Théorème de Cauchy - Lipschitz

$$(S)\begin{cases} \text{si } X'(t) = F(X(t)) & , F \text{ de classe } \mathcal{C}^1 \\ X(t_0) = X_0 \end{cases}$$

$t_0 \in I$, $X_0 \in \Omega$, alors $\exists \varepsilon_0$ tq (S) admet une unique solution de classe $\mathcal{C}^1$ ds un intervalle $]t_0-\varepsilon_0, t_0+\varepsilon_0[= \mathcal{V}(t_0)$. solu° locale.

Cas de deux racines réelles distinctes

Exemple 2.2.6 (p. 27)

$$(S)\begin{cases} x'(t) = y(t) \\ y'(t) = x(t) \end{cases} \quad \text{on pose } X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \rightsquigarrow X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}$$

$$\Rightarrow X'(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$(S) \Leftrightarrow X'(t) = A\,X(t)$ avec $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda \in \{1, -1\}$$

on pose $\lambda_1 = 1$ et $\lambda_2 = -1$.

Soit $u_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{R}^2$ tq $A u_1 = \lambda u_1 \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} b_1 \\ a_1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

$\Rightarrow b_1 = a_1 \Rightarrow u_1 = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on prend $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Soit $u_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ tq $A u_2 = \lambda u_2 \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} -a_2 \\ -b_2 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} -a_2 \\ -b_2 \end{pmatrix} \Rightarrow b_2 = -a_2 \Rightarrow u_2 = \begin{pmatrix} a_2 \\ -a_2 \end{pmatrix} = a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

on prend $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

on pose $X_1(t) = e^{\lambda_1 t} u_1 = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

$X_2(t) = e^{\lambda_2 t} u_2 = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$

on a : $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$, $\alpha_1, \alpha_2 \in \mathbb{R}$

$$= \alpha_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + \alpha_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$X(t) = \begin{pmatrix} \alpha_1 e^t + \alpha_2 e^{-t} \\ \alpha_1 e^t - \alpha_2 e^{-t} \end{pmatrix} \Rightarrow \begin{cases} x(t) = \alpha_1 e^t + \alpha_2 e^{-t} \\ y(t) = \alpha_1 e^t - \alpha_2 e^{-t} \end{cases}, \alpha_1, \alpha_2 \in \mathbb{R}$$

on a : $\begin{cases} x(0) = 0 \\ y(0) = 1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 + \alpha_2 = 0 & (1) \\ \alpha_1 - \alpha_2 = 1 & (2) \end{cases}$ $(1)+(2)$: $2\alpha_1 = 1 \Rightarrow \alpha_1 = \frac{1}{2} \Rightarrow \alpha_2 = -\frac{1}{2}$

$$\Rightarrow \begin{cases} x(t) = \frac{1}{2} e^t - \frac{1}{2} e^{-t} \\ y(t) = \frac{1}{2} e^t + \frac{1}{2} e^{-t} \end{cases} \Rightarrow \begin{cases} x(t) = 1/2\,(e^t - e^{-t}) \\ y(t) = 1/2\,(e^t + e^{-t}) \end{cases}$$

Cas de deux valeurs propres complexes

$\lambda_1 = \mu + i\nu$, si λ_1 une vp complexe de A alors $\lambda_2 = \bar{\lambda}_1$ est une vp de A

$u_1 = v + iw$, si u_1 un vect propre associé à λ_1 $\Rightarrow$ $u_2 = \bar{u}_1$ est un vect propre associé à $\lambda_2 = \bar{\lambda}_1$

on a : $X(t) = C_1 X_1(t) + C_2 X_2(t)$, $C_1 = a + ib$

$= C_1 e^{\lambda_1 t} u_1 + C_2 e^{\lambda_2 t} u_2 \in \mathbb{C}^2$

on a : $X(t) = 2e^{\mu t}\left[\left(a\cos(\nu t) - b\sin(\nu t)\right)v - \left(b\cos(\nu t) + a\sin(\nu t)\right)w\right]$ $a, b \in \mathbb{R}$

exemple

$$(S)\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases} \qquad (x(0), y(0)) = (0, 1)$$

on pose $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \Rightarrow X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$(S) \Leftrightarrow X'(t) = A\,X(t)$ avec $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda = i$ ou $\lambda = -i$

on pose $\lambda_1 = i = \mu + i\nu \longrightarrow \mu = 0$, $\nu = 1$

soit $u_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{C}^2$ tq $A u_1 = \lambda_1 u_1 \Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} i\,a_1 \\ i\,b_1 \end{pmatrix}$

