Differential Equations

Ordinary Differential Equations	Partial Differential Equations
$\frac{d^2v}{dt^2} + 6tv = 1$	$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$
involve one or more **Ordinary derivatives** of unknown functions	involve one or more **Partial derivatives** of unknown functions

Solutions of Ordinary Differential Equations

$x(t) = \cos(2t)$ is a solution to the ODE $\frac{d^2x(t)}{dt^2} + 4x(t) = 0$

All functions of the form $x(t) = \cos(2t + c)$ are solutions.

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions. Two conditions are needed to uniquely specify the solution.

$$x(0) = a \, , \, \dot{x}(0) = b$$

Classification of ODEs

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - *n*th order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

Consider $y' = f(x, y)$ with an initial condition $y = y_0$ at $x = x_0$. The function f(x,y) is linear, nonlinear or table of values. When the value of y is given at $x = x_0$ and the solution required in $x_0 \le x \le x_f$,the problem called an **initial value problem**. If y is given at $x = x_f$ and the solution required in $x_0 \le x \le x_f$ then the problem is called a **boundary value problem**.

Auxiliary Conditions

Initial-Value Problems

- All auxiliary conditions are at one point of the independent variable.

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

- All auxiliary conditions are not at one point of the independent variable.
- More difficult to solve than initial value problems.

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Numerical Methods for Solving ODE

Single-Step Methods

- Estimates of the solution at a particular step are entirely based on information on the previous step.

Multiple-Step Methods

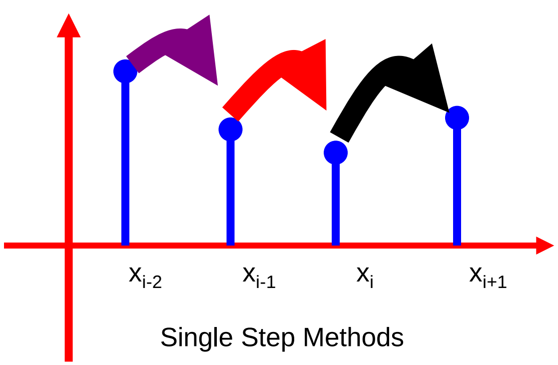
- Estimates of the solution at a particular step are based on information on more than one step.

Single Step Methods

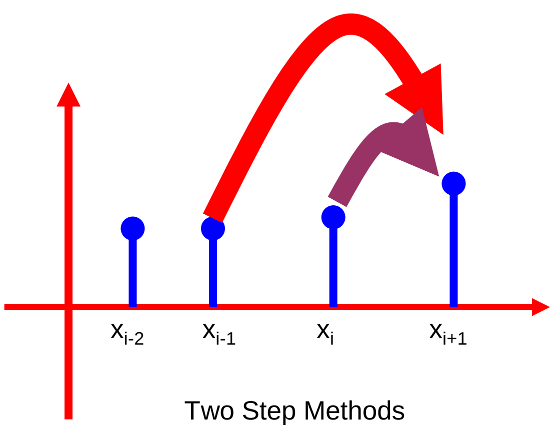
- Single Step Methods:
  - Euler and Runge-Kutta are single step methods.
  - Estimates of y_{i+1} depends only on y_i and x_i.

Multi-Step Methods

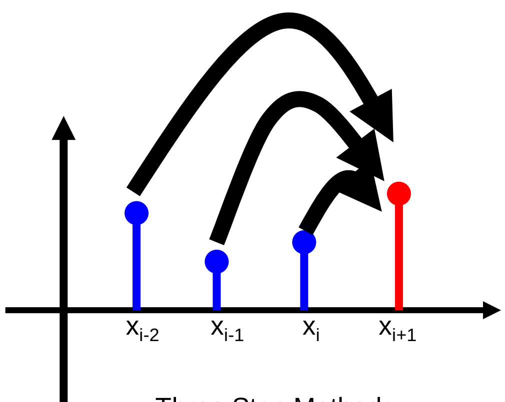
- 2-Step Methods
  - In a two-step method, estimates of y_{i+1} depends on y_i, y_{i-1}, x_i, and x_{i-1}
- 3-Step Methods
  - In an 3-step method, estimates of y_{i+1} depends on y_i ,y_{i-1} ,y_{i-2}, x_i , x_{i-1}, and x_{i-2}



Single Step Methods



Two Step Methods



Three Step Methods

PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

- This is an iterative method. Consider $\frac{dy}{dx} = f(x, y)$ with an initial condition $y = y_0$ at $x = x_0$.

  Integrating in $(x_0, x_0 + h)$

$$y(x_0 + h) = y(x_0) + \int_{x_0}^{x_0+h} f(x,y)dx$$

- This integral equation is solved by successive approximations. This process is repeated and in the n^{th} approximation, we get

$$y^{(n)} = y_0 + \int_{x_0}^{x_0+h} f(x,y^{(n-1)})dx$$

Example: Find $y(1.1)$ given that $y' = x - y$, $y(1) = 1$ by Picard's Method.

Solution: $y^{(1)}(1.1) = 1 + \int_1^{1.1} (x-1)dx = 1.005$

Successive iterations yield 1.0045, 1.0046, 1.0046. Thus **y(1.1) = 1.0046**

Exact value is **y(1.1) = 1.0048**

EXAMPLE Solve $\frac{dy}{dx} = x + y$, given $y(0) = 1$. Obtain the values of $y(0.1)$, $y(0.2)$, using Picard's method.

Solution

Here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$

The Picard's algorithm is $y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$

$$y = 1 + \int_0^x f(x, y)\, dx$$

Put $y = y_0$, we get

$$y^{(1)} = 1 + \int_0^x f(x, 1)\, dx = 1 + \int_0^x (x + 1)\, dx = 1 + x + \frac{x^2}{2}$$

Again using $y = y^{(1)}$

$$y^{(2)} = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

$$y^{(3)} = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \ldots$$

Setting $x = 0.1$, we get $y(0.1) = 1 + 0.1 + 0.01 + \frac{1}{3}(0.001) + \frac{1}{24}(0.0001)$

$$= 1 + 0.1 + 0.01 + 0.0003333 + 0.0000041$$

$$= 1.1103374$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24}$$

$$= 1.242733$$

EXAMPLE Solve $y' + y = e^x$, $y(0) = 0$, by Picard's method.

Solution

We know that $y = y_0 + \int_{x_0}^{x} f(x, y)\,dx = 0 + \int_0^x (e^x - y)\,dx$

Here $x_0 = 0, y_0 = 0$ $\quad y^{(1)} = \int_0^x (e^x - 0)\,dx = e^x - 1$

$$y^{(2)} = \int_0^x (e^x - e^x + 1)\,dx = x$$

$$y^4 = \int_0^x \left[e^x - \left(e^x - \frac{x^2}{2} - 1 \right) \right] dx = \frac{x^3}{6} + x$$

$$y^{(3)} = \int_0^x (e^x - x)\,dx = e^x - \frac{x^2}{2} - 1$$

$$y^5 = \int_0^x \left(e^x - x - \frac{x^3}{6} \right) dx = e^x - \frac{x^2}{2} - \frac{x^4}{24} - 1$$

Taylor Series Method

The problem to be solved is a first order ODE:

$$\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0$$

Estimates of the solution at different base points: $y(x_0 + h),\ y(x_0 + 2h),\ y(x_0 + 3h),\ \ldots$ are computed using the truncated Taylor series expansions.

Truncated Taylor Series Expansion

$$y(x_0 + h) \approx \sum_{k=0}^{n} \frac{h^k}{k!}\left(\left.\frac{d^k y}{dx^k}\right|_{x=x_0, y=y_0} \right) \approx y(x_0) + h \left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}} + \frac{h^2}{2!} \left.\frac{d^2 y}{dx^2}\right|_{\substack{x=x_0,\\ y=y_0}} + \ldots + \frac{h^n}{n!} \left.\frac{d^n y}{dx^n}\right|_{\substack{x=x_0,\\ y=y_0}}$$

The n^{th} order Taylor series method uses the n^{th} order Truncated Taylor series expansion.

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots$$

Assume $x = x_0 + h$

$$\boxed{\therefore\ y(x_0 + h) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \ldots}$$

Example: Solve the following boundary value problem $y' = -y$ at $x =$ 0.2, 0.4 given that $y(0) = 1$

Solution: Taylor expansion given by

$$y(x_0 + h) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \ldots$$

Consider the point of expansion zero $\quad \therefore h = x - x_0 = 0.2 - 0 = 0.2$

Evaluate the derivatives at the point of expansion zero

$$y'(x) = -y \Rightarrow y'(x_0) = -y(x_0) = -1,$$

$$y''(x) = -y'(x) \Rightarrow y''(x_0) = -y'(x_0) = 1$$

$$y'''(x) = -y''(x) \Rightarrow y'''(x_0) = -y''(x_0) = -1$$

Then $\quad y(0.2) = y(0 + 0.2) = 1 + \frac{0.2}{1!}(-1) + \frac{(0.2)^2}{2!}(1) + \frac{(0.2)^3}{3!}(-1) + \ldots\ldots = 0.81867$

$$y(0.4) = y(0.2 + 0.2)$$

$$= 0.81867 + \frac{0.2}{1!}(-0.81867) + \frac{(0.2)^2}{2!}(0.81867) + \frac{(0.2)^3}{3!}(-0.81867) + \ldots$$

$$= 0.67022$$

Example: Using the Second order Taylor Series Method solve

Solution: $\frac{dx}{dt} + 2x^2 + t = 1, \quad x(0) = 1, \qquad \text{use} \quad h = 0.01$

$$\frac{dx}{dt} = f(t,x) = 1 - 2x^2 - t \quad , \quad t_0 = 0, \quad x_0 = 1, \quad h = 0.01$$

$$\frac{d^2x(t)}{dt^2} = 0 - 4x\frac{dx}{dt} - 1 = -4x(1 - 2x^2 - t) - 1$$

$$x_{i+1} = x_i + h(1 - 2x_i^2 - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^2 - t_i))$$

$$Step\,1: \quad x_1 = 1 + 0.01(1 - 2(1)^2 - 0) + \frac{(0.01)^2}{2}(-1 - 4(1)(1 - 2 - 0)) = 0.9901$$

$$Step\,2: \quad x_2 = 0.9901 + 0.01(1 - 2(0.9901)^2 - 0.01) + \frac{(0.01)^2}{2}(-1 - 4(0.9901)(1 - 2(0.9901)^2 - 0.01))$$

$$= 0.9807$$

$$Step\,3: \quad x_3 = 0.9716$$

Summary of the results:

i	t_i	x_i
0	0.00	1
1	0.01	0.9901
2	0.02	0.9807
3	0.03	0.9716

EXAMPLE Find $y(1.1)$, given $y' = 2x - y$ and $y(1) = 3$.

Solution

Given $y' = 2x - y,\ x_0 = 1,\ y_0 = 3,\ x_1 = 1.1,\ h = 0.1$

Taylor series is given by $y_1 = y_0 + hy_0' + \dfrac{h^2}{2!} y_0'' + \dfrac{h^3}{3!} y_0''' + \ldots$

$$y' = 2x - y \quad \Rightarrow \quad y_0' = 2x_0 - y_0 = 2(1) - 3 = -1$$

$$y'' = 2 - y' \quad \Rightarrow \quad y_0'' = 2 - y_0' = 2 - (-1) = 3$$

$$y''' = -y'' \quad \Rightarrow \quad y_0''' = -y_0'' = -3$$

$$y_1 = 3 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-3) + \ldots$$

$$= 3 - 0.1 + 0.015 - 0.0005 \ldots \quad = 2.9145$$

EXAMPLE Given $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ using the Taylor series method.

Solution

Given $y' = 3x + y/2$ and $x_0 = 0,\ y_0 = 1$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$$

To find y(0.1):

$$y' = 3x + \frac{y}{2} \quad \Rightarrow \quad y_0' = 3x_0 + \frac{y_0}{2} = 3(0) + \frac{1}{2} = 0.5$$

$$y'' = 3 + \frac{y'}{2} \quad \Rightarrow \quad y_0'' = 3 + \frac{y_0'}{2} = 3 + \frac{0.5}{2} = 3.25$$

$$y''' = \frac{y''}{2} \quad \Rightarrow \quad y_0''' = \frac{y_0''}{2} = \frac{3.25}{2} = 1.625$$

$$y^{iv} = \frac{y'''}{2} \quad \Rightarrow \quad y_0^{iv} = \frac{y_0'''}{2} = \frac{1.625}{2} = 0.8125$$

$$y_1 = 1 + (0.1)\,(0.5) + \frac{(0.1)^2}{2}(3.25) + \frac{(0.1)^3}{6}(1.625) + \frac{(0.1)^4}{24}(0.8125) + \ldots$$

$$= 1.0665$$

$$y_1 = y(0.1) = 1.0665$$

Example: Find $y(0.2)$, given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: $y'' = -y$, $x_0 = 0$, $y_0 = 1$, $y_0' = 0$, $h = 0.2$.

To find y(0.2) We know that Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$y'' = -y \quad \Rightarrow \quad y_0'' = -y_0 = -1$$

$$y''' = -y' \quad \Rightarrow \quad y_0''' = -y_0' = 0$$

$$y^{iv} = -y'' \quad \Rightarrow \quad y_0^{iv} = -y_0'' = -(-1) = 1$$

$$y^{v} = -y''' \quad \Rightarrow \quad y_0^{v} = -y_0''' = 0$$

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^3}{6}(6) + \frac{(0.2)^4}{24}(1) + \ldots$$

$$= 1 + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^4}{24}(1) + \ldots$$

$$= 1 - 0.02 + 0.00006667$$

$$= 0.9800667$$

Euler Method

- First order Taylor series method is known as Euler Method.

$$y(x_0 + h) = y(x_0) + h \left.\frac{dy}{dx}\right|_{\substack{x=x_0, \\ y=y_0}} + O(h^2)$$

- Only the constant term and linear term are used in the Euler method.
- First derivative provides a direct estimate of the **slope** at x_i:

$$\int_{x_0}^{x_1} \left.\frac{dy}{dx}\right|_{\substack{x=x_i, \\ y=y_i}} dx = \int_{x_0}^{x_1} \mathrm{f}(x_i, y_i) dx = \mathrm{f}(x_i, y_i) \int_{x_0}^{x_1} dx = h\,\mathrm{f}(x_i, y_i)$$

