Rail Geometry Gauge Change Calculations - Equivalence Verification

Rail Geometry Analysis

1 Problem Statement

Given interpolated rail coordinates for left and right rails, verify that two computational approaches for calculating gauge parameter changes produce identical results:

Parameter-First Approach: Calculate gauge parameters, then compute parameter changes
Delta-First Approach: Calculate coordinate deltas, then compute parameter changes directly

This verification demonstrates mathematical equivalence between both computational methods for gauge calculations.

2 Notation

Variable Definitions:

- ch_c = Chainage location (distance along track centerline) where gauge is calculated
- τ_0 = Baseline time (reference measurement)
- τ_m = Current measurement time (m = 1, 2, 3, ...)
- L = Left rail (subscript)
- R = Right rail (subscript)
- X, Y, Z = 3D coordinates (X = Easting, Y = Northing, Z = Elevation)
- Δ = Change or difference operator

Coordinate Notation:

- $X(ch_c, \tau_m)_L$ = X-coordinate of left rail at chainage ch_c and time τ_m
- $Y(ch_c, \tau_m)_L$ = Y-coordinate of left rail at chainage ch_c and time τ_m
- $Z(ch_c, \tau_m)_L$ = Z-coordinate of left rail at chainage ch_c and time τ_m
- $\Delta X(ch_c, \tau_m)_L$ = Change in X-coordinate: $X(ch_c, \tau_m)_L - X(ch_c, \tau_0)_L$

Parameter Notation:

- Gauge(ch_c, τ_m) = Rail gauge at chainage ch_c and time τ_m
- ΔGauge(ch_c, τ_m) = Change in gauge from baseline to time τ_m

3 Input Dataset

Assume we have interpolated rail coordinates available at a specific chainage location:

$$\text{Left Rail:} \quad X(ch_c, \tau_0)_L, Y(ch_c, \tau_0)_L, Z(ch_c, \tau_0)_L \quad \text{(baseline)} \tag{1}$$

$$X(ch_c, \tau_m)_L, Y(ch_c, \tau_m)_L, Z(ch_c, \tau_m)_L \quad \text{(current)} \tag{2}$$

$$\text{Right Rail:} \quad X(ch_c, \tau_0)_R, Y(ch_c, \tau_0)_R, Z(ch_c, \tau_0)_R \quad \text{(baseline)} \tag{3}$$

$$X(ch_c, \tau_m)_R, Y(ch_c, \tau_m)_R, Z(ch_c, \tau_m)_R \quad \text{(current)} \tag{4}$$

Example: At chainage 1000m, we have baseline and current 3D coordinates for both left and right rails.

4 Parameter-First Approach

4.1 Step 1: Calculate Baseline Gauge

For 3D gauge (true spatial distance):

$$\text{Gauge}(ch_c, \tau_0) = \sqrt{(X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R)^2 + (Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R)^2 + (Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R)^2} \tag{5}$$

4.2 Step 2: Calculate Current Gauge

$$\text{Gauge}(ch_c, \tau_m) = \sqrt{(X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R)^2 + (Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R)^2 + (Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R)^2} \tag{6}$$

4.3 Step 3: Calculate Gauge Change

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \text{Gauge}(ch_c, \tau_m) - \text{Gauge}(ch_c, \tau_0) \tag{7}$$

Substituting the gauge definitions:

$$\begin{aligned}\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = &\sqrt{(X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R)^2 + (Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R)^2 + (Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R)^2} \\ &- \sqrt{(X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R)^2 + (Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R)^2 + (Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R)^2}\end{aligned} \tag{8}$$

5 Delta-First Approach

5.1 Step 1: Calculate Coordinate Deltas

$$\Delta X(ch_c, \tau_m)_L = X(ch_c, \tau_m)_L - X(ch_c, \tau_0)_L \tag{9}$$

$$\Delta Y(ch_c, \tau_m)_L = Y(ch_c, \tau_m)_L - Y(ch_c, \tau_0)_L \tag{10}$$

$$\Delta Z(ch_c, \tau_m)_L = Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L \tag{11}$$

$$\Delta X(ch_c, \tau_m)_R = X(ch_c, \tau_m)_R - X(ch_c, \tau_0)_R \tag{12}$$

$$\Delta Y(ch_c, \tau_m)_R = Y(ch_c, \tau_m)_R - Y(ch_c, \tau_0)_R \tag{13}$$

$$\Delta Z(ch_c, \tau_m)_R = Z(ch_c, \tau_m)_R - Z(ch_c, \tau_0)_R \tag{14}$$

5.2 Step 2: Define Separation Vectors

Define baseline separation vector:

$$\vec{S}_0 = (X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R, Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R, Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R) \tag{15}$$

