# TD 1



## Exercice 1 :

1) $x(t) = e^{-at}\ \varepsilon(t)\quad a \in \mathbb{R}^*_+$

$$W_x = \int_0^{+\infty} e^{-2at}\,dt = -\frac{1}{2a}\left[e^{-2at}\right]_0^{+\infty} = \frac{1}{2a} \rightarrow \boxed{W_x = \frac{1}{2a}}$$

2) $x(t) = t.e^{-at}\ \varepsilon(t),\ a \in \mathbb{R}^*_+$.

$$W_x = \int_0^{+\infty} t^2.e^{-2at}.\,dt.$$ ; par intégration par partie.

$U(t) = t^2 \longrightarrow U'(t) = 2t$

$V' = e^{-2at} \longrightarrow V = -\frac{1}{2a}\,e^{-2at}$.

$$W_x = \underbrace{\left[\frac{t^2}{-2a}.e^{-2at}\right]_0^{+\infty}}_{0} + \int_0^{+\infty} + t.\frac{1}{a}.e^{-2at}\,dt.$$

par IPP :

$U(t) = t \longrightarrow U' = 1$

$V' = e^{-2at} \longrightarrow V = -\frac{1}{2a}.e^{-2at}dt$.

$$= \frac{1}{a}\underbrace{\left[t - \frac{1}{2a}.e^{-2at}\right]_0^{+\infty}}_{0} + \frac{1}{2a}\int_0^{+\infty} e^{-2at}\,dt.$$

$$= \frac{1}{a}.\left(\frac{1}{2a}\ \frac{-1}{2a}\left[e^{-2at}\right]_0^{+\infty}\right) = \frac{1}{4a^2} \rightarrow \boxed{W_x = \frac{1}{4a^2}}$$

3) $x(t) = e^{-t^2/2a^2}.dt\ .\ \varepsilon(t)\quad a \in \mathbb{R}^*_+$.

$$W_x = \int_0^{+\infty} e^{-\frac{t^2}{a^2}}\,dt =$$

$(f \circ g)' = g' \times f'(g)$

$$\left(e^{-\frac{t^2}{a^2}}\right)' = -\frac{2t}{a^2}.e^{-\frac{t^2}{a^2}}$$

$$= \int_0^{+\infty}\left(-\frac{a^2}{2t}\left(\frac{-2t}{a^2}.e^{-\frac{t^2}{a^2}}\right)\right).dt$$

$$= x_0^2 \cdot \left[ \frac{T}{2} + \frac{1}{2} \left[ \frac{\sin(4\pi f_0 t)}{4\pi f_0} \right]_{-\frac{T}{2}}^{\frac{T}{2}} \right]$$

$= [$

IPP : $\begin{cases} u' = -\dfrac{a^2}{2t} \to u' = \dfrac{a^2}{2t^2} \\ v' = -\dfrac{2t}{a^2} \cdot e^{-\frac{t^2}{a^2}} \quad v = e^{-\frac{t^2}{a^2}} \end{cases}$

$$= \left[ \underbrace{\frac{-a^2}{2t} \cdot e^{-\frac{t^2}{a^2}}}_{0} \right]_0^{+\infty} - \int_0^{+\infty} \underbrace{\frac{a^2}{2t^2} \cdot e^{-\frac{t^2}{a^2}} \cdot dt}_{\to +\infty} \quad \to +\infty$$

Exercice ②:

$$P_x = \lim_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \cdot dt$$

1) $x(t) = x_0 \cdot e^{-2\pi j f_0 t}$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \cdot dt = x_0^2 \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-4\pi j f t} \cdot dt = x_0^2 \cdot \left[ \frac{1}{4\pi j f_0} \cdot e^{4\pi j f_0 t} \right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{x_0^2}{4\pi j f_0 T} \left( e^{2j\pi f_0 T} - e^{-2j\pi f_0 T} \right)$$

$$= \frac{x_0^2}{2\pi f_0 \cancel{T}} \cancel{T} \sin(2\pi f_0 T)$$

$$= x_0^2 \cdot T \operatorname{sinc}(2\pi f_0 T)$$

$$P = \lim_{T \to +\infty} \frac{1}{T} x_0^2 \cdot T \underbrace{\sin(f_0 T)}_{0} = 0$$

