Rail Geometry Twist Change Calculations - Equivalence Verification

Rail Geometry Analysis

1 Problem Statement

Given interpolated rail coordinates for left and right rails, verify that two computational approaches for calculating twist parameter changes produce identical results:

Parameter-First Approach: Calculate twist parameters, then compute parameter changes
Delta-First Approach: Calculate coordinate deltas, then compute parameter changes directly

This verification demonstrates mathematical equivalence between both computational methods for twist calculations.

2 Notation

Variable Definitions:

- ch_w = Chainage location where twist is calculated
- Δch = Distance interval over which twist is measured
- τ_0 = Baseline time (reference measurement)
- τ_m = Current measurement time (m = 1, 2, 3, ...)
- L = Left rail (subscript)
- R = Right rail (subscript)
- X, Y, Z = 3D coordinates (X = Easting, Y = Northing, Z = Elevation)
- Δ = Change or difference operator

Coordinate Notation:

- $Z(ch, \tau_m)_L$ = Z-coordinate of left rail at chainage ch and time τ_m
- $\Delta Z(ch, \tau_m)_L$ = Change in Z-coordinate: $Z(ch, \tau_m)_L - Z(ch, \tau_0)_L$

Parameter Notation:

- $\text{Cant}(ch, \tau_m)$ = Cant (cross-level) at chainage ch and time τ_m
- $\text{Twist}(ch_w, \Delta ch, \tau_m)$ = Twist over interval Δch starting at chainage ch_w and time τ_m
- $\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)$ = Change in twist from baseline to time τ_m

3 Input Dataset

Assume we have interpolated rail coordinates available at two chainage locations separated by distance Δch:

$$\begin{aligned}
\text{At } ch_w\text{:} \quad & Z(ch_w, \tau_0)_L, Z(ch_w, \tau_0)_R \quad \text{(baseline)} && (1)\\
& Z(ch_w, \tau_m)_L, Z(ch_w, \tau_m)_R \quad \text{(current)} && (2)
\end{aligned}$$

$$\begin{aligned}
\text{At } ch_w + \Delta ch\text{:} \quad & Z(ch_w + \Delta ch, \tau_0)_L, Z(ch_w + \Delta ch, \tau_0)_R \quad \text{(baseline)} && (3)\\
& Z(ch_w + \Delta ch, \tau_m)_L, Z(ch_w + \Delta ch, \tau_m)_R \quad \text{(current)} && (4)
\end{aligned}$$

Example: At chainages 1000m and 1014m (14m interval), we have baseline and current Z-coordinates for both rails.

4 Parameter-First Approach

4.1 Step 1: Calculate Baseline Cant Values

$$\text{Cant}(ch_w, \tau_0) = Z(ch_w, \tau_0)_L - Z(ch_w, \tau_0)_R \tag{5}$$

$$\text{Cant}(ch_w + \Delta ch, \tau_0) = Z(ch_w + \Delta ch, \tau_0)_L - Z(ch_w + \Delta ch, \tau_0)_R \tag{6}$$

4.2 Step 2: Calculate Current Cant Values

$$\text{Cant}(ch_w, \tau_m) = Z(ch_w, \tau_m)_L - Z(ch_w, \tau_m)_R \tag{7}$$

$$\text{Cant}(ch_w + \Delta ch, \tau_m) = Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_m)_R \tag{8}$$

4.3 Step 3: Calculate Baseline and Current Twist

$$\text{Twist}(ch_w, \Delta ch, \tau_0) = \text{Cant}(ch_w + \Delta ch, \tau_0) - \text{Cant}(ch_w, \tau_0) \tag{9}$$

$$\text{Twist}(ch_w, \Delta ch, \tau_m) = \text{Cant}(ch_w + \Delta ch, \tau_m) - \text{Cant}(ch_w, \tau_m) \tag{10}$$

4.4 Step 4: Calculate Twist Change

$$\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = \text{Twist}(ch_w, \Delta ch, \tau_m) - \text{Twist}(ch_w, \Delta ch, \tau_0) \tag{11}$$

Substituting the twist definitions:

$$\begin{aligned}
\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = {} & [\text{Cant}(ch_w + \Delta ch, \tau_m) - \text{Cant}(ch_w, \tau_m)]\\
& - [\text{Cant}(ch_w + \Delta ch, \tau_0) - \text{Cant}(ch_w, \tau_0)]
\end{aligned} \tag{12}$$