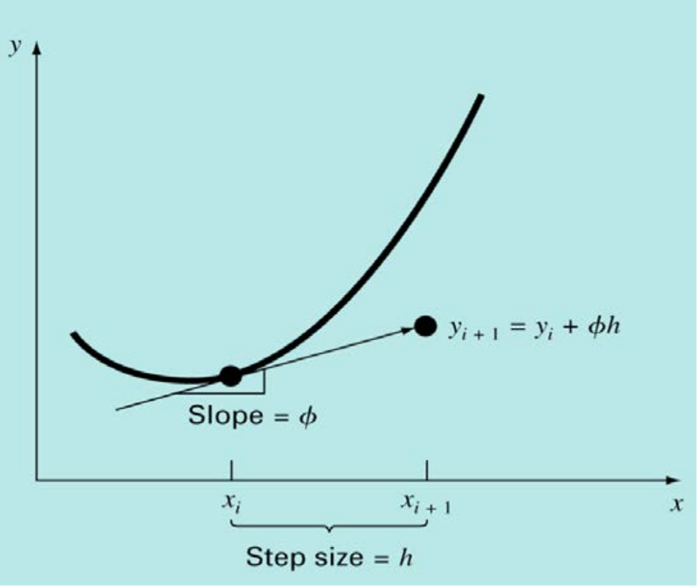
- The Euler Method is

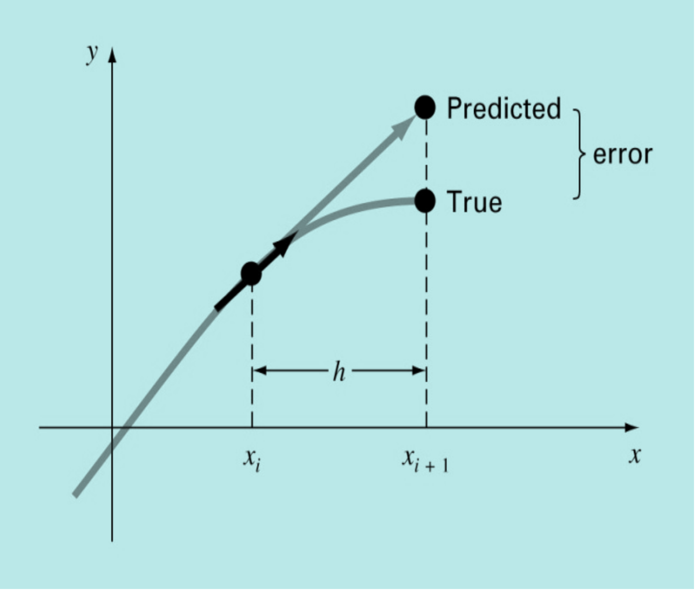
$$\boxed{y_{i+1} = y_i + h\ f(x_i, y_i) \ \Leftarrow\ y_0 = y(x_0)}$$

$$\text{New value} = \text{old value} + \text{slope} * (\text{step\_size})$$

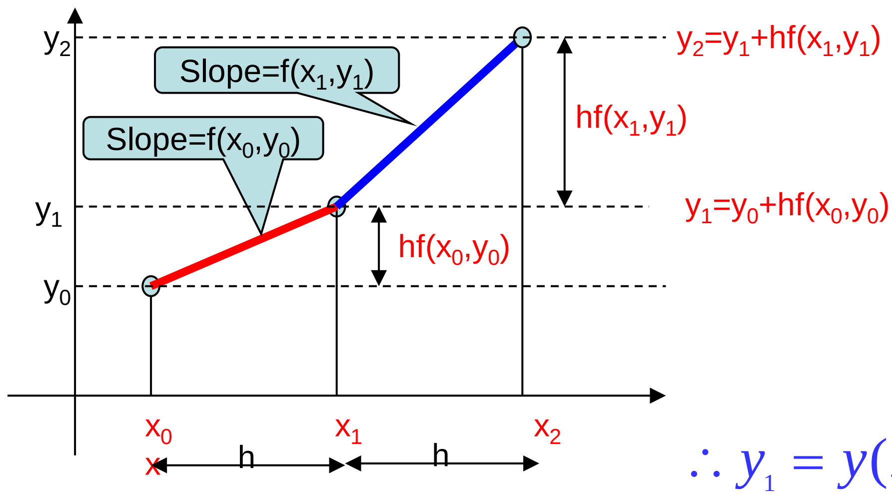
$$y_{i+1} = y_i + \phi * h$$

- The error due to the use of the truncated Taylor series is of order $O(h^2)$.





Interpretation of Euler Method



Note:

- When the step h becomes small, the numerical solution becomes nearly the same as the exact solution.
- Computing approximate values of the solution f(x) at the points: $x_1 = x_0 + h$, $x_2 = x_0+2h$, $x_3 = x_0 + 3h$…

$$\therefore y_1 = y(x_0 + h), y_2 = y(x_0 + 2h), y_3 = y(x_0 + 3h)...$$

Example: Obtain a solution between $x = 0$ to $x = 2$ with a step size of 0.5 for

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

Initial conditions : $x = 0$ to $y = 1$

Solution: $$y_{i+1} = y_i + f(x_i, y_i) \cdot h$$

Exact Solution: $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

$$y(0.5) = y(0) + f(0,1)(0.5) = 1.0 + 8.5(0.5) = 5.25$$

(true solution at x=0.5 is y(0.5) = 3.22 and e_t = -63%)

$$y(1.0) = y(0.5) + f(0.5, 5.25)(0.5) = 5.25 + (-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5)(0.5) = 5.875$$

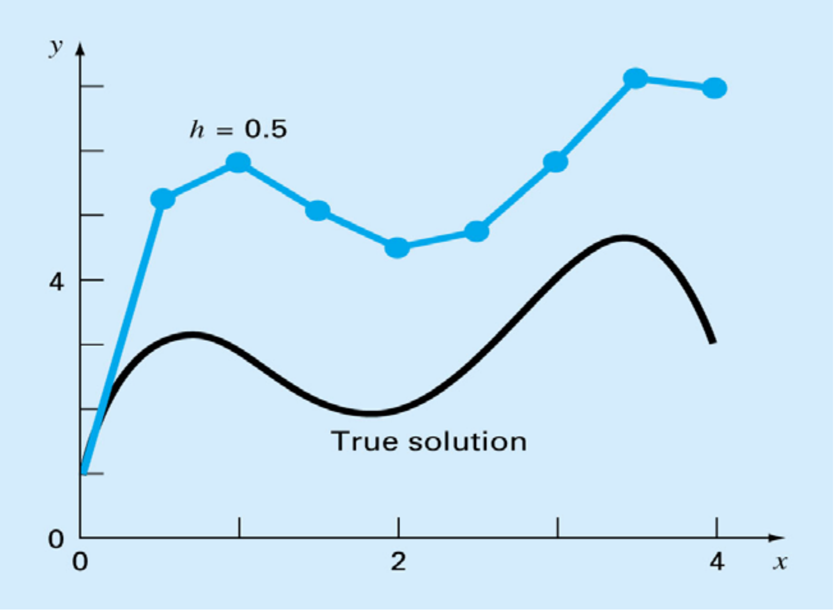
(true solution at x=1 is y(1) = 3 and e_t = -96%)

$$y(1.5) = y(1.0) + f(1.0, 5.875)(0.5) = 5.875 + (-2(1.0)^3 + 12(1.0)^2 - 20(1.0) + 8.5)(0.5) = 5.125$$

Table of Solution $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$ from $x = 0$ to $x = 2$ with step size $h = 0.5$

Initial condition: $(x = 0\ ;\ y = 1)$

X	y_{euler}	y_{true}	Error Global	Error Local
0	1	1	%	%
0.5	5.250	3.218	-63.1	-63.1
1.0	5.875	3.000	-95.8	-28
1.5	5.125	2.218	131.0	-1.41
2.0	4.500	2.000	-125.0	20.5



Example: Obtain a solution between $x = 0.02$ to $x = 0.08$ with a step size of 0.02 for $\frac{dy}{dx} = x^2 + y$ given that $y(0) = 1$

Solution: $y_{n+1} = y_n + h \cdot [x_n^2 + y_n] \Leftarrow h = x_{n+1} - x_n$

For n = 0 $\quad y_1 = y_0 + h\ f(x_0, y_0) = 1 + 0.02(0^2 + 1) = 1.02$

For n = 1 $\quad y_2 = y_1 + h\ f(x_1, y_1) = 1.02 + 0.02((0.02)^2 + 1.02) = 1.0404$

For n = 2 $y_3 = y_2 + h\ f(x_2, y_2) = 1.0404 + 0.02((0.04)^2 + 1.0404) = 1.0612$

For n = 3 $y_4 = y_3 + h\ f(x_3, y_3) = 1.0612 + 0.02((0.06)^2 + 1.0612) = 1.0825$

For n = 4 $y_5 = y_4 + h\ f(x_4, y_4) = 1.0825 + 0.02((0.08)^2 + 1.0825) = 1.1043$

We can put the results in a table:

n	x_n	y_n	y_{n+1}
0	0.00	1.000	1.0200
1	0.02	1.0200	1.0404
2	0.04	1.0404	1.0612
3	0.06	1.0612	1.0825
4	0.08	1.0825	1.1043

Example: Use Euler method to solve the ODE $\frac{dy}{dx} = 1 + x^2, \quad y(1) = -4$ to determine $y(1.01)$, $y(1.02)$ and $y(1.03)$.

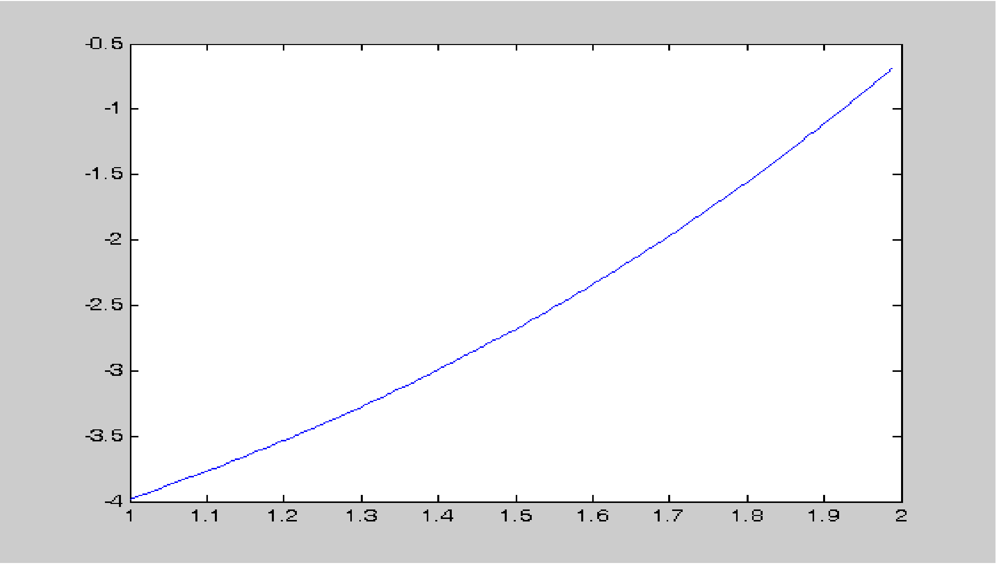
Solution: $f(x, y) = 1 + x^2, \quad x_0 = 1, \quad y_0 = -4, \quad h = 0.01$

Euler Method $y_{i+1} = y_i + h\ f(x_i, y_i)$

$Step1: \quad y_1 = y_0 + h\ f(x_0, y_0) = -4 + 0.01(1 + (1)^2) = -3.98$

$Step2: \quad y_2 = y_1 + h\ f(x_1, y_1) = -3.98 + 0.01\left(1 + (1.01)^2\right) = -3.9598$

$Step3: \quad y_3 = y_2 + h\ f(x_2, y_2) = -3.9598 + 0.01\left(1 + (1.02)^2\right) = -3.9394$



A graph of the solution of the ODE for 1<x<2

i	x_i	y_i	True value of y_i
0	1.00	-4.00	-4.00
1	1.01	-3.98	-3.97990
2	1.02	-3.9595	-3.95959
3	1.03	-3.9394	-3.93909

Comparison with true value:

Example: solve

$$y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$$

Solution: for $t_0 = 0, \quad y_0 = y(0) = 1$

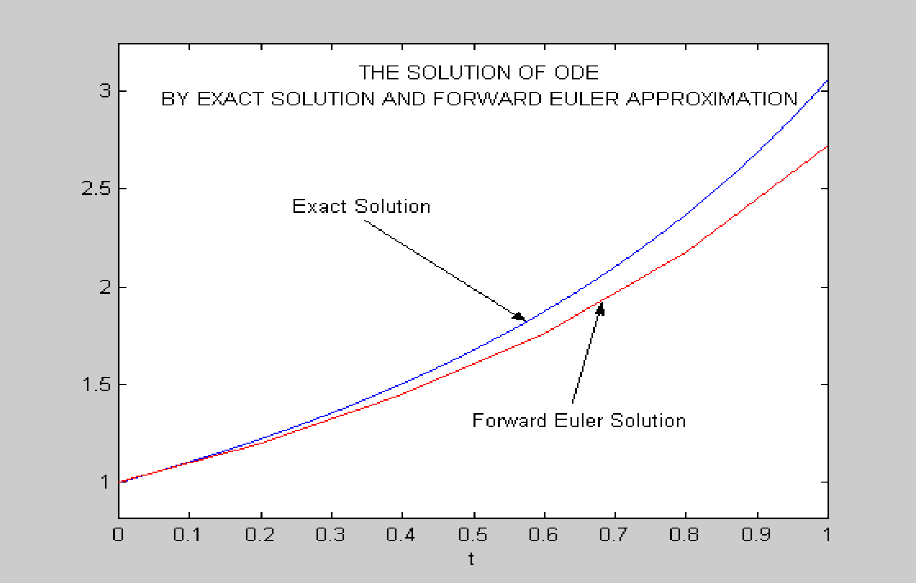
for $t_1 = 0.25,$

$$y_1 = y_0 + hy_0' = y_0 + h(t_0 y_0 + 1)$$
$$= 1 + 0.25(0 \cdot 1 + 1) = 1.25$$

for $t_2 = 0.5,$

$$y_2 = y_1 + hy_1' = y_1 + h(t_1 y_1 + 1)$$
$$= 1.25 + 0.25(0.25 \cdot 1.25 + 1) = 1.5781$$

A graph of the solution of the ODE for 0<t<1



Backward Euler Method

- Consider the backward difference approximation for first derivative

$$y_n' \cong \frac{y_n - y_{n-1}}{h}, \quad h = t_n - t_{n-1}$$

- Rewriting the above equation we have

$$y_n = y_{n-1} + hy_n', \qquad y_n' = f(y_n, t_n)$$

- So, y_n is recursively calculated as

$$\begin{aligned} y_1 &= y_0 + hy_1' = y_0 + h\, f(y_1, t_1) \\ y_2 &= y_1 + h\, f(y_2, t_2) \\ &\vdots \\ y_n &= y_{n-1} + h\, f(y_n, t_n) \end{aligned}$$

Example: Solve $y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$

Solution: Solving the problem using backward Euler method for y_n yields

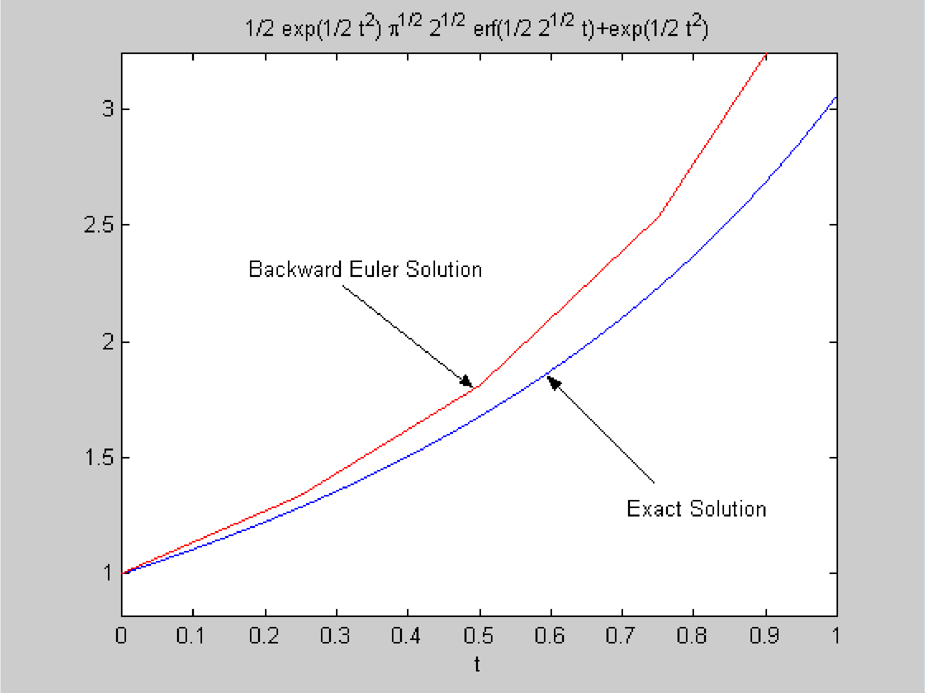
$$y_n = y_{n-1} + hy_n' = y_{n-1} + h(t_n y_n + 1) \Leftrightarrow y_n - ht_n y_n = y_{n-1} + h \Leftrightarrow y_n = \frac{y_{n-1} + h}{1 - ht_n}$$