2) $x(t) = x_0 \cdot \cos(2\pi f_0 t)$

$$P = \lim_{T \to +\infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x_0^2 \cdot \cos^2(2\pi f_0 T) \, dt$$

$$= x_0^2 \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} \cdot \cos^2(2\pi f_0 T) = x_0^2 \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1 + \cos(4\pi f_0 T)}{2} \, dt$$

$$= x_0^2 \cdot \left[ \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{dt}{2} + \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \frac{(4\pi f_0 t)}{2} \, dt \right]$$

$$= x_0^2 \cdot \left[ \frac{T}{2} + \frac{1}{2} \left[ \frac{\sin(4\pi f_0 t)}{4\pi f_0} \right]_{-\frac{T}{2}}^{\frac{T}{2}} \right]$$

$$= x_0^2 \left[ \frac{T}{2} + \frac{1}{2} \left( \frac{1}{4\pi f_0} \left( \sin(2\pi f_0 T) - \sin(2\pi f_0 T) \right) \right] \right.$$

$$= x_0^2 \left( \frac{T}{2} + \frac{1}{8\pi f_0} \left( 2 \sin(2\pi f_0 T) \right) \right]$$

$$= x_0^2 \left[ \frac{T}{2} + \frac{1}{4\pi f_0 T} \sin(2\pi f_0 T) \right]$$

$$= x_0^2 \left[ \frac{T}{2} + \frac{T}{2} \operatorname{sin_c}(2 f_0 T) \right]$$

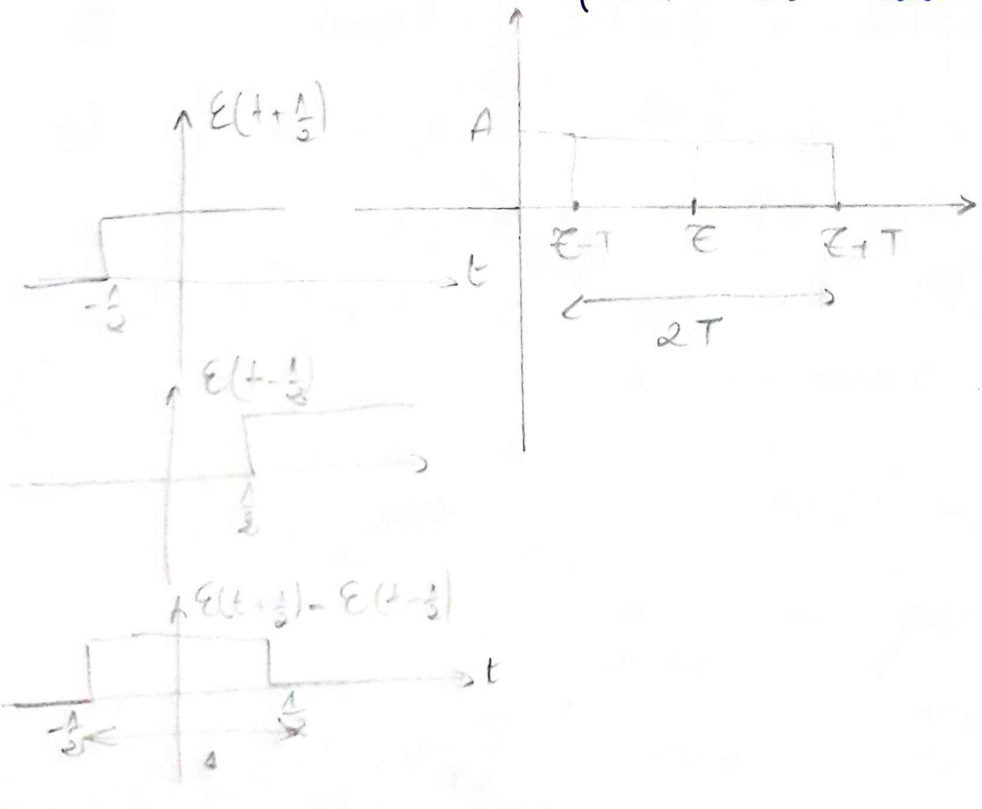
$$= \frac{x_0^2 T}{2} \left[ 1 + \operatorname{sin_c}(2 f_0 T) \right].$$

$$P = \lim_{T \to +\infty} \frac{1}{T} \cdot \frac{x_0^2 T}{2} \cdot \left[ 1 + \sin(2 f T) \right]$$

$$= \frac{x_0^2}{2}.$$

$$P = \lim_{T \to +\infty} \frac{1}{T} \int_0^{\frac{T}{2}} 1 \, dt = \lim_{T \to +\infty} \frac{1}{T} \left[ t \right]_0^{\frac{T}{2}} = \frac{1}{2}.$$

