Orthogonal Functions

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DEFINITION 1

Inner Product of Function

The ***inner product*** of two functions f_1 and f_2 on an interval [a, b] is the number

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x) dx$$

DEFINITION 2

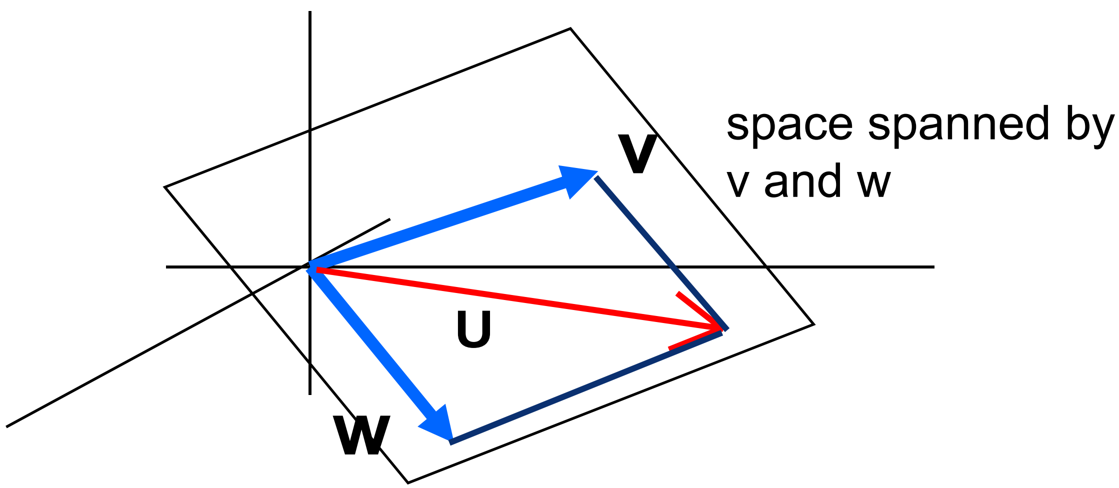
Orthogonal Function

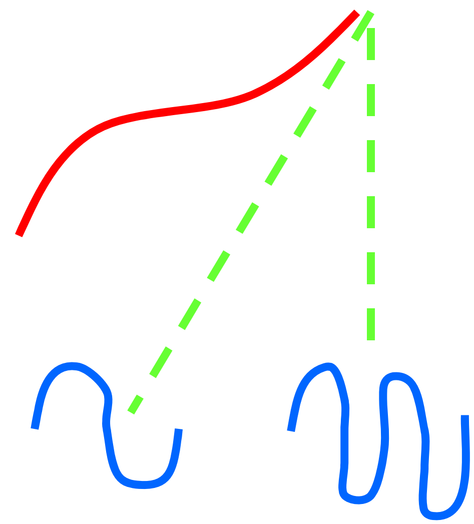
Two functions f_1 and f_2 are said to be ***orthogonal*** on an interval [a, b] if

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx = 0$$

Fourier Series

Conceptual analogy





$$\boxed{\mathbf{u} \approx < \mathbf{u}, \mathbf{v} > \mathbf{v} + < \mathbf{u}, \mathbf{w} > \mathbf{w}}$$

$$f(t) \approx c_1 e^{j\omega_0 t} + c_2 e^{j2\omega_0 t} \, where$$

$$c_k = < f(t), e^{jk\omega_0 t} > = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega_0 t} dt$$

Example:1. Show that $f(x) = x$ and $g(x) = x^2$ are orthogonal on $[-1, 1]$

Solution:

$$< f, g > \; = \int_{-1}^{1} x.x^2 dx = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0.$$

$as < f, g > = 0 \quad \Rightarrow \quad f(x) = x \;\; and \;\; g(x) = x^2 \quad are$ orthogonal on $[-1,1]$

Example:2. Find value of c for which $f(x)=x$ and $g(x)=2+3cx$ are orthogonal on $[0,1]$.

Solution:

$$\boxed{\text{Inner product } <f,g> = \int_a^b f(x)\,g(x)\,dx}$$

$$<f,g> = \int_a^b f(x)g(x)dx$$

$$= \int_0^1 \left(2x + 3cx^2\right)dx$$

$$= x^2 + cx^3\Big]_0^1 = 0$$

$$1 + c = 0 \qquad \Rightarrow \qquad c = -1$$

$f(x) = x,\ g(x) = 2\text{-}3x$ are Orthogonal function on $[0,1]$

DEFINITION 3

Inner Product of Function

A set of real-valued functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x), \phi_n(x)\, dx = 0, \quad m \neq n$$

Orthonormal Sets

- The expression ($\mathbf{u}$, $\mathbf{u}$) = $||\mathbf{u}||^2$ is called the ***square norm***. Thus we can define the square norm of a function as

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2 dx, \qquad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x) dx}$$

- If $\{\phi_n(x)\}$ is an orthogonal set on [a, b] with the property that $||\phi_n(x)|| = 1$ for all n, then it is called an ***orthonormal set*** on [a, b].

Set of orthogonal function

A set of real valued functions $f_1(x), f_2(x), \cdots$ on interval [a, b] is orthogonal if

$$< f_m, f_n > = \int_a^b f_m(x) f_n(x) dx = 0, \qquad \text{if } m \neq n$$

Set of Ortho-normal Function

1 $< f_m, f_n > = 0$

2 . $\| f_n \| = 1$

Norm of function

$$\| f_n \| = \sqrt{(f_n, f_n)}$$

$$< f_n, f_n > = \int_a^b f_n^{\,2}(x) dx$$

$$\| f_n(x) \| = \sqrt{(f_n, f_n)} = \left[\int_a^b f_n^{\,2}(x) dx \right]^{\frac{1}{2}}$$

Example 3 Show that the set {1, cos x, cos 2x, …} is orthogonal on [−π, π].

Solution

Let $\phi_0(x)$ = 1, $\phi_n(x)$ = cos nx, we show that

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos nx dx$$

$$= \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = 0, \; for \; n \neq 0$$

and $(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos mx \cdot \cos nx dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]\Bigg|_{-\pi}^{\pi} = 0, m \neq n$$

Example 4

Find the norms of each functions in Example 3.

Solution

$$\phi_0 = 1, \|\phi_0\|^2 = \int_{-\pi}^{\pi} dx = 2\pi \Rightarrow \|\phi_0\| = \sqrt{2\pi}$$

$$\phi_n = \cos nx,$$

$$\| \phi_n \|^2 = \int_{-\pi}^{\pi} \cos^2 nx dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx) dx = \pi$$

$$\| \phi_n \| = \sqrt{\pi}, n > 0$$

Vector Analogy

- Recalling from the vectors in 3-space that

  we have $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,$

  Thus we can make an analogy between vectors and functions.

$$\mathbf{u} = \frac{(\mathbf{u},\ \mathbf{v}_1)}{\|\ \mathbf{v}_1\ \|^2}\mathbf{v}_1 + \frac{(\mathbf{u},\ \mathbf{v}_2)}{\|\ \mathbf{v}_2\ \|^2}\mathbf{v}_2 + \frac{(\mathbf{u},\ \mathbf{v}_3)}{\|\ \mathbf{v}_3\ \|^2}\mathbf{v}_3 = \sum_{n=1}^{3}\frac{(\mathbf{u},\ \mathbf{v}_n)}{\|\ \mathbf{v}_n\ \|^2}\mathbf{v}_n$$

Orthogonal Series Expansion

- Suppose $\{\phi_n(x)\}$ is an orthogonal set on $[a, b]$. If $f(x)$ is defined on $[a, b]$, we first write as

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ?$$

- Then

$$\int_a^b f(x)\phi_m(x)\,dx$$

$$= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots$$

$$+ c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots$$

$$= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots$$

- Since $\{\phi_n(x)\}$ is an orthogonal set on $[a, b]$, each term on the right-hand side is zero except $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)dx = c_n(\phi_n, \phi_n) = c_n \int_a^b \phi_n^{\ 2}(x)dx$$

$$\boxed{c_n = \frac{\int_a^b f(x)\phi_n(x)dx}{\int_a^b \phi_n^{\ 2}(x)dx},} \quad n = 0,1,2,\ldots$$

In other words, $$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x),$$

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$$

Then (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x)$$

DEFINITION 4 **Orthogonal Set/Weight Function**

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on $[a, b]$, if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, m \neq n$$

- Under the condition of the above definition, we have

$$c_n = \frac{\int_a^b f(x)w(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}$$

$$\|\phi_n(x)\|^2 = \int_a^b w(x)\phi_n^2(x)\,dx$$

Generalized Fourier Series Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We ask: If $y = f(x)$ is a function defined on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$, for which $f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots$?

$$\int_a^b f(x)\phi_m(x)\,dx = c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots$$
$$= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots.$$

By orthogonality each term on the right-hand side of the last equation is zero *except* when $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients are

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 0, 1, 2, \ldots.$$

In other words, $$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x),$$

where $$c_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\|\phi_n(x)\|^2}.$$

With inner product notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}\,\phi_n(x).$$

Important Results

1. $$\int_{-\pi}^{\pi} \cos nx dx = \frac{\sin nx}{n}\int_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = \frac{1}{n}[0-0] = 0$$

$$\sin n\pi = 0 \qquad \sin(\text{-}\,\mathrm{n}\pi) = -\sin n\pi = 0$$

2. $$\int_{-\pi}^{\pi} \sin nx dx = -\frac{\cos nx}{n}\int_{-\pi}^{\pi} = \frac{1}{n}[\cos n\pi - \cos(-n\pi)] = -\frac{1}{\mathrm{n}}[\cos n\pi - \cos n\pi] = 0$$

$$\cos(-\theta) = \cos\theta$$

$$\cos n\pi = (-1)^n$$

$$\cos(-n\pi) = \cos n\pi = (-1)^n$$

$$\int_{0}^{\pi} \sin nx dx = -\frac{\cos nx}{n}\int_{0}^{\pi} = -\frac{1}{n}[\cos n\pi - \cos 0] = -\frac{1}{n}\left[(-1)^n - 1\right]$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$\int x \sin nx dx = -x \frac{\cos nx}{n} + \frac{\sin nx}{n^2}$$

$$\int x \cos nx dx = x \frac{\sin nx}{n} + \frac{\cos nx}{n^2}$$

$$\int x^2 \sin nx dx = -x^2 \frac{\cos nx}{n} + 2x \frac{\sin nx}{n^2} - 2 \frac{\cos nx}{n^3}$$

$$\int x^2 \cos nx dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3}$$

$$\sin a \ \ \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\cos \ a \ \ \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin \ a \ \ \ \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$$

$$\cos \ a \ \ \ \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$$

Example:

If $f(x) = x$ and $g(x) = x^2$ are orthogonal function on[-2,2]. Find c_1 and c_2 such that $h(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f and g on [-2,2]. Also find norms of $f(x)$ and $g(x)$

Solution:

If f and g are orthogonal to h, then

$$< f, h > = \int_{-2}^{2} x\left(x + c_1x^2 + c_2x^3\right)dx = \int_{-2}^{2}\left(x^2 + c_1x^3 + c_2x^4\right)dx = 0$$

$$= \frac{x^3}{3} + c_1\frac{x^4}{4} + c_2\frac{x^5}{5}\Big|_{-2}^{2} = 0,$$

$$= \frac{8}{3} + \frac{8}{3} + c_1\frac{16}{4} - c_1\frac{16}{4} + c_2\frac{32}{5} + c_2\frac{32}{5} = 0,$$

$$\frac{16}{3} + \frac{64}{5}c_2 = 0,$$

$$c_2 = -\frac{16}{3} + \frac{5}{64} = -\frac{5}{12} \Rightarrow c_2 = -\frac{5}{12}$$

$$< g, h > = \int_{-2}^{2} x^2\left(x + c_1x^2 + c_2x^3\right)dx = \int_{-2}^{2} \left(x^3 + c_1x^4 + c_2x^5\right)dx = 0$$

$$= \frac{x^4}{4} + c_1\frac{x^5}{5} + c_2\frac{x^6}{6}\Big|_{-2}^{2} = 0$$

$$\frac{16}{4} - \frac{16}{4} + c_1\frac{32}{5} + c_1\frac{32}{5} + c_2\frac{64}{6} - c_2\frac{64}{6} = 0$$

$$\frac{64}{5}c_1 = 0 \quad \Rightarrow \quad c_1 = 0.$$

$h(x) = x - \frac{5}{12}x^3$ is orthogonal to both $f(x) = x$ and $g(x) = x^2$

<u>NORMS.</u> $$\|f(x)\|^2 = \int_{-2}^{2} (f(x))^2 dx = \int_{-2}^{2} x^2 dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{8}{3} - \left(\frac{-8}{3}\right) = \frac{16}{3}$$

$$\|f(x)\| = \frac{4}{\sqrt{3}}$$

$$\|g(x)\|^2 = \int_{-2}^{2} (g(x))^2 dx = \int_{-2}^{2} x^4 dx = \frac{x^5}{5}\Big|_{-2}^{2} = \frac{32}{5} - \left(\frac{-32}{5}\right) = \frac{64}{5}$$

$$\|g(x)\| = \frac{8}{\sqrt{5}}$$

Example:

Show that $f(x) = \mathrm{e}^{2x}$ and $g(x) = \mathrm{e}^{-2x}(x-1)$ are orthogonal on $[0,2]$.