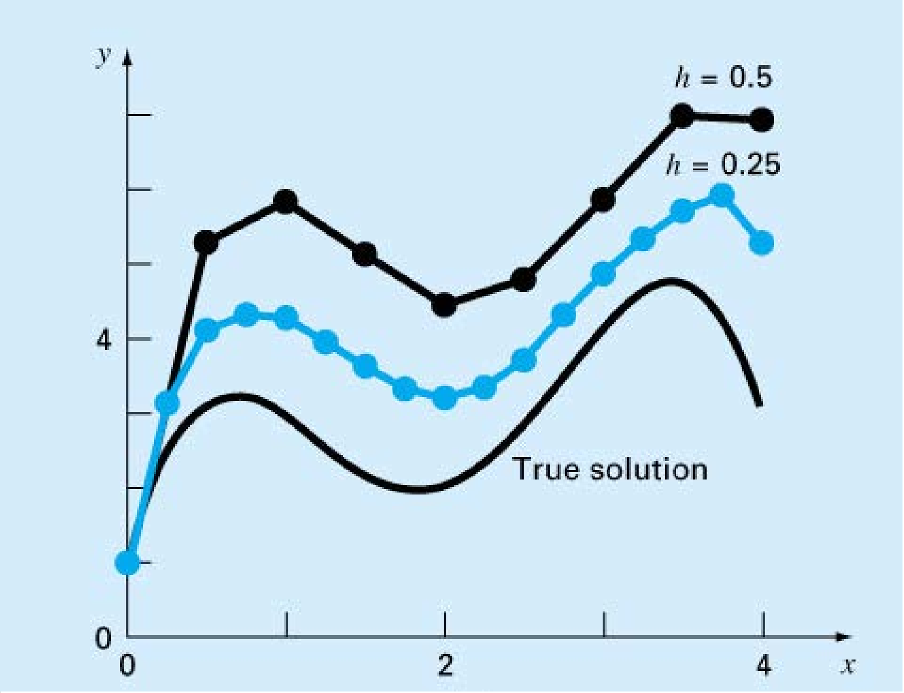
$$\text{for} \quad t_1 = 0.25, \qquad y_1 = \frac{y_0 + h}{1 - ht_1} = \frac{1 + 0.25}{1 - 0.25 * 0.25} = 1.333$$

$$\text{for} \quad t_2 = 0.5, \qquad y_2 = \frac{y_1 + h}{1 - ht_2} = \frac{1.333 + 0.25}{1 - 0.25 * 0.5} = 1.8091$$

$$\text{for} \quad t_3 = 0.75, \qquad y_3 = \frac{y_2 + h}{1 - ht_3} = \frac{1.8091 + 0.25}{1 - 0.25 * 0.75} = 2.5343$$

$$\text{for} \quad t_4 = 1, \qquad y_4 = \frac{y_3 + h}{1 - ht_4} = \frac{2.5343 + 0.25}{1 - 0.25 * 1} = 3.7142$$





- Although the computation captures the general trend solution, the error is still considerable. This error can be reduced by using a smaller step size.

Error Analysis for Euler's Method

- Numerical solutions of ODEs involves two types of error:
  - *Truncation* error
    - *Local truncation error* $E_a = O(h^2) = \frac{f'(x_i, y_i)}{2!} h^2$
    - *Global truncation error* $E_a = O(h)$
  - *Round-off* errors

Programming Euler Method

Write a MATLAB program to implement Euler method to solve:

$$v' = 1 - 2v^2 - t. \quad v(0) = 1 \quad \text{for} \quad t_i = 0.01i, \quad i = 1,2,\ldots,100$$