5 Delta-First Approach

5.1 Step 1: Calculate Z-Coordinate Deltas

$$\Delta Z(ch_w, \tau_m)_L = Z(ch_w, \tau_m)_L - Z(ch_w, \tau_0)_L \tag{13}$$

$$\Delta Z(ch_w, \tau_m)_R = Z(ch_w, \tau_m)_R - Z(ch_w, \tau_0)_R \tag{14}$$

$$\Delta Z(ch_w + \Delta ch, \tau_m)_L = Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_0)_L \tag{15}$$

$$\Delta Z(ch_w + \Delta ch, \tau_m)_R = Z(ch_w + \Delta ch, \tau_m)_R - Z(ch_w + \Delta ch, \tau_0)_R \tag{16}$$

5.2 Step 2: Calculate Cant Changes

$$\Delta\text{Cant}(ch_w, \tau_m) = \Delta Z(ch_w, \tau_m)_L - \Delta Z(ch_w, \tau_m)_R \tag{17}$$

$$\Delta\text{Cant}(ch_w + \Delta ch, \tau_m) = \Delta Z(ch_w + \Delta ch, \tau_m)_L - \Delta Z(ch_w + \Delta ch, \tau_m)_R \tag{18}$$

5.3 Step 3: Calculate Twist Change Directly

$$\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} = \Delta\text{Cant}(ch_w + \Delta ch, \tau_m) - \Delta\text{Cant}(ch_w, \tau_m) \tag{19}$$

Substituting the cant change definitions:

$$\begin{aligned}\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} = {} & [\Delta Z(ch_w + \Delta ch, \tau_m)_L - \Delta Z(ch_w + \Delta ch, \tau_m)_R] \\ & - [\Delta Z(ch_w, \tau_m)_L - \Delta Z(ch_w, \tau_m)_R]\end{aligned} \tag{20}$$

6 Equivalence Proof

6.1 Expand Parameter-First Method

Substitute cant definitions into the parameter-first result:

$$\begin{aligned}\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = {} & [\text{Cant}(ch_w + \Delta ch, \tau_m) - \text{Cant}(ch_w, \tau_m)] \\ & - [\text{Cant}(ch_w + \Delta ch, \tau_0) - \text{Cant}(ch_w, \tau_0)]\end{aligned} \tag{21}$$

Expand cant terms:

$$\begin{aligned}= {} & [[Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_m)_R] - [Z(ch_w, \tau_m)_L - Z(ch_w, \tau_m)_R]] \\ & - [[Z(ch_w + \Delta ch, \tau_0)_L - Z(ch_w + \Delta ch, \tau_0)_R] - [Z(ch_w, \tau_0)_L - Z(ch_w, \tau_0)_R]]\end{aligned} \tag{22}$$

Simplify:

$$\begin{aligned}= {} & [Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_m)_R - Z(ch_w, \tau_m)_L + Z(ch_w, \tau_m)_R] \\ & - [Z(ch_w + \Delta ch, \tau_0)_L - Z(ch_w + \Delta ch, \tau_0)_R - Z(ch_w, \tau_0)_L + Z(ch_w, \tau_0)_R]\end{aligned} \tag{23}$$

6.2 Expand Delta-First Method

Substitute coordinate delta definitions:

$$\begin{aligned}\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} = {} & [\Delta Z(ch_w + \Delta ch, \tau_m)_L - \Delta Z(ch_w + \Delta ch, \tau_m)_R] \\ & - [\Delta Z(ch_w, \tau_m)_L - \Delta Z(ch_w, \tau_m)_R]\end{aligned} \tag{24}$$

Expand delta terms:

$$\begin{aligned}= {} & [[Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_0)_L] - [Z(ch_w + \Delta ch, \tau_m)_R - Z(ch_w + \Delta ch, \tau_0)_R]] \\ & - [[Z(ch_w, \tau_m)_L - Z(ch_w, \tau_0)_L] - [Z(ch_w, \tau_m)_R - Z(ch_w, \tau_0)_R]]\end{aligned} \tag{25}$$

Simplify:

$$\begin{aligned}= {} & [Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_0)_L - Z(ch_w + \Delta ch, \tau_m)_R + Z(ch_w + \Delta ch, \tau_0)_R] \\ & - [Z(ch_w, \tau_m)_L - Z(ch_w, \tau_0)_L - Z(ch_w, \tau_m)_R + Z(ch_w, \tau_0)_R]\end{aligned} \tag{26}$$

6.3 Rearrange Terms

Rearrange parameter-first result:

$$\begin{aligned}\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = {} & Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_0)_L \\ & - Z(ch_w + \Delta ch, \tau_m)_R + Z(ch_w + \Delta ch, \tau_0)_R \\ & - Z(ch_w, \tau_m)_L + Z(ch_w, \tau_0)_L \\ & + Z(ch_w, \tau_m)_R - Z(ch_w, \tau_0)_R \end{aligned} \tag{27}$$

Rearrange delta-first result:

$$\begin{aligned}\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} = {} & Z(ch_w + \Delta ch, \tau_m)_L - Z(ch_w + \Delta ch, \tau_0)_L \\ & - Z(ch_w + \Delta ch, \tau_m)_R + Z(ch_w + \Delta ch, \tau_0)_R \\ & - Z(ch_w, \tau_m)_L + Z(ch_w, \tau_0)_L \\ & + Z(ch_w, \tau_m)_R - Z(ch_w, \tau_0)_R \end{aligned} \tag{28}$$

6.4 Final Comparison

Both expressions are identical:

$$\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = \Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} \tag{29}$$

7 Worked Example

Consider a specific numerical example to demonstrate both computational methods.

