

The Fourier Transform

- The Fourier Series deals with *periodic* signals

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jn\omega_o t}$$

$$\hat{X}_n = \frac{1}{T} \int_{t_o}^{t_o+T} x(t) e^{-jn\omega_o t} dt$$

- The Fourier Transform deals with *non-periodic* signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Fourier Integral and Transform

Fourier integral is a tool used to analyze non-periodic waveforms or non-recurring signals, such as lightning bolts. Fourier integral formula is derived from Fourier series by allowing the period to approach infinity:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t} d\omega$$

where the coefficients become a continuous function of the frequency variable ω , as in

$$F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Fourier Integral and Transform

The function $F(i\omega)$ is called the Fourier integral of $f(t)$. The function $F(i\omega)$ is also called the Fourier transform of $f(t)$. In the same spirit, $f(t)$ is referred to as the inverse Fourier transform of $F(i\omega)$.

The pair allows us to transform back and forth between the time and the frequency domains for a non-periodic signal.

The periodic function $f_T(t)$ can be represented by an exponential Fourier series.

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \quad \text{where}$$

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt \quad \text{and} \quad \omega_o = 2\pi/T$$

As the magnitude of the Fourier coefficients go to zero when the period is increased, we define

$$\omega_n \equiv n\omega_o \quad \text{and} \quad F(\omega_n) \equiv T F_n$$

The Fourier series pair become

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t} \quad \text{where} \quad F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt$$

The spacing between adjacent lines ($\Delta\omega$) in the line spectrum of $f_T(t)$ is

$$\Delta\omega = 2\pi/T$$

Therefore, we have

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \frac{\Delta\omega}{2\pi}$$

Now as T becomes very large, $\Delta\omega$ becomes smaller and the spectrum becomes denser. In the limit, the discrete lines in the spectrum of $f_T(t)$ merge and the frequency spectrum becomes continuous.

Mathematically, the infinite sum becomes an integral

$$\lim_{T \rightarrow \infty} f_T(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \Delta\omega$$

$$\Rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

← Inverse Fourier transform of $F(\omega)$

Similarly,

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

$$\Rightarrow F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_n t} dt \quad \because \omega_n \equiv n\omega_o \text{ and } F(\omega_n) \equiv TF_n$$

$$\Rightarrow \lim_{T \rightarrow \infty} F(\omega_n) = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f_T(t) e^{-jn\omega_o t} dt$$

← Fourier transform of $f(t)$

From Fourier Series to the Fourier Integral

Fourier series of $f(x)$ (period $2L$): $f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x)$, $w_n = \frac{n\pi}{L}$
 $L \rightarrow \infty$, $f(x)$?

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos w_n x \int_{-L}^L f_L(v) \cos w_n v dv + \sin w_n x \int_{-L}^L f_L(v) \sin w_n v dv \right]$$

$$\Delta w = w_{n+1} - w_n = \frac{\pi}{L} \quad \left(\frac{1}{L} = \frac{\Delta w}{\pi} \right)$$

$$\Rightarrow f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[(\cos w_n x) \Delta w \int_{-L}^L f_L(v) \cos w_n v dv + (\sin w_n x) \Delta w \int_{-L}^L f_L(v) \sin w_n v dv \right]$$

If $f(x) = \lim_{L \rightarrow \infty} f_L(x)$ is absolutely integrable, $\int_{-\infty}^{\infty} |f(x)| dx$ exists

$$L \rightarrow \infty, \text{ then } f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\cos wx \int_{-\infty}^{\infty} f(v) \cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv dv \right] dw$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv, \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv$$

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

Fourier Integral

When $f(x)$ is an even function, $f(u) \sin \omega u$ is an odd function of u ,
the **Fourier cosine integral representation** of $f(x)$

$$f(x) = \int_0^\infty A(\omega) \cos \omega x d\omega.$$

$$A(\omega) = \frac{2}{\pi} \int_0^\infty f(u) \cos \omega u du,$$

Similarly, when $f(x)$ is an odd function, $f(u) \cos \omega u$ is an odd function of u ,
the **Fourier sine integral representation** of $f(x)$ given by

$$f(x) = \int_0^\infty B(\omega) \sin \omega x d\omega.$$

$$B(\omega) = \frac{2}{\pi} \int_0^\infty f(u) \sin \omega u du,$$

the **exponential Fourier transform**, where k is an arbitrary scale factor.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\{-i\omega x\} dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp\{i\omega x\} d\omega.$$

Fourier transform pair

$$\lim_{|\omega| \rightarrow \infty} F(\omega) = 0.$$

$$\mathcal{F}\{f(x)\} = \frac{k}{2\pi} \int_{-\infty}^{\infty} f(x) \exp\{\pm i\omega x\} dx \quad \text{and}$$

$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{k} \int_{-\infty}^{\infty} F(\omega) \exp\{\mp i\omega x\} d\omega,$$

Properties and Theorems of the Fourier Transform

Linearity

$$a_1 f_1(t) + a_2 f_2(t) + \dots + a_n f_n(t) \Leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) + \dots + a_n F_n(\omega)$$

Symmetry

$$F(t) \Leftrightarrow 2\pi f(-\omega)$$

Time Scaling

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Time Shifting

$$f(t - t_0) \Leftrightarrow F(\omega) e^{-j\omega t_0}$$

Frequency Shifting

$$e^{j\omega_0 t} f(t) \Leftrightarrow F(\omega - \omega_0)$$

$$e^{j\omega_0 t} f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega - \omega_0}{a}\right)$$

Time Differentiation

$$\frac{d^n}{dt^n} f(t) \Leftrightarrow (j\omega)^n F(\omega)$$

Frequency Differentiation

$$(-jt)^n f(t) \Leftrightarrow \frac{d^n}{d\omega^n} F(\omega)$$

Time Integration

$$\int_{-\infty}^t f(\tau) d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega)$$

Conjugate Time and Frequency Functions

$$f^*(t) \Leftrightarrow F^*(-\omega)$$

Time Convolution

$$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega) F_2(\omega)$$

Frequency Convolution

$$f_1(t) f_2(t) \Leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$$

Area Under $f(t)$ If $F(\omega)$ is the Fourier transform of $f(t)$, then,

$$F(0) = \int_{-\infty}^{\infty} f(t) dt$$

that is, the area under a time function $f(t)$ is equal to the value of its Fourier transform evaluated at $\omega = 0$.

Area Under $F(\omega)$ If $F(\omega)$ is the Fourier transform of $f(t)$, then,

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) d\omega$$

that is, the value of the time function $f(t)$, evaluated at $t = 0$, is equal to the area under its Fourier transform $F(\omega)$ times $1/2\pi$.