Solution:

$$< f, g > \;= \int_0^2 e^{2x}.e^{-2x}(x-1)dx$$

$$= \int_0^2 (x-1)dx$$

$$= \frac{x^2}{2} - x \Big|_0^2 = \frac{4}{2} - 2 = 2 - 2 = 0$$

f and g are orthogonal on $[0,2]$

Example:

Show that $f(x) = \cos x$ and $g(x) = \sin^2 x$ are orthogonl on $[0,\pi]$

Solution:

$$< f, g > = \int_0^{\pi} \cos x . \sin^2 x \, dx$$

$$\boxed{\begin{array}{l} u = \sin x, du = \cos x dx \\ \int u^2 du = \dfrac{u^3}{3} = \dfrac{\sin^3 x}{3} \end{array}}$$

$$= \frac{\sin^3 x}{3} \Big|_0^{\pi}$$

$$= \frac{(\sin \pi)^3}{3} - \frac{(\sin 0)^3}{3} = 0 - 0 = 0$$

$f(x) = \cos x, g(x) = \sin^2 x$ are orthogonal on $[0,\pi]$

Example:

Find whether the functions $f(x) = e^{-x}, g(x) = \cos x$

are orthogonal on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Solution:

$$< f, g > = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^{-x} \cos x dx = \frac{e^{-x}}{1+1}(-\cos x + \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{e^{-\frac{5\pi}{4}}}{2}\left[+\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \frac{e^{-\frac{\pi}{4}}}{2}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 0 - 0 = 0$$

f and g are orthogonal on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Note : $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\cos\frac{5\pi}{4} = \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

Note: **For Ortho-normal set**

$$\text{(i)} < f_n, f_m > \; = 0. \quad (ii) \; \|f_n(x)\| = 1.$$

Note

$$\{f_n(x)\}, \quad n = 1,2,3,\ldots \quad \text{is orthogonal set on } [\mathrm{a}, \mathrm{b}] \; , if$$

$$< f_n, f_m > \; = \int_a^b f_n(x).f_m(x)dx = 0 \qquad , n \neq m$$

Norm let $\mathrm{n} = \mathrm{m}$

$$\|f_n\|^2 = \int_a^b (f_n(x))^2 dx$$

Example: Show that

sin x, sin 2x, sin 3x, ...is a set of orthogonal functions on $[0,\pi]$. Find the norm.

Solution.

$$f(x) = \{\sin nx\}, \quad \boldsymbol{n} = 1,2,3,\cdots$$

$$n \neq m \qquad (\sin nx, \sin mx) = \int_0^{\pi} \sin nx \sin mx dx$$

$$= \frac{1}{2}\int_0^{\pi}\left[\cos(n-m)x - \cos(n+m)x\right]dx$$

$$= \frac{1}{2}\left(\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right|_0^{\pi} = 0$$

$\{\sin nx\}$ is set of orthogonal function

$$\|\sin nx\|^2 = \int_0^{\pi} \sin^2 nx dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2nx) dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 2nx}{2n} \right)\Bigg|_0^{\pi} = \frac{\pi}{2}$$

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}$$

$$\boxed{\begin{array}{ll} \underline{\textit{Note}} & \text{Sin(integer) } \pi = 0 \\ & \text{Sin}0 = 0 \end{array}}$$

Example:

Show that $\{1, \cos\frac{n\pi x}{L}\}, n = 1, 2, 3, \cdots$ is a set of orthogonal functions on $[0,L]$. Find the norms.

Solution:

$$\{1, \; f_n(x)\}, \quad n = 1,2,3,\cdots$$

$$f_n(x) = \cos\frac{n\pi x}{L}$$

To discuss orthogonality, we will find inner products

$$<1, f_n>, <f_n, f_m>$$

To calculate norm, we will find $\|1\|$ and $\|f_n(x)\|$

$$\left(f_n(x), f_m(x)\right) = \int_0^L \cos\frac{n\pi x}{L}.\cos\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_0^L \left[\cos\frac{(n+m)\pi}{L}x + \cos\frac{(n-m)\pi}{L}x\right]dx$$

$$= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi}{L}x + \frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi}{L}x\right|_0^L$$

$$= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\left(\sin(n+m)\pi - \sin 0\right) + \frac{L}{(n-m)\pi}\left(\sin(n-m)\pi - \sin 0\right)\right]$$

$$= \frac{1}{2}(0+0) = 0$$

$(1, f_n)$ and (f_n, f_m) are orthogonal, hence $\left\{1, \cos\frac{n\pi}{L}x\right\}$ is orthogonal on $[0, L]$

Norms

$$\|1\|^2 = \int_0^L 1^2\,dx = \int_0^L dx = x\Big|_0^L = L, \qquad \|1\| = \sqrt{L}$$

$$\|f_n(x)\|^2 = (f_n(x), f_n(x)) = \int_0^L \cos^2\frac{n\pi}{L}x\,dx = \frac{1}{2}\int_0^L\left[1+\cos\frac{2n\pi}{L}x\right]dx$$

$$= \frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\frac{2n\pi}{L}x\Big|_0^L\right.$$

$$= \frac{L}{2}$$

$$\|f(x)\| = \sqrt{\frac{L}{2}}$$

Example: Show that the set of function

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots\right\}$$

is an orthonornal set on $[-\pi, \pi]$

Solution:

We will prove that the inner products $\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right), \left(\frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}\right), \left(\frac{\cos nx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}\right)$

$\left(\frac{\sin nx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}\right), \left(\frac{\sin nx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}\right)$ are all zero.

and to regarding norms we will prove that $\left\|\frac{1}{\sqrt{2\pi}}\right\| = 1, \left\|\frac{\cos nx}{\sqrt{\pi}}\right\| = 1, \left\|\frac{\sin x}{\sqrt{\pi}}\right\| = 1$

1. $$\left(\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right) = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos nx}{\sqrt{\pi}} dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{n} \frac{\sin nx}{\sqrt{\pi}} \Big|_{-\pi}^{\pi} = 0$$

2. $$\left(\frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}\right) = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin nx}{\sqrt{\pi}} dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{n} \frac{\cos nx}{\pi}\Big|_{-\pi}^{\pi} = 0$$

3. $$\left(\frac{\cos nx}{\pi}, \frac{\cos mx}{\pi}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx.\cos mx dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(n-m)x + \cos(n+m)x\right]dx = 0$$

4. $$\left(\frac{\sin nx}{\sqrt{\pi},}, \frac{\sin mx}{\sqrt{\pi},}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx.\sin mx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\cos(n-m)x - \cos(n+m)x\right]dx = 0$$

5. $$\left(\frac{\sin nx}{\sqrt{\pi},}, \frac{\cos mx}{\sqrt{\pi},}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \cos .mx dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sin(n-m)x + \sin(n+m)x\right]dx = 0$$

$$\left\| \frac{1}{\sqrt{2\pi}} \right\|^2 = \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2\pi}} \right)^2 dx = \frac{1}{2\pi} x \Big|_{-\pi}^{\pi} = \frac{1}{2\pi} [\pi - (-\pi)] = \frac{1}{2\pi} .2\pi = 1$$

$$\left\| \frac{\cos nx}{\sqrt{\pi}} \right\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 nx dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 + \cos 2nx] dx$$

$$= \frac{1}{2\pi} \left[x + \frac{\sin 2nx}{2n} \Big|_{-\pi}^{\pi} \right. = \frac{1}{2\pi} .2\pi = 1$$

$$\left\| \frac{\sin nx}{\sqrt{\pi}} \right\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 nx dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} [1 - \cos 2nx] dx$$

$$= \frac{1}{2\pi} \left[x + \frac{\sin 2nx}{2n} \Big|_{-\pi}^{\pi} \right. = \frac{1}{2\pi} .2\pi = 1$$

$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\text{cosnx}}{\sqrt{\pi}}, \frac{\text{sinnx}}{\sqrt{\pi}} \right\}$, $\quad n = 1, 2, 3, \cdots$ is ortho-normal set.

Fourier Series

The Fourier Series

Jean Baptiste Joseph Fourier
1768 to 1830
French Mathematician and Physicist

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

The Fourier Series

Definition

$$N \to \infty$$

A ***Fourier Series*** is an accurate representation of a periodic signal and consists of the sum of sinusoids at the fundamental and harmonic frequencies. One of the advantages of a Fourier representation over some other representation, such as a Taylor series, is that it may represent a discontinuous function, for example the saw-tooth wave . The waveform *f(t)* depends on the ***amplitude*** and ***phase*** of every harmonic components, and we can generate any non-sinusoidal waveform by an appropriate combination of sinusoidal functions.

To be described by the Fourier Series the waveform *f(t)* must satisfy the following mathematical properties (The Dirichlet conditions):

1. *f(t)* is a ***single-value function*** except at possibly a finite number of points.
2. The integral $\int_{t_0}^{t_0+T} |f(t)|\, dt < \infty$ for any t_0.
3. *f(t)* has a finite number of ***discontinuities*** within the period *T*.
4. *f(t)* has a finite number of ***maxima*** and ***minima*** within the period *T*.

THEOREM 1 **Criterion for Convergence**

Let f and f' be piecewise continuous on the interval $(-p, p)$; That is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuous at these points. Then the Fourier series of f on the interval converge to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

The Fourier Series of a function

Let $f(x)$ is periodic fuction of period $2T$ on the interval $-T \le x \le T$,
then $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T} \right) \rightarrow 1$$

where the fourier coeficients a_0 , a_n and b_n are

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx \qquad \rightarrow 2$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x\,dx \qquad \rightarrow 3$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x\,dx \qquad \rightarrow 4 \qquad n = 1,2,3,\cdots$$

Note.1. The Fourier series **(1)** converges to $f(x)$ at all points where $f(x)$ is continuous.

Note.2. The Fourier series **(1)** converges to

$\frac{1}{2}\left[f(x+0)+f(x-0)\right]$ at all points of jump discontinity of *f(x)*

Note.3. Periodic Function : Any function which repeats itself after a certain period is known as perdiodic function.

Examples :

$$\sin(\mathrm{x}+2\pi)=\sin x \qquad \text{, Period } 2\pi$$

$$\cos(\mathrm{x}+2\pi)=\cos x \qquad \text{, Period } 2\pi$$

$$f(x+2T)=f(x) \qquad \text{, Period } 2T$$

$$\sin(\mathrm{x}+2\mathrm{n}\pi)=\sin x \qquad \text{, Period } 2\pi$$

$$f(x)=f(x+P)=f(x+2P)=f(x+3P)=\cdots, \qquad \text{Period P.}$$

Special Case

If $f(x)$ is a periodic function of period 2π on interval $[-\pi, \pi]$, the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rightarrow 5$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \qquad \rightarrow 6$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \qquad \rightarrow 7$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \qquad \rightarrow 8 \qquad n = 1, 2, 3, \cdots$$

Fourier Series of Periodic Functions

A function $f(t)$ is said to be periodic with period T if there is a number $T > 0$ such that

$$f(t+T) = f(t)$$

for all t. Every integer multiple of the period is also a period:

$$f(t+nT) = f(t)\,, \quad n = 0, \pm 1, \pm 2, \ldots$$

Consider the function

$$f(t) = \cos(2\pi t) + \frac{1}{2}\cos(4\pi t)$$

The individual terms are periodic with periods 1 and ½, respectively, but the sum is periodic with period 1.

Fourier Series of Periodic Functions

$$
\begin{aligned}
f(t+1) &= \cos(2\pi(t+1)) + \frac{1}{2}\cos(4\pi(t+1)) \\
&= \cos(2\pi t + 2\pi) + \frac{1}{2}\cos(4\pi t + 4\pi) \\
&= \cos(2\pi t)\cos(2\pi) - \sin(2\pi t)\sin(2\pi) \\
&\quad + \frac{1}{2}[\cos(4\pi t)\cos(4\pi) - \sin(4\pi t)\sin(4\pi)] \\
&= \cos(2\pi t) + \frac{1}{2}\cos(4\pi t) \\
&= f(t)
\end{aligned}
$$

Fourier Series of Periodic Functions

Suppose that $f(t)$ is a function defined on the real line such that $f(t+T) = f(t)$ for all *t*. Such functions are said to be periodic with period *T*, or *T*-periodic. A continuous **Fourier series** of a function with period *T* can be written:

$$f(t) = a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t) + a_2 \cos(2\omega t) + \\ b_2 \sin(2\omega t) + a_3 \cos(3\omega t) + b_3 \sin(3\omega t) + \dots$$

or more concisely,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

where $\omega = \dfrac{2\pi}{T}$ is called the fundamental frequency and its constant multiples $2\omega, 3\omega,$ etc are called harmonics.

Fourier Series of Periodic Functions

Recall the formulas

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Hence

$$\cos(n\omega t) = \frac{e^{in\omega t} + e^{-in\omega t}}{2} \text{ and } \sin(n\omega t) = \frac{e^{in\omega t} - e^{-in\omega t}}{2i}$$

Also the formulas

$$e^{in\omega t} = \cos(n\omega t) + i\sin(n\omega t)$$

$$e^{-in\omega t} = \cos(n\omega t) - i\sin(n\omega t)$$

Fourier Series of Periodic Functions

The series in (13.21) can be rewritten as

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{in\omega t}+e^{-in\omega t}}{2}\right)+b_n\left(\frac{e^{in\omega t}-e^{-in\omega t}}{2i}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{in\omega t}\left(\frac{a_n}{2}+\frac{b_n}{2i}\right)+e^{-in\omega t}\left(\frac{a_n}{2}-\frac{b_n}{2i}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{in\omega t}\left(\frac{a_n - ib_n}{2}\right)+e^{-in\omega t}\left(\frac{a_n + ib_n}{2}\right)\right]$$

Now, letting

$$\boxed{c_0 = a_0\,, \quad c_n = \frac{a_n - ib_n}{2}\,, \quad c_{-n} = \frac{a_n + ib_n}{2}}$$

Fourier Series of Periodic Functions

The expression for $f(t)$ becomes

$$\boxed{f(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\omega t} + c_{-n} e^{-in\omega t}\right)}$$

The coefficients c_n are complex numbers and they satisfy

$$c_{-n} = \overline{c_n}$$

Notice that when $n = 0$ we have $c_0 = \overline{c_0}$ which implies that c_0 is a real number.

