Twist Change Calculation Methods - Equivalence Verification

Rail Geometry Analysis

1 Problem Statement

Given interpolated rail coordinates at chainages ch_w and $ch_w + \Delta ch$ for left and right rails, verify that two methods for calculating twist change produce identical results:

Method A: Calculate twist, then compute change in twist

Method B: Calculate coordinate deltas, then compute twist change directly

2 Input Dataset

Assume we have interpolated rail coordinates at two chainages separated by interval Δch:

At chainage ch_w:

$$\text{Left Rail:} \quad Z(ch_w, 0)_\mathrm{L}, Z(ch_w, m)_\mathrm{L} \tag{1}$$
$$\text{Right Rail:} \quad Z(ch_w, 0)_\mathrm{R}, Z(ch_w, m)_\mathrm{R} \tag{2}$$

At chainage $ch_w + \Delta ch$:

$$\text{Left Rail:} \quad Z(ch_w + \Delta ch, 0)_\mathrm{L}, Z(ch_w + \Delta ch, m)_\mathrm{L} \tag{3}$$
$$\text{Right Rail:} \quad Z(ch_w + \Delta ch, 0)_\mathrm{R}, Z(ch_w + \Delta ch, m)_\mathrm{R} \tag{4}$$

3 Method A: Twist-First Approach

3.1 Step 1: Calculate Baseline Twist

First, calculate cant at both chainages for baseline:

$$\mathrm{Cant}(ch_w, 0) = Z(ch_w, 0)_\mathrm{L} - Z(ch_w, 0)_\mathrm{R} \tag{5}$$
$$\mathrm{Cant}(ch_w + \Delta ch, 0) = Z(ch_w + \Delta ch, 0)_\mathrm{L} - Z(ch_w + \Delta ch, 0)_\mathrm{R} \tag{6}$$

Then calculate baseline twist:

$$\mathrm{Twist}(ch_w, \Delta ch, 0) = \mathrm{Cant}(ch_w + \Delta ch, 0) - \mathrm{Cant}(ch_w, 0) \tag{7}$$

3.2 Step 2: Calculate Current Twist

Calculate cant at both chainages for current time:

$$\mathrm{Cant}(ch_w, m) = Z(ch_w, m)_\mathrm{L} - Z(ch_w, m)_\mathrm{R} \tag{8}$$
$$\mathrm{Cant}(ch_w + \Delta ch, m) = Z(ch_w + \Delta ch, m)_\mathrm{L} - Z(ch_w + \Delta ch, m)_\mathrm{R} \tag{9}$$

Then calculate current twist:

$$\mathrm{Twist}(ch_w, \Delta ch, m) = \mathrm{Cant}(ch_w + \Delta ch, m) - \mathrm{Cant}(ch_w, m) \tag{10}$$

3.3 Step 3: Calculate Twist Change

$$\Delta \mathrm{Twist}(ch_w, \Delta ch, m)_A = \mathrm{Twist}(ch_w, \Delta ch, m) - \mathrm{Twist}(ch_w, \Delta ch, 0) \tag{11}$$

Substituting the twist definitions:

$$\begin{aligned} \Delta \mathrm{Twist}(ch_w, \Delta ch, m)_A = {} & [\mathrm{Cant}(ch_w + \Delta ch, m) - \mathrm{Cant}(ch_w, m)] & (12) \\ & - [\mathrm{Cant}(ch_w + \Delta ch, 0) - \mathrm{Cant}(ch_w, 0)] & (13) \end{aligned}$$

4 Method B: Delta-First Approach

4.1 Step 1: Calculate Z-Coordinate Deltas

$$\begin{aligned} \Delta Z(ch_w, m)_\mathrm{L} &= Z(ch_w, m)_\mathrm{L} - Z(ch_w, 0)_\mathrm{L} & (14) \\ \Delta Z(ch_w, m)_\mathrm{R} &= Z(ch_w, m)_\mathrm{R} - Z(ch_w, 0)_\mathrm{R} & (15) \\ \Delta Z(ch_w + \Delta ch, m)_\mathrm{L} &= Z(ch_w + \Delta ch, m)_\mathrm{L} - Z(ch_w + \Delta ch, 0)_\mathrm{L} & (16) \\ \Delta Z(ch_w + \Delta ch, m)_\mathrm{R} &= Z(ch_w + \Delta ch, m)_\mathrm{R} - Z(ch_w + \Delta ch, 0)_\mathrm{R} & (17) \end{aligned}$$

4.2 Step 2: Calculate Cant Changes

$$\begin{aligned} \Delta \mathrm{Cant}((, ch_{)} w, m) &= \Delta Z(ch_w, m)_\mathrm{L} - \Delta Z(ch_w, m)_\mathrm{R} & (18) \\ \Delta \mathrm{Cant}((, ch_{)} w + \Delta ch, m) &= \Delta Z(ch_w + \Delta ch, m)_\mathrm{L} - \Delta Z(ch_w + \Delta ch, m)_\mathrm{R} & (19) \end{aligned}$$