Parseval's Theorem

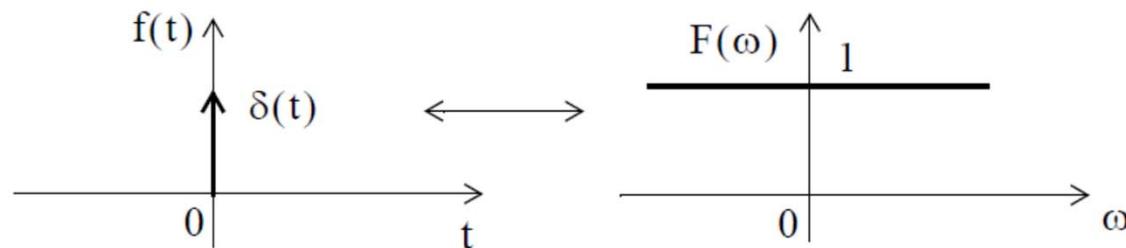
$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

the integral is called *the energy of the signal*.

The Delta Function Pair

$$\delta(t) \Leftrightarrow 1$$

$$\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}$$

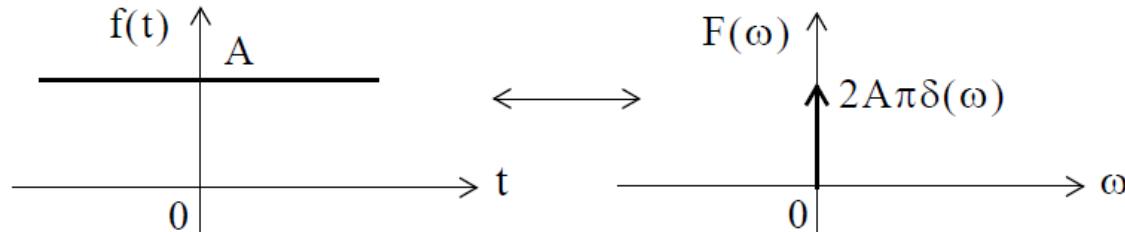


The Fourier transform of the delta function

The Constant Function Pair

$$A \Leftrightarrow 2A\pi\delta(\omega)$$

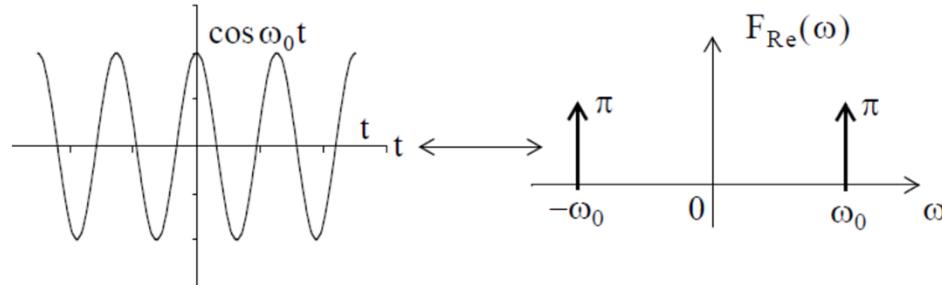
$$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0)$$



The Fourier transform of constant A

The Cosine Function Pair

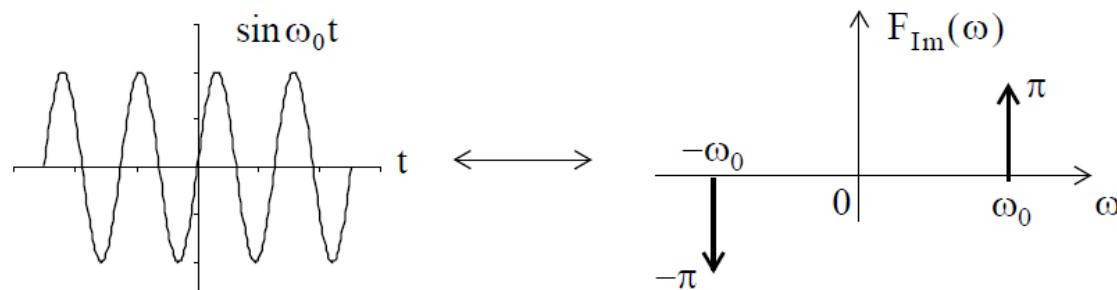
$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \Leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



The Fourier transform of $f(t) = \cos \omega_0 t$

The Sine Function Pair

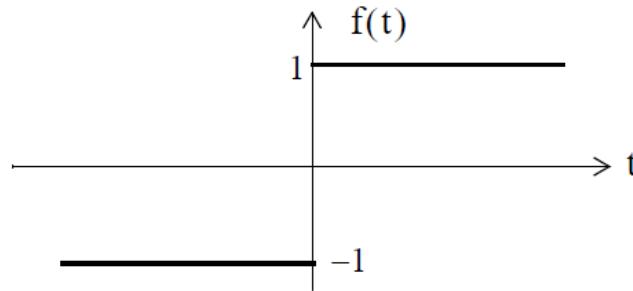
$$\sin \omega_0 t = \frac{1}{j2}(e^{j\omega_0 t} - e^{-j\omega_0 t}) \Leftrightarrow j\pi\delta(\omega - \omega_0) - j\pi\delta(\omega + \omega_0)$$



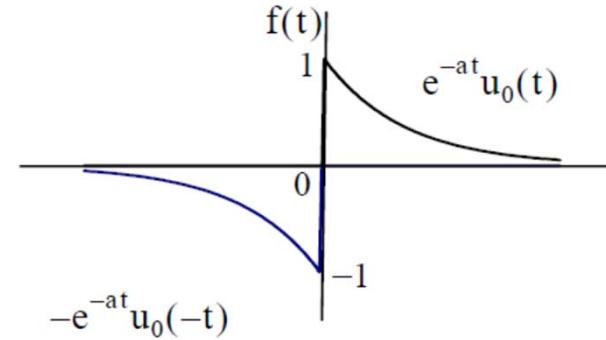
The Fourier transform of $f(t) = \sin \omega_0 t$

The Signum Function Pair

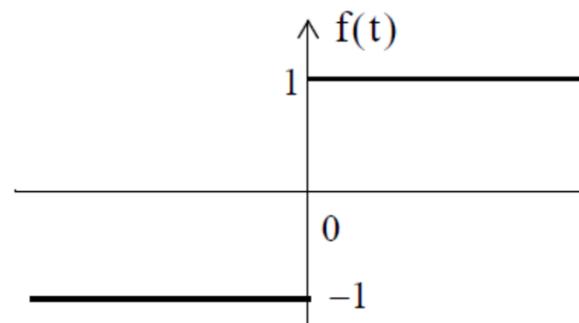
$$\operatorname{sgn}(t) = u_0(t) - u_0(-t) \Leftrightarrow \frac{2}{j\omega}$$



