Rail Geometry Parameter Change Calculations - Equivalence Verification

Rail Geometry Analysis

1 Problem Statement

Given interpolated rail coordinates for left and right rails, verify that two computational approaches for calculating geometry parameter changes produce identical results:

Parameter-First Approach: Calculate geometry parameters, then compute parameter changes
Delta-First Approach: Calculate coordinate deltas, then compute parameter changes directly

This verification covers cant, gauge, twist, and horizontal versine parameters, demonstrating mathematical equivalence between both computational methods.

2 Notation

Variable Definitions:

- ch_c = Chainage location (distance along track centerline) where cant/gauge are calculated
- ch_w = Chainage location where twist is calculated
- τ_0 = Baseline time (reference measurement)
- τ_m = Current measurement time (m = 1, 2, 3, ...)
- L = Left rail (subscript)
- R = Right rail (subscript)
- X, Y, Z = 3D coordinates (X = Easting, Y = Northing, Z = Elevation)
- Δ = Change or difference operator

Coordinate Notation:

- $X(ch_c, \tau_m)_L$ = X-coordinate of left rail at chainage ch_c and time τ_m
- $\Delta X(ch_c, \tau_m)_L$ = Change in X-coordinate: $X(ch_c, \tau_m)_L - X(ch_c, \tau_0)_L$

Parameter Notation:

- $\text{Cant}(ch_c, \tau_m)$ = Cant (cross-level) at chainage ch_c and time τ_m
- $\Delta\text{Cant}(ch_c, \tau_m)$ = Change in cant from baseline to time τ_m
- $\text{Gauge}(ch_c, \tau_m)$ = Rail gauge at chainage ch_c and time τ_m
- $\text{Twist}(ch_w, \Delta ch, \tau_m)$ = Twist over interval Δch starting at chainage ch_w
- $\text{Versine}(ch_v, \tau_m)$ = Horizontal versine at chainage ch_v and time τ_m

3 Input Dataset

Assume we have interpolated rail coordinates available at a specific chainage location:

$$\text{Left Rail:} \quad X(ch_c, \tau_0)_\text{L}, Y(ch_c, \tau_0)_\text{L}, Z(ch_c, \tau_0)_\text{L} \quad \text{(baseline measurement)} \tag{1}$$

$$X(ch_c, \tau_m)_\text{L}, Y(ch_c, \tau_m)_\text{L}, Z(ch_c, \tau_m)_\text{L} \quad \text{(current measurement)} \tag{2}$$

$$\text{Right Rail:} \quad X(ch_c, \tau_0)_\text{R}, Y(ch_c, \tau_0)_\text{R}, Z(ch_c, \tau_0)_\text{R} \quad \text{(baseline measurement)} \tag{3}$$

$$X(ch_c, \tau_m)_\text{R}, Y(ch_c, \tau_m)_\text{R}, Z(ch_c, \tau_m)_\text{R} \quad \text{(current measurement)} \tag{4}$$

Example: At chainage 1000m, we have baseline coordinates from January 2024 (τ_0) and current coordinates from June 2024 (τ_m) for both left and right rails.

4 Cant Change Calculations

4.1 Parameter-First Approach

4.1.1 Step 1: Calculate Baseline Cant

$$\text{Cant}(ch_c, \tau_0) = Z(ch_c, \tau_0)_\text{L} - Z(ch_c, \tau_0)_\text{R} \tag{5}$$

4.1.2 Step 2: Calculate Current Cant

$$\text{Cant}(ch_c, \tau_m) = Z(ch_c, \tau_m)_\text{L} - Z(ch_c, \tau_m)_\text{R} \tag{6}$$

4.1.3 Step 3: Calculate Cant Change

$$\Delta\text{Cant}(ch_c, \tau_m)_\text{param} = \text{Cant}(ch_c, \tau_m) - \text{Cant}(ch_c, \tau_0) \tag{7}$$

Substituting the cant definitions:

$$\Delta\text{Cant}(ch_c, \tau_m)_\text{param} = [Z(ch_c, \tau_m)_\text{L} - Z(ch_c, \tau_m)_\text{R}] - [Z(ch_c, \tau_0)_\text{L} - Z(ch_c, \tau_0)_\text{R}] \tag{8}$$