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for i=1:100
   v=v+h*f(t,v)
   t=t+h;
   T(i+1)=t;
   V(i+1)=v;
end
```

Definition of the ODE

Initial condition

Storing information

Main loop

Euler method

Runge-Kutta Methods

- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.

- **First order RK** method with $n = 1$ is **Euler's method**.
  - Global error is proportional to $O(h)$
- **Second order RK methods:** $y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$
  - Error is proportional to $O(h^2)$

- Given $\dfrac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0$ Taylor series expansion to the second order term is $y_{i+1} = y_i + h\dfrac{dy}{dx} + \dfrac{h^2}{2}\dfrac{d^2y}{dx^2} + O(h^3)$

or $$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + O(h^3)$$

but $$f'(x_i, y_i) = \frac{\partial f(x_i, y_i)}{\partial x} + \frac{\partial f(x_i, y_i)}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x, y)$$

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i)\right)\frac{h^2}{2!} + O(h^3)}$$

where $k_1 = f(x_i, y_i) \quad , \quad k_2 = f(x_i, y_i) + \dfrac{\partial f}{\partial x}\alpha\, h + \dfrac{\partial f}{\partial y}\beta\, k_1 h$

- Values of a_1, a_2, α, and β are evaluated by setting the second order equation to Taylor series expansion.
- We replace k_1 and k_2 in $y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$ to get

$$y_{i+1} = y_i + \left\{ a_1 f(x_i, y_i) + a_2 \left(f(x_i, y_i) + \frac{\partial f}{\partial x} \alpha\, h + \frac{\partial f}{\partial y} \beta\, k_1 h \right) \right\} h$$

or $$y_{i+1} = y_i + a_1 h\, f(x_i, y_i) + a_2 h\, f(x_i, y_i) + a_2\, \alpha\, h^2 \frac{\partial f}{\partial x} + a_2\, \beta\, f(x_i, y_i) h^2 \frac{\partial f}{\partial y}$$

Compare with $$y_{i+1} = y_i + f(x_i, y_i)h + \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i) \right] \frac{h^2}{2!}$$

Obtaining **three** equations to evaluate **four** unknowns constants.

$$a_1 + a_2 = 1 \quad ; \quad a_2 \alpha = \frac{1}{2} \quad ; \quad a_2 \beta = \frac{1}{2}$$

- Then $a_1 = 1 - a_2$, $\alpha = \dfrac{1}{2a_2}$, $\beta = \dfrac{1}{2a_2}$ where $a_2 \neq 0$.
- Because we can choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.
- Every version would yield exactly the same results if the solution to ODE were quadratic, linear or a constant. However, they yield different results if the solution is more complicated (typically the case).
- Three of the most commonly used methods are:

Case 1: $a_2 = ½$ This is **Heun's Method** with a single corrector. Note that k_1 is the slope at the beginning of the interval and k_2 is the slope at the end of the interval.

$$a_1 = 1 - a_2 = 1 - 1/2 = 1/2 \quad , \quad a_2\alpha = \frac{1}{2} \quad , \quad a_2\beta = \frac{1}{2}$$

$$\alpha = \beta = \frac{1}{2a_2} = 1 \Rightarrow \boxed{y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h}$$

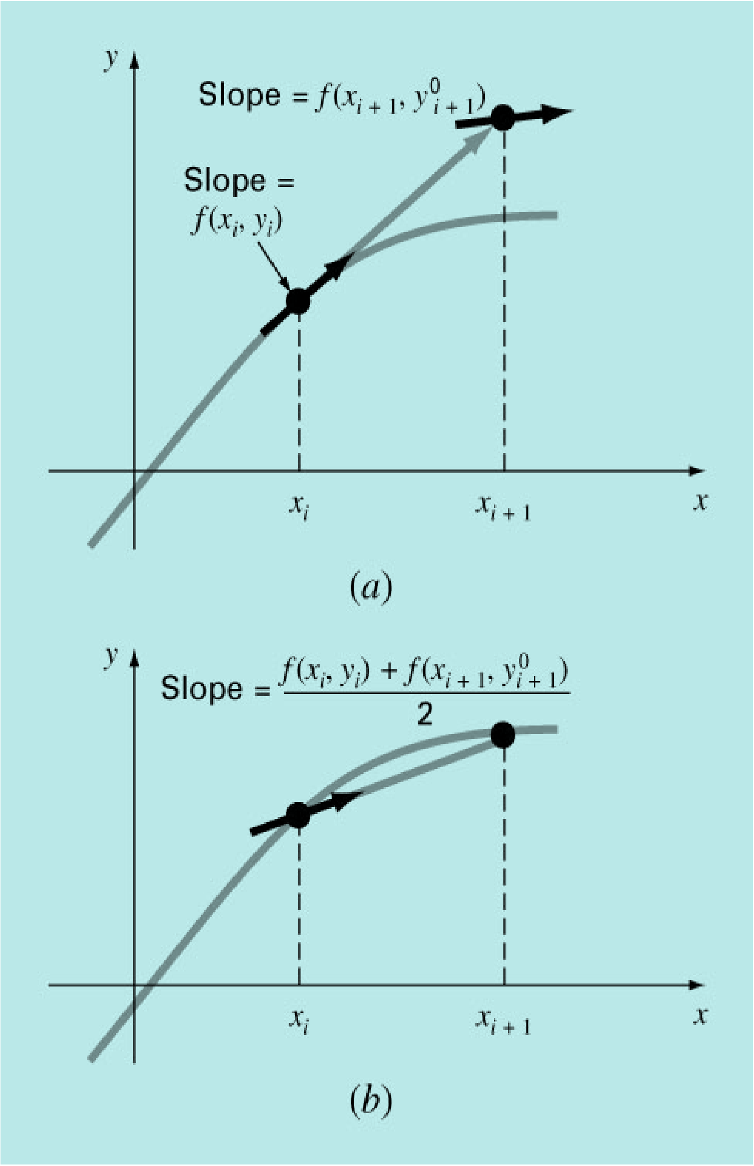
where $\quad k_1 = f(x_i, y_i) \quad , \quad k_2 = f(x_i + h, y_i + k_1 h)$

Local error is $O(h^3)$ and global error is $O(h^2)$

- To improve the estimate of the slope, determine two derivatives for the interval:
  - At the initial point
  - At the end point
- The two derivatives are then averaged to obtain an improved estimate of the slope for the entire interval.
- If *f* is linear in *y*, we can solved similar as backward Euler method
- If *f* is nonlinear in *y*, we necessary to used the method for solving nonlinear equations i.e. successive substitution method (*fixed point*)

$$\textbf{\textit{Predictor}}: \quad y_{i+1}^0 = y_i + f(x_i, y_i)h$$

$$\textbf{\textit{Corrector}}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}h$$



Case 2: $a_2 = 1$ This is the **Improved Polygon Method**. In this method points are averaged instead of slopes.

$$a_1 = 1 - a_2 = 1 - 1 = 0 \quad , \quad a_2\alpha = \frac{1}{2} \quad , \quad a_2\beta = \frac{1}{2}$$

$$\alpha = \beta = \frac{1}{2a_2} = \frac{1}{2} \Rightarrow \boxed{y_{i+1} = y_i + k_2 h}$$

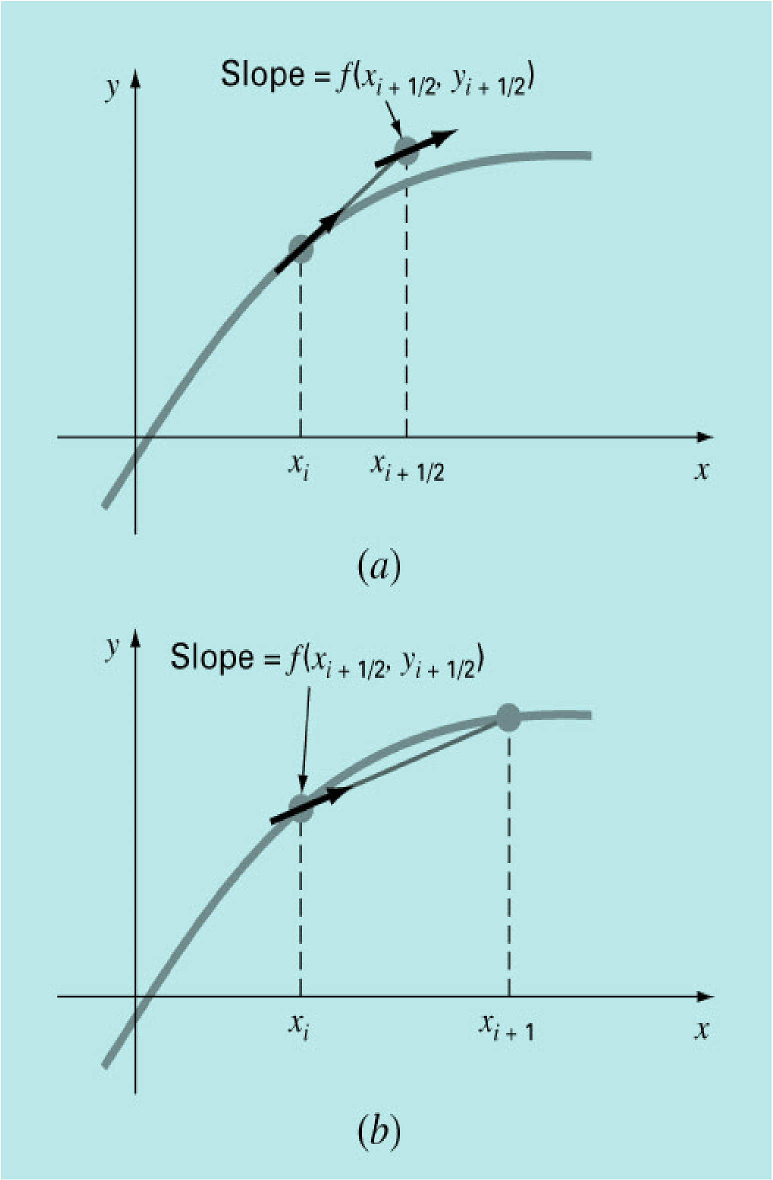
where $\quad k_1 = f(x_i, y_i) \; , \; k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$

Local error of order $O(h^3)$ and Global error is $O(h^2)$

- Euler method is used to estimate the solution at the midpoint.
- The value of the rate function $f(x,y)$ at the mid point is calculated.

$$\boldsymbol{Predictor}: \quad y^0_{i+\frac{1}{2}} = y_i + f(x_i, y_i)\frac{h}{2}$$

$$\boldsymbol{Corrector}: \quad y_{i+1} = y_i + f(x_{i+\frac{1}{2}}, y^0_{i+\frac{1}{2}})h$$



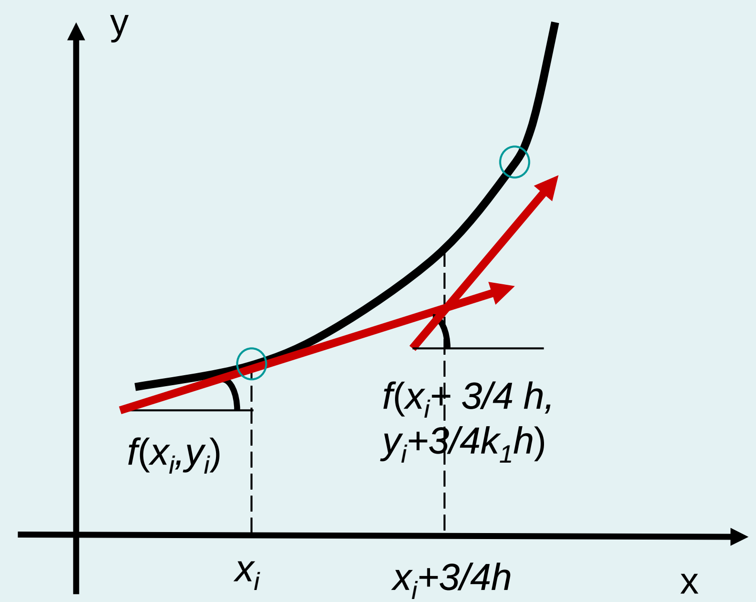
Case 3: a_2 = 2/3 This is the **Ralston's Method**.

$$a_1 = 1 - a_2 = 1 - \frac{2}{3} = \frac{1}{3} \quad , \; a_2\alpha = \frac{1}{2} \quad , \; a_2\beta = \frac{1}{2}$$

$$\alpha = \beta = \frac{1}{2a_2} = \frac{3}{4} \Rightarrow \boxed{y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h}$$

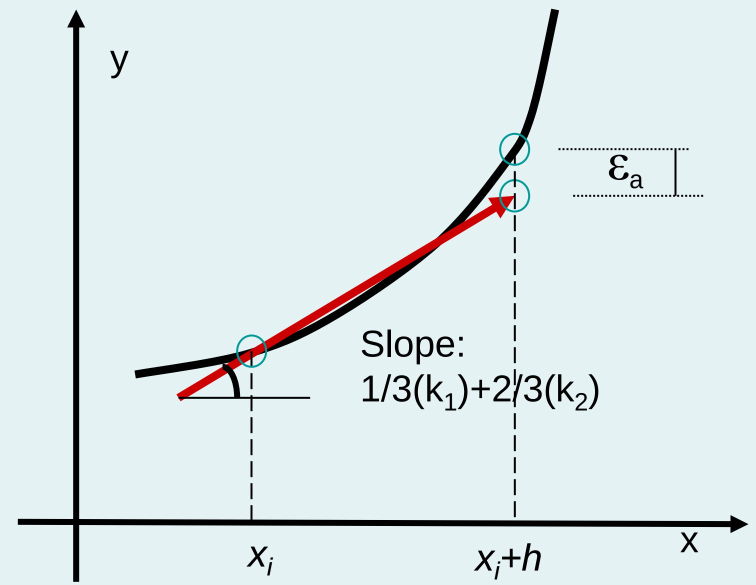
where $k_1 = f(x_i, y_i) \; , \; k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$

Local error of order $O(h^3)$ and Global error is $O(h^2)$



$$\textbf{\textit{Predictor}}: \quad y^0_{i+1} = y_i + f(x_i, y_i)\frac{3}{4}h$$

$$\textbf{\textit{Corrector}}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i)}{3}h + \frac{2f(x_{i+\frac{3}{4}}, y^0_{i+\frac{3}{4}})}{3}h$$



Summary

Three of the most commonly used methods are:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \qquad k_1 = f(x_i, y_i)$$

- Heun Method with a Single Corrector (a_2=1/2)

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \qquad k_2 = f\left(x_i + h, y_i + k_1 h\right)$$

- The Midpoint Method (a_2= 1)

$$y_{i+1} = y_i + k_2 h \qquad k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

- Raltson's Method (a_2= 2/3)

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h \qquad k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

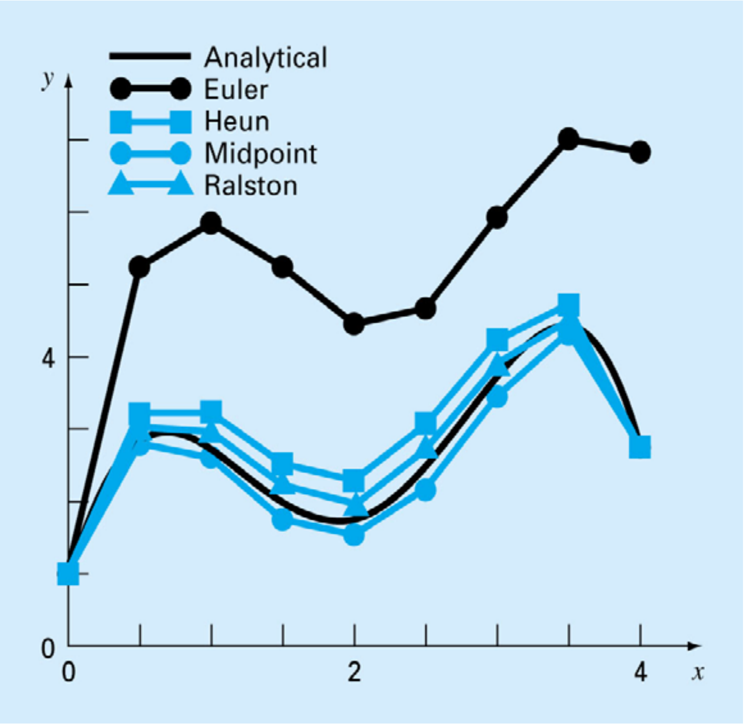
- Euler, Midpoint and Heun's methods are similar in the following sense:

$$y_{i+1} = y_i + h \times slope$$

- Different methods use different estimates of the slope. Both Midpoint and Heun's methods are comparable in accuracy to the second order Taylor series method.

Comparison of Various Second-Order RK Methods

Method	Local error	Global error
Euler Method $\quad y_{i+1} = y_i + h\ f(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor: $y^0_{i+1} = y_i + h\ f(x_i, y_i)$ Corrector: $y^{k+1}_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y^k_{i+1})\right)$	$O(h^3)$	$O(h^2)$
Midpoint $\quad y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\ f(x_i, y_i)$ $y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$



Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1k_1 + a_2k_2 + \cdots + a_nk_n$$

Increment function

$$k_1 = f(x_i, y_i)$$

a's, p's and q's are constants

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1}h, y_i + q_{n-1}k_1h + q_{n-1,2}k_2h + \cdots + q_{n-1,n-1}k_{n-1}h)$$

- ***k*'s** are recurrence functions. Because each ***k*** is a functional evaluation, this recurrence makes RK methods efficient for computer calculations.
- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.

Example: Solve the following system to find *x*(1.02) using Raltson's Method

$$\dot{x}(t) = 1 + x^2 + t^3, \qquad x(1) = -4, \; h = 0.01$$

Solution: STEP1

$$k_1 = f(t_0 = 1, x_0 = -4) = 1 + x_0^{\,2} + t_0^{\,3} = 18$$

$$k_2 = f(t_0 + \frac{3h}{4}, x_0 + k_1\frac{3h}{4})$$

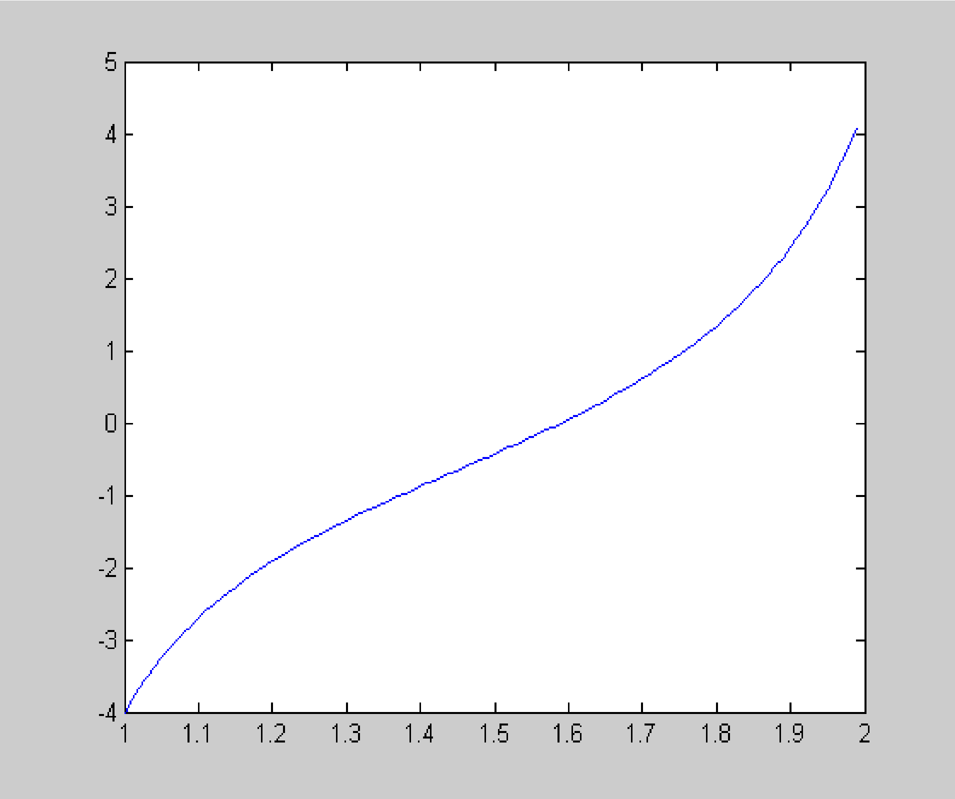
$$= 1 + \left(x_0 + 0.18(\frac{3}{4})\right)^2 + (t_0 + \frac{.03}{4})^3$$

$$= 16.96089417$$

$$x(1+0.01) = x(1) + h\left(\frac{k_1}{3} + \frac{2k_2}{3}\right)$$

$$= -4 + (\frac{0.18}{3} + \frac{0.3392178834}{3})$$

$$= -3.826927372$$



STEP 2

$$k_1 = f(t_1 = 1.01, x_1 = -3.8269) = 1 + x_1^{\,2} + t_1^{\,3} = 16.68$$

$$k_2 = f(t_1 + \frac{3h}{4}, x_1 + k_1\frac{3h}{4}) = 1 + \left(x_1 + 0.1668(\frac{3}{4})\right)^2 + (t_1 + \frac{.03}{4})^3 = 15.75674735$$

$$x(1.01+0.01) = x(1.01) + h\left(\frac{k_1}{3} + \frac{2k_2}{3}\right) = -3.8269 + (\frac{0.1668}{3} + \frac{0.315134947}{3}) = -3.666255018$$

Example: Solve the following system to find $y(1.01)$, $y(1.02)$ using Heun Method

Solution: $$\frac{dy}{dx} = 1 + y^2 + x^3 = f(x,y), \qquad y(1) = -4 \quad , \quad h = 0.01$$

$$\text{Step 1:} \qquad x_0 = 1 \quad , \quad y_0 = -4$$

$$k_1 = f(x_0, y_0) = (1 + y_0^{\,2} + x_0^{\,3}) = 18.0$$

$$k_2 = f(x_0 + h, y_0 + k_1 h) = (1 + (y_0 + 0.18)^2 + (x_0 + .01)^3) = 16.92$$

$$y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = -4 + \frac{0.01}{2}(18 + 16.92) = -3.8254$$

$$\text{Step 2:} \quad x_1 = 1.01 \quad , \quad y_1 = -3.8254$$

$$k_1 = f(x_1, y_1) = (1 + y_1^{\,2} + x_1^{\,3}) = 16.66$$

$$k_2 = f(x_1 + h, y_1 + k_1 h)$$

$$= (1 + (y_1 + 0.1666)^2 + (x_1 + .01)^3) = 15.45$$

$$y_2 = y_1 + \frac{h}{2}(k_1 + k_2)$$

$$= -3.8254 + \frac{0.01}{2}(16.66 + 15.45) = -3.6648$$

i	x_i	y_i
0	1.00	-4.0000
1	1.01	-3.8254
2	1.02	-3.6648

Summary of the solution

Example: Find $y(0.1)$ and $y(0.2)$ if $y' = y - x$, $y(0) = 2$ take $h = 0.1$, three decimal places are required. (using Heun Method)

Solution: To get $y_1 = y(0.1)$

$$y_1^0 = y_0 + h \cdot f(x_0, y_0) = 2 + 0.1(2-0) = 2.2$$

self iteration

$$y_1^1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^0)] = 2 + \frac{0.1}{2}[(2-0) + (2.2-0.1)] = 2.205$$

self iteration

$$y_1^2 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^1)] = 2 + \frac{0.1}{2}[(2-0) + (2.205-0.1)] = \underline{2.205}25$$

$\boxed{\text{Then } y(0.1) = 2.205}$

To get $y_2 = y(0.2)$

$$y_2^1 = y_1 + h \cdot f(x_1, y_1) = 2.205 + 0.1(2.205 - 0.1) = 2.416$$

$$y_2^1 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^0)] = 2.205 + \frac{0.1}{2}[(2.205-0.1) + (2.416-0.2)] = \underline{2.421}05$$

$$y_2^2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^1)] = 2.205 + \frac{0.1}{2}[(2.205-0.1) + (2.421-0.2)] = \underline{2.421}3$$

$\boxed{\text{Then } y(0.2) = 2.241}$

Example: From $f(x,y)=1+x^2+y,\; y_0=y(0)=1, h=0.1$ find $y(0.2)$

Solution: Step 1:

$$y_{0+\frac{1}{2}}=y_0+\frac{h}{2}\,f(x_0,y_0)=1+0.05(1+0+1)=1.1$$