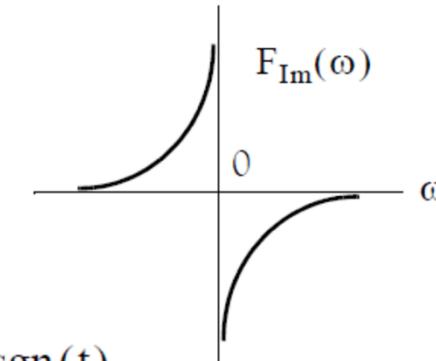
The signum function



The signum function as an exponential approaching a limit



The Fourier transform of $\operatorname{sgn}(t)$

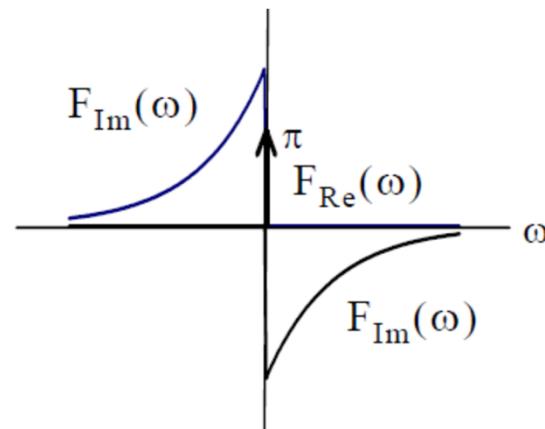
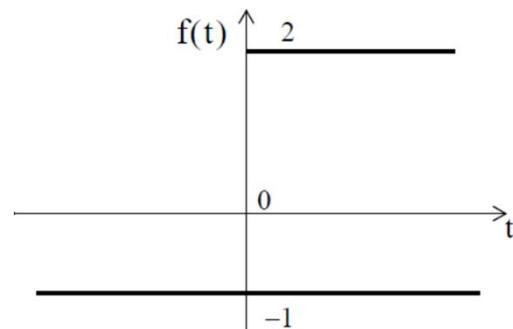
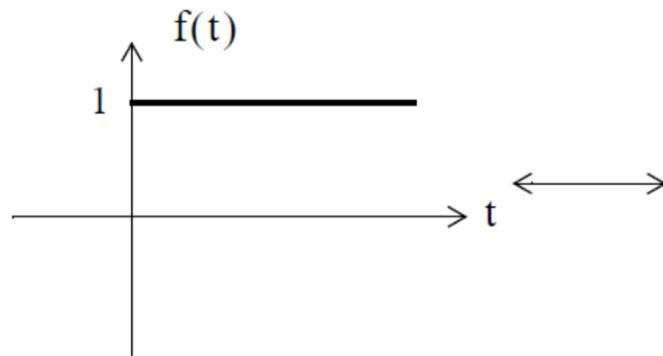


The Unit Step Function Pair

$$u_0(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\operatorname{sgn}(t) = 2u_0(t) - 1$$

Alternate expression for the signum function



The Fourier transform of the unit step function

By the convolution integral, $u_0(t)*f(t) = \int_{-\infty}^t f(\tau)u_0(t-\tau)d\tau$

Next, by the time convolution property, $u_0(t)*f(t) \Leftrightarrow U_0(\omega) \cdot F(\omega)$

$$U_0(\omega) \cdot F(\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega} \right) F(\omega) = \pi\delta(\omega)F(\omega) + \frac{F(\omega)}{j\omega} = \pi F(0)\delta(\omega) + \frac{F(\omega)}{j\omega}$$

The $e^{-j\omega_0 t} u_0(t)$ Function Pair

$$e^{-j\omega_0 t} u_0(t) \Leftrightarrow \pi\delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)}$$

The $(\cos \omega_0 t) u_0(t)$ Function Pair

$$\begin{aligned} (\cos \omega_0 t)(u_0 t) &\Leftrightarrow \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{1}{2j(\omega - \omega_0)} + \frac{1}{2j(\omega + \omega_0)} \\ &\Leftrightarrow \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2} \end{aligned}$$

The $(\sin \omega_0 t) u_0(t)$ Function Pair

$$(\sin \omega_0 t)(u_0 t) \Leftrightarrow \frac{\pi}{j2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{\omega^2}{\omega_0^2 - \omega^2}$$

Derivation of the Fourier Transform from the Laplace Transform

If a time function $f(t)$ is zero for $t \leq 0$, we can obtain the Fourier transform of $f(t)$ from the one-sided Laplace transform of $f(t)$ by substitution of s with $j\omega$.

Example

It is known that $\mathcal{L} [e^{-\alpha t} u_0(t)] = \frac{1}{s + \alpha}$. Compute $\mathcal{F} \{ e^{-\alpha t} u_0(t) \}$

Solution:

$$\mathcal{F} \{ e^{-\alpha t} u_0(t) \} = \mathcal{L} [e^{-\alpha t} u_0(t)] \Big|_{s=j\omega} = \frac{1}{s + \alpha} \Big|_{s=j\omega} = \frac{1}{j\omega + \alpha}$$

Thus, we have obtained the following Fourier transform pair.

$$e^{-\alpha t} u_0(t) \Leftrightarrow \frac{1}{j\omega + \alpha}$$

Example

It is known that

$$\mathcal{L} [(e^{-\alpha t} \cos \omega_0 t) u_0(t)] = \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$$

Compute $\mathcal{F}\{(e^{-\alpha t} \cos \omega_0 t) u_0(t)\}$

Solution:

$$\begin{aligned}\mathcal{F}\{(e^{-\alpha t} \cos \omega_0 t) u_0(t)\} &= \mathcal{L}[(e^{-\alpha t} \cos \omega_0 t) u_0(t)] \Big|_{s=j\omega} \\ &= \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2} \Big|_{s=j\omega} = \frac{j\omega + \alpha}{(j\omega + \alpha)^2 + \omega_0^2}\end{aligned}$$

Thus, we have obtained the following Fourier transform pair.

$$(e^{-\alpha t} \cos \omega_0 t) u_0(t) \Leftrightarrow \frac{j\omega + \alpha}{(j\omega + \alpha)^2 + \omega_0^2}$$

We can also find the Fourier transform of a time function $f(t)$ that has non-zero values for $t < 0$, and it is zero for all $t > 0$. But because the one-sided Laplace transform does not exist for $t < 0$, we must first express the negative time function in the $t > 0$ domain, and compute the one-sided Laplace transform. Then, the Fourier transform of $f(t)$ can be found by substituting s with $-j\omega$. In other words, when $f(t) = 0$ for $t \geq 0$, and $f(t) \neq 0$ for $t < 0$, we use the substitution

$$\mathcal{F}\{f(t)\} = \mathcal{L}[f(-t)] \Big|_{s=-j\omega}$$

Example

Compute the Fourier transform of $f(t) = e^{-a|t|}$

- using the Fourier transform definition
- by substitution into the Laplace transform equivalent