For any value of n, the magnitudes of c_n and c_{-n} are equal:

$$|c_n| = |c_{-n}|$$

Fourier Series of Periodic Functions

Then the series

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

can be written as

$$\boxed{f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega t}}$$

where, they are both called the **Fourier series** of *f*.

Fourier Series of Periodic Functions

Hence, the only term in the series that survives the integration is the term with $n = k$, and we obtain

$$\int_0^T f(t)e^{-ik\omega t}dt = c_n T$$

In other words, relabeling the integer k as n, we have the formula for the coefficients c_n:

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt}$$

which are the coefficients for the series It is then straightforward to find the coefficients

Example 0

Expand $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

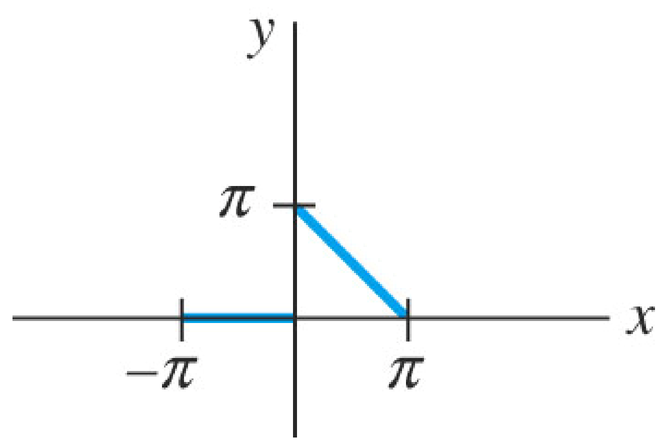
in a Fourier series.

Solution

The graph of f is shown in the Figure with $p = \pi$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 dx + \int_{0}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\sin nx\, dx\right]$$

$$= -\frac{1}{n\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi}$$

$$= \frac{-\cos n\pi + 1}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$ ←cos $n\pi = (-1)^n$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx dx = \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}$$

Example: 1.

Find the Fourier series of the function

$$f(x) = x \quad , \qquad -\pi < x < \pi \quad , \qquad f(x+2\pi) = f(x)$$

Solution:

$$\text{Fouries series is } \; \mathrm{f(x)} = \frac{\mathrm{a}_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dn = \frac{1}{\pi}.\frac{x^2}{2}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = \frac{1}{\pi}.\left[x\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}.\left[\pi\frac{\sin n\pi}{n} - \pi\frac{\sin(-n\pi)}{n}\right]$$

$$+ \frac{1}{\pi}.\left[\frac{\cos n\pi}{n^2} - \frac{\cos(-n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}[0-0] + \frac{1}{\pi}[0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx = \frac{1}{\pi}.\left[-x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}.\left[-x\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + 0$$

$$b_n = \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}\right] = \frac{-2}{n}(-1)^n$$

Substituting values of a_0, a_n and b_n

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx, \text{ is Fourier series of } \quad f(x) = x, \ -\pi \leq x \leq \pi$$

Example:2

Find the Fourier senie of the function

$$f(x) = x^2 \qquad , \ -1 \le x \le 1 \ , \ f(x+2) = f(x)$$

Solution:

$$2T = 2 \Rightarrow T = 1$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x)$$

$$\text{where} \quad a_0 = \int_{-1}^{1} x^2 dx$$

$$a_n = \int_{-1}^{1} x^2 \cos n\pi x \, dx$$

$$b_n = \int_{-1}^{1} x^2 \sin n\pi x dx$$

$$a_0 = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$a_n = \int_{-1}^{1} x^2 \cos n\pi x dx = \left[x^2 \frac{\sin n\pi x}{n\pi} + 2x \frac{\cos n\pi x}{n^2\pi^2} - 2\frac{\sin n\pi x}{n^3\pi^3}\right|_{-1}^{1}$$

$$= \frac{4\cos \mathrm{n}\pi}{\mathrm{n}^2\pi^2} = \frac{4(-1)^{\mathrm{n}}}{\mathrm{n}^2\pi^2}, \qquad n = 1,2,3$$

$$b_n = \int_{-1}^{1} x^2 \sin n\pi x dx = \left[-x^2 \frac{\cos n\pi x}{n\pi} + 2x \frac{\sin n\pi x}{n^2\pi^2} + 2\frac{\cos n\pi x}{n^3\pi^3}\right|_{-1}^{1} = 0$$

The Fouries series of $f(x) = x^2, \quad -1 < x < 1$ *is*

$$f(x) \approx \frac{1}{3} + \sum_{\mathrm{n}=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x$$

$$\approx \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

Example:3.

Find the Fourier series for the square wave

$$f(x)=\begin{cases} 0, & -\pi < x < 0 \\ 1\,, & 0 \le x < \pi \end{cases}$$

Solution:

$$f(x) \approx \frac{a_\circ}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} 1.dx + = \right] \frac{1}{\pi}\int_{0}^{\pi} dx = \frac{1}{\pi} x\Big|_0^{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos xdx = \frac{1}{\pi}\int_{0}^{\pi} \cos nxdx = \frac{1}{\pi}\frac{\sin nx}{n}\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx = -\frac{1}{\pi}\frac{\cos nx}{n}\Big|_0^{\pi}$$

$$= \frac{-\left[(-1)^n - 1\right]}{n\pi}$$

The Fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1\ , & 0 \le x < \pi \end{cases} \qquad n = 1, 2, 3, \cdots$

is $\qquad f(x) \approx \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-\left[(-1)^n - 1\right]}{n\pi} \sin nx$

$$\approx \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n\pi} \sin nx$$

Example: 4.

Find the Fourier series of

$$f(x) = e^{-x} \quad , \ -\pi < x < \pi$$

Solution:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\,dx = \frac{1}{\pi}(-e^{-x})\Big|_{-\pi}^{\pi} = \frac{1}{\pi}[-e^{-\pi} + e^{\pi}]$$

$$a_0 = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-\pi}}{1+n^2}\left(-\cos n\pi + n\sin n\pi\right) - \frac{e^{\pi}}{1+n^2}\left(-\cos\left(-n\pi\right) + n\sin\left(-n\pi\right)\right)\right]$$

$$= \frac{1}{\pi\left(1+n^2\right)}\left[e^{-\pi}.\left[-\left(-1\right)^n\right] - e^{\pi}\left(-\left(-1\right)^n\right)\right]$$

$$= \frac{\left(-1\right)^n}{\pi\left(1+n^2\right)}\left[e^{\pi} - e^{-\pi}\right] \qquad \mathrm{n} = 1,2,3,\cdots$$

$$\mathrm{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-\pi} \sin nx dx = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx + n\cos nx\right)\Big|_{-\pi}^{\pi} \right.$$

$$= \frac{-n}{\pi\left(1+n\right)^2}\left[e^{-x} \cos nx \Big|_{-\pi}^{\pi} \right.$$

$$= \frac{-n}{\pi\left(1+n^2\right|}\left[e^{-\pi} \cos n\pi - e^{\pi} \cos\left(-n\pi\right) \right]$$

$$= \frac{-n}{\pi\left(1+n^2 \right)}\left[e^{-\pi} \left(-1\right)^n - e^{\pi} \left(-1\right)^n \right]$$

$$= \frac{n\left(-1\right)^n}{\pi\left(1+n^2\right)}\left[e^{\pi} - e^{-\pi} \right] \qquad n = 1, 2, 3, \cdots$$

$$f\left(x\right) \approx \frac{1}{2}\left[\frac{e^{\pi} - e^{-\pi}}{\pi} \right]$$

$$+ \sum_{n=1}^{\infty}\left[\frac{\left(-1\right)^n}{\pi\left(1+n^2\right)}\left[e^{\pi} - e^{-\pi} \right]\cos nx + \frac{n\left(-1\right)^n}{\pi\left(1+n^2\right)}\left[e^{\pi} - e^{-\pi} \right]\sin nx \right]$$

Even and Odd functions

Even function

A function $f(x)$ is an even function, if $f(-x) = f(x)$.

Odd function

A function $f(x)$ is an odd function, if $f(-x) = -f(x)$.

Fourier Series of Periodic Functions

A useful observation is

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ is even} \\ 0 & \text{if } F \text{ is odd} \end{cases}$$

Function F is **even** if $F(-x) = F(x)$ and **odd** if $F(-x) = -F(x)$.

Since $\cos(n\omega t)$ is even and $\sin(n\omega t)$ is odd, we have:

if f is even: $a_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega t)\,dt \quad \text{and} \quad b_n = 0$

if f is odd: $a_n = 0 \quad \text{and} \quad b_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\omega t)\,dt$

Example

Use the (continuous) Fourier series to approximate the square or rectangular wave function

$$f(t) = \begin{cases} -1 & \text{for } -T/2 < t < -T/4 \\ 1 & \text{for } -T/4 < t < T/4 \\ -1 & \text{for } T/4 < t < T/2 \end{cases}$$

<u>Solution</u>:

We first calculate

$$a_0 = \frac{1}{T}\left(\int_{-T/2}^{-T/4} -dt + \int_{-T/4}^{T/4} dt - \int_{T/4}^{T/2} dt\right) = 0$$

Example

Then the coefficient a_n:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$= \frac{2}{T}\left[-\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt + \int_{-T/4}^{T/4}\cos(n\omega t)\,dt\right.$$

$$\left.-\int_{T/4}^{T/2}\cos(n\omega t)\,dt\right]$$

Solving term by term:

$$-\frac{2}{T}\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt = -\frac{2}{T}\cdot\frac{1}{n\omega}\Big[\sin(n\omega t)\Big]_{-T/2}^{-T/4}$$

Example

$$-\frac{2}{T}\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt = -\frac{2}{nT\omega}\left[\sin\left(-\frac{n\omega T}{4}\right) - \sin\left(-\frac{n\omega T}{2}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\sin\left(-\frac{n\pi}{2}\right) - \sin(-n\pi)\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

from which, the solutions for each sequence:

$n = 1 \rightarrow \frac{1}{\pi}$ $\quad n = 3 \rightarrow -\frac{1}{3\pi}$ $\quad n = 5 \rightarrow \frac{1}{5\pi}$ $\quad n = 7 \rightarrow -\frac{1}{7\pi}$

$n = 2 \rightarrow 0$ $\quad n = 4 \rightarrow 0$ $\quad n = 6 \rightarrow 0$ $\quad n = 8 \rightarrow 0$

Example

The middle term gives

$$\frac{2}{T}\int_{-T/4}^{T/4}\cos(n\omega t)\,dt = \frac{2}{nT\omega}\left[\sin\left(\frac{n\omega T}{4}\right) - \sin\left(-\frac{n\omega T}{4}\right)\right]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Checking the solutions for each n:

$$n = 1 \rightarrow \frac{2}{\pi} \qquad n = 3 \rightarrow -\frac{2}{3\pi} \qquad n = 5 \rightarrow \frac{2}{5\pi} \qquad n = 7 \rightarrow -\frac{2}{7\pi}$$

$$n = 2 \rightarrow 0 \qquad n = 4 \rightarrow 0 \qquad n = 6 \rightarrow 0 \qquad n = 8 \rightarrow 0$$

Example

And the last term gives

$$-\frac{2}{T}\int_{T/4}^{T/2}\cos(n\omega t)\,dt = -\frac{2}{nT\omega}\left[\sin\left(\frac{n\omega T}{2}\right) - \sin\left(\frac{n\omega T}{4}\right)\right]$$

$$= -\frac{1}{n\pi}\left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Checking the solutions for each n:

$n = 1 \rightarrow \frac{1}{\pi}$ $\quad n = 3 \rightarrow -\frac{1}{3\pi}$ $\quad n = 5 \rightarrow \frac{1}{5\pi}$ $\quad n = 7 \rightarrow -\frac{1}{7\pi}$

$n = 2 \rightarrow 0$ $\quad n = 4 \rightarrow 0$ $\quad n = 6 \rightarrow 0$ $\quad n = 8 \rightarrow 0$

Example

Collecting all solutions from all terms gives us

$$\text{for } n = 1 \longrightarrow \qquad a_1 = \frac{1}{\pi} + \frac{2}{\pi} + \frac{1}{\pi} = \frac{4}{\pi}$$

$$\text{for } n = 3 \longrightarrow \qquad a_3 = -\frac{1}{3\pi} - \frac{2}{3\pi} - \frac{1}{3\pi} = -\frac{4}{3\pi}$$

$$\text{for } n = 5 \longrightarrow \qquad a_5 = \frac{1}{5\pi} + \frac{2}{5\pi} + \frac{1}{5\pi} = \frac{4}{5\pi}$$

$$\text{for } n = 7 \longrightarrow \qquad a_7 = -\frac{1}{7\pi} - \frac{2}{7\pi} - \frac{1}{7\pi} = -\frac{4}{7\pi}$$

$$\text{for } n = 9 \longrightarrow \qquad a_9 = \frac{1}{9\pi} + \frac{2}{9\pi} + \frac{1}{9\pi} = \frac{4}{9\pi}$$

etc.