$$y_1=y_0+h\;f(x_{0+\frac{1}{2}},y_{0+\frac{1}{2}})=1+0.1(1+0.0025+1.1)=1.2103$$

Step 2:

$$y_{1+\frac{1}{2}}=y_1+\frac{h}{2}\,f(x_1,y_1)=1.2103+.05(1+0.01+1.2103)=1.3213$$

$$y_2=y_1+h\;f(x_{1+\frac{1}{2}},y_{1+\frac{1}{2}})=1.2103+0.1(2.3438)=1.4446$$

Midpoint Method

Step 1:

$$\text{Predictor: } y_1^0=y_0+h\;f(x_0,y_0)=1+0.1(2)=1.2$$

$$\text{Corrector: } y_1^1=y_0+\frac{h}{2}\left(f(x_0,y_0)+f(x_1,y_1^0)\right)=1.2105$$

Step 2:

$$\text{Predictor: } y_2^0=y_1+h\;f(x_1,y_1)=1.4326$$

$$\text{Corrector: } y_2^1=y_1+\frac{h}{2}\left(f(x_1,y_1)+f(x_2,y_2^0)\right)=1.4452$$

Heun Method

Example: solve $y' = ty + 1, \quad y_0 = y(0) = 1, \quad 0 \le t \le 1, \quad h = 0.25$

Solution: f is linear in y. So, solving the problem using modified Euler method for y_n yields

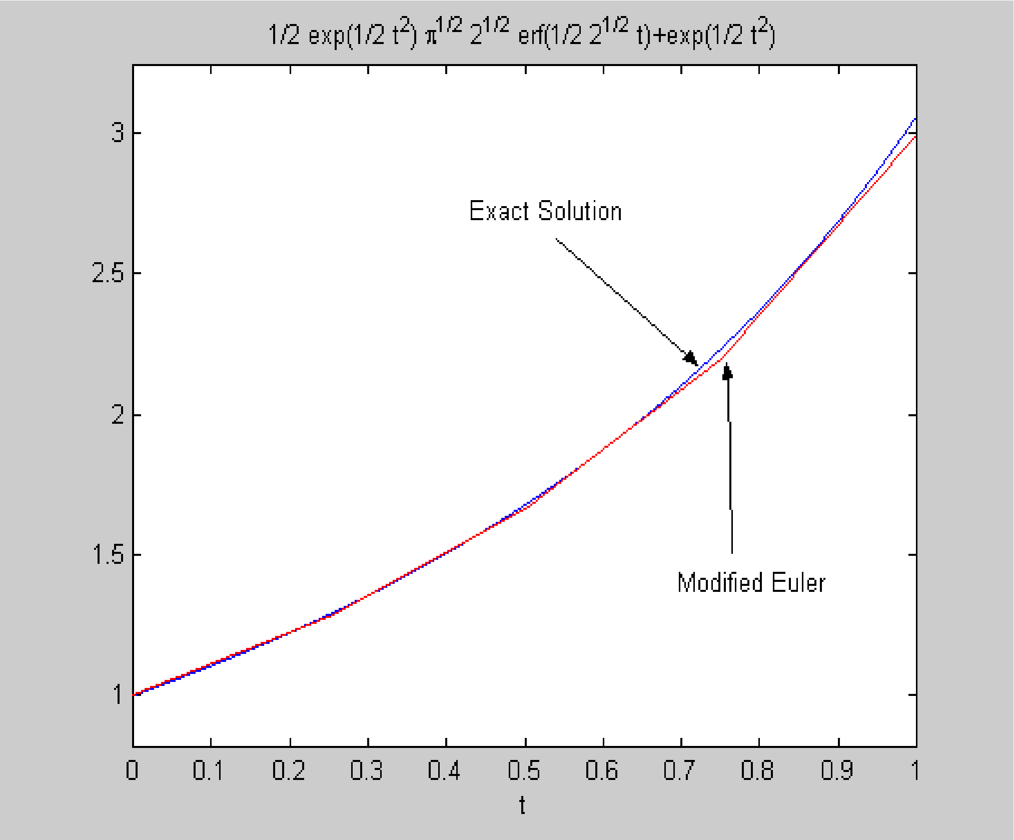
$$y_n = y_{n-1} + \frac{h}{2}(y'_{n-1} + y'_n)$$

$$= y_{n-1} + \frac{h}{2}(t_{n-1}y_{n-1} + 1 + t_n y_n + 1)$$

Written in iterative form:

$$y_n(1 - \frac{h}{2}t_n) = y_{n-1}(1 + \frac{h}{2}t_{n-1}) + h$$

$$y_n = \frac{(1 + \frac{h}{2}t_{n-1})}{(1 - \frac{h}{2}t_n)} y_{n-1} + h$$



Exercises: Solve the following ODE $\dot{y}(x) = 1 + x^2 + y \quad , \quad y(0) = 1$

Use $h = 0.1$. Determine y(0.1) and y(0.2)

a) Use the Heun's Method

b) Use the Midpoint Method

Third Order Runge-Kutta Method

- The third order Runge-Kutta (RK-3) method is derived by applying the Simpson's 1/3 rule to integrating $y' = f(x, y)$ over $[x_n, x_{n+1}]$.
- So, we have

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)dt = y_n + \frac{h}{6}\left(f(x_n, y_n) + 4f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + f(x_{n+1}, y_{n+1}) \right)$$

- We estimate $y_{n+1/2}$ by the forward Euler method.
- The estimate y_{n+1} may be obtained by forward difference method, central difference method for *h*/2, or linear combination both forward and central difference method. One of RK-3 scheme is written as

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$$

$$\text{where} \quad k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

Local error is $O(h^4)$ and Global error is $O(h^3)$

Fourth Order Runge Kutta Method

- The fourth order Runge-Kutta (RK-4) method is derived by applying the Simpson's 1/3 or Simpson's 3/8 rule to integrating $y' = f(x, y)$ over the interval $[x_n, x_{n+1}]$. The formula of RK-4 based on **the Simpson's 1/3** is written as

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$\text{where} \quad k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f\left(x_i + h, y_i + hk_3\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

- Note that for ODE that are a function of x alone that this is also the equivalent of Simpson's 1/3 Rule

- The fourth order Runge-Kutta (RK-4) method is derived based on **Simpson's 3/8 rule** is written as

$$y_{n+1} = y_n + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

$$\text{where} \quad k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1)$$

$$k_3 = f(x_n + \frac{2}{3}h, y_n + \frac{1}{3}hk_1 + \frac{1}{3}hk_2,)$$

$$k_4 = f(x_n + h, y_n + 3hk_1 - 3hk_2 + hk_3,)$$

Example: Find $y(0.1)$ and $y(0.2)$ if $y' = y - x, \quad y(0) = 2, \quad h = 0.1$ four decimal places are required.

Solution: To get $y_1 = y(0.1)$

$$k_1 = hf(x_n, y_n) = 0.1(2-0) = 0.2$$

$$k_2 = hf\left[(x_n + \frac{h}{2}), (y_n + \frac{k_1}{2})\right] = 0.1(2.1 - 0.05) = 0.205$$

$$k_3 = hf\left[(x_n + \frac{h}{2}), (y_n + \frac{k_2}{2})\right] = 0.1(2.1025 - 0.05) = 0.2053$$

$$k_4 = hf\left[(x_n + h), (y_n + k_3)\right] = 0.1(2.2053 - 0.1) = 0.2105$$

$$\therefore\ y = y_0 + \frac{k_1 + 2(k_2 + k_3) + k_4}{6} = 2 + \frac{0.2 + 2(0.205 + 0.2053) + 0.2105}{6} = \underline{2.2051}833.$$

Then y(0.1) = 2.2052

To get $y_2 = y(0.2)$

$$k_1 = hf(x_n, y_n) = 0.1(2.2052 - 0.1) = 0.2105$$

$$k_2 = hf\left[(x_n + \frac{h}{2}), (y_n + \frac{k_1}{2})\right] = 0.1(2.3105 - 0.15) = 0.2161$$

$$k_3 = hf\left[(x_n + \frac{h}{2}), (y_n + \frac{k_2}{2})\right] = 0.1(2.3133 - 0.15) = 0.2163$$

$$k_4 = hf\left[(x_n + h), (y_n + k_3)\right] = 0.1(2.4215 - 0.2) = 0.2222$$

$$\therefore\ y_2 = y_1 + \frac{k_1 + 2(k_2 + k_3) + k_4}{6} = 2.2052 + \frac{0.2105 + 2(0.2161 + 0.2163) + 0.2222}{6} = \underline{2.4214}5$$

Then y(0.2) = 2.4215

Example: Use the RK4 method with h = 0.1 to obtain $y(1.5)$ for the solution of $y' = 2xy, \quad y(1) = 1$.

Solution: We first compute the case n = 0.

$$k_1 = f(x_0, y_0) = 2x_0 y_0 = 2$$

$$k_2 = f(x_0 + \tfrac{1}{2}(0.1), y_0 + \tfrac{1}{2}(0.1)2)$$

$$= 2(x_0 + \tfrac{1}{2}(0.1))(y_0 + \tfrac{1}{2}(0.2)) = 2.31$$

$$k_3 = (x_0 + \tfrac{1}{2}(0.1), y_0 + \tfrac{1}{2}(0.1)2.31)$$

$$= 2(x_0 + \tfrac{1}{2}(0.1))(y_0 + \tfrac{1}{2}(0.231)) = 2.34255$$

$$k_4 = f(x_0 + 0.1, y_0 + (0.1)2.34255)$$

$$= 2(x_0 + 0.1)(y_0 + 0.234255) = 2.715361$$

$$y_1 = y_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{0.1}{6}(2 + 2(2.31) + 2(2.34255) + 2.715361)$$

$$= 1.23367435$$

Comparison of Numerical Methods with $h = 0.1$

x_n	Euler	Improved Euler	RK4	Actual Value
1.00	1.0000	1.0000	1.0000	1.0000
1.10	1.2000	1.2320	1.2337	1.2337
1.20	1.4640	1.5479	1.5527	1.5527
1.30	1.8154	1.9832	1.9937	1.9937
1.40	2.2874	2.5908	2.6116	2.6117
1.50	2.9278	3.4509	3.4902	3.4904

Comparison of Numerical Methods with $h = 0.05$

x_n	Euler	Improved Euler	RK4	Actual Value
1.00	1.0000	1.0000	1.0000	1.0000
1.05	1.1000	1.1077	1.1079	1.1079
1.10	1.2155	1.2332	1.2337	1.2337
1.15	1.3492	1.3798	1.3806	1.3806
1.20	1.5044	1.5514	1.5527	1.5527
1.25	1.6849	1.7531	1.7551	1.7551
1.30	1.8955	1.9909	1.9937	1.9937
1.35	2.1419	2.2721	2.2762	2.2762
1.40	2.4311	2.6060	2.6117	2.6117
1.45	2.7714	3.0038	3.0117	3.0117
1.50	3.1733	3.4795	3.4903	3.4904

Example: Use the RK4 method to find $y(0.2)$ and $y(0.4)$ for the solution of

Solution: $$\frac{dy}{dx} = 1 + y + x^2 \quad , \quad y(0) = 0.5 \ , \ h = 0.2$$

Step 1

$$\begin{cases} k_1 = f(x_0, y_0) = (1 + y_0 + x_0^{\ 2}) = 1.5 \\ k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + (y_0 + 0.15) + (x_0 + 0.1)^2 = 1.64 \\ k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + (y_0 + 0.164) + (x_0 + 0.1)^2 = 1.654 \\ k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + (y_0 + 0.16545) + (x_0 + 0.2)^2 = 1.7908 \\ y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8293 \end{cases}$$

Step 2
$y(0.2) = 0.8293$

$$\begin{cases} k_1 = f(x_1, y_1) = 1.7893 \\ k_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h) = 1.9182 \\ k_3 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h) = 1.9311 \\ k_4 = f(x_1 + h, y_1 + k_3 h) = 2.0555 \\ y_2 = y_1 + \frac{0.2}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2141 \end{cases}$$

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141

Summary of the solution

Exercise: Use 4th-Order Runge-Kutta Method to compute $y(0.2)$ and $y(0.4)$

$$\frac{dy}{dx} = 1 + y + x^2 \quad , \quad y(0) = 0.5 \quad , \quad h = 0.1$$

Summary

- Runge Kutta methods generate an accurate solution without the need to calculate high order derivatives.
- Second order RK have local truncation error of order $O(h^3)$ and global truncation error of order $O(h^2)$.
- Higher order RK have better local and global truncation errors.
- N function evaluations are needed in the N^{th} order RK method.
- Using smaller values of h and using Euler's or Heun's methods probably gives better accuracy than using RK4 with larger values of h.

Higher-Order Runge-Kutta

- Higher order Runge-Kutta methods are available.
- Derived similar to second-order Runge-Kutta.
- Higher order methods are more accurate but require more calculations.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

Increment function(representative slope over the interval)

$$\boxed{\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n} \; ; \; a's = \text{constants}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

$$p's \text{ and } q's \text{ are constants}$$

Fifth-Order Runge-Kutta Method

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h\right)$$

$$k_6 = f\left(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h\right)$$

$$y_{i+1} = y_i + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$$

Taylor Series in One Variable

The n^{th} order Taylor Series expansion of $f(x)$

$$f(x+h) = \boxed{\sum_{i=0}^{n} \frac{h^i}{i!} f^{(i)}(x)} + \boxed{\frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\bar{x})}$$

Error

Approximation

where $\bar{x}$ is between x and $x+h$

Taylor Series in Two Variables

The n^{th} order Taylor Series expansion of $f(x)$

Error

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\, \partial y} \right) + \ldots$$

$$= \boxed{\sum_{i=0}^{n} \frac{1}{i!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^i f(x,y)} + \boxed{\frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(\bar{x}, \bar{y})}$$

Approximation

$(\bar{x}, \bar{y})$ is on the line joining between (x,y) and $(x+h, y+k)$

Systems of Equations

- Methods discussed earlier such as Euler, Runge-Kutta,… are used to solve first order ordinary differential equations.
- The same formulas will be used to solve a system of first order ODEs.
  - In this case, the differential equation is a vector equation and the dependent variable is a vector variable.
- Many practical problems in engineering and science require the solution of a system of simultaneous ordinary differential equations (ODEs) rather than a single equation:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$
$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$
$$\vdots$$
$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

- Solution requires that *n* initial conditions be known at the starting value of *x*.

  i.e. $(x_0, y_1(x_0), y_2(x_0), \ldots, y_n(x_0))$
- At iteration *i*, *n* values $(y_1(x_i), y_2(x_i), \ldots, y_n(x_i))$ are computed.

System Ordinary Differential Equation

Taylor Series Method for Simultaneous First-order Differential Equations

EXAMPLE: Solve the differential equations using the Taylor series

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy, \quad \text{for } x = 0.3 \;, \; x = 0, \; y = 0, \; z = 1.$$