Solution:

- Using the Fourier transform definition, we obtain

$$\begin{aligned}\mathcal{F}\{e^{-a|t|}\} &= \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_{-\infty}^0 e^{(a-j\omega)t} dt + \int_0^{\infty} e^{-(a+j\omega)t} dt \\ &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2}{\omega^2 + a^2}\end{aligned}$$

and thus we have the transform pair

$$e^{-a|t|} \Leftrightarrow \frac{2}{\omega^2 + a^2}$$

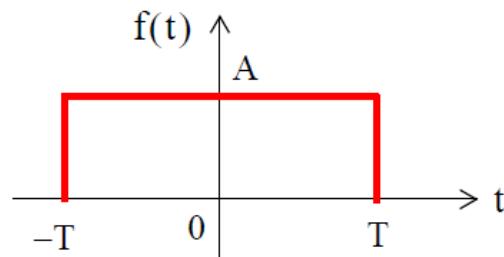
- By substitution into the Laplace transform equivalent, we obtain

$$\mathcal{F}\{e^{-a|t|}\} = \mathcal{L}[e^{-at}] \Big|_{s=j\omega} + \mathcal{L}[e^{at}] \Big|_{s=-j\omega} = \frac{1}{s+a} \Big|_{s=j\omega} + \frac{1}{s+a} \Big|_{s=-j\omega}$$

$$= \frac{1}{j\omega + a} + \frac{1}{-j\omega + a} = \frac{2}{\omega^2 + a^2}$$

and this result is the same as (8.79). We observe that since $f(t)$ is real and even, $F(\omega)$ is also real and even.

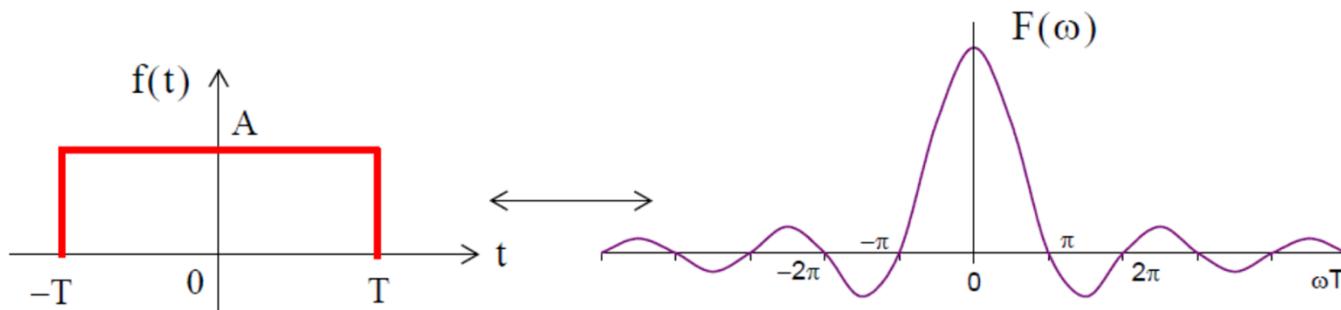
The Transform of $f(t) = A[u_0(t+T) - u_0(t-T)]$



Rectangular pulse waveform $f(t) = A[u_0(t+T) - u_0(t-T)]$

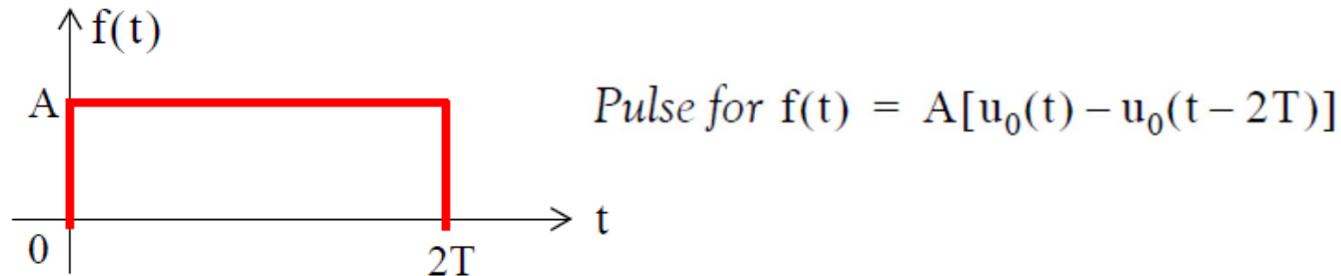
Thus, we have the waveform pair

$$A[u_0(t+T) - u_0(t-T)] \Leftrightarrow 2AT \frac{\sin \omega T}{\omega T}$$



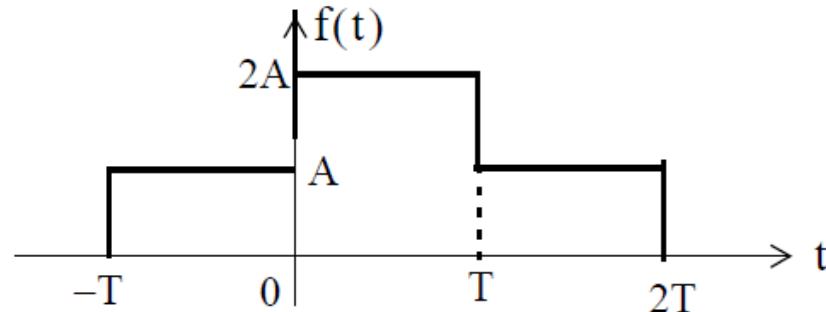
The waveform $f(t) = A[u_0(t+T) - u_0(t-T)]$ and its Fourier transform

The Transform of $f(t) = A[u_0(t) - u_0(t - 2T)]$



$$f(t - t_0) \Leftrightarrow F(\omega) e^{-j\omega t_0} \quad \text{Thus, multiplying } 2AT \frac{\sin \omega T}{\omega T} \text{ by } e^{-j\omega T}$$

The Transform of $f(t) = A[u_0(t + T) + u_0(t) - u_0(t - T) - u_0(t - 2T)]$



Waveform for $f(t) = A[u_0(t + T) + u_0(t) - u_0(t - T) - u_0(t - 2T)]$

$$F(\omega) = F_1(\omega) + F_2(\omega) = 2AT \frac{\sin \omega T}{\omega T} + 2AT e^{-j\omega T} \left(\frac{\sin \omega T}{\omega T} \right)$$

$$F(\omega) = 4AT e^{-j\frac{\omega T}{2}} \cos \left(\frac{\omega T}{2} \right) \frac{\sin \omega T}{\omega T}$$

