Coding and Cryptography Notes

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1 Theorem and Algorithms

1.1 Euclids's Algorithm

1.1.1 Statement

To find the GCD of two numbers, say a and b ($b < a$), do the following:

$$
\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}
$$

Since the remainder is decreasing with every step but cannot be negative, there will eventually be a remainder r_N which is equal to zero, at which point the algorithm stops. The remainder r_{N-1} is the GCD of a and b. If ($a < b$) swap a and b before beginning the algorithm.

1.1.2 Euclid's Algorithm for finding modular inverses

Assume we have two integers a and b which are co-prime. We want to find the inverse of a $(mod\ b)$, denoted $\overline{a}$. We know from the definition of the modular inverse that $a\overline{a} \equiv 1\ (mod\ b)$. Hence our task is to solve the equation $ax + by = 1$ for x and y which will give us $x = \overline{a}$.

We do this by using Euclid's Algorithm and stopping when the remainder is equal to 1:

$$\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\cdot \\
&\cdot \\
&\cdot \\
r_{N-3} &= q_{N-1} r_{N-2} + r_{N-1}
\end{aligned}$$

We know that $r_{N-1} = 1$ because a and b are co-prime and hence their GCD is 1. We now re-write the equations and subtitute back until we end with an equation of the form $1 = ax + by$ and we will have $\overline{a} \equiv x \ (mod\ b)$.

1.2 Chinese Remainder Theorem

1.2.1 Statement

The Chinese Remainder Theorem states that the system of congruences,

$$\begin{aligned}
x &\equiv a_1 \ (mod\ m_1) \\
x &\equiv a_2 \ (mod\ m_2) \\
&\cdot \\
&\cdot \\
&\cdot \\
x &\equiv a_r \ (mod\ m_r)
\end{aligned}$$

where the modulo $m_1, m_2, ..., m_r$ are all pairwise coprime has a unique solution modulo M given by

$$x \equiv a_1(M_1\overline{M_1}) + a_2(M_2\overline{M_2}) + ... + a_r(M_r\overline{M_r}) \ (mod\ M)$$

where

- $M = m_1 m_2 ... m_r$
- $M_i = \frac{M}{m_i}$
- $M_i\overline{M_i} \equiv 1 \ (mod\ m_i)$

1.2.2 Example 1

Let

$$\begin{aligned}
x &\equiv 0 \ (mod\ 3) \\
x &\equiv 3 \ (mod\ 4) \\
x &\equiv 4 \ (mod\ 5)
\end{aligned}$$

Find $x \ (modM)$

- $M = m_1 m_2 m_3 = (3)(4)(5) = 60$

- $M_1 = \frac{M}{m_1} = \frac{60}{3} = 20$
  $M_2 = \frac{M}{m_2} = \frac{60}{4} = 15$
  $M_3 = \frac{M}{m_3} = \frac{60}{5} = 12$

- $$M_1\overline{M_1} \equiv 1 \ (mod\ m_i)$$
  $$\Rightarrow 20\overline{M_1} \equiv 1 \ (mod\ 3)$$
  Using Euclid's Algorithm :
  Solving $1 = 20\overline{M_1} - 3y$
  $$20 = 3(6) + 2$$
  $$3 = 2(1) + 1$$
  Substituting back :
  $$3 = (20 - 3(6))(1) + 1$$
  $$3 = 20(1) - 3(6) + 1$$
  $$1 = 3(7) + 20(-1)$$
  Hence $\overline{M_1} \equiv -1 \ (mod\ 3)$
  $\Rightarrow \overline{M_1} \equiv 2 \ (mod\ 3)$

1.3 Modular Exponentiation Algorithm

1.3.1 Statement

To find the least postive residue of $a^n \ (modm)$ where n is a large number do the following:

- Express n in binary to get $n = (b_0)2^0 + (b_1)2^1 + (b_2)2^2 + ... + (b_k)2^k$ $(b_i \in [0, 1])$
- Calculate the least positive residues modulo n for $a^{2^0}, \ a^{2^1}, ..., \ a^{2^k}$
- Multiply the least positive residues whose corresponding bit is 1 in the first step.
- Find the least positive residue of the result above.

1.3.2 Example

What is the least positive residue of $3^2 31$ *modulo* 49?

- 231 in binary:
  $$231 = (1)2^7 + (1)2^6 + (1)2^5 + (0)2^4 + (0)2^3 + (1)2^2 + (1)2^1 + (1)2^0$$
  Hence $231 = (11100111)_2$

- Least positive residues modulo 49 for $3^{2^0}, \ 3^{2^1}, ..., \ 3^{2^7}$:
  $b_0 = 1 \qquad 3^{2^0} \equiv 3^1 \equiv 3 \ (mod\ 49)$
  $b_1 = 1 \qquad 3^{2^1} \equiv 3^2 \equiv 9 \ (mod\ 49)$
  $b_2 = 1 \qquad 3^{2^2} \equiv 3^4 \equiv 9^2 \equiv 32 \ (mod\ 49)$