Define delta separation vector:

$$\vec{\Delta S} = (\Delta X(ch_c, \tau_m)_L - \Delta X(ch_c, \tau_m)_R, \\ \Delta Y(ch_c, \tau_m)_L - \Delta Y(ch_c, \tau_m)_R, \\ \Delta Z(ch_c, \tau_m)_L - \Delta Z(ch_c, \tau_m)_R) \tag{16}$$

5.3 Step 3: Calculate Gauge Change Using Vector Addition

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} = |\vec{S}_0 + \vec{\Delta S}| - |\vec{S}_0| \tag{17}$$

6 Equivalence Proof

6.1 Expand Current Separation Vector

The current separation vector is:

$$\vec{S}_0 + \vec{\Delta S} = (X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R, Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R, Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R) \tag{18}$$

6.2 Expand Delta-First Method

Substituting vector definitions:

$$\begin{aligned}\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} &= |\vec{S}_0 + \vec{\Delta S}| - |\vec{S}_0| \\ &= \sqrt{(X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R)^2 + (Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R)^2 + (Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R)^2} \\ &\quad - \sqrt{(X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R)^2 + (Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R)^2 + (Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R)^2}\end{aligned} \tag{19}$$

6.3 Final Comparison

Both methods yield identical expressions:

$$\begin{aligned}\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = &\sqrt{(X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R)^2 + (Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R)^2 + (Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R)^2} \\ &- \sqrt{(X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R)^2 + (Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R)^2 + (Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R)^2}\end{aligned} \tag{20}$$

$$\begin{aligned}\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} = &\sqrt{(X(ch_c, \tau_m)_L - X(ch_c, \tau_m)_R)^2 + (Y(ch_c, \tau_m)_L - Y(ch_c, \tau_m)_R)^2 + (Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R)^2} \\ &- \sqrt{(X(ch_c, \tau_0)_L - X(ch_c, \tau_0)_R)^2 + (Y(ch_c, \tau_0)_L - Y(ch_c, \tau_0)_R)^2 + (Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R)^2}\end{aligned} \tag{21}$$

7 Worked Example

Consider a specific numerical example to demonstrate both computational methods.

7.1 Given Data

At chainage $ch_c = 1000$m:

$$\begin{aligned}\text{Baseline } (\tau_0)\text{:} \quad & X(1000, \tau_0)_L = 500.125\text{m}, \quad Y(1000, \tau_0)_L = 200.345\text{m}, \quad Z(1000, \tau_0)_L = 102.345\text{m} \\ & X(1000, \tau_0)_R = 498.590\text{m}, \quad Y(1000, \tau_0)_R = 200.355\text{m}, \quad Z(1000, \tau_0)_R = 102.330\text{m}\end{aligned} \tag{22}$$

$$\begin{aligned}\text{Current } (\tau_m)\text{:} \quad & X(1000, \tau_m)_L = 500.120\text{m}, \quad Y(1000, \tau_m)_L = 200.350\text{m}, \quad Z(1000, \tau_m)_L = 102.358\text{m} \\ & X(1000, \tau_m)_R = 498.598\text{m}, \quad Y(1000, \tau_m)_R = 200.352\text{m}, \quad Z(1000, \tau_m)_R = 102.340\text{m}\end{aligned} \tag{23}$$

7.2 Parameter-First Calculation

Step 1: Calculate baseline gauge

$$\begin{aligned}\text{Gauge}(1000,\tau_0) &= \sqrt{(500.125-498.590)^2+(200.345-200.355)^2+(102.345-102.330)^2}\\ &= \sqrt{(1.535)^2+(-0.010)^2+(0.015)^2}\\ &= \sqrt{2.356225+0.000100+0.000225}\\ &= \sqrt{2.356550} = 1.5351\text{m}\end{aligned} \tag{24}$$