The Transform of $f(t) = A \cos \omega_0 t [u_0(t+T) - u_0(t-T)]$

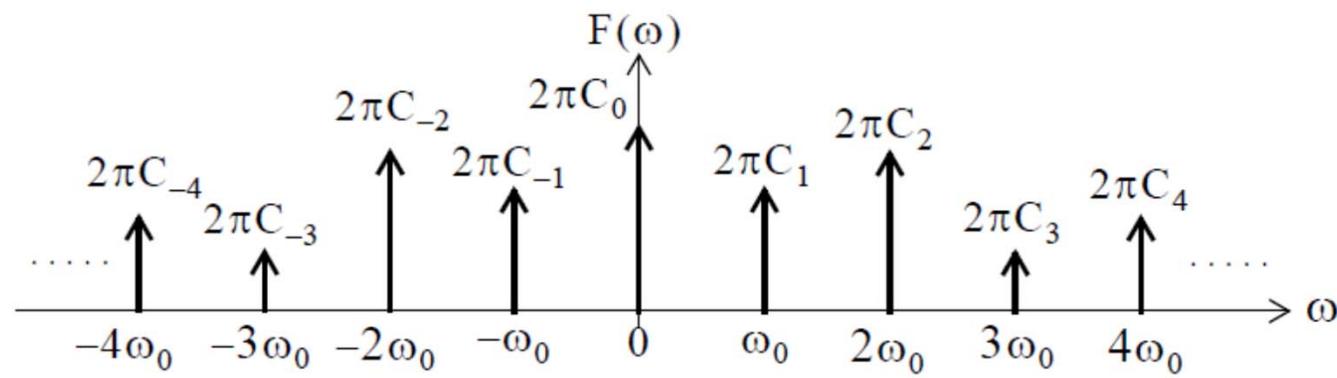
$$A \cos \omega_0 t [u_0(t+T) - u_0(t-T)] \Leftrightarrow AT \left[\frac{\sin[(\omega - \omega_0)T]}{(\omega - \omega_0)T} + \frac{\sin[(\omega + \omega_0)T]}{(\omega + \omega_0)T} \right]$$

The Transform of a Periodic Time Function with Period T

the exponential Fourier series $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

we obtain

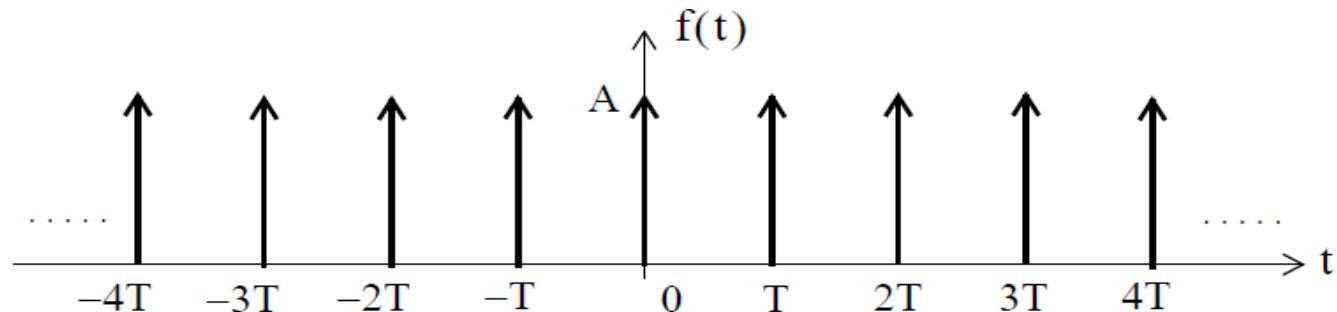
$$\mathcal{F}\{f(t)\} = \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right\} = \sum_{n=-\infty}^{\infty} C_n \mathcal{F}\{e^{jn\omega_0 t}\} = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$



Line spectrum

The Transform of the Periodic Time Function $f(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT)$

The periodic time function $f(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT)$



Waveform for $f(t) = A \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$F(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0) \quad \text{where} \quad C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

$$\text{By substitution} \quad C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T}$$

$$\text{we obtain} \quad F(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

THEOREM

Fourier transform of $x^n f(x)$ Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

$$(a) \quad \mathcal{F}\{x f(x)\} = i \frac{d}{d\omega} [F(\omega)]$$

and

$$(b) \quad \mathcal{F}\{x^n f(x)\} = i^n \frac{d^n}{d\omega^n} [F(\omega)],$$

for all n such that $\lim_{|\omega| \rightarrow \infty} F^{(n)}(\omega) = 0$.

THEOREM

Fourier transform of $x^m f^{(n)}(x)$ Let $f(x)$ be a continuous n times differentiable function. Furthermore, let $x^m f^{(r)}(x)$ for $r = 1, 2, \dots, n$ satisfy Dirichlet conditions and be absolutely integrable over $(-\infty, \infty)$, and let $\omega^n F(\omega)$ possess an m times differentiable inverse Fourier transform. Then, provided $\lim_{|x| \rightarrow \infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\{x^m f^{(n)}(x)\} = (i)^{m+n} \frac{d^m}{d\omega^m} [\omega^n F(\omega)].$$

EXAMPLE

Find the Fourier transforms of

$$(a) \quad f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a, \end{cases} \quad (b) \quad g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise,} \end{cases} \quad (c) \quad p(x) = \frac{1}{x^2 + a^2}$$

by making use of the standard integral $\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$ ($a > 0$) and (d) $q(x) = \begin{cases} e^{i\omega x}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$. In each case confirm that the Fourier transform vanishes as $\omega \rightarrow \pm\infty$.

Solution

$$\begin{aligned} (a) \quad F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{-i\omega x} dx = \frac{1}{\omega\sqrt{2\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{i} \right] \\ &= \frac{1}{\omega} \sqrt{\frac{2}{\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i} \right] = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}. \end{aligned}$$

As $\sin \omega a$ is bounded, it follows directly that $\lim_{|\omega| \rightarrow \infty} F(\omega) = 0$.

$$(b) \quad G(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega a}}{i\omega} \right).$$

As the numerator of $G(\omega)$ is bounded, it follows that $\lim_{|\omega| \rightarrow \infty} G(\omega) = 0$. This example shows that although $f(x)$ may be real, its Fourier transform can be complex.

$$(c) \quad P(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega x}{x^2 + a^2} dx.$$

The integrand of the second integral is odd, so the value of the integral is zero. Using the standard result

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-|\omega|a}$$

in the remaining integral on the right, we find that

$$P(\omega) = \sqrt{\frac{\pi}{2}} \frac{e^{-|\omega|a}}{a} \quad (a > 0).$$

In this case the factor $e^{-|\omega|a}$ ensures that $\lim_{|\omega| \rightarrow \infty} P(\omega) = 0$.

$$(d) \quad Q(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i(\omega-a)x} dx \\ = \frac{i}{\sqrt{2\pi}} \left(\frac{1 - e^{-i(\omega-a)}}{a - \omega} \right).$$

As the numerator of the Fourier transform is bounded, the denominator causes the transform to vanish as $|\omega| \rightarrow \infty$. This example shows that a complex function can also have a Fourier transform and, in general, that the transform will be complex.