7.1 Given Data

At chainages $ch_w = 1000\text{m}$ and $ch_w + \Delta ch = 1014\text{m}$ ($\Delta ch = 14\text{m}$):

$$\begin{aligned}\text{Baseline } (\tau_0)\text{:} \quad & Z(1000, \tau_0)_L = 102.345\text{m}, \quad Z(1000, \tau_0)_R = 102.330\text{m} \\ & Z(1014, \tau_0)_L = 102.355\text{m}, \quad Z(1014, \tau_0)_R = 102.338\text{m}\end{aligned} \tag{30}$$

$$\begin{aligned}\text{Current } (\tau_m)\text{:} \quad & Z(1000, \tau_m)_L = 102.358\text{m}, \quad Z(1000, \tau_m)_R = 102.340\text{m} \\ & Z(1014, \tau_m)_L = 102.368\text{m}, \quad Z(1014, \tau_m)_R = 102.346\text{m}\end{aligned} \tag{31}$$

7.2 Parameter-First Calculation

Step 1: Calculate baseline cant values

$$\text{Cant}(1000, \tau_0) = 102.345 - 102.330 = 0.015\text{m} \tag{32}$$

$$\text{Cant}(1014, \tau_0) = 102.355 - 102.338 = 0.017\text{m} \tag{33}$$

Step 2: Calculate current cant values

$$\text{Cant}(1000, \tau_m) = 102.358 - 102.340 = 0.018\text{m} \tag{34}$$

$$\text{Cant}(1014, \tau_m) = 102.368 - 102.346 = 0.022\text{m} \tag{35}$$

Step 3: Calculate baseline and current twist

$$\text{Twist}(1000, 14, \tau_0) = 0.017 - 0.015 = 0.002\text{m} \tag{36}$$

$$\text{Twist}(1000, 14, \tau_m) = 0.022 - 0.018 = 0.004\text{m} \tag{37}$$

Step 4: Calculate twist change

$$\Delta\text{Twist}(1000, 14, \tau_m)_{\text{param}} = 0.004 - 0.002 = 0.002\text{m} \tag{38}$$

7.3 Delta-First Calculation

Step 1: Calculate coordinate deltas

$$\Delta Z(1000, \tau_m)_L = 102.358 - 102.345 = 0.013\text{m} \tag{39}$$
$$\Delta Z(1000, \tau_m)_R = 102.340 - 102.330 = 0.010\text{m} \tag{40}$$
$$\Delta Z(1014, \tau_m)_L = 102.368 - 102.355 = 0.013\text{m} \tag{41}$$
$$\Delta Z(1014, \tau_m)_R = 102.346 - 102.338 = 0.008\text{m} \tag{42}$$

Step 2: Calculate cant changes

$$\Delta\text{Cant}(1000, \tau_m) = 0.013 - 0.010 = 0.003\text{m} \tag{43}$$
$$\Delta\text{Cant}(1014, \tau_m) = 0.013 - 0.008 = 0.005\text{m} \tag{44}$$

Step 3: Calculate twist change directly

$$\Delta\text{Twist}(1000, 14, \tau_m)_{\text{delta}} = 0.005 - 0.003 = 0.002\text{m} \tag{45}$$

7.4 Verification

Both methods yield identical results:

$$\Delta\text{Twist}(1000, 14, \tau_m)_{\text{param}} = \Delta\text{Twist}(1000, 14, \tau_m)_{\text{delta}} = 0.002\text{m} \tag{46}$$

8 Conclusion

$$\boxed{\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = \Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}}} \tag{47}$$

VERIFIED: Both approaches produce identical results for twist change calculations.

Mathematical Basis: The equivalence holds because twist is a linear combination of cant values, and the distributive property of subtraction ensures identical results.

9 Implementation Notes

- **Parameter-First**: More intuitive, provides intermediate cant and twist values for analysis
- **Delta-First**: More computationally efficient, reuses coordinate deltas across multiple parameters
- Both methods are mathematically equivalent and produce identical numerical results
- Twist calculations require coordinates at two chainage locations separated by the measurement interval
- Choice depends on computational efficiency needs and whether intermediate parameter values are required