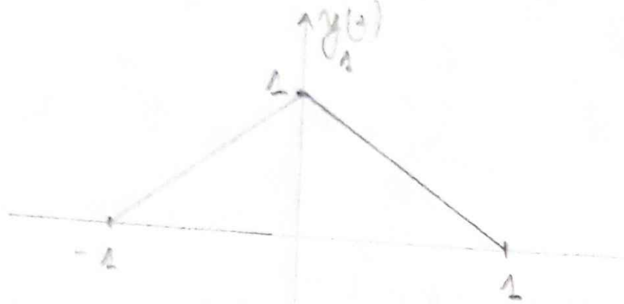
Exercice 3 : (démonstration dans le cours).



Exercice (4) : $y_1(t) = tri(t)$. 

$$tri(t) = \begin{cases} 1-|t| & si\ |t|<1 \\ 0 & si\ |t|>1 \end{cases}$$

$$= \begin{cases} 1-t & si\ 0<t<1 \\ 1+t & si\ -1<t<0 \\ 0 & sinon. \end{cases}$$

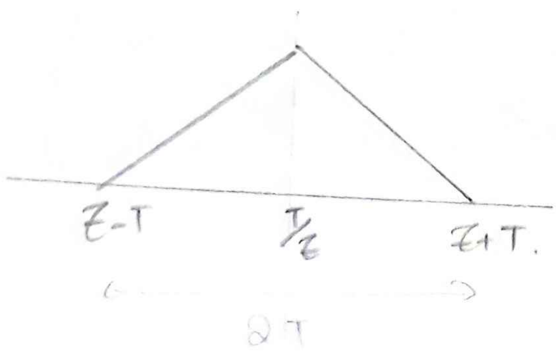



2) $y_2(t) = tri\left(\frac{t-\varepsilon}{T}\right)$ (forme généralisé).

$$= \begin{cases} * \dfrac{t-\varepsilon}{T} + 1 & \varepsilon-T < t < \varepsilon. \\ = \dfrac{t-\varepsilon}{T} + 1 & \\ * \dfrac{-(t-\varepsilon)}{T} + 1 & \varepsilon < t < \varepsilon+T. \end{cases}$$

$t$

$$= \begin{cases} 1 - \left|\dfrac{t-\varepsilon}{T}\right| & si\ \left|\dfrac{t-\varepsilon}{T}\right| < 1 \\ 0\ sinon. & \end{cases}$$



③

c) $x_1(t) = A \cdot \text{rect}\left(\frac{t}{\Delta}\right) = \begin{cases} A & \text{si } -\frac{\Delta}{2} < t < \frac{\Delta}{2} \\ 0 & \text{sinon.} \end{cases}$

$$\int_{-\infty}^{+\infty} x_1(t)\, dt = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} A\, dt = A\left(\frac{\Delta}{2} + \frac{\Delta}{2}\right) = A\Delta$$

$y_1(t) = A \cdot \text{tri}\left(\frac{t}{\Delta}\right) = \begin{cases} \frac{t}{\Delta} + 1 & \text{si } -\Delta < t < 0 \\ -\frac{t}{\Delta} + 1 & \text{si } 0 < t < \Delta. \end{cases}$

$$= A\left(\int_{-\Delta}^{0} \frac{t}{\Delta} + 1\, dt + \int_{0}^{\Delta} \frac{t}{\Delta} + 1\, dt.\right)$$

$$= A\left(\left[\frac{1}{2\Delta} t^2 + t\right]_{-\Delta}^{0} + \left[-\frac{1}{2\Delta} t^2 + t\right]_{0}^{\Delta}\right)$$

$$= A\left(-\left(\frac{\Delta}{2} - \Delta\right) + \left(-\frac{\Delta}{2} + \Delta\right)\right)$$

$$= A.\left(\frac{-3\Delta}{2} + \frac{\Delta}{2}\right) = A.\Delta.$$

d) voir cours.