$\Leftrightarrow \begin{cases} -b_1 = i a_1 \\ a_1 = i b_1 \end{cases} \Rightarrow b_1 = -i a_1 \Rightarrow u_2 = \begin{pmatrix} a_1 \\ -i a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$

on prend $u_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} = v + iw = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{v} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{w}$

on a : $X(t) = 2e^{\mu t}\left[\left(a\cos(\nu t) - b\sin(\nu t)\right)v - \left(b\cos(\nu t) + a\sin(\nu t)\right)w\right]$

$\Rightarrow X(t) = 2\left[\left(a\cos(t) - b\sin(t)\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left(b\cos(t) + a\sin(t)\right)\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$

$\Rightarrow X(t) = \begin{pmatrix} 2\left(a\cos(t) - b\sin(t)\right) \\ 2\left(b\cos(t) + a\sin(t)\right) \end{pmatrix} \Rightarrow \begin{cases} x(t) = 2\left(a\cos(t) - b\sin(t)\right) \\ y(t) = 2\left(b\cos(t) + a\sin(t)\right) \end{cases}$

on a : $\begin{cases} x(0) = 0 \\ y(0) = 1 \end{cases} \Rightarrow \begin{cases} 2a = 0 \\ 2b = 1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x(t) = -\sin(t) \\ y(t) = \cos(t) \end{cases}$

Cas d'une racine double :

si λ v.p double de A

- trouver N tq $A = N + \lambda I$ avec $N^2 = 0$ (N nilpotente d'ordre 2)
- trouver u_1 tq $N u_1 = 0$ (vect. propre associé à λ)
- trouver u_2 indépendant de u_1 tq $N u_2 = u_1$
- on pose $X_1(t) = e^{\lambda t} u_1$, $X_2(t) = e^{\lambda t}(u_2 + t\, u_1)$

on a : $X(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$, $\alpha_1, \alpha_2 \in \mathbb{R}$

exercices (p. 36)

7/ (S) : $X'(t) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} X(t) + \begin{pmatrix} 1 \\ t \end{pmatrix}$ $\rightarrow$ (S) $X'(t) = A X(t) + B(t)$

avec $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ et $B(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$

(H) : $X'(t) = A X(t)$

$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 = 0 \iff \boxed{\lambda = 1}$

Soit $N = A - \lambda I = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$N^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow N$ nilpotente d'ordre 2.

soit $u_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \mathbb{R}^2$ tq $N u_1 = 0 \iff \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2 b_1 = 0 \\ 0 = 0 \end{cases} \Rightarrow b_1 = 0$

$u_1 = \begin{pmatrix} a_1 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on prend $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Soit $u_2 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ tq $N u_2 = u_1 \iff \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \iff \{ 2 b_2 = 1 \Rightarrow b_2 = \frac{1}{2}$

$\Rightarrow u_2 = \begin{pmatrix} a_2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} a_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = a_2 \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{u_1} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \rightarrow$ on prend $u_2 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

on pose $X_1(t) = e^{\lambda t} u_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

$X_2(t) = e^{\lambda t}(u_2 + t\, u_1) = e^t\left(\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} t e^t \\ \frac{e^t}{2} \end{pmatrix}$

on a $X_H(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t)$, $\alpha_1, \alpha_2 \in \mathbb{R}$

$X_H(t) = \alpha_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} t e^t \\ \frac{e^t}{2} \end{pmatrix} = \begin{pmatrix} \alpha_1 e^t + \alpha_2 t e^t \\ \alpha_2 \frac{e^t}{2} \end{pmatrix} = \begin{pmatrix} (\alpha_1 + t\alpha_2) e^t \\ \frac{\alpha_2}{2} \frac{e^t}{2} \end{pmatrix}$