Solution: Taylors series for y' is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$y' = 1 + xz \quad \Rightarrow \quad y_0' = 1 + x_0 z_0 = 1 + (0)(1) = 1$$

$$y'' = xz' + z \quad \Rightarrow \quad y_0'' = x_0 z_0' + z_0$$

$$= x_0(-x_0 y_0) + z_0 = 0 + 1 = 1$$

$$y''' = x z'' + z' + z' \quad \Rightarrow \quad y_0''' = x_0 \; z_0'' + 2z_0'$$

$$= x_0(-x_0 y_0' - y_0) + 2(x_0 y_0) = 0$$

$$y_1 = 0 + (0.3)(1) + \frac{(0.3)^2}{2}(1) + \frac{(0.3)^3}{6}(0) + \ldots = 0.3 + 0.045 = 0.345$$

$$\boxed{y_1 = y(0.3) = 0.345}$$

Taylor series for z' is given by

$$z_1 = z_0 + h z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

$$z' = -xy \quad \Rightarrow \quad z_0' = -x_0 y_0 = 0$$

$$z'' = -(xy' + y) \quad \Rightarrow \quad z_0'' = -x_0 y_0' - y_0$$

$$= -(0)(1) - 0 = 0$$

$$z''' = -[xy'' + y' + y'] \quad \Rightarrow \quad z_0''' = -x_0 y_0'' - 2y_0'$$

$$= -(0)(1) - 2(1) = -2$$

$$z_1 = 1 + (0.3)(0) + \frac{(0.3)^2}{2}(0) - \frac{(0.3)^3}{6}(2) + \ldots$$

$$= 1 - 0.009 = 0.991$$

$$\boxed{z_1 = z(0.3) = 0.991}$$

- Numerical methods for ordinary differential equations calculate solution on the points $t_n = t_{n-1} + h$ where *h* is the steps size.

$$x' = f(t, x, y) \qquad x(t_0) = x_0, \quad y(t_0) = y_0$$

$$y' = g(t, x, y)$$

Solving by the RK4 method looks like this

$$x_{n+1} = x_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$m_1 = f(t_n, x_n, y_n) \qquad k_1 = g(t_n, x_n, y_n)$$

$$m_2 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_1, y_n + \tfrac{1}{2}hk_1) \qquad k_2 = g(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_1, y_n + \tfrac{1}{2}hk_1)$$

$$m_3 = f(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_2, y_n + \tfrac{1}{2}hk_2) \qquad k_3 = g(t_n + \tfrac{1}{2}h, x_n + \tfrac{1}{2}hm_2, y_n + \tfrac{1}{2}hk_2)$$

$$m_4 = f(t_n + h, x_n + hm_3, y_n + hk_3) \qquad k_4 = g(t_n + h, x_n + hm_3, y_n + hk_3)$$

Example: Use the RK4 method to approximate $x(0.6)$ and $y(0.6)$ with $h = 0.2$ and $h = 0.1$. Consider

$$x' = 2x + 4y$$

$$y' = -x + 6y$$

$$x(0) = -1, \quad y(0) = 6$$

Solution: With $h = 0.2$ and the given data

$$m_1 = f(t_0, x_0, y_0) = f(0, -1, 6) = 2(-1) + 4(6) = 22$$

$$k_1 = g(t_0, x_0, y_0) = g(0, -1, 6) = -1(-1) + 6(6) = 37$$

$$m_2 = f(t_0 + \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}hm_1,\ y_0 + \tfrac{1}{2}hk_1) = f(0.1, 1.2, 9.7) = 41.2$$

$$k_2 = g(t_0 + \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}hm_1,\ y_0 + \tfrac{1}{2}hk_1) = g(0.1, 1.2, 9.7) = 57$$

$$m_3 = f(t_0 + \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}hm_2,\ y_0 + \tfrac{1}{2}hk_2) = f(0.1, 3.12, 11.7) = 53.04$$

$$k_3 = g(t_0 + \tfrac{1}{2}h,\ x_0 + \tfrac{1}{2}hm_2,\ y_0 + \tfrac{1}{2}hk_2) = g(0.1, 3.12, 11.7) = 67.08$$

$$m_4 = f(t_0 + h,\ x_0 + hm_3,\ y_0 + hk_3) = f(0.2, 9.608, 19.416) = 96.88$$

$$k_4 = g(t_0 + h,\ x_0 + hm_3,\ y_0 + hk_3) = g(0.2, 9.608, 19.416) = 106.888$$

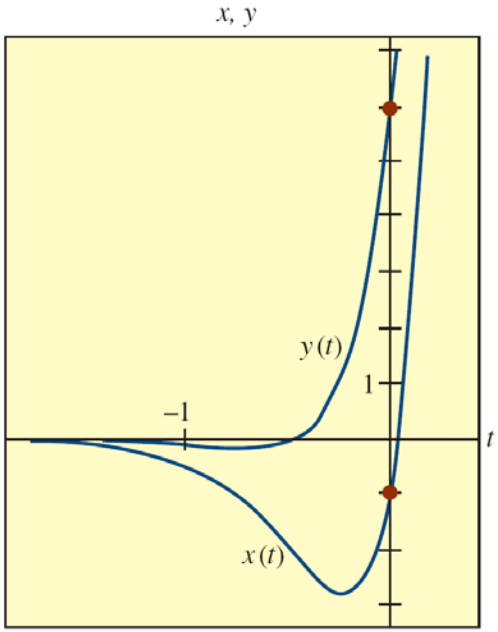


FIGURE 6.4.2 Numerical solution curves for IVP in Example 3

Therefore, we get

$$x_1 = x_0 + \frac{0.2}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$= -1 + \frac{0.2}{6}(2.2 + 2(41.2) + 2(53.04) + 96.88) = 9.2453$$

$$y_1 = y_0 + \frac{0.2}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 6 + \frac{0.2}{6}(37 + 2(57) + 2(67.08) + 106.888) = 19.0683$$

$h = 0.1$

t_n	x_n	y_n
0.00	−1.0000	6.0000
0.10	2.3840	10.8883
0.20	9.3379	19.1332
0.30	22.5541	32.8539
0.40	46.5103	55.4420
0.50	88.5729	93.3006
0.60	160.7563	152.0025

$h = 0.2$

t_n	x_n	y_n
0.00	−1.0000	6.0000
0.20	9.2453	19.0683
0.40	46.0327	55.1203
0.60	158.9430	150.8192

Example: Use the Runge-Kutta method to find the solution at x=0.1

$$\frac{dy_1}{dx} = y_1 - y_2, \qquad y_1(0) = 0$$

$$\frac{dy_2}{dx} = -y_1 + y_2, \qquad y_2(0) = 1$$

at $x = 0.1$, taking $h = 0.1$.

Solution: Here $h = 0.1$, $x_{10} = 0$, $y_{10} = 0.0$, $y_{20} = 1.0$

$$f(x, y_1, z_2) = \frac{dy_1}{dx} = y_1 - y_2$$

$$g(x, y_1, y_2) = -y_1 + y_2$$

To find $y_1(0.1)$:

$$k_1 = hf(x_{10}, y_{10}, z_{10}) = (0.1)\,(y_{10} - y_{20}) = (0.1)\,(0.0 - 1.0) = -0.1$$

$$k_2 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}, y_{20} + \frac{l_1}{2}\right) = (0.1)\,f(0.05, -0.05, 1.05) = -0.11$$

$$k_3 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}, y_{20} + \frac{l_2}{2}\right) = hf(0.05, -0.055, 1.055) = -0.1110$$

$$k_4 = hf(x_{10} + h, y_{10} + k_3, y_{20} + l_3) = (0.1)\,f(0.1, -0.111, 1.111) = -0.1222$$

$$\Delta y_1 = 1/6\,(k_1 + 2k_2 + 2k_3 + k_4) \quad = -\,0.1107$$

$$y_1(0.1) = y_{10} + \Delta y_1 \quad = 0 - 0.1107 \quad = -\,0.1107$$

$$l_1 = hg(x_{10}, y_{10}, y_{20}) = (0.1)\,(-\,y_{10} + y_{20}) \;\; = (0.1)\,(-\,0 + 1) \;\; = 0.1$$

$$l_2 = hg\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}, y_{20} + \frac{l_1}{2}\right) = (0.1)\; g(0.05, -\,0.05, 1.05)$$

$$= (0.1)\,[-\,(-\,0.05) + 1.05] \;\; = 0.11$$

$$l_3 = hg\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}, y_{20} + \frac{l_2}{2}\right) = (0.1)\; g(0.05, -\,0.055, 1.055)$$

$$= (0.1)\,(0.055 + 1.055) = 0.1110$$

$$l_4 = hg(x_{10} + h, y_{10} + k_3, y_{20} + l_3) = (0.1)\; g(0.1, -\,0.111, 1.111)$$

$$= (0.1)\,(0.111 + 1.111) \;\; = 0.1222$$

$$\Delta y_2 = 1/6\,(l_1 + 2l_2 + 2l_3 + l_4) \;\; = 0.1107$$

$$y_2(0.1) = y_{20} + \Delta y_2 \quad = 1 + 0.1107 \;\; = 1.1107$$

$$\boxed{y_1(0.1) = -\,0.1107 \qquad\qquad y_2(0.1) = 1.1107}$$

Systems of High Order ODEs

High order ODE → Convert → System of first order ODE → Solve

- Any n^{th} order ODE is converted to a system of n first order ODEs.
- There are an infinite number of ways to select the new variables. As a result, for each high order ODE there are an infinite number of set of equivalent first order systems of ODEs.
- Use a table to make the conversion easier.

Conversion Procedure

1. **Select the dependent variables.** Take the original dependent variables and their derivatives up to one degree less than the highest order derivative for each variable.
2. **Write the Differential Equations** in terms of the new variables. The equations come from the way the new variables are defined or from the original equation.
3. **Express the equations in a matrix form.**

The general approach to solve high order ODE

High order ODE → Convert → System of first order ODEs → Solve

$$\ddot{y} + 3\dot{y} + 6y = 1$$
$$\dot{y}(0) = 1;\ y(0) = 4$$

Second order ODE

Convert →

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 - 3y_2 - 6y_1 \end{bmatrix},\quad Y(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Two first order ODEs

→ Solve

- Exactly the same formula is used but the scalar variables and functions are replaced by vector variables and vector values functions.
- ***Y*** is a vector of length *n.*
- ***F***(***Y***,*x*) is a vector valued function.

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \dots \\ y_n(x) \end{bmatrix} \quad Y \text{ is } n \times 1 \ vector \Rightarrow \frac{d\,Y(x)}{dx} = \begin{bmatrix} \frac{d\,y_1}{dx} \\ \frac{d\,y_2}{dx} \\ \dots \\ \frac{d\,y_n}{dx} \end{bmatrix} = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \dots \\ f_n(Y,x) \end{bmatrix} = F(Y,x)$$

Example of Conversion:

Convert $\ddot{y}+3\dot{y}+6y=1, \quad \dot{y}(0)=1; \quad y(0)=4$ to a system of first order ODEs

- Select a new set of variables (Second order ODE $\Rightarrow$ We need two variables)

$$y_1 = y \quad , \quad y_2 = \dot{y}$$

One degree less than the highest order derivative

old name	new name	Initial cond.	Equation
y	y_1	4	$\dot{y}_1 = y_2$
$\dot{y}$	y_2	1	$\dot{y}_2 = 1-3y_2-6y_1$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 1-3y_2-6y_1 \end{bmatrix}, Y(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Example of Conversion:

Convert $\dddot{y}+2\ddot{y}+7\dot{y}+8y=0 \quad \ddot{y}(0)=9, \ \dot{y}(0)=1; \quad y(0)=4$

- Select a new set of variables (3 of them)

$$y_1 = y \ , \ y_2 = \dot{y} \ , \ y_3 = \ddot{y}$$

One degree less than the highest order derivative

old name	new name	Initial cond.	Equation
y	y_1	4	$\dot{y}_1 = y_2$
$\dot{y}$	y_2	1	$\dot{y}_2 = y_3$
$\ddot{y}$	y_3	9	$\dot{y}_3 = -2y_3 - 7y_2 - 8y_1$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ -2y_3 - 7y_2 - 8y_1 \end{bmatrix} \;,\; Y(0) = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

Example of Conversion:

Convert $\dddot{x} + 5\ddot{x} + 2\dot{x} + 8y = 0 \;,\; x(0) = 4 \;;\; \dot{x}(0) = 2 \;;\; \ddot{x}(0) = 9;$

$\ddot{y} + 2xy + \dot{x} = 2 \qquad ,\; y(0) = 1 \;;\; \dot{y}(0) = -3$

- Select a new set of variables $((3+2)$ variables)

$z_1 = x \;,\; z_2 = \dot{x} \;,\; z_3 = \ddot{x} \;,\; z_4 = y \;,\; z_5 = \dot{y}$

One degree less than the highest order derivative

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	2	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -5z_3 - 2z_2 - 8z_4$
y	z_4	1	$\dot{z}_4 = z_5$
$\dot{y}$	z_5	-3	$\dot{z}_5 = 2 - z_2 - 2z_1 z_4$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -5z_3 - 2z_2 - 8z_4 \\ z_5 \\ 2 - z_2 - 2z_1 z_4 \end{bmatrix} , \quad Z(0) = \begin{bmatrix} 4 \\ 2 \\ 9 \\ 1 \\ -3 \end{bmatrix}$$

Methods for Solving a System of First Order ODEs

- We have extended Euler and RK2 methods to solve systems of first order ODEs.
- Other methods used to solve first order ODE can be easily extended to solve systems of first order ODEs.

Euler Method for Solving a System of First Order ODEs

Recall Euler method for solving a first order ODE:

$$\boxed{Given\quad \frac{dy(x)}{dx} = f(y,x), \quad y(a) = y_a}$$

$$Euler\ Method:$$

$$y(a+h) = y(a) + h\, f(y(a),a)$$

$$y(a+2h) = y(a+h) + h\, f(y(a+h),a+h)$$

$$y(a+3h) = y(a+2h) + h\, f(y(a+2h),a+2h)$$

Euler method to solve a system of *n* first order ODEs:

$$\boxed{Given\quad \frac{dY(x)}{dx} = F(Y,x) = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \dots \\ f_n(Y,x) \end{bmatrix}, \quad Y(a) = \begin{bmatrix} y_1(a) \\ y_2(a) \\ \dots \\ y_n(a) \end{bmatrix}}$$

$$Euler\ \ Method:$$

$$Y(a+h) = Y(a) + h\ \ F(Y(a),a)$$

$$Y(a+2h) = Y(a+h) + h\, F(Y(a+h),a+h)$$

$$Y(a+3h) = Y(a+2h) + h\, F(Y(a+2h),a+2h)$$

Example of a Second Order ODE: Solve the equation using Euler method. Use h = 0.1 to find $x(0.2)$