EXAMPLE Find the Fourier transform of $f(x) = \exp(-a^2x^2)$ ($a > 0$).

Solution The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} |\exp(-a^2x^2)| dx = \int_{-\infty}^{\infty} \exp(-a^2x^2) dx = \frac{1}{a} \int_{-\infty}^{\infty} \exp(-u^2) du = \frac{\sqrt{\pi}}{a},$$

where we have made use of the standard integral $\int_{-\infty}^{\infty} \exp(-u^2) du = \sqrt{\pi}$. This shows that $f(x)$ is absolutely integrable over the interval $(-\infty, \infty)$, and so $f(x)$ has a Fourier transform. A straightforward calculation establishes that $f(x)$ satisfies the differential equation

$$f' + 2a^2 x f = 0.$$

Taking the Fourier transform of this equation using Theorem 10.2 gives

$$\mathcal{F}\{f'(x)\} + 2a^2 \mathcal{F}\{xf(x)\} = 0.$$

Applying Theorem 10.2 to the first term and Theorem 10.2 to the second term and cancelling a factor i reduces this to the variables separable equation for $F(\omega)$,

$$2a^2 F' + \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-a^2x^2) e^{-i\omega x} dx.$$

When variables are separated, the equation becomes

$$\int \frac{F'}{F} d\omega = -\frac{1}{2a^2} \int \omega d\omega,$$

so

$$\ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A, \quad \text{or} \quad F(\omega) = A \exp\left[-\frac{\omega^2}{4a^2}\right],$$

where, for convenience, the arbitrary integration constant has been written in the form $\ln A$. To determine A we use the fact that $A = F(0)$, but

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-a^2 x^2) dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{a} = \frac{1}{a\sqrt{2}},$$

and so

$$\mathcal{F}\{\exp(-a^2 x^2)\} = F(\omega) = \frac{1}{a\sqrt{2}} \exp\left\{-\frac{\omega^2}{4a^2}\right\} \quad (a > 0).$$

When working with Fourier integrals, as with the Laplace transform, it is useful to introduce the convolution operation to establish the relationship between the functions $f(x)$ and $g(x)$ and their respective Fourier transforms $F(\omega)$ and $G(\omega)$.

The **convolution** of functions $f(x)$ and $g(x)$ denoted by $f * g$ is a function of x , and if the dependence on a variable x in the convolution is to be emphasized,

it is then denoted by $(f * g)(x)$. The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)dt = \int_{-\infty}^{\infty} f(x - t)g(t)dt.$$

A slightly different definition of the convolution operation for the Fourier transform is also to be found in the literature, where it is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x - t)dt.$$

THEOREM

The convolution theorem for Fourier transforms Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

(a) $\mathcal{F}\{(f * g)(x)\} = 2\pi \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$, or $\mathcal{F}\{f * g\} = 2\pi F(\omega)G(\omega)$

and, conversely,

(b) $(f * g)(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega.$

THEOREM

The Parseval relation for the Fourier transform If $f(x)$ has the Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

THEOREM

Fourier transforms involving scaling x by a , shifting x by a , and shifting ω by λ If $f(x)$ has a Fourier transform $F(\omega)$, then

(i) $\mathcal{F}\{f(ax)\} = \frac{1}{a} F(\omega/a) \quad (a > 0)$

(ii) $\mathcal{F}\{f(x - a)\} = e^{-i\omega a} F(\omega)$

(iii) $\mathcal{F}\{e^{i\lambda x} f(x)\} = F(\omega - \lambda)$

EXAMPLE Using the function $f(x)$ and its Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}$.
(a) $\mathcal{F}\{f(2x)\}$, (b) $\mathcal{F}\{f(x - \pi)\}$, and (c) $\mathcal{F}\{e^{ix} f(x)\}$.

Solution Using the results of Theorem

$$\mathbf{(a)} \quad \mathcal{F}\{f(2x)\} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega a/2)}{(\omega/2)} \right) = \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega a/2)}{\omega} \right)$$

$$\mathbf{(b)} \quad \mathcal{F}\{f(x - \pi)\} = e^{-i\pi\omega} \sqrt{\frac{2}{\pi}} \left(\frac{\sin \omega a}{\omega} \right)$$

$$\mathbf{(c)} \quad \mathcal{F}\{e^{ix} f(x)\} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega - 1)a}{\omega - 1} \right)$$

The **Dirac delta function** $\delta(x)$ was introduced in connection with the Laplace transform, where it was recognized that it is not a function in the usual sense, but an *operation* that only has meaning when it appears in the integrand of a definite integral. Because of its many uses in connection with physical problems described by differential equations, we now extend its definition in a way that is suitable for use with Fourier transforms. This is accomplished by defining $\delta(x - a)$ in a symmetrical manner about $x = a$ in terms of the integrals

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = \int_{-\infty}^{\infty} \delta(a - x) f(x) dx = f(a),$$

where a is any real number.

This definition allows the Fourier transform of $\delta(x - a)$ to be represented as

$$\mathcal{F}\{\delta(x - a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - a) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} e^{-i\omega a}.$$

EXAMPLE Find the Fourier transform of $f(x) = \delta(x - a) \exp[-b^2 x^2]$ ($b > 0$).

Solution By definition

$$\begin{aligned}\mathcal{F}\{\delta(x - a) \exp[-b^2 x^2]\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - a) \exp[-b^2 x^2] e^{-i\omega x} dx \\ &= \frac{1}{\sqrt{2\pi}} \exp[-(a^2 b^2 + i\omega a)].\end{aligned}$$

Fourier Transforms of Partial Derivatives with Respect to x of a Function $f(x, t)$

The Fourier transform with respect to x of a function $f(x, t)$ of two independent variables x and t , denoted by $F(\omega, t)$, is defined as

$${}_x \mathcal{F}\{f(x, t)\} = F(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, t) e^{-i\omega x} dx,$$

where the prefix suffix x shows the variable that is being transformed.

and the Fourier transforms of the partial derivatives of $f(x, t)$ with respect to x are given by

$${}_x\mathcal{F} \left\{ \frac{\partial^n}{\partial x^n} [f(x, t)] \right\} = (i\omega)^n F(\omega, t)$$

$${}_x\mathcal{F} \{ x^n f(x, t) \} = i^n \frac{\partial^n}{\partial \omega^n} [F(\omega, t)]$$

$${}_x\mathcal{F} \left\{ x^m \frac{\partial^n}{\partial x^n} [f(x, t)] \right\} = i^{m+n} \frac{\partial^m}{\partial \omega^m} [\omega^n F(\omega, t)].$$

