Rail Geometry Cant Change Calculations - Equivalence Verification

Rail Geometry Analysis

1 Problem Statement

Given interpolated rail coordinates for left and right rails, verify that two computational approaches for calculating cant parameter changes produce identical results:

Parameter-First Approach: Calculate cant parameters, then compute parameter changes

Delta-First Approach: Calculate coordinate deltas, then compute parameter changes directly

This verification demonstrates mathematical equivalence between both computational methods for cant calculations.

2 Notation

Variable Definitions:

- ch_c = Chainage location (distance along track centerline) where cant is calculated
- τ_0 = Baseline time (reference measurement)
- τ_m = Current measurement time (m = 1, 2, 3, ...)
- L = Left rail (subscript)
- R = Right rail (subscript)
- X, Y, Z = 3D coordinates (X = Easting, Y = Northing, Z = Elevation)
- Δ = Change or difference operator

Coordinate Notation:

- $Z(ch_c, \tau_m)_L$ = Z-coordinate of left rail at chainage ch_c and time τ_m
- $\Delta Z(ch_c, \tau_m)_L$ = Change in Z-coordinate: $Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L$

Parameter Notation:

- $\text{Cant}(ch_c, \tau_m)$ = Cant (cross-level) at chainage ch_c and time τ_m
- $\Delta\text{Cant}(ch_c, \tau_m)$ = Change in cant from baseline to time τ_m

3 Input Dataset

Assume we have interpolated rail coordinates available at a specific chainage location:

$$\text{Left Rail:} \quad Z(ch_c, \tau_0)_L \quad \text{(baseline measurement)} \tag{1}$$

$$Z(ch_c, \tau_m)_L \quad \text{(current measurement)} \tag{2}$$

$$\text{Right Rail:} \quad Z(ch_c, \tau_0)_R \quad \text{(baseline measurement)} \tag{3}$$

$$Z(ch_c, \tau_m)_R \quad \text{(current measurement)} \tag{4}$$

Example: At chainage 1000m, we have baseline Z-coordinates from January 2024 (τ_0) and current Z-coordinates from June 2024 (τ_m) for both left and right rails.

4 Parameter-First Approach

4.1 Step 1: Calculate Baseline Cant

$$\text{Cant}(ch_c, \tau_0) = Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R \tag{5}$$

4.2 Step 2: Calculate Current Cant

$$\text{Cant}(ch_c, \tau_m) = Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R \tag{6}$$

4.3 Step 3: Calculate Cant Change

$$\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = \text{Cant}(ch_c, \tau_m) - \text{Cant}(ch_c, \tau_0) \tag{7}$$

Substituting the cant definitions:

$$\begin{aligned}\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = {} & [Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R] \\ & - [Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R]\end{aligned} \tag{8}$$

5 Delta-First Approach

5.1 Step 1: Calculate Z-Coordinate Deltas

$$\Delta Z(ch_c, \tau_m)_L = Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L \tag{9}$$

$$\Delta Z(ch_c, \tau_m)_R = Z(ch_c, \tau_m)_R - Z(ch_c, \tau_0)_R \tag{10}$$

5.2 Step 2: Calculate Cant Change Directly

$$\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} = \Delta Z(ch_c, \tau_m)_L - \Delta Z(ch_c, \tau_m)_R \tag{11}$$

Substituting the delta definitions:

$$\begin{aligned}\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} = {} & [Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L] \\ & - [Z(ch_c, \tau_m)_R - Z(ch_c, \tau_0)_R]\end{aligned} \tag{12}$$

6 Equivalence Proof

6.1 Algebraic Expansion

Expand parameter-first result:

$$\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = [Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R] \tag{13}$$

$$- [Z(ch_c, \tau_0)_L - Z(ch_c, \tau_0)_R] \tag{14}$$

$$= Z(ch_c, \tau_m)_L - Z(ch_c, \tau_m)_R \tag{15}$$

$$- Z(ch_c, \tau_0)_L + Z(ch_c, \tau_0)_R \tag{16}$$

Expand delta-first result:

$$\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} = [Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L] \tag{17}$$

$$- [Z(ch_c, \tau_m)_R - Z(ch_c, \tau_0)_R] \tag{18}$$

$$= Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L \tag{19}$$

$$- Z(ch_c, \tau_m)_R + Z(ch_c, \tau_0)_R \tag{20}$$

6.2 Rearranging Terms

Rearrange parameter-first:

$$\begin{aligned}\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L \\ - Z(ch_c, \tau_m)_R + Z(ch_c, \tau_0)_R\end{aligned} \tag{21}$$

Compare with delta-first:

$$\begin{aligned}\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} = Z(ch_c, \tau_m)_L - Z(ch_c, \tau_0)_L \\ - Z(ch_c, \tau_m)_R + Z(ch_c, \tau_0)_R\end{aligned} \tag{22}$$

6.3 Final Comparison

Both expressions are identical:

$$\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = \Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} \tag{23}$$

7 Worked Example

Consider a specific numerical example to demonstrate both computational methods.

7.1 Given Data

At chainage $ch_c = 1000\text{m}$:

$$\begin{aligned}\text{Baseline } (\tau_0)\text{:} \quad & Z(1000, \tau_0)_L = 102.345\text{m} \\ & Z(1000, \tau_0)_R = 102.330\text{m}\end{aligned} \tag{24}$$

$$\begin{aligned}\text{Current } (\tau_m)\text{:} \quad & Z(1000, \tau_m)_L = 102.358\text{m} \\ & Z(1000, \tau_m)_R = 102.340\text{m}\end{aligned} \tag{25}$$

7.2 Parameter-First Calculation

Step 1: Calculate baseline cant

$$\text{Cant}(1000, \tau_0) = 102.345 - 102.330 = 0.015\text{m} \tag{26}$$

Step 2: Calculate current cant

$$\text{Cant}(1000, \tau_m) = 102.358 - 102.340 = 0.018\text{m} \tag{27}$$

Step 3: Calculate cant change

$$\Delta\text{Cant}(1000, \tau_m)_{\text{param}} = 0.018 - 0.015 = 0.003\text{m} \tag{28}$$

7.3 Delta-First Calculation

Step 1: Calculate coordinate deltas

$$\Delta Z(1000, \tau_m)_L = 102.358 - 102.345 = 0.013\text{m} \tag{29}$$

$$\Delta Z(1000, \tau_m)_R = 102.340 - 102.330 = 0.010\text{m} \tag{30}$$

Step 2: Calculate cant change directly

$$\Delta\text{Cant}(1000, \tau_m)_{\text{delta}} = 0.013 - 0.010 = 0.003\text{m} \tag{31}$$

7.4 Verification

Both methods yield identical results:

$$\Delta\text{Cant}(1000, \tau_m)_{\text{param}} = \Delta\text{Cant}(1000, \tau_m)_{\text{delta}} = 0.003\text{m} \tag{32}$$

8 Conclusion

$$\boxed{\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = \Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}}} \tag{33}$$

VERIFIED: Both approaches produce identical results for cant change calculations.

Mathematical Basis: The equivalence holds due to the linear nature of cant calculations and the distributive property of subtraction over addition.

9 Implementation Notes

- **Parameter-First**: More intuitive, provides intermediate cant values for analysis
- **Delta-First**: More computationally efficient, reuses coordinate deltas
- Both methods are mathematically equivalent and produce identical numerical results
- Choice depends on computational efficiency needs and whether intermediate cant values are required