Solution: $\ddot{x} + 2\dot{x} + 8x = 2 \;,\; x(0) = 1 \;;\; \dot{x}(0) = -2$

Select a new set of variables: $z_1 = x \;,\; z_2 = \dot{x}$

The second order equation is expressed as:

$$\dot{Z} = F(Z) = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix} ,\; Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$F(Z) = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix} ,\; Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} ,\; h = 0.1$$

$$Z(0+0.1) = Z(0) + hF(Z(0)) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 0.1 \begin{bmatrix} -2 \\ 2 - 2(-2) - 8(1) \end{bmatrix} = \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix}$$

$$Z(0.2) = Z(0.1) + hF(Z(0.1)) = \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix} + 0.1 \begin{bmatrix} -2.2 \\ 2 - 2(-2.2) - 8(0.8) \end{bmatrix} = \begin{bmatrix} 0.58 \\ -2.2 \end{bmatrix}$$

Example: Use the Euler's method to obtain $y(0.2)$, where

Solution: $y'' + xy' + y = 0 \,,\quad y(0) = 1 \,,\quad y'(0) = 2$

Let $y' = u$, then $\qquad y' = u \quad , \quad u' = -xu - y$

$$y_{n+1} = y_n + hu_n$$

$$u_{n+1} = u_n + h[-x_n u_n - y_n]$$

Using h = 0.1, y_0 = 1, u_0 = 2, we find

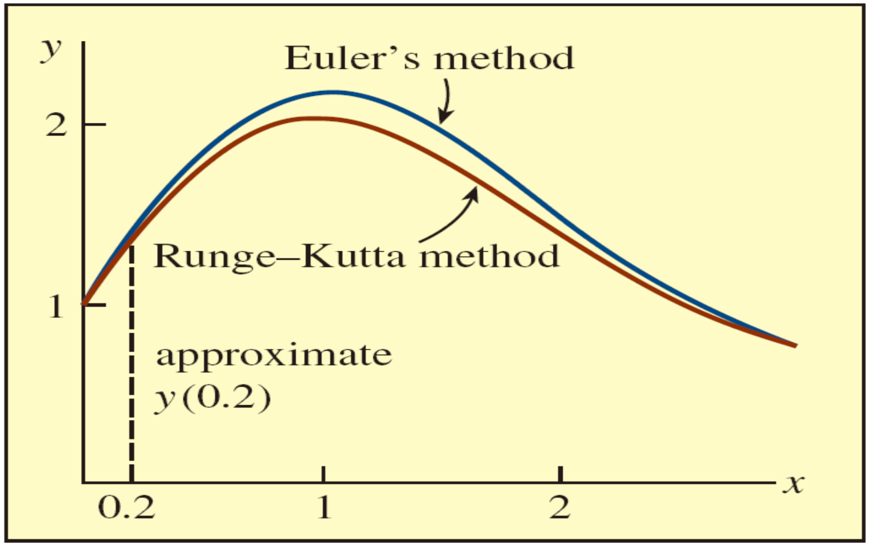
$$y_1 = y_0 + (0.1)u_0 = 1 + (0.1)2 = 1.2$$

$$u_1 = u_0 + (0.1)[-x_0 u_0 - y_0] = 2 + (0.1)[-(0)(2) - 1] = 1.9$$

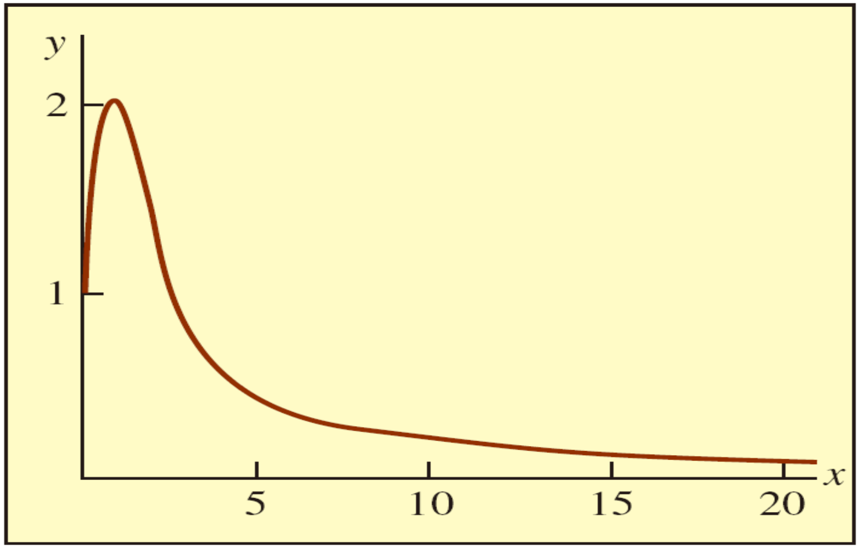
$$y_2 = y_1 + (0.1)u_1 = 1.2 + (0.1)(1.9) = 1.3$$

$$u_2 = u_1 + (0.1)[-x_1 u_1 - y_1] = 1.9 + (0.1)[-(0.1)(1.9) - 1.2] = 1.761$$

In the words, $y(0.2) \approx 1.39$ and $y'(0.2) \approx 1.761$.



(a) Euler's method (blue)
Runge–Kutta method (red)



(b) Runge–Kutta

The comparison of results between by Euler's method and by the RK4 method.

Example : Solve the equation using Euler method to find $y(0.2)$.

Solution: $$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1-y_1 \end{bmatrix} = F(Y,x) \ , \ Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two steps of Euler Method with $h = 0.1$

$$STEP\,1: \quad Y(0+h) = Y(0) + h\,F(Y(0),0)$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0) \\ 1-y_1(0) \end{bmatrix} = \begin{bmatrix} -1+0.1 \\ 1+0.1(1+1) \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}$$

$$STEP\,2: \quad Y(0+2h) = Y(h) + h\,F(Y(h),h)$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0.1) \\ 1-y_1(0.1) \end{bmatrix} = \begin{bmatrix} -0.9+0.12 \\ 1.2+.1(1+0.9) \end{bmatrix} = \begin{bmatrix} -0.78 \\ 1.39 \end{bmatrix}$$

Example : Solve the equation using RK2 method to find $y(0.2)$.

Solution: $$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1-y_1 \end{bmatrix} = F(Y,x) \ , \ Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two steps of second order Runge – Kutta Method with $h = 0.1$

$$STEP\,1: \quad Y(0+h) = Y(0) + 0.5(k_1 + k_2)$$

$$k_1 = h\,F(Y(0),0) = 0.1 \begin{bmatrix} y_2(0) \\ 1-y_1(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$k_2 = hF(Y(0)+k_{1,0}+h) = 0.1\begin{bmatrix} y_2(0)+0.2 \\ 1-(y_1(0)+0.1) \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix}$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix}\right) = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix}$$

$$STEP\ 2: \quad Y(0.1+h) = Y(0.1) + 0.5(k_1 + k_2)$$

$$k_1 = hF(Y(0.1), 0.1) = 0.1\begin{bmatrix} y_2(0.1) \\ 1-y_1(0.1) \end{bmatrix} = \begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix}$$

$$k_2 = hF(Y(0.1)+k_{1,0.1}+h) = 0.1\begin{bmatrix} y_2(0.1)+0.189 \\ 1-(y_1(0.1)+0.1195) \end{bmatrix} = \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix}$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix} + \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix}\right) = \begin{bmatrix} -0.7611 \\ 1.3780 \end{bmatrix}$$

Example : Solve the equation using RK4 method.

Solution:

$$\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$$

Using the Runge–Kutta method for $x = 0.2$, with initial conditions $x = 0, y = 1, y' = 0$.

We know that $\dfrac{dy}{dx} = z$ and $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

$$\frac{dz}{dx} - xz^2 + y^2 = 0 \quad \Longrightarrow \quad \frac{dz}{dx} = xz^2 - y^2$$

$$\frac{dy}{dx} = z \quad , \quad \frac{dz}{dx} = xz^2 - y^2 \quad \Longrightarrow \quad x_0 = 0,\ y_0 = 1,\ y_0' = 0,\ h = 0.2$$

$$y' = z \quad \Rightarrow \quad y_0' = z_0 = 0$$

To find y(0.2):

$$k_1 = hf(x_0, y_0, z_0) = hz_0 = (0.2)\ (0) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.2)\left(z_0 + \frac{(-0.2)}{2}\right)$$

$$= (0.2)\ (0 - 0.1) = -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = h\left(z_0 + \frac{l_2}{2}\right)$$

$$= (0.2)\left(0 - \frac{0.1998}{2}\right) = -0.01998$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = h(z_0 + l_3) = (0.2)\ (0 - 0.1958)$$

$$= -0.0392$$

$$l_1 = hg(x_0, y_0, z_0) = h(x_0 z_0{}^2 - y_0{}^2) = (0.2)\,(0 - 1) \;= -\,0.2$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.2)\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_1}{2}\right)^2 - \left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left[\left(0 + \frac{0.2}{2}\right)\left(0 - \frac{0.2}{2}\right)^2 - \left(1 + \frac{0}{2}\right)^2\right] = -\,0.1998$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = (0.2)\left[\left(0 + \frac{0.2}{2}\right)\left(0 - \frac{0.1998}{2}\right)^2 - \left(1 - \frac{0.02}{2}\right)^2\right]$$

$$= -\,0.1958$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.2)\,[(0.2)\,(0 - 0.1958)^2 - (1 - 0.01998)^2]$$

$$= -\,0.1906$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0 + 2(-\,0.02) + 2(-\,0.01998) - 0.0392]$$

$$= -\,0.0199$$

$$y(0.2) = y_1 = y_0 + \Delta y \;= 1 - 0.0199 \;= 0.9801$$

$$y(0.2) = 0.9801$$

Summary

- Formulas used in solving a first order ODE are used to solve systems of first order ODEs.
  - Instead of scalar variables and functions, we have vector variables and vector functions.
- High order ODEs are converted to a set of first order ODEs.

Exercise: solve a second order ODE with RK4

$$\ddot{x} + 3\dot{x} + 6x = 1 \ , \ x(0) = 1 \ ; \ \dot{x}(0) = 2 \ ; \ h = 0.1$$

To find $x(0.2)$

Solution of Boundary-Value Problems

Shooting Method for Boundary-Value Problems

1. Guess a value for the auxiliary conditions at one point of time.
2. Solve the initial value problem using Euler, Runge-Kutta, …
3. Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.

- Use interpolation in updating the guess.
- It is an iterative procedure and can be efficient in solving the BVP.

Boundary-Value Problem → convert → **Initial-value Problem**

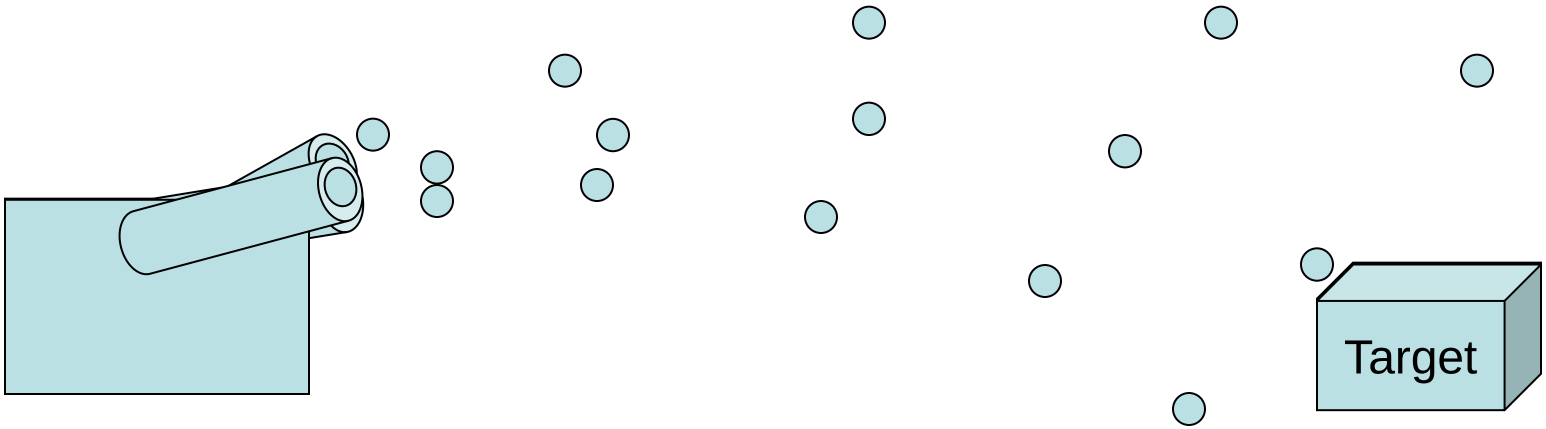
Find $y(x)$ to solve BVP

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

1. **Convert the ODE to a system of first order ODEs.**
2. **Guess the initial conditions that are not available.**
3. **Solve the Initial-value problem.**
4. **Check if the known boundary conditions are satisfied.**
5. **If needed modify the guess and resolve the problem again.**

The Shooting Method

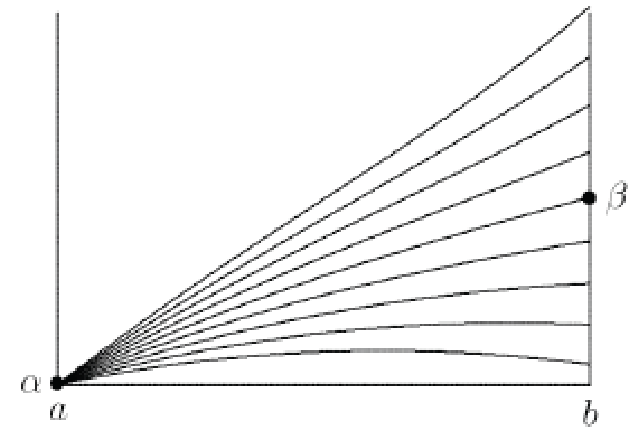


- In statement of two-point BVP, we are given value of $u(a)$
- If we also knew value of $u'(a)$, then we would have IVP that we could solve by methods discussed previously
- Lacking that information, we try sequence of increasingly accurate guesses until we find value for $u'(a)$ such that when we solve resulting IVP, approximate solution value at $t = b$ matches desired boundary value, $u(b) = \beta$
- For given γ, value at b of solution $u(b)$ to IVP $u'' = f(t, u, u')$

  with initial conditions $u(a) = \alpha, \qquad u'(a) = \gamma$

  can be considered as function of γ, say $g(\gamma)$

- Then BVP becomes problem of solving equation $g(\gamma) = \beta$
- One-dimensional zero finder can be used to solve this scalar equation



Example :

- Consider two-point BVP for second-order ODE

$$u'' = 6t, \qquad 0 < t < 1$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- For each guess for $u'(0)$, we will integrate resulting IVP using classical fourth-order Runge-Kutta method to determine how close we come to hitting desired solution value at $t = 1$
- For simplicity of illustration, we will use step size $h = 0.5$ to integrate IVP from $t = 0$ to $t = 1$ in only two steps
- First, we transform second-order ODE into system of two first-order ODEs $\boldsymbol{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} y_2 \\ 6t \end{bmatrix}$
- We first try guess for initial slope of $y_2(0) = 1$

$$
\begin{aligned}
\boldsymbol{y}^{(1)} &= \boldsymbol{y}^{(0)} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} 1.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix} \right) = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{y}^{(2)} = \begin{bmatrix} 0.625 \\ 1.750 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} 1.75 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 2.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 2.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
$$

- So we have hit $y_1(1) = 2$ instead of desired value $y_1(1) = 1$
- We try again, this time with initial slope $y_2(0) = -1$

$$
\begin{aligned}
\boldsymbol{y}^{(1)} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1.0 \\ 1.5 \end{bmatrix} + 2\begin{bmatrix} -0.625 \\ 1.500 \end{bmatrix} + \begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} \right) \\
&= \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{y}^{(2)} &= \begin{bmatrix} -0.375 \\ -0.250 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -0.25 \\ 3.00 \end{bmatrix} + 2\begin{bmatrix} 0.5 \\ 4.5 \end{bmatrix} + 2\begin{bmatrix} 0.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\end{aligned}
$$

- So we have hit $y_1(1) = 0$ instead of desired value $y_1(1) = 1$, but we now have initial slope bracketed between -1 and 1

- We omit further iterations necessary to identify correct initial slope, which turns out to be $y_2(0) = 0$

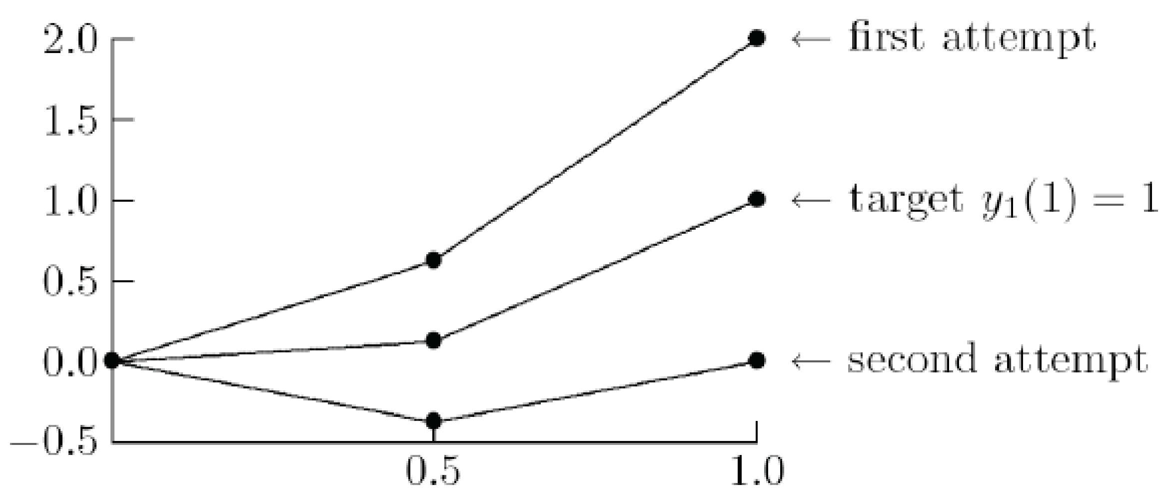
$$\boldsymbol{y}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{0.5}{6} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0.0 \\ 1.5 \end{bmatrix} + 2 \begin{bmatrix} 0.375 \\ 1.500 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix}$$

