Rail Geometry Monitoring - Advanced Mathematical Framework

Rail Geometry Analysis

1 Mathematical Foundation

1.1 Coordinate System Definition

Let us define the fundamental coordinate system for prism P_p at time m:

$$(X(P_p, m), Y(P_p, m), Z(P_p, m)) \tag{1}$$

1.2 Baseline Establishment

The baseline coordinates are established at 0:

$$X_0(P_p) = X(P_p, 0) \tag{2}$$
$$Y_0(P_p) = Y(P_p, 0) \tag{3}$$
$$Z_0(P_p) = Z(P_p, 0) \tag{4}$$

Building upon this foundation, we define displacement vectors:

$$\Delta X(P_p, m) = X(P_p, m) - X_0(P_p) \tag{5}$$
$$\Delta Y(P_p, m) = Y(P_p, m) - Y_0(P_p) \tag{6}$$
$$\Delta Z(P_p, m) = Z(P_p, m) - Z_0(P_p) \tag{7}$$

2 Coordinate Transformation

2.1 Prism-to-Rail Transformation

Transform prism coordinates to rail running edge using offset corrections:

$$X_{\text{rail}}(P_p, m) = X_{\text{prism}}(P_p, m) + \Delta X_{\text{off}}(P_p) \tag{8}$$
$$Y_{\text{rail}}(P_p, m) = Y_{\text{prism}}(P_p, m) + \Delta Y_{\text{off}}(P_p) \tag{9}$$
$$Z_{\text{rail}}(P_p, m) = Z_{\text{prism}}(P_p, m) + \Delta Z_{\text{off}}(P_p) \tag{10}$$

3 Interpolation Framework

3.1 Bounding Point Selection

For target chainage ch_{target}, find bounding points:

$$ch_{\text{last}} = \max\{ch_p : ch_p \leq ch_{\text{target}}\} \tag{11}$$
$$ch_{\text{next}} = \min\{ch_p : ch_p \geq ch_{\text{target}}\} \tag{12}$$

3.2 Interpolation Ratio

Define the fundamental interpolation parameter:

$$r = \frac{ch_{\text{target}} - ch_{\text{last}}}{ch_{\text{next}} - ch_{\text{last}}} \tag{13}$$

3.3 Linear Interpolation

Using the ratio r, interpolated coordinates become:

$$X_{\text{intp}}(ch_{\text{target}}, m) = X_{\text{last}} + r \cdot (X_{\text{next}} - X_{\text{last}}) \tag{14}$$

$$Y_{\text{intp}}(ch_{\text{target}}, m) = Y_{\text{last}} + r \cdot (Y_{\text{next}} - Y_{\text{last}}) \tag{15}$$

$$Z_{\text{intp}}(ch_{\text{target}}, m) = Z_{\text{last}} + r \cdot (Z_{\text{next}} - Z_{\text{last}}) \tag{16}$$

4 Geometry Parameter Calculations

4.1 Cant Definition

For corresponding left and right rail points at chainage ch_c:

$$\text{Cant}(ch_c, m) = Z_{\text{intp}}(ch_c, m)_{\text{L}} - Z_{\text{intp}}(ch_c, m)_{\text{R}} \tag{17}$$

4.2 Gauge Calculations

4.2.1 3D Gauge (True Spatial Distance)

$$\text{Gauge}_{3D}(ch_c, m) = \sqrt{(X_{\text{intp}}(ch_c, m)_{\text{L}} - X_{\text{intp}}(ch_c, m)_{\text{R}})^2 + (Y_{\text{intp}}(ch_c, m)_{\text{L}} - Y_{\text{intp}}(ch_c, m)_{\text{R}})^2 + (Z_{\text{intp}}(ch_c, m)_{\text{L}} - Z_{\text{intp}}(ch_c,} \tag{18}$$