Fourier Cosine and Sine Transforms

Fourier cosine transform

$$F_C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \omega x dx.$$

$$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx,$$

Fourier cosine inversion integral

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_C(\omega) \cos \omega x d\omega.$$

Fourier sine transform

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx,$$

Fourier cosine inversion integral

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_S(\omega) \sin \omega x d\omega.$$

EXAMPLE

Find $\mathcal{F}_C\{e^{-ax}\}$ and $\mathcal{F}_S\{e^{-ax}\}$ when $a > 0$, and use the results with the Fourier cosine and sine inversion integrals and an interchange of variables to show that

$$\mathcal{F}_C\left\{\frac{1}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} e^{-a\omega}.$$

Solution By definition

$$\begin{aligned} \mathcal{F}_C\{e^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos \omega x dx \\ &= \operatorname{Re} \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-ax} e^{i\omega x} dx \right\} = \sqrt{\frac{2}{\pi}} \operatorname{Re} \left\{ \frac{1}{a - i\omega} \right\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{\omega^2 + a^2} \right). \end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}_S\{e^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin \omega x dx \\ &= \text{Im} \sqrt{\frac{2}{\pi}} \left\{ \int_0^\infty e^{-ax} e^{i\omega x} dx \right\} = \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{\omega^2 + a^2} \right).\end{aligned}$$

Using these results in the Fourier cosine and sine inversion integrals gives

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2} d\omega = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega x}{\omega^2 + a^2} d\omega, \text{ for } a > 0,$$

so after x and ω are interchanged, these results become

$$e^{-a\omega} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \omega x}{x^2 + a^2} dx = \frac{2}{\pi} \int_0^\infty \frac{x \cos \omega x}{x^2 + a^2} dx.$$

However,

$$\mathcal{F}_C\left\{\frac{1}{x^2 + a^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos \omega x}{x^2 + a^2} dx \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2 + a^2}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin \omega x}{x^2 + a^2} dx,$$

so combining results gives

$$\mathcal{F}_C\left\{\frac{1}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} e^{-a\omega}.$$

THEOREM

Fourier cosine and sine transforms of derivatives Let $f(x)$ be continuous and absolutely integrable over $[0, \infty)$ and such that $\lim_{x \rightarrow \infty} f(x) = 0$. Then if $f'(x)$ and $f''(x)$ are piecewise continuous on each finite subinterval of $[0, \infty)$,

(i)
$$\mathcal{F}_C\{f'(x)\} = \omega \mathcal{F}_S\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

(ii)
$$\mathcal{F}_S\{f'(x)\} = -\omega \mathcal{F}_C\{f(x)\}$$

(iii)
$$\mathcal{F}_C\{f''(x)\} = -\omega^2 \mathcal{F}_C\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

(iv)
$$\mathcal{F}_S\{f''(x)\} = -\omega^2 \mathcal{F}_S\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0).$$

The Parseval relation for the Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then the Parseval relation for the Fourier cosine transform is

$$\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx,$$

and the Parseval relation for the Fourier sine transform is

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.$$

THEOREM

Shifting ω and scaling x in Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then

- (i) $\mathcal{F}_C\{\cos(ax)f(x)\} = \frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
- (ii) $\mathcal{F}_C\{\sin(ax)f(x)\} = \frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
- (iii) $\mathcal{F}_S\{\cos(ax)f(x)\} = \frac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$
- (iv) $\mathcal{F}_S\{\sin(ax)f(x)\} = \frac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$
- (v) $\mathcal{F}_C\{f(ax)\} = \frac{1}{a}F_C(\omega/a) \quad (a > 0)$
- (vi) $\mathcal{F}_S\{f(ax)\} = \frac{1}{a}F_S(\omega/a) \quad (a > 0).$

Fourier cosine and sine transforms of partial derivatives of a function $f(x, y)$

$${}_x\mathcal{F}_C\{f'(x, t)\} = \omega F_S(\omega, t) - \sqrt{\frac{2}{\pi}} f(0, t)$$

$${}_x\mathcal{F}_S\{f'(x, t)\} = -\omega F_C(\omega, t)$$

$${}_x\mathcal{F}_C\{f''(x, t)\} = -\omega^2 F_S(\omega, t) - \sqrt{\frac{2}{\pi}} f'(0, t)$$

$${}_x\mathcal{F}_S\{f''(x, t)\} = -\omega^2 F_C(\omega, t) + \sqrt{\frac{2}{\pi}} \omega f(0, t)$$

EXAMPLE

Given $f(x) = e^{-ax}$ with $a > 0$, use the results of Theorem find (a) $\mathcal{F}_C\{\cos bx f(x)\}$ and (b) $\mathcal{F}_S\{f(bx)\}$, when $b > 0$.

Solution

(a) Using Theorem (i) with

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{\omega^2 + a^2} \right),$$

gives

$$\begin{aligned}\mathcal{F}_C\{\cos bx e^{-ax}\} &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega+b)^2+a^2}\right) + \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega-b)^2+a^2}\right) \\ &= \sqrt{\frac{2}{\pi}}\frac{a(\omega^2+a^2+b^2)}{[(\omega+b)^2+a^2][(\omega-b)^2+a^2]}.\end{aligned}$$

(b) Using Theorem (vi) with

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

gives

$$\mathcal{F}_S\{f(bx)\} = \mathcal{F}_S\{e^{-abx}\} = \frac{1}{b}\sqrt{\frac{2}{\pi}}\left(\frac{\omega/b}{(\omega/b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2b^2}\right).$$

TABLE

Fourier Transform Pairs

$$f(x)$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$1. af(x) + bg(x)$$

$$aF(\omega) + bG(\omega)$$

$$2. f^{(n)}(x)$$

$$(i\omega)^n F(\omega)$$

$$3. x^n f(x)$$

$$(i)^n \frac{d^n}{d\omega^n} [F(\omega)]$$

$$4. x^m f^{(n)}(x)$$

$$(i)^{m+n} \frac{d^m}{d\omega^m} [\omega^n F(\omega)]$$

$$5. f(ax) (a > 0)$$

$$\frac{1}{a} F(\omega/a)$$

$$6. f(x - a)$$

$$e^{-i\omega a} F(\omega)$$

$$7. e^{i\lambda x} f(x)$$

$$F(\omega - \lambda)$$

$$8. (f * g)(x)$$

$$\sqrt{2\pi} F(\omega) G(\omega)$$

(convolution theorem)

$$9. \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

(Parseval relation)

$$10. \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \quad (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{\sin a\omega}{\omega} \right)$$

$$11. \frac{\sin ax}{x} \quad (a > 0)$$

$$\begin{cases} \sqrt{\frac{\pi}{2}}, & |\omega| < a \\ 0, & |\omega| > a \end{cases}$$

$$12. \begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (0 < a < b)$$

$$\frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega} \right)$$

$$13. \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos \omega a}{\omega^2} \right)$$