e) $$\int_{-\infty}^{+\infty} \text{tri}(t) \cdot e^{-2\pi j f t} \cdot dt = \int_{-1}^{0} (1+t) \cdot e^{-2\pi j f t} dt + \int_{0}^{1} (1-t) \cdot e^{-2\pi j f t} \cdot dt$$

$$= 2 \cdot \int_{0}^{1} (1-t) \cdot e^{-j2\pi f t}\, dt = 2 \int_{0}^{1} e^{-j2\pi f t} - t e^{-j2\pi f t}\, dt$$

$\begin{cases} u = 1 - t \rightarrow u' = -1 \\ v' = e^{-2\pi j f t} \quad v = \frac{-1}{2\pi j f} \cdot e^{-j2\pi f t}. \end{cases}$

$$= 2 \cdot \left(\left[(1-t) \cdot \frac{-1}{2\pi j f} \cdot e^{-2\pi f t}\right]_0^1 - \int_0^1 \frac{1}{2j\pi f} \cdot e^{-2j\pi f t}\, dt\right.$$

$$= 2 \cdot \left( \frac{1}{2\pi j f} - \left[ \frac{1}{2} \right.\right.$$

$(\cdots)$ (Fi dan).

Exercice 6 :

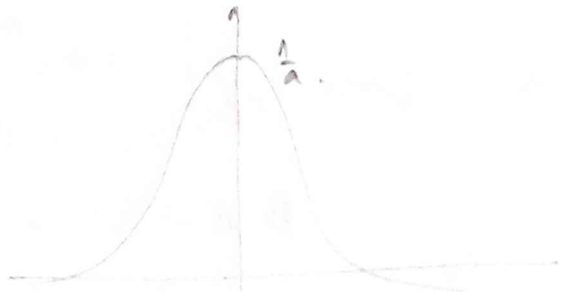
$$x(t) = e^{-at} \cdot \varepsilon(t).$$

1) $$\int_{-\infty}^{+\infty} e^{-at} \cdot e^{-2\pi j f t} \cdot \varepsilon(t).$$

$$= \int_{0}^{+\infty} e^{-t(a+2\pi j f)} \, dt.$$

$$= \frac{-1}{a+2j\pi f} \cdot \left[ e^{-t(a+j2\pi f)} \right]_0^{+\infty} = \frac{1}{a+2j\pi f}.$$

$$|X(f)| = \frac{1}{\sqrt{a^2+4\pi^2 f^2}}$$




| | |
|---|---|
| $x(t) * \delta(t-t_0)$ | $\varepsilon(t) = x(t) \cdot y(t)$ |
| $x(t-t_0)$ | $\varepsilon(f) = X(f) * Y(f)$ |

Exercice 7 :



**Rappel :**

$$x(t) * y(t) = x * y(t) = \int_{-\infty}^{+\infty} x(t) . y(t - \vartheta) . d\vartheta$$

$$= \int_{-\infty}^{+\infty} x(t - \vartheta) \; y(\vartheta) . d\vartheta .$$

$$x(t) * y(t - t_0) = \int_{-\infty}^{+\infty} x(\vartheta) y(t - \vartheta - t_0) . d\vartheta = (y * x)(t - t_0)$$

$$x(t) * y(at + b) = \int_{-\infty}^{+\infty} x(\vartheta) y(a(t - \vartheta) + b . d\vartheta .$$

A°/ Propriétés :

$$x * y = y * x .$$

$$[x * y] * z = x * [y * z] = x * y * z .$$

$$x * [y + z] = x * y + x * z .$$

$$x(t_0) = \int_{-\infty}^{+\infty} x(t) . \delta(t - t_0) . dt .$$

$$x(t) * \delta(t) = x(t) .$$

$$x(t) * \delta(t - t_0) = x(t - t_0) .$$

$$x(t - t_1) * \delta(t - t_1) = x(t - t_1 - t_0) .$$

$$\delta(at) = \frac{1}{a} \delta(t) .$$

B°/ Calcul de produit de convolut° :

