# Introduction to Quantum Computing

Quantum Computing Summer School 2019

University of the Witwatersrand

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*'Anyone who is not shocked by quantum theory has not understood it.'*
*- Niels Bohr*

Quantum Computers perform quantum information processing using a non-classical model of computation. Quantum computing exploits aspects of quantum mechanics to expand our computational horizon. The complexity of an algorithm or computation depends on the amount of resources used by a computer to find a solution to a problem or to complete a task. Two particularly important resources are time and space - computer memory. An approximate measurement on complexity allows for elaboration on the robustness of computing models. The big $\mathcal{O}$ notation expresses the resource requirements.

# 1 Fundamentals of Quantum Mechanics

The basic unit of quantum information is known as a qubit. A qubit can be represented by a Bloch-sphere:

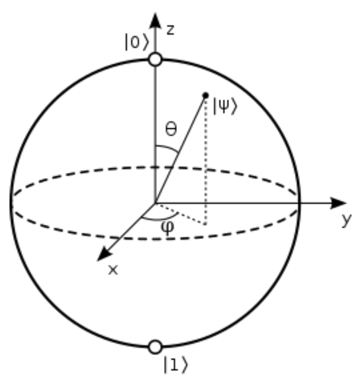


Figure 1: The Bloch Sphere

Dirac Notation is often used in quantum mechanics. The symbol representing the vector is known as the 'ket' which is denoted as $|a\rangle$, and the transpose complex conjugate of the *ket* is given by the 'bra' which is denoted as $\langle b|$. The inner product will be written as bra-kets, $\langle a|b\rangle$.

The vector spaces considered are over the complex numbers, and are finite dimensional. These vector spaces are members of a class of vector spaces called Hilbert spaces, $\mathcal{H}$.

Considering how to represent two-state quantum states (one of two distinguishable states) using bra-ket notation, the '0' state is represented by the ket $|0\rangle$ and as a vector as $\begin{bmatrix}1\\0\end{bmatrix}$, and the '1' state as, $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$.

## 1.1 Properties & Principles

**Heisenburg's Uncertainty Principle**
Can never simultaneously know the following for an object:

- Exact position.
- Exact speed.

### 1.1.1 Superposition

Superposition refers to a `combination of states` which would typically be described independently. Mathematically, superposition can be described by taking a linear combination of the state vectors for the '0' and '1' states. This can be represented using bra-ket notation as: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$, and as vectors by $\alpha_0 \begin{bmatrix}1\\0\end{bmatrix} + \alpha_1 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}$.

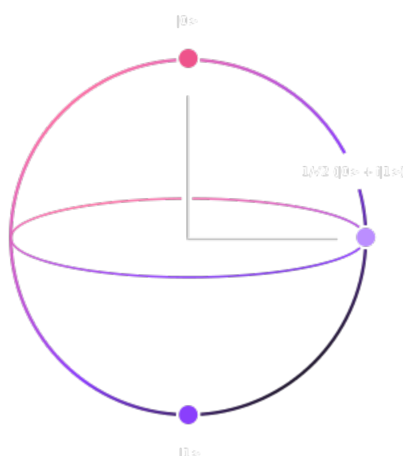


Figure 2: Illustration of superposition

### 1.1.2 Entanglement

Entangled particles `behave together as a system` in ways that cannot be explained using classical logic. A minimum of two qubits are required to describe this property.

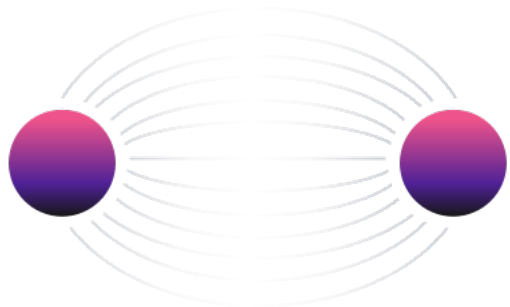

Figure 3: Illustration of entanglement

Teleportation exhibits the power of entanglement for quantum computing and quantum communication.

### 1.1.3 Interference

Quantum states can undergo interference due to a phenomenon known as phase. Quantum interference can be understood similarly to wave interference; when two waves are in phase, their amplitudes add, and when they are out of phase, their amplitudes cancel.