4.2 Delta-First Approach

4.2.1 Step 1: Calculate Z-Coordinate Deltas

$$\Delta Z(ch_c, \tau_m)_\text{L} = Z(ch_c, \tau_m)_\text{L} - Z(ch_c, \tau_0)_\text{L} \tag{9}$$

$$\Delta Z(ch_c, \tau_m)_\text{R} = Z(ch_c, \tau_m)_\text{R} - Z(ch_c, \tau_0)_\text{R} \tag{10}$$

4.2.2 Step 2: Calculate Cant Change Directly

$$\Delta\text{Cant}(ch_c, \tau_m)_\text{delta} = \Delta Z(ch_c, \tau_m)_\text{L} - \Delta Z(ch_c, \tau_m)_\text{R} \tag{11}$$

Substituting the delta definitions:

$$\Delta\text{Cant}(ch_c, \tau_m)_\text{delta} = [Z(ch_c, \tau_m)_\text{L} - Z(ch_c, \tau_0)_\text{L}] - [Z(ch_c, \tau_m)_\text{R} - Z(ch_c, \tau_0)_\text{R}] \tag{12}$$

4.3 Cant Equivalence Proof

4.3.1 Algebraic Expansion

Expand parameter-first result:

$$\begin{aligned}
\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} &= [Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_m)_{\text{R}}] && (13)\\
&\quad - [Z(ch_c, \tau_0)_{\text{L}} - Z(ch_c, \tau_0)_{\text{R}}] && (14)\\
&= Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_m)_{\text{R}} && (15)\\
&\quad - Z(ch_c, \tau_0)_{\text{L}} + Z(ch_c, \tau_0)_{\text{R}} && (16)
\end{aligned}$$

Expand delta-first result:

$$\begin{aligned}
\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} &= [Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_0)_{\text{L}}] && (17)\\
&\quad - [Z(ch_c, \tau_m)_{\text{R}} - Z(ch_c, \tau_0)_{\text{R}}] && (18)\\
&= Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_0)_{\text{L}} && (19)\\
&\quad - Z(ch_c, \tau_m)_{\text{R}} + Z(ch_c, \tau_0)_{\text{R}} && (20)
\end{aligned}$$

4.3.2 Rearranging Terms

Rearrange parameter-first:

$$\begin{multline}
\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_0)_{\text{L}} \\
- Z(ch_c, \tau_m)_{\text{R}} + Z(ch_c, \tau_0)_{\text{R}} \qquad (21)
\end{multline}$$

Compare with delta-first:

$$\begin{multline}
\Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}} = Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_0)_{\text{L}} \\
- Z(ch_c, \tau_m)_{\text{R}} + Z(ch_c, \tau_0)_{\text{R}} \qquad (22)
\end{multline}$$

4.3.3 Conclusion

$$\boxed{\Delta\text{Cant}(ch_c, \tau_m)_{\text{param}} = \Delta\text{Cant}(ch_c, \tau_m)_{\text{delta}}} \qquad (23)$$

VERIFIED: Both approaches produce identical results for cant change.

5 Gauge Change Calculations

Gauge uses the same coordinate inputs as cant but involves distance calculations rather than simple differences.

5.1 Gauge Definition

For 3D gauge (true spatial distance), first define separation components:

$$\begin{aligned}
\Delta X_{\text{sep}}(ch_c, \tau_m) &= X(ch_c, \tau_m)_{\text{L}} - X(ch_c, \tau_m)_{\text{R}} && (24)\\
\Delta Y_{\text{sep}}(ch_c, \tau_m) &= Y(ch_c, \tau_m)_{\text{L}} - Y(ch_c, \tau_m)_{\text{R}} && (25)\\
\Delta Z_{\text{sep}}(ch_c, \tau_m) &= Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_m)_{\text{R}} && (26)
\end{aligned}$$

Then gauge becomes:

$$\text{Gauge}(ch_c, \tau_m) = \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \qquad (27)$$

5.2 Parameter-First Approach

5.2.1 Step 1: Calculate Baseline Gauge

$$\text{Gauge}(ch_c, \tau_0) = \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_0)^2} \tag{28}$$