1) $\Rightarrow$ $x(t) = A[\delta(t + t_0) + \delta(t - t_0)]$ ; $y(t) = B\left[\delta(t) + \frac{1}{2}[\delta(t + t_1) + \delta(t - t_1)]\right]$.

$$x(t) * y(t) = A[\delta(t + t_0) + \delta(t - t_0)] * B\left[\delta(t) + \frac{1}{2}[\delta(t + t_1) + \delta(t - t_1)]\right]$$

$$= A . B\Big[\delta(t + t_0) * \delta(t) + \frac{1}{2}[\delta(t + t_0) * \delta(t + t_1) + \delta(t + t_0) * \delta(t - t_1)$$

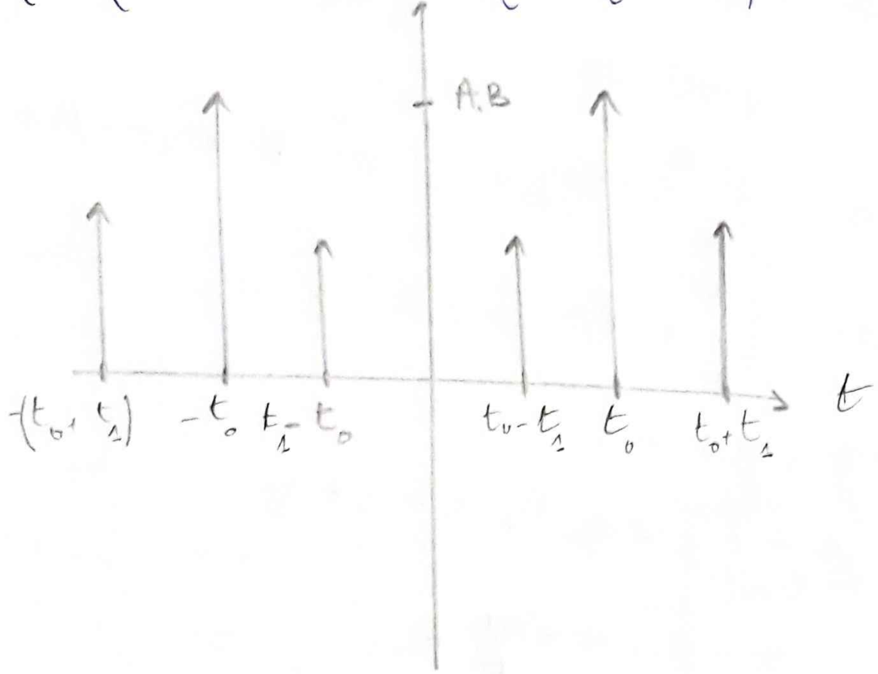
$$+ \delta(t - t_0) * \delta(t) + \frac{1}{2}[\delta(t - t_0) * \delta(t + t_1) + \delta(t - t_0) * \delta(t - t_1)]\Big] .$$

$$= A . B\Big[\delta(t + t_0) + \frac{1}{2}[\delta(t + t_0 + t_1) + \delta(t + t_0 - t_1)] + \delta(t - t_0) +$$

$$\frac{1}{2}[\delta(t - t_0 + t_1) + \delta(t - t_0 - t_1)]\Big]$$

$$= A.B\left[\delta(t+t_0) + \delta(t-t_0) + \frac{1}{2}\left(\delta(t+(t_0+t_1)) + \delta(t-(t_0+t_1))\right)\right.$$

$$\left. + \frac{1}{2}\left(\delta(t+(t_0-t_1)) + \delta(t-(t_0-t_1))\right)\right]$$




$x(t) = \cos\left(\frac{\pi t}{T}\right) . \text{rect}\left(\frac{t}{T}\right)$ ; $y(t) = A . \delta_T(t)$

$$x(t) * y(t) ? \quad = \begin{cases} \cos\left(\frac{\pi t}{T}\right) & \text{si } -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{sinon} \end{cases} \qquad = A \sum_{-\infty}^{+\infty} \delta(t-KT)$$

$$x(t) * y(t) = \int_{-\infty}^{+\infty} A \cos\left(\frac{\pi\tau}{T}\right) \text{rect}\left(\frac{\tau}{T}\right) \sum_{-\infty}^{+\infty} \delta(t-\tau-KT) . d\tau$$

$$= \sum_{-\infty}^{+\infty} A \int_{-\infty}^{+\infty} \cos\left(\left(\frac{\pi\tau}{T}\right)\right) \text{rect}\left(\frac{\tau}{T}\right) \delta(t-(\tau-KT)) . d\tau ,$$

