 = P_1 - K_2 C_2 \cdot P_1 . \end{cases}$$

$$K_2 = P_1 \cdot C_2^T \left(R + C_2 P_1 C_2^T \right)^{-1}.$$

$$= \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(0{,}01 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0{,}4 & 0 \\ 0 & 0{,}4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{-1}.$$

$$= \begin{bmatrix} 0{,}4 \\ 0{,}8 \end{bmatrix} \left(0{,}01 + 2 \right)^{-1}$$

$$= \begin{bmatrix} 0{,}19 \\ 0{,}39 \end{bmatrix}$$

$$\hat{\theta}_2 = \hat{\theta}_1 + K_2 \left(y_2 - C_2 \hat{\theta}_1 \right).$$

$$= \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix} + \begin{bmatrix} 0{,}19 \\ 0{,}39 \end{bmatrix} \left(10{,}8 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4{,}41 \\ 4{,}41 \end{bmatrix} \right)$$