Example

In general

$$a_n = \begin{cases} \dfrac{4}{n\pi} & \text{for } n = 1, 5, 9, \dots \\ -\dfrac{4}{n\pi} & \text{for } n = 3, 7, 11, \dots \\ 0 & \text{for } n = \text{ even integers} \end{cases}$$

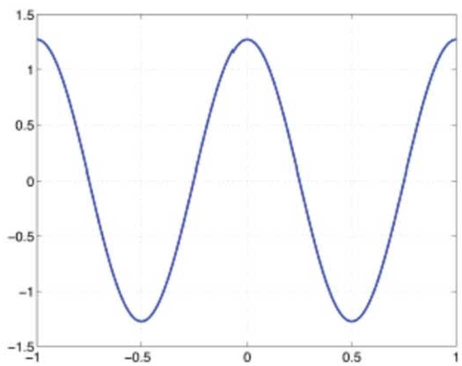
Similarly, it can be determined that all the b's = 0. Therefore the Fourier series approximation is

$$f(t) = \frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \frac{4}{5\pi}\cos(5\omega t) + \dots$$

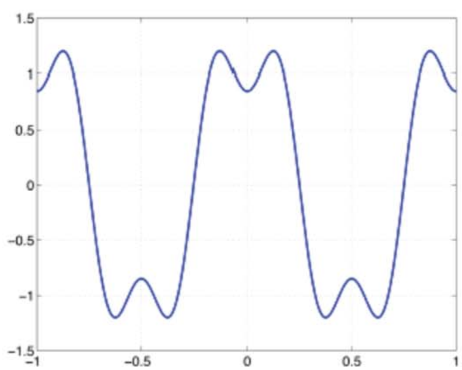
Example

$$f(t) = \frac{4}{\pi}\cos(\omega t) - \frac{4}{3\pi}\cos(3\omega t) + \frac{4}{5\pi}\cos(5\omega t) + \ldots$$

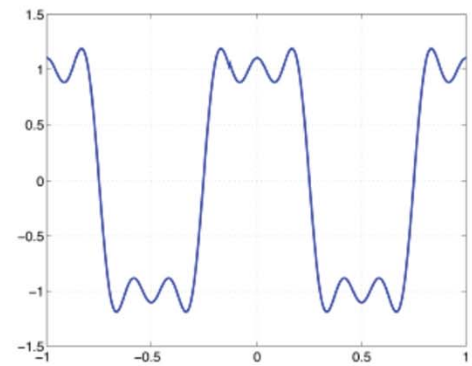
$n = 1$



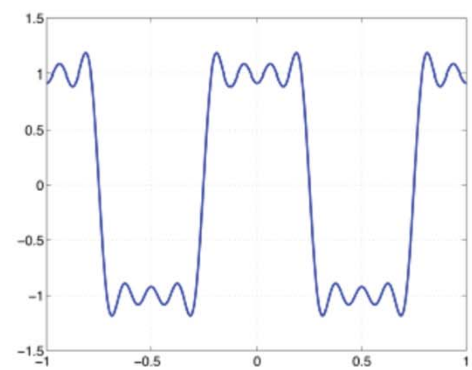
$n = 3$



$n = 5$



$n = 7$



Example: Test the nature of function

(i) $f(x) = x^2$

(ii) $f(x) = x^3$

(iii) $f(x) = \cos x$

Solution:

$(i) \quad f(x) = x^2$

$$f(-x) = (-x)^2 = x^2 \Rightarrow f(x) = x^2 \qquad \text{is even.}$$

$(ii) \quad f(x) = x^3$

$$f(-x) = (-x)^3 = -x^3 \Rightarrow f(x) = x^3 \qquad \text{is odd.}$$

$(iii) \quad f(x) = \cos x$

$$f(-x) = \cos(-x) = \cos x \Rightarrow f(x) = \cos x \qquad \text{is even.}$$

Function	*Classification*
x^2	**even**
x^3	**odd**
$\cos x$	**even**
$\sin x$	**odd**
$\lvert x \rvert$	**even**
$3x+2$	**neither**
$x + x^2 \sin x$	**odd**
$\sin x + \cos x$	**neither**

THEOREM 2 Properties of Even/Odd Functions

(a) The product of two even functions is even.

(b) The product of two odd functions is even.

(c) The product of an even function and an odd Function is odd.

(d) The sum (difference) of two even functions is even.

(e) The sum (difference) of two odd functions is odd.

(f) If *f* is even then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(g) If *f* is odd then $\int_{-a}^{a} f(x)dx = 0$

DEFINITION 6

Fourier Cosine and Sine Series

(i) The Fourier series of an even function *f* on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

DEFINITION 6

Fourier Cosine and Sine Series

(ii) The Fourier series of an odd function *f* on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{i=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

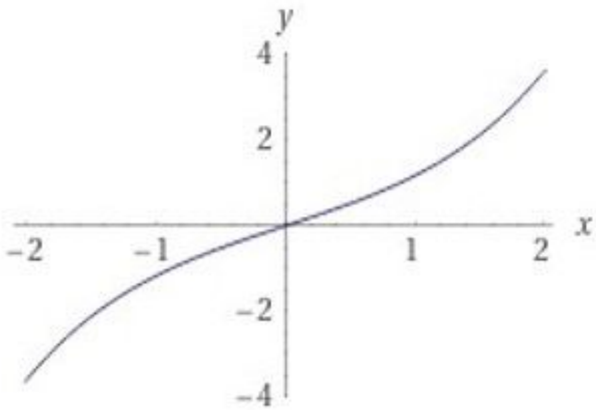
where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x \, dx$$

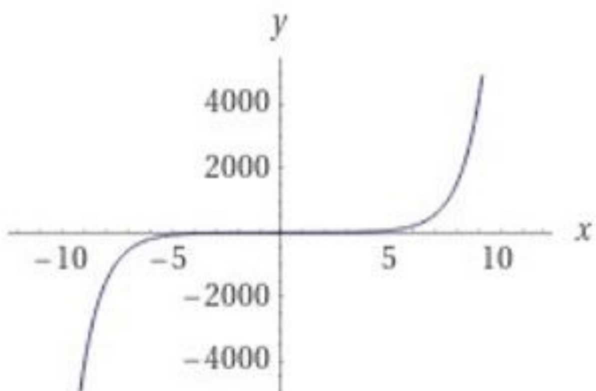
Input:

$\sinh(x)$

Plots:



(x from -2 to 2)



(x from -12 to 12)

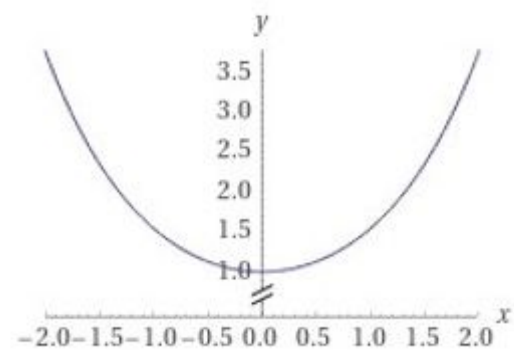
cosh(x)

Extended Keyboard   Upload

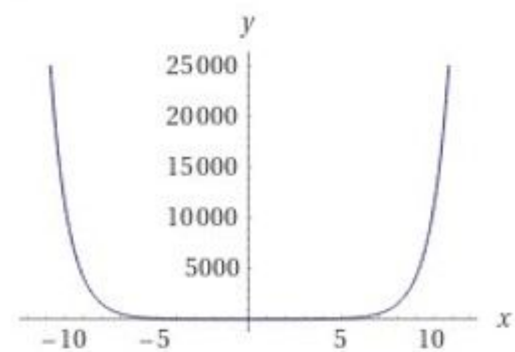
Input:

$\cosh(x)$

Plots:

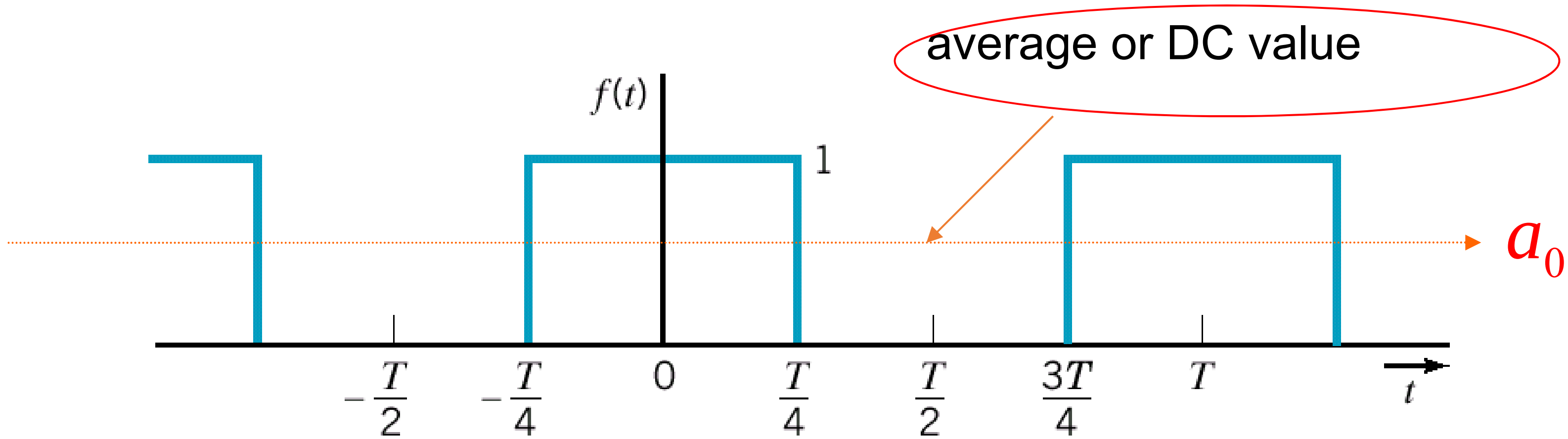


(x from -2 to 2)



(x from -12 to 12)

Example determine Fourier Series and plot for N = 7



$$a_0 = \frac{1}{2}$$

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\int_{-T/4}^{T/4} 1\,dt = \frac{1}{2}$$

An ***even function*** exhibits symmetry around the vertical axis at *t = 0* so that *f(t) = f(-t).*

$$\therefore \quad b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\, dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} 1 \sin n\omega_0 t\, dt = 0$$

Determine only a_n

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} 1 \cos n\omega_0 t\, dt$$

$$= \frac{2}{T\omega_0 n}\sin n\omega_0 t \Big|_{-T/4}^{T/4}$$

$$a_n = \frac{1}{\pi n}\left[\sin\left(\frac{\pi n}{2}\right) - \sin\left(\frac{-\pi n}{2}\right)\right]$$

$$a_n = 0 \quad \text{when} \quad n = 2,\ 4,\ 6,\ \ldots$$

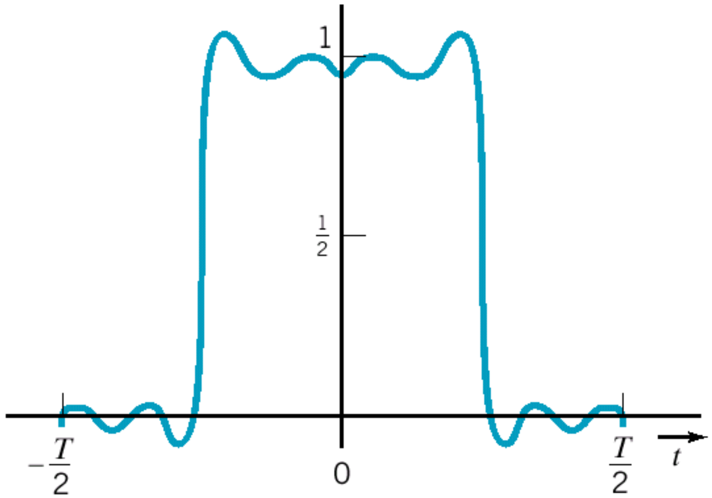
and

$$a_n = \frac{2(-1)^q}{\pi n} \quad \text{when} \quad n = 1,\ 3,\ 5,\ \ldots$$

where

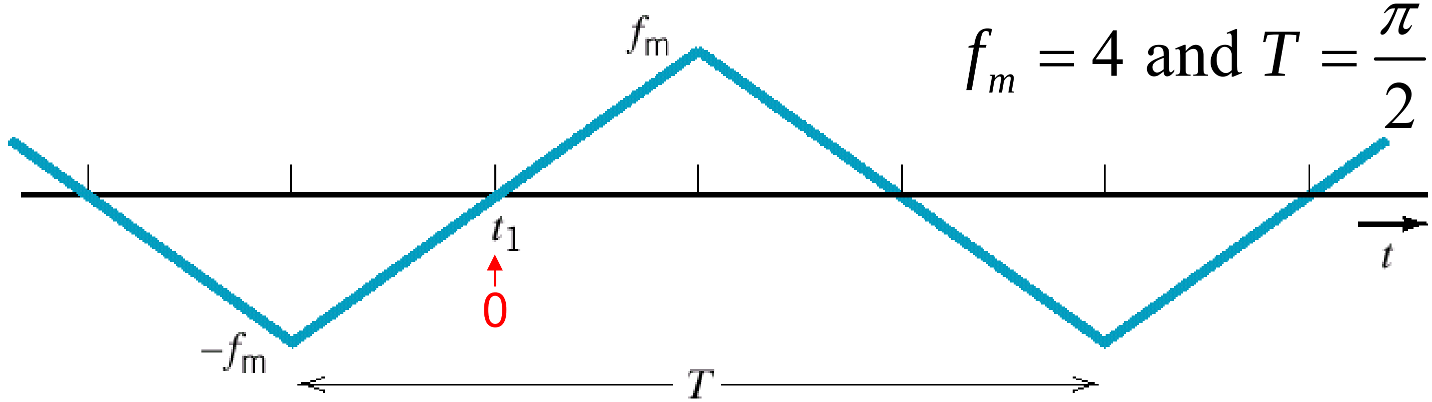
$$q = \frac{(n-1)}{2}$$

$$f(t) = \frac{1}{2} + \sum_{n=1,odd}^{N} \frac{2(-1)^q}{\pi n} \cos n\omega_0 t$$



$$a_1 = \frac{2}{\pi}, a_3 = \frac{-2}{3\pi}, a_5 = \frac{2}{5\pi}, a_7 = \frac{-2}{7\pi}$$