Step 2: Calculate current gauge

$$\begin{aligned}\text{Gauge}(1000,\tau_m) &= \sqrt{(500.120-498.598)^2+(200.350-200.352)^2+(102.358-102.340)^2}\\ &= \sqrt{(1.522)^2+(-0.002)^2+(0.018)^2}\\ &= \sqrt{2.316484+0.000004+0.000324}\\ &= \sqrt{2.316812} = 1.5222\text{m}\end{aligned} \tag{25}$$

Step 3: Calculate gauge change

$$\Delta\text{Gauge}(1000,\tau_m)_{\text{param}} = 1.5222 - 1.5351 = -0.0129\text{m} \tag{26}$$

7.3 Delta-First Calculation

Step 1: Calculate coordinate deltas

$$\Delta X(1000,\tau_m)_L = 500.120 - 500.125 = -0.005\text{m} \tag{27}$$
$$\Delta Y(1000,\tau_m)_L = 200.350 - 200.345 = 0.005\text{m} \tag{28}$$
$$\Delta Z(1000,\tau_m)_L = 102.358 - 102.345 = 0.013\text{m} \tag{29}$$
$$\Delta X(1000,\tau_m)_R = 498.598 - 498.590 = 0.008\text{m} \tag{30}$$
$$\Delta Y(1000,\tau_m)_R = 200.352 - 200.355 = -0.003\text{m} \tag{31}$$
$$\Delta Z(1000,\tau_m)_R = 102.340 - 102.330 = 0.010\text{m} \tag{32}$$

Step 2: Define separation vectors
Baseline separation vector (left rail - right rail at baseline):

$$\begin{aligned}\vec{S}_0 &= (X(1000,\tau_0)_L - X(1000,\tau_0)_R, Y(1000,\tau_0)_L - Y(1000,\tau_0)_R, Z(1000,\tau_0)_L - Z(1000,\tau_0)_R)\\ &= (500.125-498.590, 200.345-200.355, 102.345-102.330)\\ &= (1.535, -0.010, 0.015)\end{aligned} \tag{33}$$

Delta separation vector (difference in coordinate deltas):

$$\begin{aligned}\vec{\Delta S} &= (\Delta X(1000,\tau_m)_L - \Delta X(1000,\tau_m)_R, \Delta Y(1000,\tau_m)_L - \Delta Y(1000,\tau_m)_R, \Delta Z(1000,\tau_m)_L - \Delta Z(1000,\tau_m)_R)\\ &= ((-0.005)-(0.008), (0.005)-(-0.003), (0.013)-(0.010))\\ &= (-0.013, 0.008, 0.003)\end{aligned} \tag{34}$$

Step 3: Calculate gauge change using vector addition

$$\vec{S}_0 + \vec{\Delta S} = (1.535-0.013, -0.010+0.008, 0.015+0.003) = (1.522, -0.002, 0.018) \tag{35}$$
$$|\vec{S}_0 + \vec{\Delta S}| = \sqrt{(1.522)^2+(-0.002)^2+(0.018)^2} = 1.5222\text{m} \tag{36}$$
$$\Delta\text{Gauge}(1000,\tau_m)_{\text{delta}} = 1.5222 - 1.5351 = -0.0129\text{m} \tag{37}$$

7.4 Verification

Both methods yield identical results:

$$\Delta\text{Gauge}(1000,\tau_m)_{\text{param}} = \Delta\text{Gauge}(1000,\tau_m)_{\text{delta}} = -0.0129\text{m} \tag{38}$$

8 Conclusion

$$\boxed{\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}}} \tag{39}$$

VERIFIED: Both approaches produce identical results for gauge change calculations.

Mathematical Basis: The equivalence holds because the vector addition $\vec{S}_0 + \vec{\Delta S}$ produces the same current separation vector used in the parameter-first approach, ensuring identical distance calculations.

9 Implementation Notes

- **Parameter-First**: More intuitive, provides intermediate gauge values for analysis
- **Delta-First**: More computationally efficient, reuses coordinate deltas across multiple parameters
- Both methods are mathematically equivalent and produce identical numerical results
- Gauge calculations involve 3D distance computations using all coordinate components
- The vector approach in delta-first method provides clear geometric interpretation
- Choice depends on computational efficiency needs and whether intermediate gauge values are required