5.2.2 Step 2: Calculate Current Gauge

$$\text{Gauge}(ch_c, \tau_m) = \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \tag{29}$$

5.2.3 Step 3: Calculate Gauge Change

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \text{Gauge}(ch_c, \tau_m) - \text{Gauge}(ch_c, \tau_0) \tag{30}$$

5.3 Delta-First Approach

5.3.1 Step 1: Calculate Coordinate Deltas

Using the same coordinate deltas as calculated for cant:

$$\Delta X(ch_c, \tau_m)_{\text{L}} = X(ch_c, \tau_m)_{\text{L}} - X(ch_c, \tau_0)_{\text{L}} \tag{31}$$
$$\Delta Y(ch_c, \tau_m)_{\text{L}} = Y(ch_c, \tau_m)_{\text{L}} - Y(ch_c, \tau_0)_{\text{L}} \tag{32}$$
$$\Delta Z(ch_c, \tau_m)_{\text{L}} = Z(ch_c, \tau_m)_{\text{L}} - Z(ch_c, \tau_0)_{\text{L}} \tag{33}$$

$$\Delta X(ch_c, \tau_m)_{\text{R}} = X(ch_c, \tau_m)_{\text{R}} - X(ch_c, \tau_0)_{\text{R}} \tag{34}$$
$$\Delta Y(ch_c, \tau_m)_{\text{R}} = Y(ch_c, \tau_m)_{\text{R}} - Y(ch_c, \tau_0)_{\text{R}} \tag{35}$$
$$\Delta Z(ch_c, \tau_m)_{\text{R}} = Z(ch_c, \tau_m)_{\text{R}} - Z(ch_c, \tau_0)_{\text{R}} \tag{36}$$

5.3.2 Step 2: Calculate Gauge Change Using Vector Addition

Define baseline and delta separation vectors:

$$\vec{S}_0 = (\Delta X_{\text{sep}}(ch_c, \tau_0), \Delta Y_{\text{sep}}(ch_c, \tau_0), \Delta Z_{\text{sep}}(ch_c, \tau_0)) \tag{37}$$

$$\begin{split}\Delta\vec{S} = (&\Delta X(ch_c, \tau_m)_{\text{L}} - \Delta X(ch_c, \tau_m)_{\text{R}},\\ &\Delta Y(ch_c, \tau_m)_{\text{L}} - \Delta Y(ch_c, \tau_m)_{\text{R}},\\ &\Delta Z(ch_c, \tau_m)_{\text{L}} - \Delta Z(ch_c, \tau_m)_{\text{R}})\end{split} \tag{38}$$

Then:

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} = |\vec{S}_0 + \Delta\vec{S}| - |\vec{S}_0| \tag{39}$$

5.4 Gauge Equivalence Proof

5.4.1 Expand Parameter-First Method

Substitute gauge definitions:

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \text{Gauge}(ch_c, \tau_m) - \text{Gauge}(ch_c, \tau_0) \tag{40}$$
$$= \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \tag{41}$$
$$- \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_0)^2} \tag{42}$$

5.4.2 Expand Delta-First Method

The current separation vector becomes:

$$\vec{S}_0 + \vec{\Delta S} = (\Delta X_{\text{sep}}(ch_c, \tau_m), \Delta Y_{\text{sep}}(ch_c, \tau_m), \Delta Z_{\text{sep}}(ch_c, \tau_m)) \tag{43}$$

Therefore:

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} = |\vec{S}_0 + \vec{\Delta S}| - |\vec{S}_0| \tag{44}$$

$$= \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \tag{45}$$

$$- \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_0)^2} \tag{46}$$

5.4.3 Final Comparison

Both methods yield identical expressions:

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \tag{47}$$

$$- \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_0)^2} \tag{48}$$

$$\Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}} = \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_m)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_m)^2} \tag{49}$$

$$- \sqrt{\Delta X_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Y_{\text{sep}}(ch_c, \tau_0)^2 + \Delta Z_{\text{sep}}(ch_c, \tau_0)^2} \tag{50}$$

5.4.4 Conclusion

$$\boxed{\Delta\text{Gauge}(ch_c, \tau_m)_{\text{param}} = \Delta\text{Gauge}(ch_c, \tau_m)_{\text{delta}}} \tag{51}$$

VERIFIED: Both approaches produce identical results for gauge change.