Example determine Fourier Series



$f_m = 4$ and $T = \dfrac{\pi}{2}$ s

$$T = \frac{\pi}{2} \qquad \therefore \quad \omega_0 = \frac{2\pi}{T} = 4 \text{ rad/s}$$

To obtain the most advantages form of symmetry, we choose $t_1 = 0$ s $\Rightarrow$ **Odd & Quarter-wave**

All $a_n = 0$ and $b_n = 0$ for even values of n and $a_0 = 0$

$$b_n = \frac{8}{T}\int_0^{T/4} f(t) \sin n\omega_0 t \, dt \quad ; \text{for odd } n$$

$$4$$

$$f(t) = \frac{f_m}{T/4}t = \frac{4f_m}{T}t \qquad ; 0 \le t \le T/4$$

$$\frac{\pi}{2}$$

$$\therefore \quad f(t) = \frac{32}{\pi}t \qquad ; 0 \le t \le T/4$$

$$b_n = \frac{8}{T}\left(\frac{32}{\pi}\right)\int_0^{T/4} t \sin n\omega_0 t \, dt$$

$$= \frac{512}{\pi^2}\left[\frac{\sin n\omega_0 t}{n^2\omega_0^2} - \frac{t\cos n\omega_0 t}{n\omega_0}\right]_0^{T/4}$$

$$= \frac{32}{\pi^2 n^2}\sin\frac{n\pi}{2} \qquad ; \text{ for odd } n$$

The Fourier Series is

$$f(t) = 3.24\sum_{n=1}^{N}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin n\omega_0 t \quad ;\ \text{for odd } n$$

$$\frac{32}{\pi^2}$$

The first 4 terms (upto and including N = 7)

$$f(t) = 3.24(\sin 4t - \frac{1}{9}\sin 12t + \frac{1}{25}\sin 20t - \frac{1}{49}\sin 28t)$$

Next harmonic is for N = 9 which has magnitude
3.24/81 = 0.04 < 2 % of b_1 (= 3.24)

Therefore the first 4 terms (including N = 7) is enough
for the desired approximation

Fourier series expansion of the function f(x)

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} [a_r \cos(\frac{2\pi r x}{L}) + b_r \sin(\frac{2\pi r x}{L})]$$

a_0, a_r, b_r are called Fourier coefficien ts, obtained by orthogonal properties

$$\boxed{a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)dx}$$

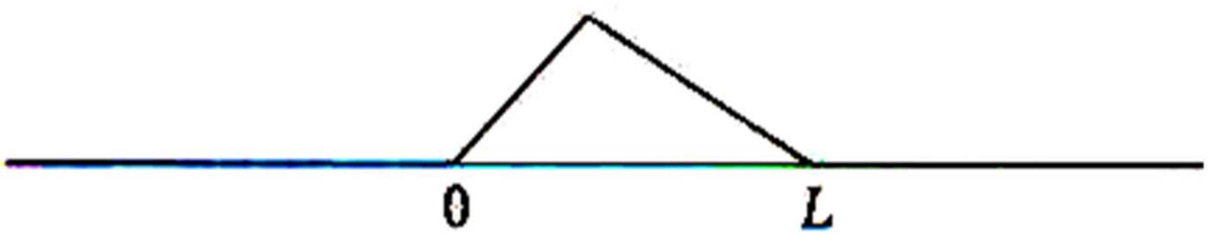
Similarly

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos(\frac{2\pi r x}{L})dx \quad \text{for } r \geq 0$$

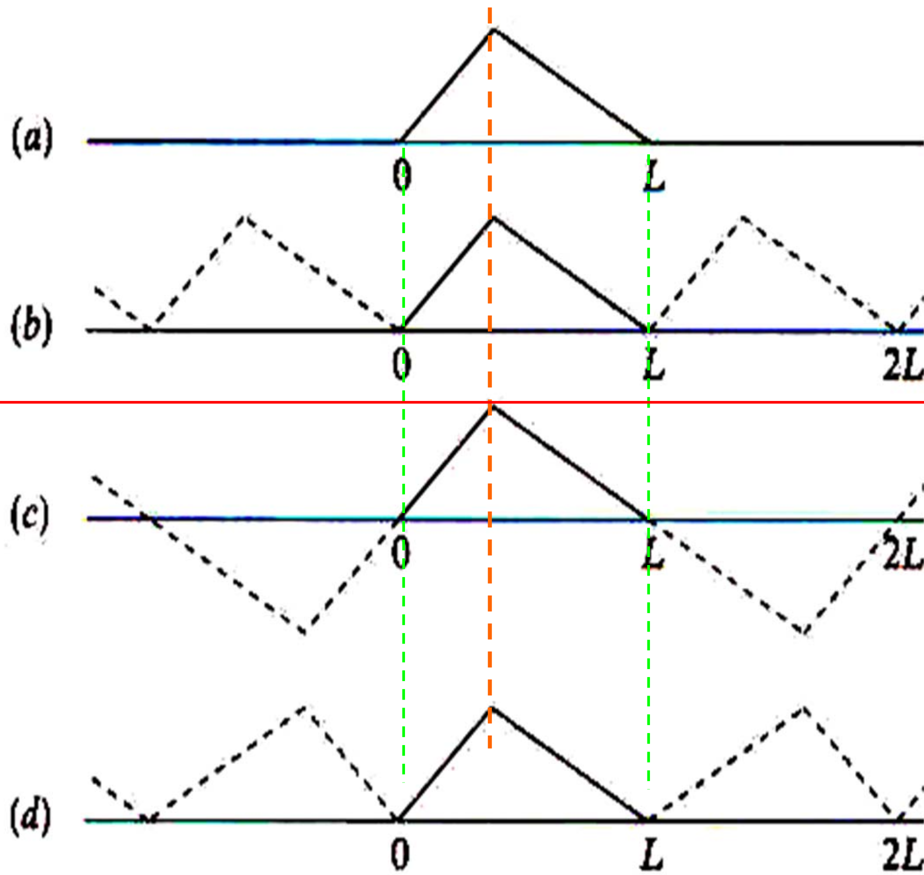
$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin(\frac{2\pi r x}{L})dx \quad \text{for } r \geq 0$$

Non-periodic functions:

(a)



Non-periodic functions: (cont.)



period=L, no particular symmetry

period=2L, anti-symmetry; odd fun

period=2L, symmetry; even fun

Half Range Fourier Series

If a function is defined over half the range, say 0 to *L, instead of the full range from -L to L, it may be expanded in a series of sine terms only or of cosine terms only.* The series produced is then called a **half range Fourier series.**

Conversely, the Fourier Series of an even or odd function can be analysed using the half range definition.

Even Function and Half Range Cosine Series

An even function can be expanded using half its range from

- 0 to *L or*
- *-L to 0 or*
- *L to 2L*

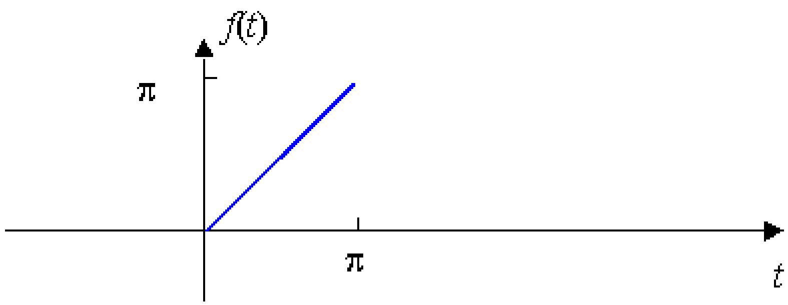
That is, the range of integration = *L. The Fourier series of the* ***half range even*** **function** is given by:

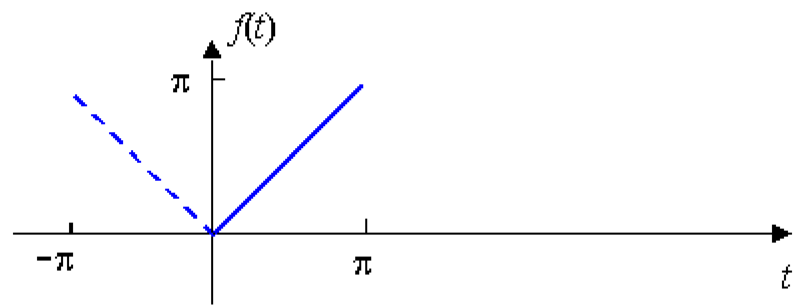
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L}$$ for $n = 1, 2, 3, \ldots$, *where*

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

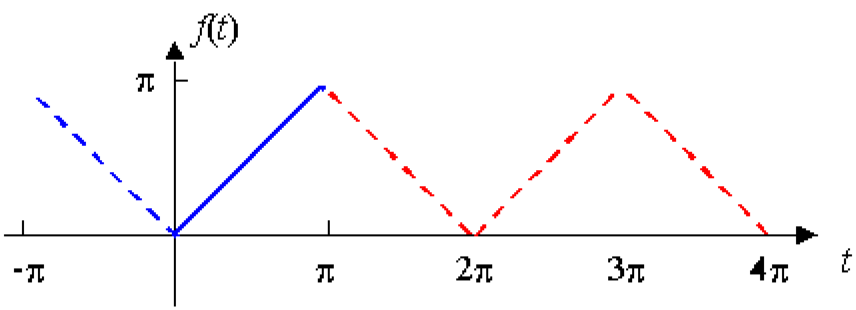
$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt \qquad b_n = 0$$

In the figure below, *f(t) = t is sketched from t = 0 to t = π.*





An even function means that it must be symmetrical about the *f(t) axis and this is* shown in the following figure by the broken line between *t = -π and t = 0.*



It is then assumed that the "triangular wave form" produced is periodic with period 2π outside of this range as shown by the red dotted lines.

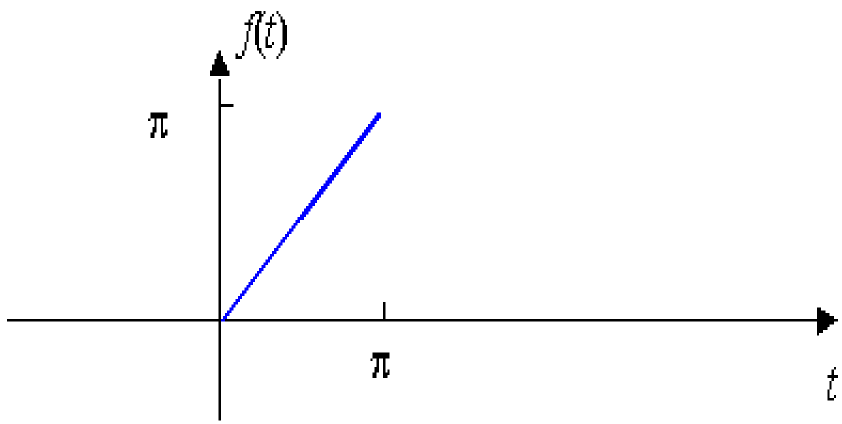
Odd Function and Half Range Sine Series

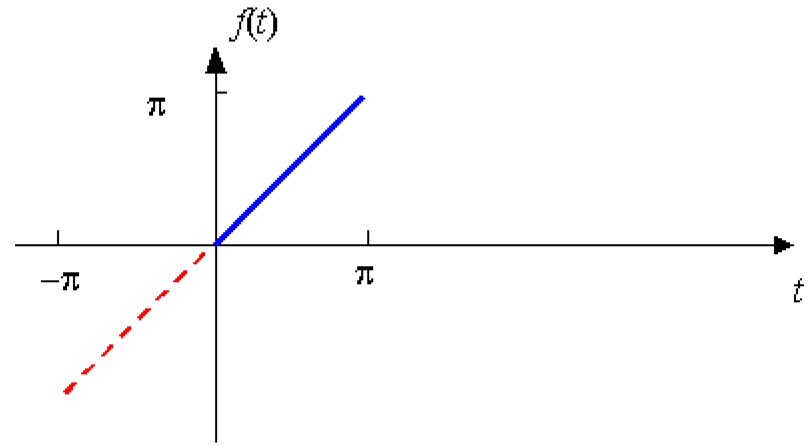
An odd function can be expanded using half its range from 0 to *L, i.e. the range of* integration = *L. The Fourier series of the odd function is:*