$$14. \frac{1}{a^2 + x^2} \quad (a > 0)$$

$$\sqrt{\frac{\pi}{2}} \frac{e^{-a|\omega|}}{a}$$

$$15. \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases} \quad (a > 0)$$

$$\frac{1}{\sqrt{2\pi}} \left(\frac{1}{a + i\omega} \right)$$

$$16. \begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases} \quad (a > 0)$$

$$\frac{1}{\sqrt{2\pi}} \left[\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega} \right]$$

$$f(x)$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$17. e^{-a|x|} \quad (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + \omega^2} \right)$$

$$18. xe^{-a|x|} \quad (a > 0)$$

$$-\sqrt{\frac{2}{\pi}} \frac{2ia\omega}{(a^2 + \omega^2)^2}$$

$$19. \begin{cases} e^{i\omega x}, & |x| < b \\ 0, & |x| > b \end{cases}$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{\sin b(\omega - a)}{\omega - a} \right)$$

$$20. \exp(-a^2 x^2) \quad (a > 0)$$

$$\frac{1}{a\sqrt{2}} \exp \left\{ -\frac{\omega^2}{4a^2} \right\}$$

$$21. \begin{cases} e^{-x} x^a, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$\frac{\Gamma(a)}{\sqrt{2\pi} (1 + i\omega)^a}$$

$$22. J_0(ax) \quad (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \frac{H(a - |\omega|)}{(a^2 - \omega^2)^{1/2}}$$

$$23. \delta(x - a) \quad (a \text{ real})$$

$$\frac{1}{\sqrt{2\pi}} e^{-ia\omega}$$

TABLE

Fourier Cosine Transform Pairs

$$f(x)$$

$$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx$$

1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
4. $f(ax)$	$\frac{1}{a} F_C\left(\frac{\omega}{a}\right) (a > 0)$
5. $f'(x)$	$\omega F_S(\omega) - \sqrt{\frac{2}{\pi}} f(0)$
6. $f''(x)$	$-\omega^2 F_C(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$
7. $\int_0^\infty f(x) ^2 dx$	$\int_0^\infty F(\omega) ^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x)g(x)dx$	$\int_0^\infty F_C(\omega)G_C(\omega)d\omega$

-
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$ $\sqrt{\frac{2}{\pi}} \left(\frac{\sin a\omega}{\omega} \right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$ $\sqrt{\frac{2}{\pi}} \left(\frac{\sin b\omega - \sin a\omega}{\omega} \right)$
11. $x^{\alpha-1} (0 < \alpha < 1)$ $\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \cos \frac{\alpha\pi}{2}$
12. $\begin{cases} x, & a < x < b \\ 0, & \text{otherwise} \end{cases}$ $\sqrt{\frac{2}{\pi}} \left(\frac{\cos b\omega + b\omega \sin b\omega - \cos a\omega - a\omega \sin a\omega}{\omega^2} \right)$
13. $e^{-ax} (a > 0)$ $\sqrt{\frac{2}{\pi}} \left(\frac{a}{\omega^2 + a^2} \right)$
14. $xe^{-ax} (a > 0)$ $\sqrt{\frac{2}{\pi}} \frac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
15. $\exp\{-ax^2\} (a > 0)$ $\frac{1}{\sqrt{2a}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
16. $\frac{1}{x^2 + a^2} (a > 0)$ $\sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a}$
-

$$f(x)$$

$$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx$$

$$17. J_0(ax) (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \frac{H(a - \omega)}{(a^2 - \omega^2)^{1/2}}$$

$$18. \frac{\sin ax}{x} (a > 0)$$

$$\sqrt{\frac{2}{\pi}} H(a - \omega)$$

TABLE Fourier Sine Transform Pairs

$$f(x)$$

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$$

$$1. af(x) + bg(x)$$

$$aF(\omega) + bG(\omega)$$

$$2. \cos(ax) f(x)$$

$$\frac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$$

$$3. \sin(ax) f(x)$$

$$\frac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$$

$$4. f(ax)$$

$$\frac{1}{a} F_S\left(\frac{\omega}{a}\right) \quad (a > 0)$$

TABLE

Fourier Sine Transform Pairs

$$f(x)$$

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$$

$$5. \quad f'(x)$$

$$-\omega F_C(\omega)$$

$$6. \quad f''(x)$$

$$-\omega^2 F_S(\omega) + \sqrt{\frac{2}{\pi}} \omega f'(0)$$

$$7. \quad \int_0^\infty |f(x)|^2 dx$$

$$\int_0^\infty |F(\omega)|^2 d\omega$$

(Parseval relation)

$$8. \quad \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty F_S(\omega)G_S(\omega)d\omega$$

$$9. \quad \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega} \right)$$

$$10. \quad \begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$\sqrt{\frac{2}{\pi}} \left(\frac{\cos a\omega - \cos b\omega}{\omega} \right)$$

$$f(x)$$

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$$

$$11. \ x^{\alpha-1} \ (0 < \alpha < 1)$$

$$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \sin \frac{\alpha\pi}{2}$$

$$12. \ e^{-ax} \ (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2 + a^2)}$$

$$13. \ xe^{-ax} \ (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \frac{2a\omega}{(\omega^2 + a^2)^2}$$

$$14. \ x \exp\{-ax^2\} \ (a > 0)$$

$$\frac{\omega}{(2a)^{3/2}} \exp\left\{-\frac{\omega^2}{4a}\right\}$$

$$15. \ \frac{x}{x^2 + a^2} \ (a > 0)$$

$$\sqrt{\frac{\pi}{2}} e^{-a\omega}$$

$$16. \ \frac{\cos ax}{x} \ (a > 0)$$

$$\sqrt{\frac{\pi}{2}} H(\omega - a)$$

$$17. \ \operatorname{erfc}\left\{\frac{x}{2a}\right\} \ (a > 0)$$

$$\sqrt{\frac{2}{\pi}} \left\{ \frac{1 - \exp(-a^2\omega^2)}{\omega} \right\}$$

Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a - j\omega}$	$a > 0$
3	$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	
12	$\text{sgn } t$	$\frac{2}{j\omega}$	

Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$
13	$\cos \omega_0 t u(t)$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
14	$\sin \omega_0 t u(t)$	$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
15	$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$ $a > 0$
16	$e^{-at} \cos \omega_0 t u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$ $a > 0$
17	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau \text{sinc}\left(\frac{\omega\tau}{2}\right)$
18	$\frac{W}{\pi} \text{sinc}(Wt)$	$\text{rect}\left(\frac{\omega}{2W}\right)$
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\omega\tau}{4}\right)$
20	$\frac{W}{2\pi} \text{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$
21	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$ $\omega_0 = \frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi} e^{-\sigma^2\omega^2/2}$