Note: The parameterized separation components ΔX_{sep}(, ,) ΔY_{sep}(, ,) ΔZ_{sep}(, g)reatly simplify the expressions while maintaining mathematical rigor.

6 Twist Change Calculations

Since twist is the difference in cant over a distance interval, the same equivalence principle applies.

6.1 Twist Definition

For chainages ch_w and $ch_w + \Delta ch$:

$$\text{Twist}(ch_w, \Delta ch, \tau_m) = \text{Cant}(ch_w + \Delta ch, \tau_m) - \text{Cant}(ch_w, \tau_m) \tag{52}$$

6.2 Twist Change Methods

Parameter-First Approach:

$$\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = \text{Twist}(ch_w, \Delta ch, \tau_m) - \text{Twist}(ch_w, \Delta ch, \tau_0) \tag{53}$$

Delta-First Approach:

$$\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}} = \Delta\text{Cant}(ch_w + \Delta ch, \tau_m) - \Delta\text{Cant}(ch_w, \tau_m) \tag{54}$$

where ΔCant(, v)alues are calculated using the cant change methods above.

6.3 Twist Equivalence

Since twist is a linear combination of cant values, and cant change calculations are equivalent:

$$\boxed{\Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{param}} = \Delta\text{Twist}(ch_w, \Delta ch, \tau_m)_{\text{delta}}} \tag{55}$$

VERIFIED: Twist change methods are equivalent by transitivity of the cant equivalence.

7 Horizontal Versine Change Calculations

Horizontal versines measure track deviation from straight line using three-point geometry. They require coordinates at three chainages: ch_{v-s}, ch_v, and ch_{v+s} where s is the half-chord length.

7.1 Input Dataset for Versines

Assume we have interpolated rail coordinates at three chainages:

$$\text{At } ch_{v-s}: \quad X(ch_{v-s}, \tau_0), Y(ch_{v-s}, \tau_0) \quad \text{and} \quad X(ch_{v-s}, \tau_m), Y(ch_{v-s}, \tau_m) \tag{56}$$

$$\text{At } ch_{v}: \quad X(ch_{v}, \tau_0), Y(ch_{v}, \tau_0) \quad \text{and} \quad X(ch_{v}, \tau_m), Y(ch_{v}, \tau_m) \tag{57}$$

$$\text{At } ch_{v+s}: \quad X(ch_{v+s}, \tau_0), Y(ch_{v+s}, \tau_0) \quad \text{and} \quad X(ch_{v+s}, \tau_m), Y(ch_{v+s}, \tau_m) \tag{58}$$

7.2 Versine Definition

Horizontal versine is the perpendicular offset from the chord connecting the end points:

$$\text{Versine}(ch_v, \tau_m) = \frac{\text{perpendicular distance from chord}}{\text{chord length}} \tag{59}$$

7.3 Parameter-First Approach

7.3.1 Step 1: Calculate Baseline Versine

$$\text{Versine}(ch_v, \tau_0) = f(X(ch_{v-s}, \tau_0), Y(ch_{v-s}, \tau_0), X(ch_v, \tau_0), Y(ch_v, \tau_0), X(ch_{v+s}, \tau_0), Y(ch_{v+s}, \tau_0)) \tag{60}$$

7.3.2 Step 2: Calculate Current Versine

$$\text{Versine}(ch_v, \tau_m) = f(X(ch_{v-s}, \tau_m), Y(ch_{v-s}, \tau_m), X(ch_v, \tau_m), Y(ch_v, \tau_m), X(ch_{v+s}, \tau_m), Y(ch_{v+s}, \tau_m)) \tag{61}$$

7.3.3 Step 3: Calculate Versine Change

$$\Delta\text{Versine}(ch_v, \tau_m)_{\text{param}} = \text{Versine}(ch_v, \tau_m) - \text{Versine}(ch_v, \tau_0) \tag{62}$$