4.2.2 2D Gauge (Traditional Horizontal)

$$\text{Gauge}_{2D}(ch_c, m) = \sqrt{(X_{\text{intp}}(ch_c, m)_{\text{L}} - X_{\text{intp}}(ch_c, m)_{\text{R}})^2 + (Y_{\text{intp}}(ch_c, m)_{\text{L}} - Y_{\text{intp}}(ch_c, m)_{\text{R}})^2} \tag{19}$$

4.3 Twist Calculation

Building on the cant definition, twist over interval Δch is:

$$\text{Twist}(ch_w, \Delta ch, m) = \text{Cant}(ch_w + \Delta ch, m) - \text{Cant}(ch_w, m) \tag{20}$$

The twist rate per unit distance becomes:

$$\frac{d\,\text{Cant}}{dch}(ch_w, m) = \frac{\text{Twist}(ch_w, \Delta ch, m)}{\Delta ch} \tag{21}$$

5 Versine Analysis

5.1 Horizontal Versine

For three points at chainages ch_{v-s}, ch_v, ch_{v+s}:

5.1.1 Perpendicular Offset

$$\text{Offset}_{\text{hz}}(ch_v) = \frac{|(Y_{v+s} - Y_{v-s}) \cdot X_v - (X_{v+s} - X_{v-s}) \cdot Y_v + X_{v+s} \cdot Y_{v-s} - X_{v-s} \cdot Y_{v+s}|}{\sqrt{(X_{v+s} - X_{v-s})^2 + (Y_{v+s} - Y_{v-s})^2}} \tag{22}$$

5.1.2 Signed Horizontal Versine

$$\text{Versine}_{\text{hz}}(ch_v) = \text{Offset}_{\text{hz}}(ch_v) \times \text{sign} \tag{23}$$

5.2 Vertical Versine

Using chainage-based interpolation for the expected elevation:

$$Z_{\text{expected}} = Z_{v-s} + \frac{ch_v - ch_{v-s}}{ch_{v+s} - ch_{v-s}} \cdot (Z_{v+s} - Z_{v-s}) \tag{24}$$

Therefore:

$$\text{Versine}_{\text{vt}}(ch_v) = Z_v - Z_{\text{expected}} \tag{25}$$

6 Change Analysis Framework

6.1 Parameter Change Definitions

For any geometry parameter P, define baseline and change:

$$P_0(ch) = P(ch, 0) \tag{26}$$

$$\Delta P(ch, m) = P(ch, m) - P_0(ch) \tag{27}$$

6.2 Specific Parameter Changes

$$\Delta\,\text{Cant}(ch_c, m) = \text{Cant}(ch_c, m) - \text{Cant}(ch_c, 0) \tag{28}$$

$$\Delta\,\text{Gauge}(ch_c, m) = \text{Gauge}(ch_c, m) - \text{Gauge}(ch_c, 0) \tag{29}$$

$$\Delta\,\text{Twist}(ch_w, m) = \text{Twist}(ch_w, m) - \text{Twist}(ch_w, 0) \tag{30}$$

6.3 Rate of Change

The temporal rate of parameter change:

$$\frac{dP}{dt}(ch, m) = \frac{\Delta P(ch, m)}{m - 0} \tag{31}$$

6.4 Threshold Analysis

Define alert conditions:

$$\text{Alert}_P(ch, m) = \begin{cases} \text{CRITICAL} & \text{if } |\Delta P(ch, m)| > T_{\text{critical}} \\ \text{WARNING} & \text{if } |\Delta P(ch, m)| > T_{\text{warning}} \\ \text{NORMAL} & \text{otherwise} \end{cases} \tag{32}$$

7 Summary

This advanced mathematical framework provides a systematic approach to rail geometry monitoring using:

- Custom LaTeX commands for consistent notation
- Progressive formula building from basic definitions
- Reusable mathematical operators
- Comprehensive change analysis framework

The framework enables automated calculation of all geometry parameters and their changes over time for effective rail infrastructure monitoring.