4.3 Step 3: Calculate Twist Change Directly

$$\Delta \mathrm{Twist}(ch_w, \Delta ch, m)_B = \Delta \mathrm{Cant}((, ch_{)} w + \Delta ch, m) - \Delta \mathrm{Cant}((, ch_{)} w, m) \tag{20}$$

5 Equivalence Proof

5.1 Expand Method A

Substitute cant definitions into Method A:

$$\begin{aligned} \Delta \mathrm{Twist}(ch_w, \Delta ch, m)_A = {} & [(Z(ch_w + \Delta ch, m)_\mathrm{L} - Z(ch_w + \Delta ch, m)_\mathrm{R}) - (Z(ch_w, m)_\mathrm{L} - Z(ch_w, m)_\mathrm{R})] & (21) \\ & - [(Z(ch_w + \Delta ch, 0)_\mathrm{L} - Z(ch_w + \Delta ch, 0)_\mathrm{R}) - (Z(ch_w, 0)_\mathrm{L} - Z(ch_w, 0)_\mathrm{R})] & (22) \end{aligned}$$

Expanding:

$$\begin{aligned} \Delta \mathrm{Twist}(ch_w, \Delta ch, m)_A = {} & Z(ch_w + \Delta ch, m)_\mathrm{L} - Z(ch_w + \Delta ch, m)_\mathrm{R} - Z(ch_w, m)_\mathrm{L} + Z(ch_w, m)_\mathrm{R} & (23) \\ & - Z(ch_w + \Delta ch, 0)_\mathrm{L} + Z(ch_w + \Delta ch, 0)_\mathrm{R} + Z(ch_w, 0)_\mathrm{L} - Z(ch_w, 0)_\mathrm{R} & (24) \end{aligned}$$

5.2 Expand Method B

Substitute delta definitions into Method B:

$$\Delta \mathrm{Twist}(ch_w, \Delta ch, m)_B = [\Delta Z(ch_w + \Delta ch, m)_\mathrm{L} - \Delta Z(ch_w + \Delta ch, m)_\mathrm{R}] - [\Delta Z(ch_w, m)_\mathrm{L} - \Delta Z(ch_w, m)_\mathrm{R}] \tag{25}$$

Substituting delta definitions:

$$\begin{aligned} \Delta \mathrm{Twist}(ch_w, \Delta ch, m)_B = {} & [Z(ch_w + \Delta ch, m)_\mathrm{L} - Z(ch_w + \Delta ch, 0)_\mathrm{L}] - [Z(ch_w + \Delta ch, m)_\mathrm{R} - Z(ch_w + \Delta ch, 0)_\mathrm{R}] & (26) \\ & - [Z(ch_w, m)_\mathrm{L} - Z(ch_w, 0)_\mathrm{L}] + [Z(ch_w, m)_\mathrm{R} - Z(ch_w, 0)_\mathrm{R}] & (27) \end{aligned}$$

Expanding:

$$\Delta\text{Twist}(ch_w, \Delta ch, m)_B = Z(ch_w + \Delta ch, m)_\text{L} - Z(ch_w + \Delta ch, 0)_\text{L} - Z(ch_w + \Delta ch, m)_\text{R} + Z(ch_w + \Delta ch, 0)_\text{R} \tag{28}$$

$$- Z(ch_w, m)_\text{L} + Z(ch_w, 0)_\text{L} + Z(ch_w, m)_\text{R} - Z(ch_w, 0)_\text{R} \tag{29}$$

5.3 Rearranging Terms

Rearrange Method A:

$$\Delta\text{Twist}(ch_w, \Delta ch, m)_A = Z(ch_w + \Delta ch, m)_\text{L} - Z(ch_w + \Delta ch, 0)_\text{L} - Z(ch_w + \Delta ch, m)_\text{R} + Z(ch_w + \Delta ch, 0)_\text{R} \tag{30}$$

$$- Z(ch_w, m)_\text{L} + Z(ch_w, 0)_\text{L} + Z(ch_w, m)_\text{R} - Z(ch_w, 0)_\text{R} \tag{31}$$

Compare with Method B - they are identical!

5.4 Conclusion

$$\boxed{\Delta\text{Twist}(ch_w, \Delta ch, m)_A = \Delta\text{Twist}(ch_w, \Delta ch, m)_B} \tag{32}$$

VERIFIED: Both methods produce identical results.

6 General Formula

The equivalent expressions can be written as:

$$\Delta\text{Twist}(ch_w, \Delta ch, m) = \left[\Delta\text{Cant}((, ch_)w + \Delta ch, m) - \Delta\text{Cant}((, ch_)w, m)\right] = [\text{Twist}(ch_w, \Delta ch, m) - \text{Twist}(ch_w, \Delta ch, 0)] \tag{33}$$

7 Implementation Notes

- **Method A** follows the conceptual definition: calculate twist parameter, then find change
- **Method B** is more direct: work with coordinate differences to compute twist change
- Both methods are mathematically equivalent due to linearity of cant and twist calculations
- Method B may be computationally more efficient as it avoids intermediate cant calculations