$b_3 = 0 \qquad 3^{2^3} \equiv 3^8 \equiv 32^2 \equiv 44 \ (mod\ 49)$
$b_4 = 0 \qquad 3^{2^4} \equiv 3^{16} \equiv 44^2 \equiv 25 \ (mod\ 49)$
$b_5 = 1 \qquad 3^{2^5} \equiv 3^{32} \equiv 25^2 \equiv 37 \ (mod\ 49)$
$b_6 = 1 \qquad 3^{2^6} \equiv 3^{64} \equiv 37^2 \equiv 46 \ (mod\ 49)$
$b_7 = 1 \qquad 3^{2^7} \equiv 3^{128} \equiv 46^2 \equiv 9 \ (mod\ 49)$

- Multiply results where $b_i = 1$:
  $(3)(9)(32)(37)(46)(9) = 13234752$
- Find least positive residue of above result:
  $3^{231} \equiv (3^{2^7})(3^{2^6})(3^{2^5})(3^{2^2})(3^{2^1})(3^{2^0}) \equiv 13234752 \equiv 48 \ (mod\ 49)$

1.3.3 Explanation of Algorithm

We know from the definition of the mod function that:

$$a \equiv r_1 \ (mod\ m)$$
$$b \equiv r_2 \ (mod\ m)$$
$$\Rightarrow ab \equiv r_1 r_2 \ (mod\ m)$$

This is what allows us to complete step 2 in the algorithm.

Furthermore, we know that $a^x a^y = a^{x+y}$ which is what allows us to split up the work in the algorithm and then recombine the results.

1.3.4 Fast-tracking the algorithm

Before applying the Modular Exponentiation Algorithm, we can eliminate a lot of the work by applying Euler's Theorem:
For any modulus n and integer a which are co-prime (ie GCD(a,n)=1) we have that:

$$a^{\varphi(m)} \equiv 1 \ (mod\ m)$$

Hence, to aid in computing the least positive residue we can express the exponent in the original problem as $n = q\varphi(m) + r$. This helps us in that we can now write $a^{q\varphi(m)+r}$ and since we know that $a^{\varphi} \equiv 1 \ (mod\ m)$ the problem is reduced to finding the least positive residue of $a^r \ (mod\ m)$. This is because $a^n \equiv a^{q\varphi(m)+r} \equiv a^{q\varphi(m)} a^r \equiv a^r \ (mod\ m)$.

1.4 Fermat's Factorization

1.4.1 Statement

Fermat's Factorization allows us to factorize an *odd* integer using the difference of two squares. Given an odd integer n, and two factors c and d s.t. $n = cd$ we know that:

$$n = (\tfrac{c+d}{2})^2 - (\tfrac{c-d}{2})^2$$

The method is as follows:

- We start by choosing $a = \lceil \sqrt{n} \rceil$
- We then calculate $n - a^2 = z$
- If z is a perfect square, we have found $b = \sqrt{z}$
- If z is not a perfect square, we increment a, and try again until we get a z which is a perfect square or we ascertain that n is a prime.
- Once we have a and b, the factors of n are calculated as:
  $c = a + b$
  $d = a - b$

2 Cryptosystems

2.1 RSA Encryption

2.1.1 How it works

Assume Bob wants to send Alice an encrypted message.

- Alice begins by choosing two large primes, p and q
- Alice then computes $n = pq$
- Alice then chooses e s.t. $e \in (1, \varphi(n))$ and GCD(e,$\varphi(n)$)= 1
- Alice's **PUBLIC KEY** is (e,n)
- Alice then computes her **PRIVATE KEY** d using $de \equiv 1 \ (mod\ \varphi(n))$ (ie d is the inverse of e under modulus $\varphi(n)$)
- Alice then shares her **PUBLIC KEY** with Bob
- Bob encrypts his message P using Alice's public key (e,n) in the formula:
  $C \equiv P^e \ (mod\ n)$
- Alice can decrypt the ciphertext C using her private key d the formula:
  $P \equiv C^d \ (mod\ n)$

  This works because:
  $C^d \ (mod\ n)$
  $\equiv (P^e)^d \ (mod\ n)$
  $\equiv P^{ed} \ (mod\ n)$
  $\equiv P^1 \ (mod\ n)$
  $\equiv P \ (mod\ n)$

2.1.2 Signed Messages

Assume Alice wants to send Bob a signed message.
Alice has public key (e_A,n_A) and private key d_A.
Bob has public key (e_B,n_B) and private key d_B.
Alice's signature is the plaintext P.

- Alice calculates $S \equiv P^{d_A} \ (mod\ n_A)$
- Alice then computes $C \equiv S^{e_B} \ (mod\ n_B)$ and sends c to Bob
- Bob first performs the usual dechiphering $S \equiv C^{d_B} \ (mod\ n_B)$
- Bob then computes $P \equiv S^{e_A} \ (mod\ n_A)$

Only Alice can perform the first step since only she knows her private key. The plaintext P will only be recovered correctly if Alice did indeed send the message (since Bob will be using her public key to decrypt). It provides more understanding to notice the following:

2.1.3 Evaluation

If someone came into posession of a ciphertext C and the sender's public key (e,n) they would not be able to easily decipher the ciphertext C since they would need to find the inverse of e modulo $\varphi(n)$. This would be an easy task but finding $\varphi(n)$ is very difficult to do when the prime factorization of n is not known. It is an unproved assumption that there is no easier way to decipher C than by first factoring $n = pq$ and then using this to determine $\varphi(n)$ and in turn using this to find the inverse of e modulo $\varphi(n)$.

2.2 Elgamal Encryption

2.2.1