Since a_o *= 0 and an = 0, we have:* $f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$ for *n = 1, 2, 3, ...*

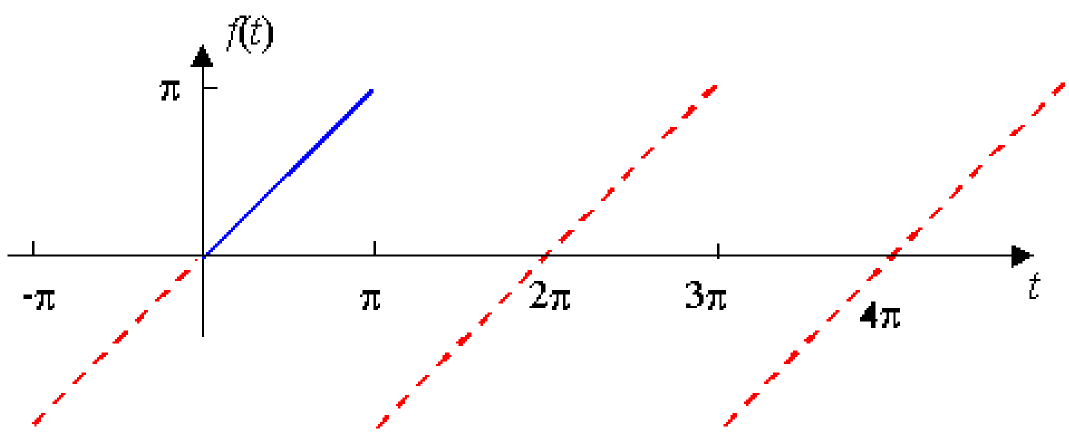
$$b_n = \frac{2}{L}\int_0^L f(t) \sin \frac{n\pi t}{L} dt$$

In the figure below, *f(t) = t is sketched from t = 0 to t = π, as before.*



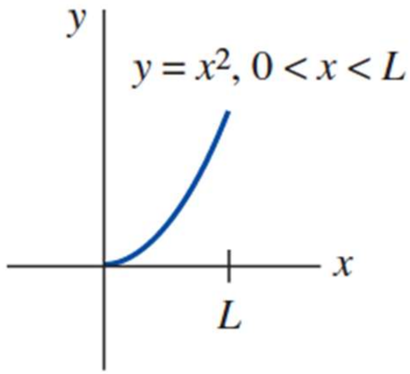


An **odd function means that it is symmetrical about the origin and this is shown by** the red broken lines between *t = -π and t = 0.*



Expansion in Three Series

Expand $f(x) = x^2$, $0 < x < L$, **(a)** in a cosine series, **(b)** in a sine series, **(c)** in a Fourier series.



SOLUTION The graph of the function is given in **FIGURE**

(a) We have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n. Thus

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x. \tag{8}$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1].$$

Hence

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x. \tag{9}$$

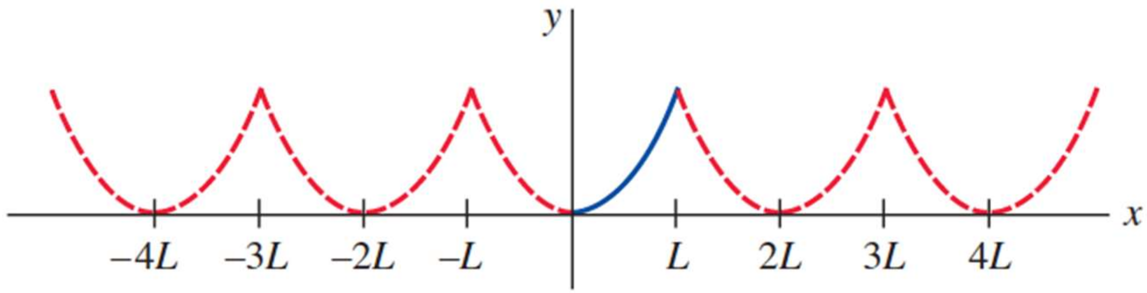
(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2}$$

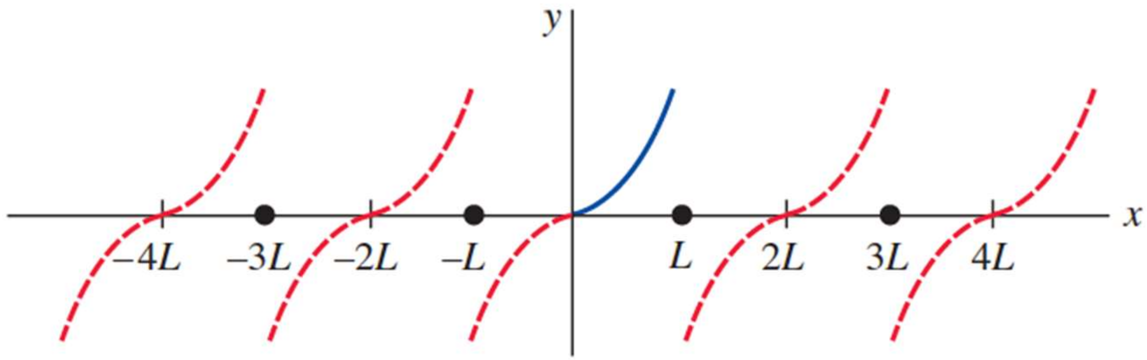
and
$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x\,dx = -\frac{L^2}{n\pi}.$$

Therefore
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}. \tag{10}$$

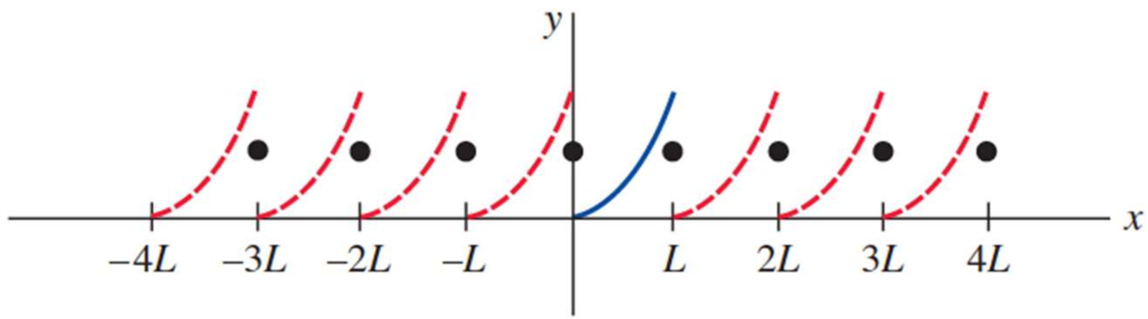
The series (8), (9), and (10) converge to the $2L$-periodic even extension of f, the $2L$-periodic odd extension of f, and the L-periodic extension of f, respectively. The graphs of these periodic extensions are shown in **FIGURE**



(a) Cosine series



(b) Sine series



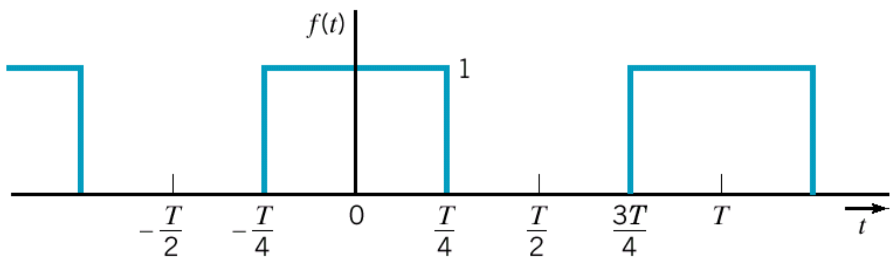
(c) Fourier series

Symmetry of the Function

Four types

1. Even-function symmetry
2. Odd-function symmetry
3. Half-wave symmetry
4. Quarter-wave symmetry

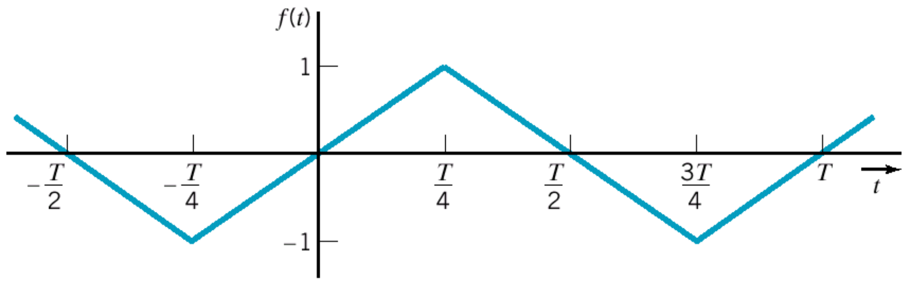
Even function $f(t) = f(-t)$ All $b_n = 0$



$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt$$

Symmetry of the Function

Odd function $f(t) = -f(-t)$ All $a_n = 0$



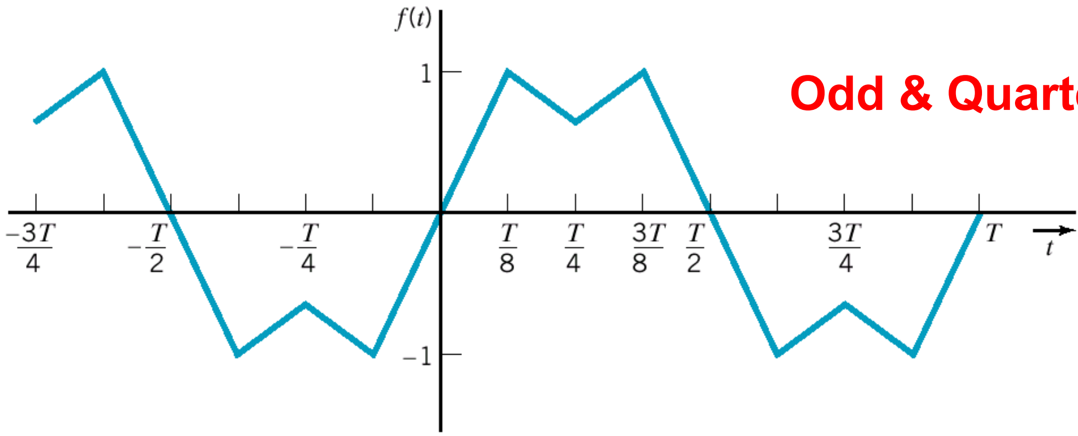
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

Half-wave symmetry $f(t) = -f(t - \frac{T}{2})$

a_n and b_n = 0 for even values of n and $a_0 = 0$

Symmetry of the Function

Quarter-wave symmetry



Odd & Quarter-wave

All $a_n = 0$ and $b_n = 0$ for even values of n and $a_0 = 0$

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega_0 t\, dt \quad ; \text{for odd } n$$

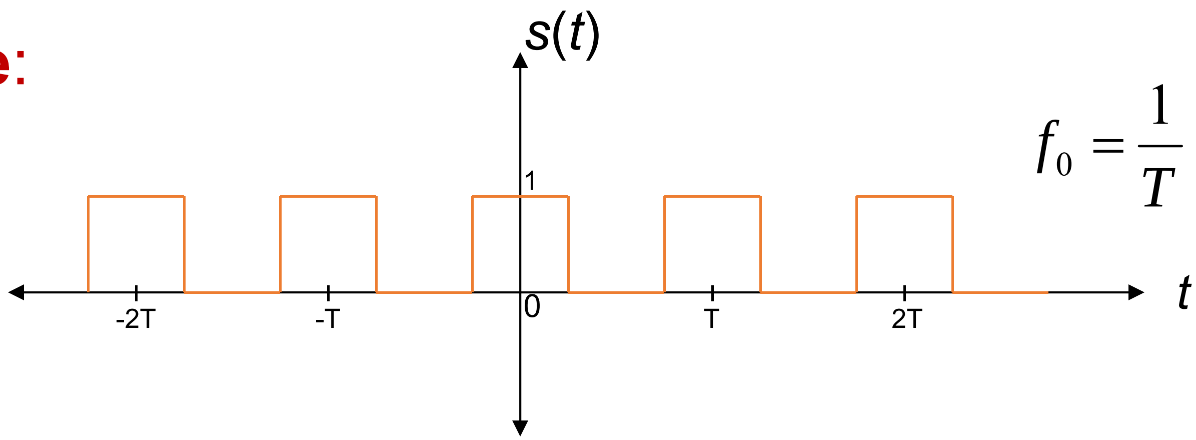
For Even & Quarter-wave

All $b_n = 0$ and $a_n = 0$ for even values of n and $a_0 = 0$

$$a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega_0 t\, dt \quad ; \text{for odd } n$$