$$\boldsymbol{y}^{(2)} = \begin{bmatrix} 0.125 \\ 0.750 \end{bmatrix} + \frac{0.5}{6} \left(\begin{bmatrix} 0.75 \\ 3.00 \end{bmatrix} + 2 \begin{bmatrix} 1.5 \\ 4.5 \end{bmatrix} + 2 \begin{bmatrix} 1.875 \\ 4.500 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

- So we have indeed hit target solution value $y_1(1) = 1$

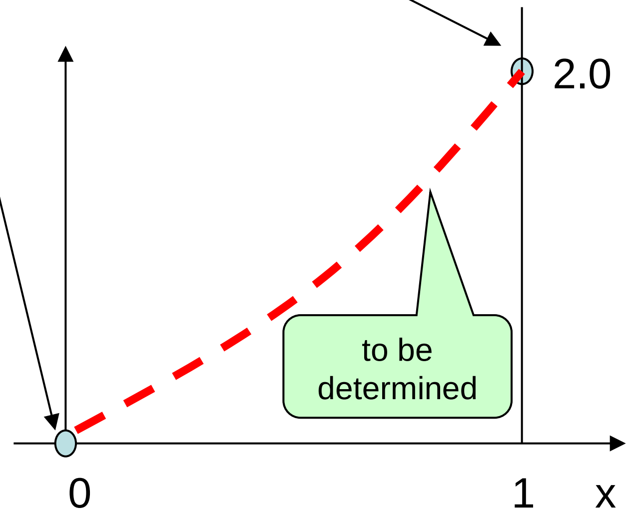


Example : Given $\ddot{y} - 4y + 4x = 0$, $y(0) = 0$, $y(1) = 2$

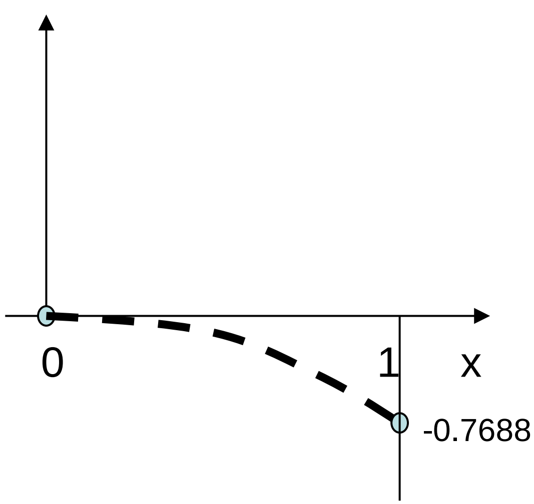
Step1: Convert to a System of First Order ODEs

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 4(y_1 - x) \end{bmatrix} \quad , \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

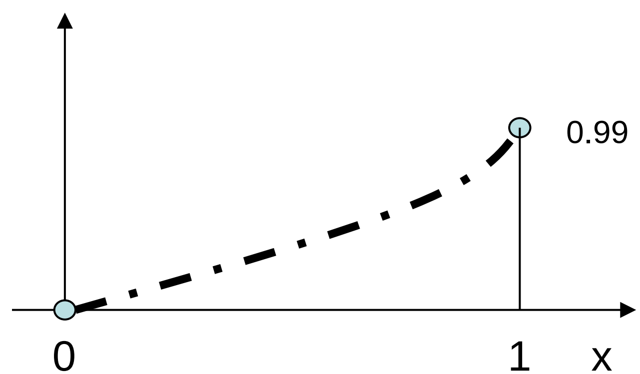
The problem will be solved using RK2 with $h = 0.01$ for different values of $y_2(0)$ until we have $y(1) = 2$



$$Guess\#1 \Rightarrow \dot{y}(0) = 0$$

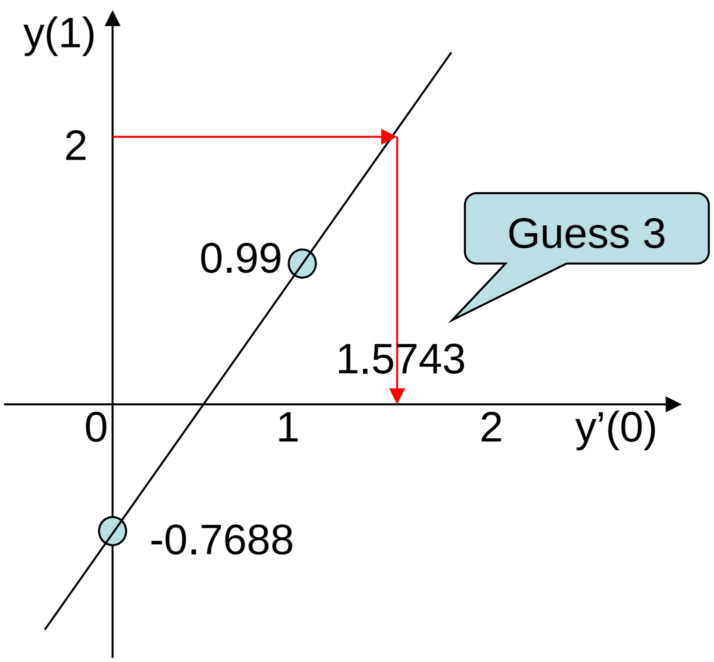


$$Guess\#2 \Rightarrow \dot{y}(0) = 1$$

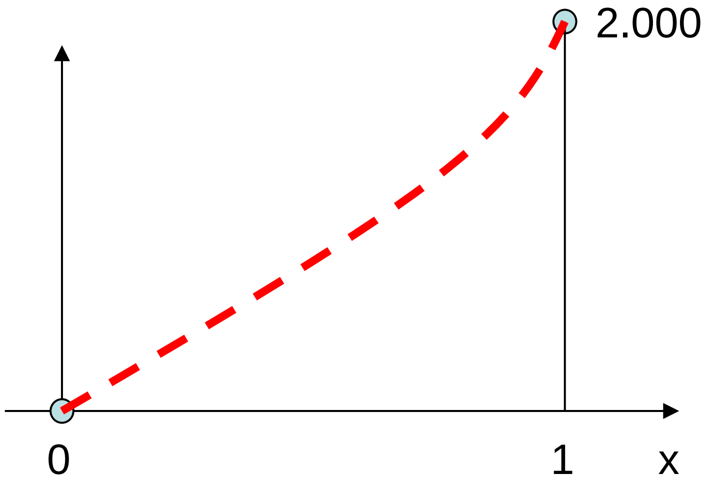


Interpolation for *Guess* # 3

Guess	$\dot{y}(0)$	$y(1)$
1	0	-0.7688
2	1	0.9900



$$Guess\#3 \Rightarrow \dot{y}(0) = 1.5743$$



y(1) = 2.00

This is the solution to the boundary value problem.

Summary

- Guess the unavailable values for the auxiliary conditions at one point of the independent variable.
- Solve the initial value problem.
- Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.
- Repeat until the boundary conditions are satisfied.
- Using interpolation to update the guess often results in few iterations before reaching the solution.
- The method can be cumbersome for high order BVP because of the need to guess the initial condition for more than one variable.

Solution of Boundary-Value Problems

Finite Difference Method

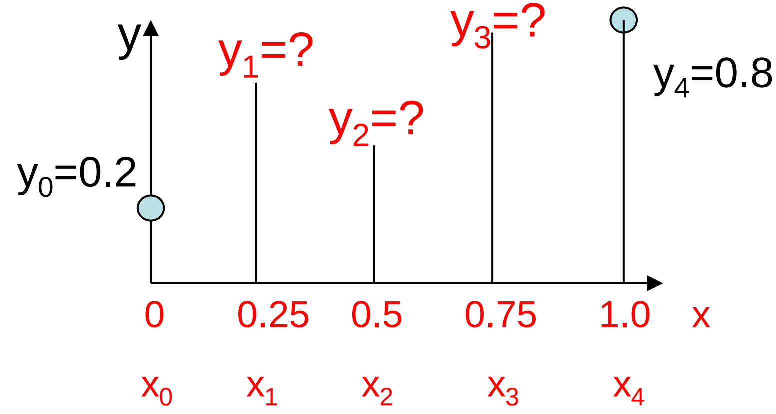
Boundary-Value Problems → convert → **Algebraic Equations**

Find $y(x)$ to solve BVP

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Find the unknowns y_1, y_2, y_3



- Divide the interval into *n* sub-intervals.
- The solution of the BVP is converted to the problem of determining the value of function at the base points.
- Use finite approximations to replace the derivatives.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.

Finite difference method

- *Finite difference method* converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximations
- For example, to solve two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

  with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

  we introduce mesh points $t_i = a + ih$, $i = 0, 1, \ldots, n+1$, where $h = (b-a)/(n+1)$
- We already have $y_0 = u(a) = \alpha$ and $y_{n+1} = u(b) = \beta$ from BC, and we seek approximate solution value $y_i \approx u(t_i)$ at each interior mesh point t_i, $i = 1, \ldots, n$
- We replace derivatives by finite difference approximations such as

$$u'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

- This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

to be solved for unknowns y_i, $i = 1, \ldots, n$

- System of equations may be linear or nonlinear, depending on whether f is linear or nonlinear
- For these particular finite difference formulas, system to be solved is tridiagonal, which saves on both work and storage compared to general system of equations
- This is generally true of finite difference methods: they yield sparse systems because each equation involves few variables

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

Tridiagonal Systems

- A tridiagonal system has a bandwidth of 3.
- An efficient LU decomposition method, called *Thomas algorithm*, can be used to solve such an equation. The algorithm consists of three steps: decomposition, forward and back substitution, and has all the advantages of LU decomposition.

Example :

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1$$

with BC

$$u(0) = 0, \qquad u(1) = 1$$

- To keep computation to minimum, we compute approximate solution at one interior mesh point, $t = 0.5$, in interval $[0, 1]$
- Including boundary points, we have three mesh points, $t_0 = 0$, $t_1 = 0.5$, and $t_2 = 1$
- From BC, we know that $y_0 = u(t_0) = 0$ and $y_2 = u(t_2) = 1$, and we seek approximate solution $y_1 \approx u(t_1)$
- Replacing derivatives by standard finite difference approximations at t_1 gives equation

$$\frac{y_2 - 2y_1 + y_0}{h^2} = f\left(t_1, y_1, \frac{y_2 - y_0}{2h}\right)$$

- Substituting boundary data, mesh size, and right hand side for this example we obtain

$$\frac{1 - 2y_1 + 0}{(0.5)^2} = 6t_1$$

  or

$$4 - 8y_1 = 6(0.5) = 3$$

  so that

$$y(0.5) \approx y_1 = 1/8 = 0.125$$

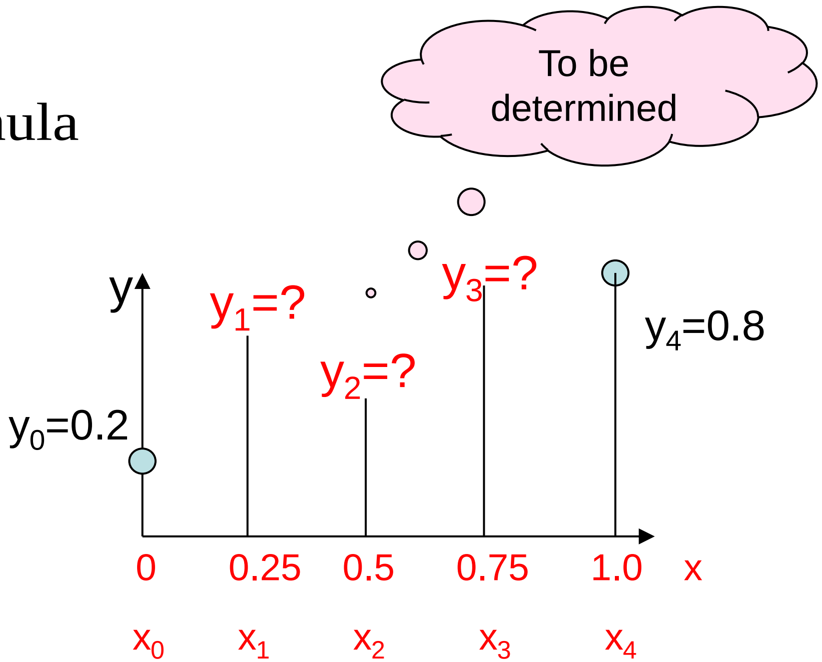
- In a practical problem, much smaller step size and many more mesh points would be required to achieve acceptable accuracy

- We would therefore obtain *system* of equations to solve for approximate solution values at mesh points, rather than single equation as in this example

Example : $\ddot{y}+2\dot{y}+y=x^2$, $y(0)=0.2$, $y(1)=0.8$

Solution: Divide the interval [0,1] into n = 4 intervals. Base points are

$x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1.0$



Replaced by Central difference formula

$$\ddot{y}=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$$

$$\dot{y}=\frac{y_{i+1}-y_{i-1}}{2h}$$

$\ddot{y}+2\dot{y}+y=x^2$ Becomes

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+2\frac{y_{i+1}-y_{i-1}}{2h}+y_i=x_i^2$$

Replaced by Forward difference formula

$$\ddot{y}\approx\frac{y(x+h)-2y(x)+y(x-h)}{h^2}=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}$$

$$\dot{y}\approx\frac{y(x+h)-y(x)}{h}=\frac{y_{i+1}-y_i}{h}$$

$$\ddot{y} + 2\dot{y} + y = x^2 \quad \text{becomes}$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^2 \quad i = 1,2,3$$

$$x_0 = 0 \ , \quad x_1 = 0.25\,, x_2 = 0.5\,, x_3 = 0.75\,, x_4 = 1$$

$$y_0 = 0.2,\ y_1 = ?\ , \quad y_2 = ? \ ,\ y_3 = ?\ , \quad y_4 = 0.8$$

$$16(y_{i+1} - 2y_i + y_{i-1}) + 8(y_{i+1} - y_i) + y_i = x_i^2$$

$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^2$$

$$i = 1 \quad 24y_2 - 39y_1 + 16y_0 = x_1^2$$

$$i = 2 \quad 24y_3 - 39y_2 + 16y_1 = x_2^2$$

$$i = 3 \quad 24y_4 - 39y_3 + 16y_2 = x_3^2$$

$$\begin{bmatrix} -39 & 24 & 0 \\ 16 & -39 & 24 \\ 0 & 16 & -39 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.25^2 - 16(0.2) \\ 0.5^2 \\ 0.75^2 - 24(0.8) \end{bmatrix}$$

Solution $\quad y_1 = 0.4791, y_2 = 0.6477, y_3 = 0.7436$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 \Rightarrow n = 100$$

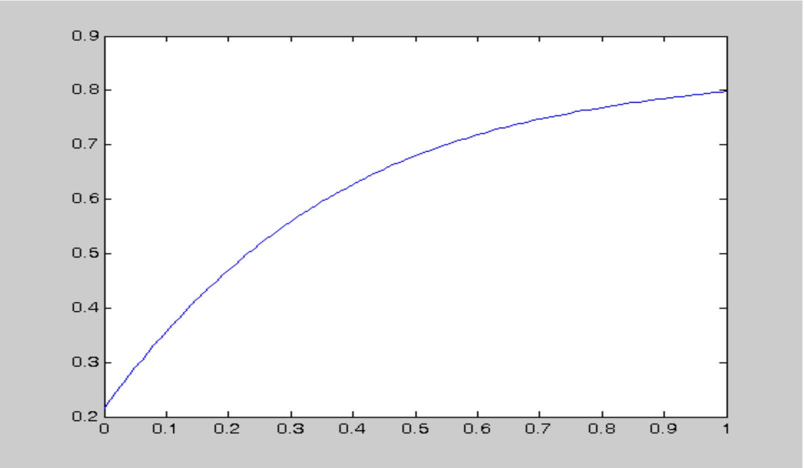
$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_i}{h} + y_i = x_i^2 \quad i = 1,2,\ldots,100$$

$$x_0 = 0,\ x_1 = 0.01, x_2 = 0.02 \ \ \ldots \ \ x_{99} = 0.99, x_{100} = 1$$

$$y_0 = 0.2,\ y_1 = ?, y_2 = ?,\ \ldots \ \ y_{99} = ?, \ \ y_{100} = 0.8$$

$$10000(y_{i+1} - 2y_i + y_{i-1}) + 200(y_{i+1} - y_i) + y_i = x_i^2$$

$$10200y_{i+1} - 20199y_i + 10000y_{i-1} = x_i^2$$



Example : Using the Finite Difference Method with n = 4 to approximate the solution of the BVP

$$y'' - 4y = 0\,, \quad y(0) = 0\,, \quad y(1) = 5$$

Solution: We have $P(x) = 0$, $Q(x) = -4$, $F(x) = 0$, $h = (1 - 0)/4 = 1/4$.

Hence we have $\quad y_{i+1} - 2.25y_i + y_{i-1} = 0$

The interior points are $x_1 = 0 + 1/4$, $x_2 = 0 + 2/4$, $x_3 = 0 + 3/4$ then

$$y_2 - 2.25y_1 + y_0 = 0$$
$$y_3 - 2.25y_2 + y_1 = 0$$
$$y_4 - 2.25y_3 + y_2 = 0$$

Together with $y_0 = 0$, $y_4 = 5$, then

$$\begin{aligned} -2.25y_1 + \quad y_2 \qquad\qquad &= 0 \\ y_1 - 2.25y_2 + \quad y_3 &= 0 \\ y_2 - 2.25y_3 &= -5 \end{aligned}$$

We obtain $y_1 = 0.7256$, $y_2 = 1.6327$, $y_3 = 2.9479$.

Example : Using the Finite Difference Method with n = 10 to approximate the solution of the BVP

$$y'' + 3y' + 2y = 4x^2 \quad , \quad y(1) = 1 \ , \ y(2) = 6$$

Solution: We have $P(x) = 3$, $Q(x) = 2$, $F(x) = 4x^2$, $h = (2 - 1)/10 = 0.1$, hence

$$1.15y_{i+1} - 1.98y_i + 0.85y_{i-1} = 0.04x_i^2$$

The interior points are $x_1 = 1.1$, $x_2 = 1.2$, …, $x_9 = 1.9$.
For i = 1, 2, …, 9 and $y_0 = 1$, $y_{10} = 6$ then

$$\begin{aligned}
1.15y_2 - 1.98y_1 \phantom{{}+0.85y_0} &= -0.8016 \\
1.15y_3 - 1.98y_2 + 0.85y_1 &= 0.0576 \\
1.15y_4 - 1.98y_3 + 0.85y_2 &= 0.0676 \\
1.15y_5 - 1.98y_4 + 0.85y_3 &= 0.0784 \\
1.15y_6 - 1.98y_5 + 0.85y_4 &= 0.0900 \\
1.15y_7 - 1.98y_6 + 0.85y_5 &= 0.1024 \\
1.15y_8 - 1.98y_7 + 0.85y_6 &= 0.1156 \\
1.15y_9 - 1.98y_8 + 0.85y_7 &= 0.1296 \\
1.98y_9 + 0.85y_8 &= -6.7556
\end{aligned}$$

Then we can solve the above system of equations to obtain y_1, y_2, …, y_9.

Example : Using the Finite Difference Method with $n = 4$ to approximate the solution of the BVP

Solution: $$\frac{d^2y}{dx^2} = y \quad \text{given } y(0) = 0,\ y(2) = 3.63 \ ,\ h = \frac{1}{2}$$

Replacing the derivative y'' by differences, we get

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} = y_i$$

$$y_{i+1} - (2 + h^2)y_i + y_{i-1} = 0,\ i = 1, 2, 3$$

$$y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0, \quad i = 1, 2, 3$$

Hence the equations to be solved are:

$$y_2 - \frac{9}{4} y_1 + y_0 = 0$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0$$

$$y_4 - \frac{9}{4} y_3 + y_2 = 0$$

Using $y_0 = 0$, $y(2) = y_4 = 3.63$, we have

$$y_2 - \frac{9}{4} y_1 = 0$$

$$y_3 - \frac{9}{4} y_2 + y_1 = 0$$

$$3.63 - \frac{9}{4} y_3 + y_2 = 0$$

Eliminating y_1

$$\frac{9}{4} y_3 - \frac{81}{16} y_2 + y_2 = 0 \Longrightarrow \boxed{\frac{9}{4} y_3 - \frac{65}{16} y_2 = 0}$$

$$\boxed{\left(1 - \frac{65}{16}\right) y_2 = -3.63} \Longrightarrow \boxed{y_2 = 1.1853}$$

$$\boxed{y_3 = 2.1401} \quad , \quad \boxed{y_1 = \frac{4}{9} y_2} \boxed{= 0.5268}$$

Taking the values, the solution is

x	0	0.5	1	1.5	2
y	0	0.5268	1.1853	2.1401	3.63

Summary of the Discretization Methods

- Select the base points.
- Divide the interval into *n* sub-intervals.
- Different formulas can be used for approximating the derivatives such as finite approximations to replace the derivatives.
- Different formulas lead to different solutions. All of them are approximate solutions.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.
- For linear second order cases, this reduces to tri-diagonal system.