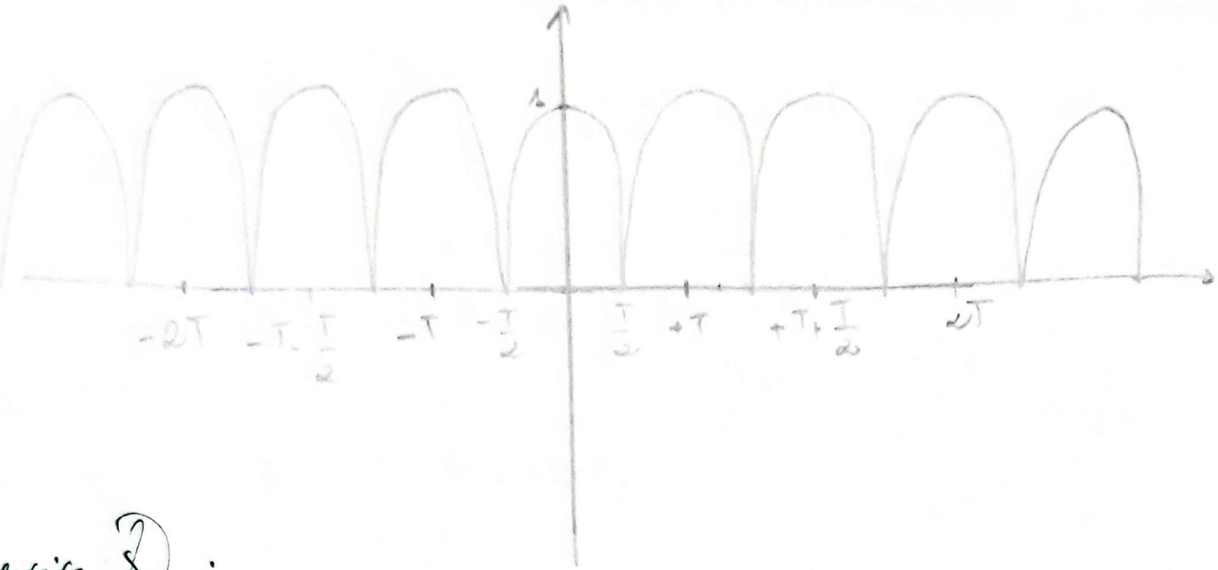
$$= \sum_{-\infty}^{+\infty} A . \cos\left(\frac{\pi(\tau-(t-KT))}{T}\right) . \text{rect}\left(\frac{\tau-(t-KT)}{T}\right)$$

autrement : $x(t) * y(t) = A \cos\left(\frac{\pi t}{T}\right) . \text{rect}\left(\frac{t}{T}\right) * \sum_{-\infty}^{+\infty} \delta(t-KT)$

$$= A . \sum_{-\infty}^{+\infty} \cos\left(\frac{\pi t}{T}\right) \text{rect}\left(\frac{t}{T}\right) * \delta(t-KT)$$

$$= A \sum_{-\infty}^{+\infty} \cos\left(\frac{\pi t}{T}\right) . \text{rect}\left(\frac{t}{T}\right) * \delta(t-KT)$$

$$= A . \sum_{-\infty}^{+\infty} \cos\left(\frac{\pi(t-KT)}{T}\right) . \text{rect}\left(\frac{t-KT}{T}\right)$$




Exercice 8 :

A- Classificat° de signaux (voir cours).

B- Classificat° énergétique de signaux simples :

valeur moyenne : $\bar{x} = \lim\limits_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$.

Energie : $W_x = \int_{-\infty}^{+\infty} |x(t)|^2\,dt$ (complexe réel)

Puissance moyenne : $\lim\limits_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt$ ($\mathbb{C}$ et $\mathbb{R}$).