Fourier Series

Example:



$$f_0 = \frac{1}{T}$$

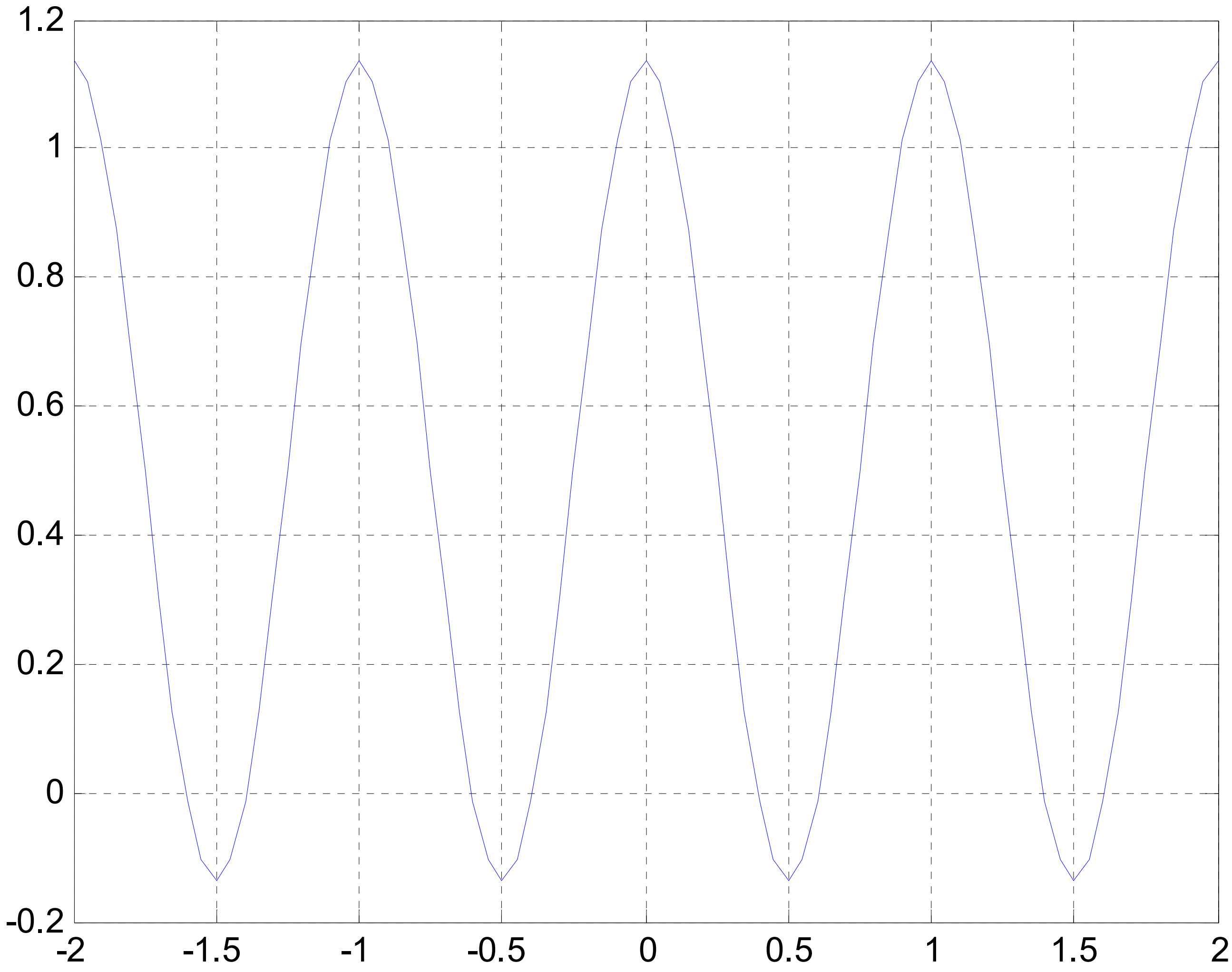
$$a_0 = \frac{1}{T}\int_0^T s(t)dt = \frac{1}{T}\int_0^{T/2} 1.dt = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T s(t)\cos 2\pi n f_0 t dt = \frac{2\sin(n\pi/2)}{n\pi}$$

$$b_n = \frac{2}{T}\int_0^T s(t)\sin 2\pi n f_0 t dt = 0$$

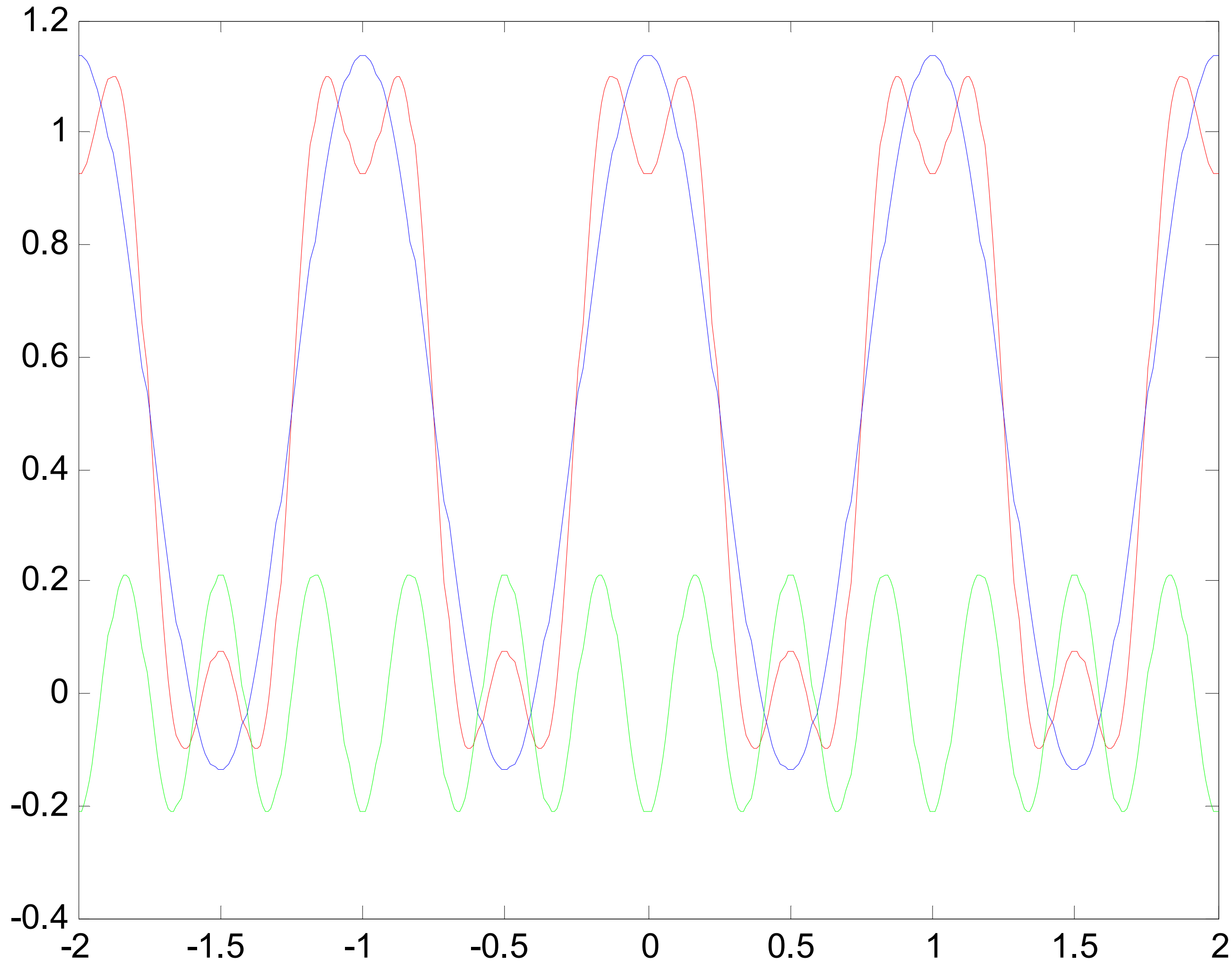
$$s(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \ldots\right)$$

Fourier Series



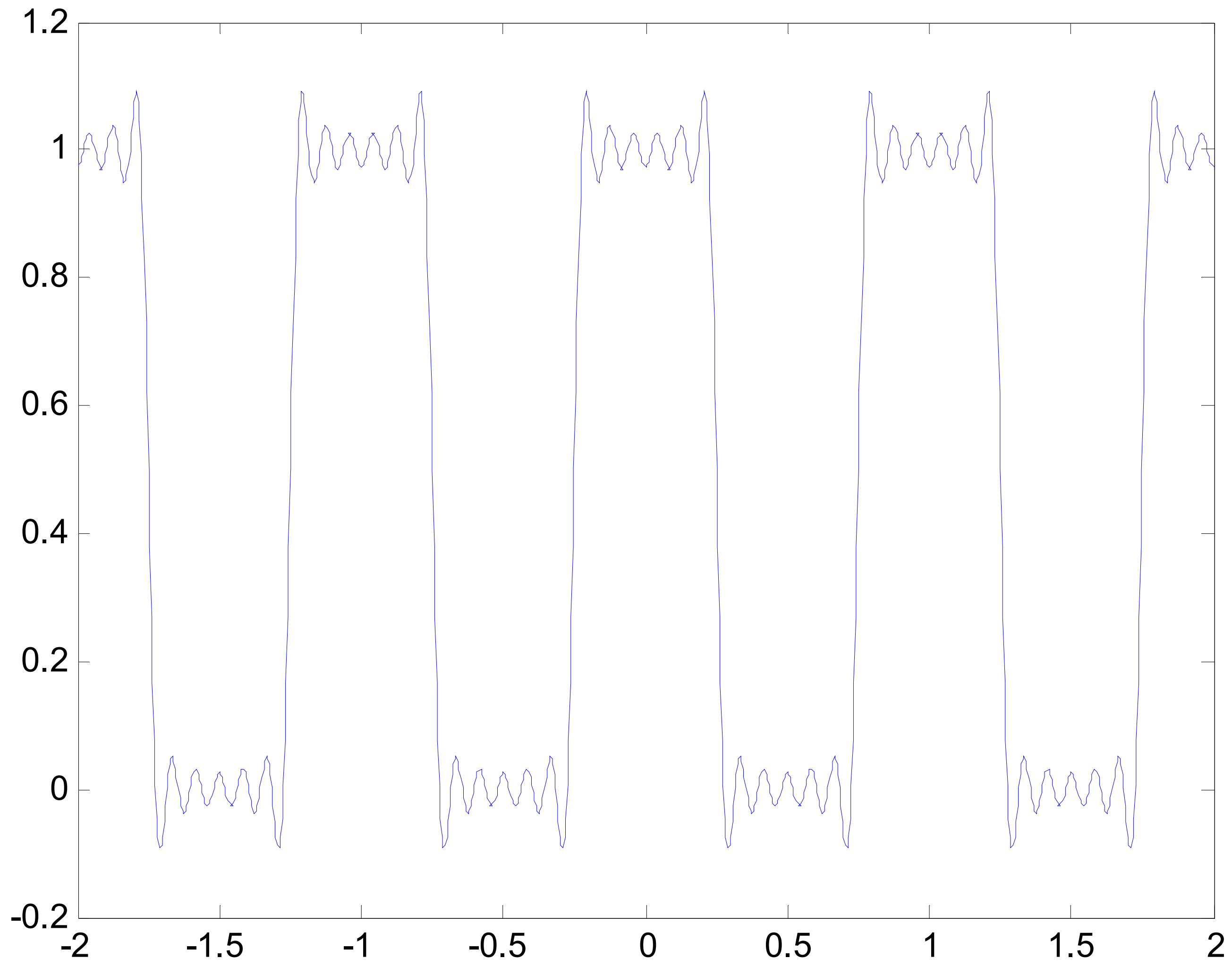
Summation of dc term plus the first harmonic

Fourier Series



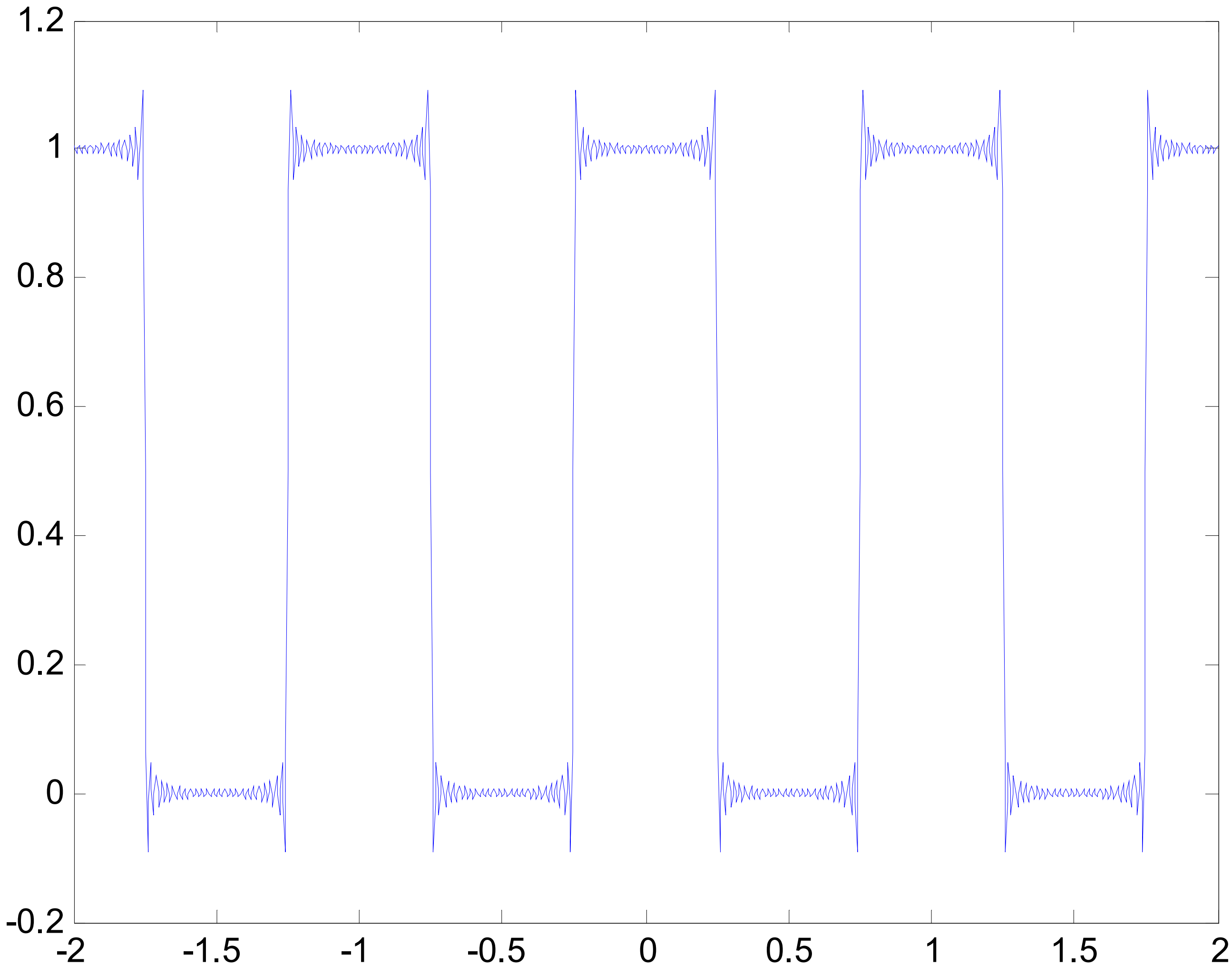
Summation of dc term plus the first three harmonics