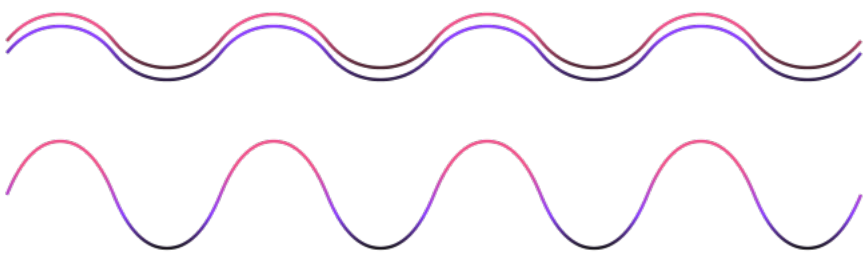

Figure 4: Illustration of interference

## 1.2 Measurement

At the instance when the physical measurement of a qubit is taken, the superposition state of the qubit collapses to a basis state. The measurement of each of the states are the absolute sum squared. For example, given a qubit in superposition $\alpha_0 |0\rangle + \alpha_1 |1\rangle$, when measures will have a probability of being in the '0' state of $|\alpha_0|^2$ and the '1' state as $|\alpha_1|^2$, such that $|\alpha_0|^2 + |\alpha_1|^2 = 1$

# 2 Linear Algebra and Dirac Notation

This section is to be used as a reminder of some prevalent Linear Algebra and Dirac Notation concepts and operations.

## 2.1 Matrix Operations

### 2.1.1 Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

### 2.1.2 Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

### 2.1.3 Multiplication (Dot Product)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \cdot e + b \cdot g & a \cdot f + b \cdot h \\ c \cdot e + d \cdot g & c \cdot f + d \cdot h \end{bmatrix}$$

### 2.1.4 Direct Sum

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{bmatrix}$$

### 2.1.5 Tensor Product

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a\,[B] & b\,[B] \\ c\,[B] & d\,[B] \end{bmatrix} = \begin{bmatrix} a \cdot e & a \cdot f & b \cdot e & b \cdot f \\ a \cdot g & a \cdot g & b \cdot g & b \cdot h \\ c \cdot e & c \cdot f & d \cdot e & d \cdot f \\ c \cdot g & c \cdot h & d \cdot g & d \cdot h \end{bmatrix}$$

## 2.2 Dirac Notation

### 2.2.1 Hadamard Basis

Considering a Hilbert space $\mathcal{H}$ of dimension 2, $\mathcal{H}_2$ , which is an orthonormal basis for $\mathcal{H}$. An example is the Hadamard basis, $|+\rangle$ and $|-\rangle$. These basis vectors can be represented as follows:

$$|+\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right),$$

$$|-\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle - |1\rangle \right).$$

The normality of these basis vectors is shown by:

$$\begin{aligned} \langle + | - \rangle &= \frac{1}{2} \left( \langle 0| + \langle 1| \right) \left( |0\rangle - |1\rangle \right) \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= 0. \end{aligned}$$

The orthogonality of these bases can be checked by:

$$\begin{aligned} \left\| \, |+\rangle \, \right\|^2 &= \langle + | + \rangle \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= 1 \\ \implies \left\| \, |+\rangle \, \right\|^2 &= 1. \end{aligned}$$

### 2.2.2 Unitary

An operator $U$ is called a unitary if $U^\dagger = U^{-1}$, where $U^{-1}$ is the inverse of $U$, such that $U \cdot U^\dagger = I$. Where $I$ is the identity operator.

### 2.2.3 Hermitian

An operator $T$ in $\mathcal{H}$ is the self-joint or Hermitian operator if $T^\dagger = T$, that is it is equal to its own Hermitian conjugate.

## 2.3 Applying Operators (Example)

Consider an operator $\mathbf{X}$ which acts on the basis state $|0\rangle$. The matrix representation of the $\mathbf{X}$ operator is given by:

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then,

$$\mathbf{X}\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |1\rangle$$

# 3 Further Introductory Notes

The field of quantum computing, and its applications, leverage knowledge from a multitude of disciplines. However, the fundamentals rely heavily on linear algebra and the specifics of its notation and nomenclature. As further study in preparation for the summer school, please familiarise yourself with the content in the Qiskit Textbook, Chapter 0.2 (`https://community.qiskit.org/textbook/ch-prerequisites/linear_algebra.html`) [1].

## References

[1] Abraham Asfaw, Luciano Bello, Yael Ben-Haim, Sergey Bravyi, Lauren Capelluto, Almudena Carrera Vazquez, Jack Ceroni, Jay Gambetta, Shelly Garion, Leron Gil, Salvador De La Puente Gonzalez, David McKay, Zlatko Minev, Paul Nation, Anna Phan, Arthur Rattew, Javad Shabani, John Smolin, Kristan Temme, Madeleine Tod, and James Wootton. Learn quantum computation using qiskit, 2019.

[2] Chris Fisher. Learn more about quantum computing fundamentals, Apr 2009.

[3] Michele Mosca, Raymond Laflamme, and Phillip Kaye. *Introduction to Quantum Computing.* Oxford University Press, 2007.

[4] Noson S Yanofsky and Mirco A Mannucci. *Quantum computing for computer scientists.* Cambridge University Press, 2008.