7.4 Delta-First Approach

7.4.1 Step 1: Calculate Coordinate Deltas

$$\Delta X(ch_{v-s}, \tau_m) = X(ch_{v-s}, \tau_m) - X(ch_{v-s}, \tau_0) \tag{63}$$

$$\Delta Y(ch_{v-s}, \tau_m) = Y(ch_{v-s}, \tau_m) - Y(ch_{v-s}, \tau_0) \tag{64}$$

$$\Delta X(ch_{v}, \tau_m) = X(ch_{v}, \tau_m) - X(ch_{v}, \tau_0) \tag{65}$$

$$\Delta Y(ch_{v}, \tau_m) = Y(ch_{v}, \tau_m) - Y(ch_{v}, \tau_0) \tag{66}$$

$$\Delta X(ch_{v+s}, \tau_m) = X(ch_{v+s}, \tau_m) - X(ch_{v+s}, \tau_0) \tag{67}$$

$$\Delta Y(ch_{v+s}, \tau_m) = Y(ch_{v+s}, \tau_m) - Y(ch_{v+s}, \tau_0) \tag{68}$$

7.4.2 Step 2: Calculate Versine Change Using Coordinate Deltas

The versine change is computed by applying the coordinate deltas to the baseline coordinates:

$$\Delta\text{Versine}(ch_v, \tau_m)_{\text{delta}} = f(\text{baseline coords} + \text{deltas}) - f(\text{baseline coords}) \tag{69}$$

This complex expression simplifies to the same result as the parameter-first approach due to the algebraic properties of coordinate transformations.

7.5 Versine Equivalence

While the algebraic expansion is significantly more complex than cant or gauge due to the non-linear versine formula, both methods are mathematically equivalent:

$$\boxed{\Delta\text{Versine}(ch_v, \tau_m)_{\text{param}} = \Delta\text{Versine}(ch_v, \tau_m)_{\text{delta}}} \tag{70}$$

VERIFIED: Both approaches produce identical results for versine change.

Note: Unlike cant (linear) and gauge (square root), versines involve complex geometric calculations with absolute values and ratios, making the delta-first approach computationally intensive but mathematically equivalent.

8 General Formula

The equivalent expressions can be written as:

$$\begin{aligned}
\Delta\text{Cant}(ch_c, \tau_m) &= [\Delta Z(ch_c, \tau_m)_{\text{L}} - \Delta Z(ch_c, \tau_m)_{\text{R}}] && (71)\\
&= [\text{Cant}(ch_c, \tau_m) - \text{Cant}(ch_c, \tau_0)] && (72)\\
\Delta\text{Gauge}(ch_c, \tau_m) &= |\vec{S}_0 + \vec{\Delta S}| - |\vec{S}_0| && (73)\\
&= \text{Gauge}(ch_c, \tau_m) - \text{Gauge}(ch_c, \tau_0) && (74)\\
\Delta\text{Twist}(ch_w, \Delta ch, \tau_m) &= [\Delta\text{Cant}(ch_w + \Delta ch, \tau_m) - \Delta\text{Cant}(ch_w, \tau_m)] && (75)\\
&= [\text{Twist}(ch_w, \Delta ch, \tau_m) - \text{Twist}(ch_w, \Delta ch, \tau_0)] && (76)
\end{aligned}$$

9 Computational Efficiency Analysis

9.1 Parameter-First Approach

- **Conceptual clarity**: Follows natural definition sequence
- **Intermediate values**: Provides cant and twist values for analysis
- **Computational cost**: Higher due to intermediate parameter calculations

9.2 Delta-First Approach

- **Computational efficiency**: Direct calculation from coordinate deltas
- **Reusability**: ΔX, ΔY, ΔZ calculations can be reused across parameters
- **Memory efficiency**: Avoids storing intermediate parameter values
- **Particularly efficient for twist**: Reuses cant change calculations

10 Implementation Recommendations

- Choose **parameter-first** for conceptual clarity and when intermediate parameter values are needed
- Choose **delta-first** for computational efficiency, especially when calculating multiple derived parameters
- Both approaches are mathematically equivalent due to linearity of geometry parameter calculations
- Consider hybrid approaches: use delta-first for efficiency, compute parameters when needed for analysis

11 Summary

Cant, gauge, twist, and horizontal versine change calculations all demonstrate mathematical equivalence between parameter-first and delta-first approaches. The choice depends on computational efficiency needs, conceptual clarity preferences, and whether intermediate parameter values are required for analysis.