Fourier Series



Summation of dc term plus the first 11 harmonics

Fourier Series



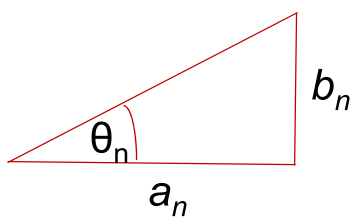
Summation of dc term plus the first 50 harmonics

The Fourier Series

The expression for a ***Fourier Series*** is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t \qquad \omega_0 = \frac{2\pi}{T}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$



$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t \right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

The Fourier Series

An alternative form

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

C_0 is the average (or DC) value of *f(t)*

harmonic amplitude phase angle

Fourier Series = a finite sum of harmonically related sinusoids

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \qquad C_0 = \frac{a_0}{2} \qquad C_n = \sqrt{a_n^2 + b_n^2}$$

For *n* = 1 the corresponding sinusoid is called ***the fundamental***

$$C_1 \cos(\omega_0 t + \theta_1)$$

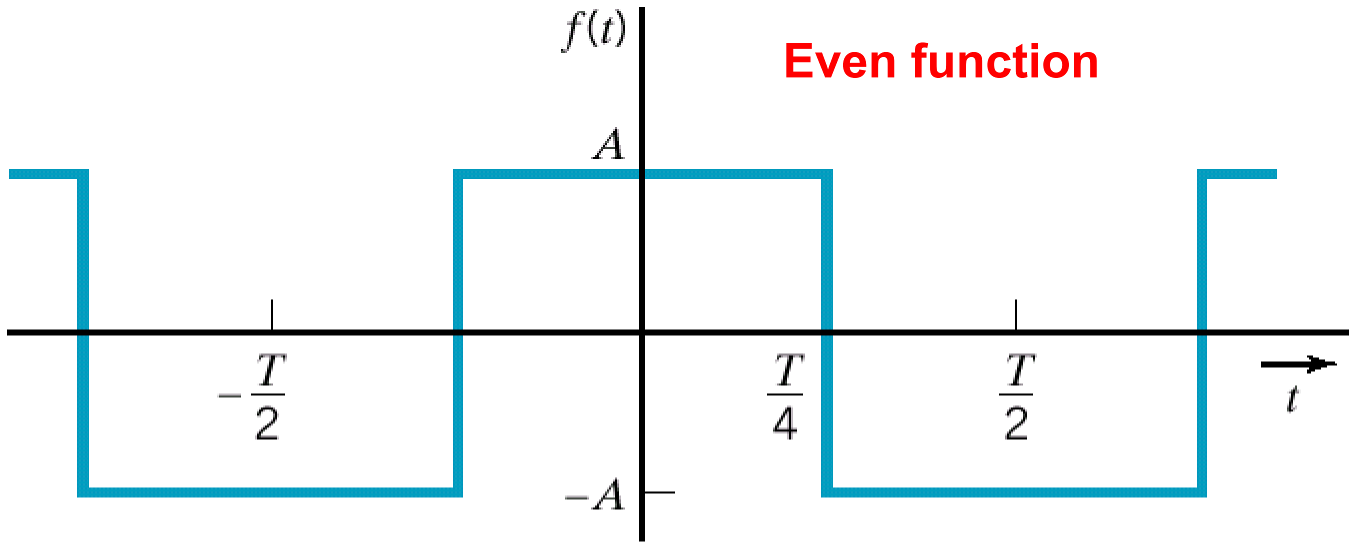
For *n* = *k* the corresponding sinusoid is called ***the kth harmonic term***

$$C_k \cos(k\omega_0 t + \theta_k)$$

Similarly, ω_0 is call the ***fundamental frequency***

$k\omega_0$ is called the ***k^{th} harmonic frequency***

Example determine complex Fourier Series



The average value of *f(t)* is zero $\therefore \quad C_0 = 0$

$$\mathbf{C}_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}dt$$

We select $t_0 = -\frac{T}{2}$ and define $jn\omega_0 = m$

$$\mathbf{C}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\int_{-T/2}^{-T/4} -Ae^{-mt}dt + \frac{1}{T}\int_{-T/4}^{T/4} Ae^{-mt}dt + \frac{1}{T}\int_{T/4}^{T/2} -Ae^{-mt}dt$$

$$= \frac{A}{mT}\left(e^{-mt}\Big|_{-T/2}^{-T/4} - e^{-mt}\Big|_{-T/4}^{T/4} + e^{-mt}\Big|_{T/4}^{T/2}\right)$$

$$= \frac{A}{jn\omega_0 T}\left(2e^{jn\pi/2} - 2e^{-jn\pi/2} + e^{-jn\pi} - e^{jn\pi}\right)$$

$$= \frac{A}{2\pi n}\left(4\sin\frac{n\pi}{2} - 2\sin(n\pi)\right) = \begin{cases} 0 & \text{; for even } n \\ \dfrac{2A}{n\pi}\sin n\dfrac{\pi}{2} & \text{; for odd } n \end{cases}$$

$$= A\frac{\sin x}{x} \quad \text{where } x = \frac{n\pi}{2}$$

Since *f(t)* is even function, all $\boldsymbol{C}_n$ are real and = 0 for *n* even

For *n* = 1

$$\mathbf{C}_1 = \frac{A \sin \pi / 2}{\pi / 2} = \frac{2A}{\pi} = \mathbf{C}_{-1}$$

For *n* = 2

$$\mathbf{C}_2 = A \frac{\sin \pi}{\pi} = 0 = \mathbf{C}_{-2}$$

For *n* = 3

$$\mathbf{C}_3 = \frac{A \sin(3\pi / 2)}{3\pi / 2} = \frac{-2A}{3\pi} = \mathbf{C}_{-3}$$

The complex Fourier Series is

$$f(t) = \ldots + \frac{-2A}{3\pi} e^{-j3\omega_0 t} + \frac{2A}{\pi} e^{-j\omega_0 t} + \frac{2A}{\pi} e^{j\omega_0 t} + \frac{-2A}{3\pi} e^{j3\omega_0 t} + \ldots$$

$$= \frac{2A}{\pi}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{-2A}{3\pi}\left(e^{j3\omega_0 t} + e^{-j3\omega_0 t}\right) + \ldots$$

$$e^{jx} + e^{-jx} = 2\cos x$$

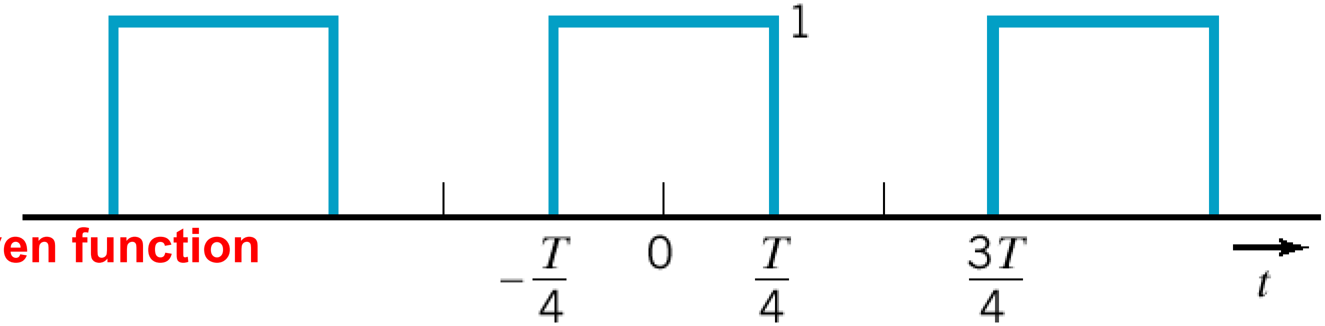
$$= \frac{4A}{\pi}\cos\omega_0 t - \frac{4A}{3\pi}\cos 3\omega_0 t + \ldots$$

$$e^{jx} - e^{-jx} = 2j\sin x$$

$$= \frac{4A}{\pi}\sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{(-1)^q}{n}\cos n\omega_0 t$$ where $q = \frac{n-1}{2}$

For real *f(t)* $\Rightarrow$ $\left|\mathbf{C}_n\right| = \left|\mathbf{C}_{-n}\right|$

Example determine complex Fourier Series



Even function

Use $jn\omega_0 = m$

$$\mathbf{C}_n = \frac{1}{T}\int_{-T/4}^{T/4} 1e^{-mt}dt$$

$$= \frac{1}{-mT} e^{-mt} \Big|_{-T/4}^{T/4}$$

$$= \frac{1}{-mT}\left(e^{-mT/4} - e^{+mT/4}\right)$$

$$\mathbf{C}_n = \frac{1}{-jn2\pi}\left(e^{-jn\pi/2} - e^{+jn\pi/2}\right)$$

$$= \begin{cases} 0 & ;\ n \text{ even}, n \neq 0 \\ (-1)^{(n-1)/2} & ;\ n \text{ odd} \end{cases}$$

To find C_0

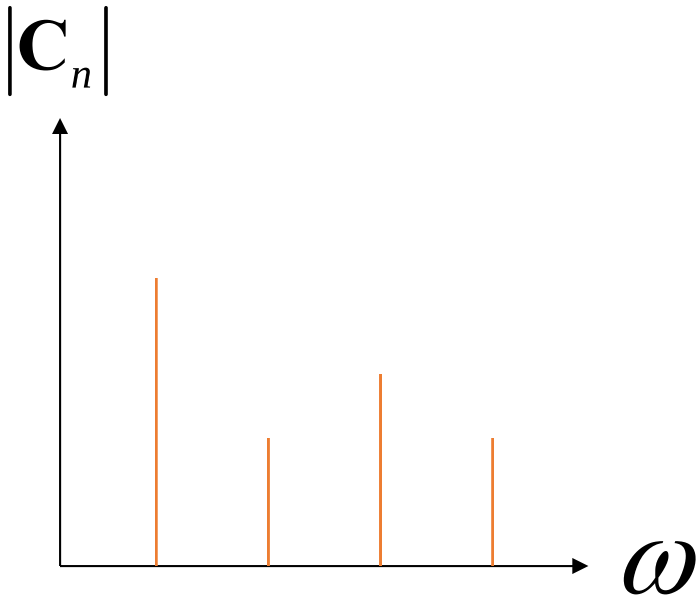
$$C_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$= \frac{1}{T}\int_{-T/4}^{T/4} 1\,dt = \frac{1}{2}$$

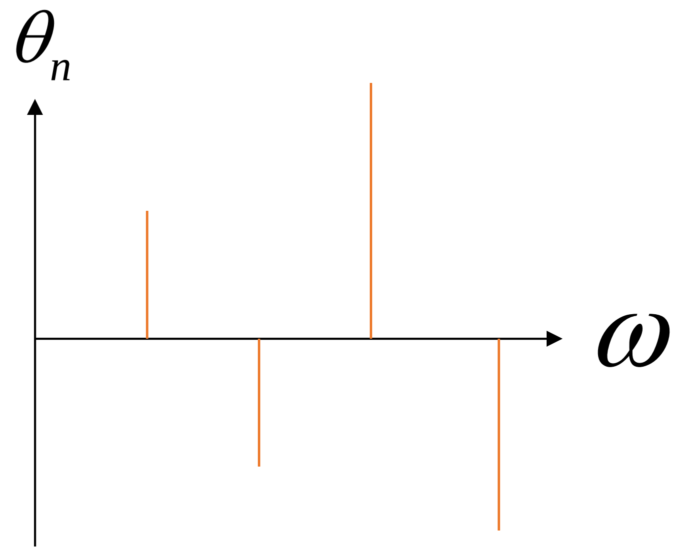
The Fourier Spectrum

The complex Fourier coefficients

$$\mathbf{C}_n = \left|\mathbf{C}_n\right| \angle \theta_n$$



Amplitude spectrum



Phase spectrum