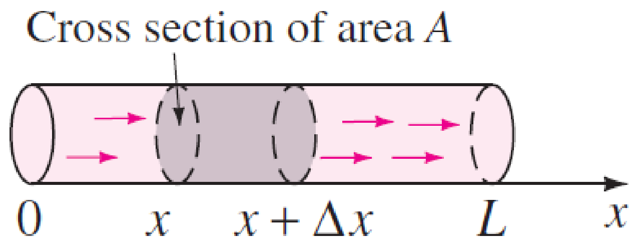
Heat Equation: Initial and Boundary Conditions

Assumptions:

- Heat only flows in x-direction.
- No heat escapes from the surface.
- No heat is generated in the rod.
- Rod is homogeneous with density ρ.

$$ku_{xx} = u_t, \quad 0 < x < L, \quad t > 0$$



Initial Condition:
Provides the spatial distribution of the temperature at time $t = 0$.

$$u(x, 0) = f(x), \ 0 < x < L$$

Boundary Conditions:
At the end points $x = 0$ and $x = L$, give the constraints on

- u: (Dirchlet condition), for example, (u_0: constant)

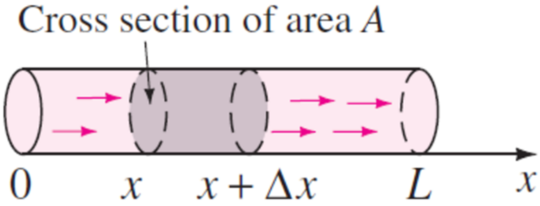
  $$u(L, t) = u_0 \quad \text{Temperature at the right end is held at constant.}$$

- u_x: (Neumann condition), for example,

  $$u_x(L, t) = 0 \quad \text{The right end is insulated.}$$

- $u_x + hu$: (Robin condition), for example, ($h > 0$, u_m: constants)

  $$u_x(L, t) = -h\{u(L, t) - u_m\} \quad \text{Heat is lost from the right end.}$$



- A problem involving both initial and boundary conditions is called an **initial boundary-value** problem
- At the two boundaries $x - 0$ and $x = L$, one can use different kinds of conditions.

Examples:

$$ku_{xx} = u_t, \quad 0 < x < L, \quad t > 0$$

Heat equation

$$u(0, t) = u_0, \quad u_x(L, t) = -h\left\{u(L, t) - u_m\right\}, \quad t > 0$$

Boundary condition

$$u(x, 0) = f(x), \quad 0 < x < L$$

Initial condition

$$ku_{xx} = u_t, \quad 0 < x < L, \quad t > 0$$

Heat equation

$$u_x(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$$

Boundary condition

$$u(x, 0) = f(x), \quad 0 < x < L$$

Initial condition

Modifications of Heat and Wave Equations

In the derivation of the heat equation and the wave equation, we assume that there is no internal or external influences. For example, *no heat escapes from the surface, no heat is generated in the rod, no external force act on the string, etc.*

Taking external and internal influences into account, more general forms of the heat equation and the wave equation are the following:

$$ku_{xx} + G(x, t, u, u_x) = u_t \qquad \text{Heat Equation}$$

$$a^2 u_{xx} + F(x, t, u, u_t) = u_{tt} \qquad \text{Wave Equation}$$

Example:

$$ku_{xx} - h(u - u_m) = u_t$$

heat transfers from the surface to an environment with constant temperature u_m

$$a^2 u_{xx} + f(x, t) = u_{tt}$$

External force f acts on the string

Example 1-D Heat equation

The heat equation can be described by the following

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \qquad 0 < x < L \quad , \qquad t > 0$$

$$u(0,\ t) = 0 \quad , \qquad u(L,\ t) = 0 \quad , \qquad t > 0$$

$$u(x,\ 0) = f(x) \quad , \qquad 0 < x < L$$

Solution of the BVP

Using $u(x,\ t) = X(x)T(t)$, and $-\lambda$ as the separation constant:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad \Longrightarrow \quad X'' + \lambda X = 0 \qquad T' + k\lambda T = 0$$

Now the boundary conditions become $u(0,\ t) = X(0)T(t) = 0$ and $u(L,\ t) = X(L)T(t) = 0$.

Then we can have $X(0) = X(L) = 0$ and $X'' + \lambda X = 0,\ \ X(0) = 0,\ \ X(L) = 0$

From the previous discussions, we have

$$X(x) = c_1 + c_2 x, \qquad \lambda = 0$$
$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \qquad \lambda = -\alpha^2 < 0$$
$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \qquad \lambda = \alpha^2 > 0$$

When the boundary conditions $X(0) = X(L) = 0$ are applied, these solutions are only $X(x) = 0$.

Applying the first condition gives $c_1 = 0$. Therefore $X(x) = c_2 \sin \alpha x$. The condition $X(L) = 0$

implies that $X(L) = c_2 \sin \alpha L = 0$

We have $\sin \alpha L = 0$ for $c_2 \neq 0$ and $\alpha = n\pi/L$, $n = 1, 2, 3, \ldots$

The values $\lambda_n = \alpha_n^2 = (n\pi/L)^2$, $n = 1, 2, 3, \ldots$ and the corresponding solutions

$$X(x) = c_2 \sin \frac{n\pi}{L} x, \qquad n = 1, 2, 3, \ldots$$

are the eigenvalues and eigenfunctions, respectively. The general solution is

$$T = c_3 e^{-k(n^2\pi^2/L^2)t}$$

and so $u_n = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x$ where $A_n = c_2 c_3$.

Now using the initial conditions $u(x, 0) = f(x)$, $0 < x < L$, we have

$$u_n(x,\ 0) = f(x) = A_n \sin\frac{n\pi}{L}x$$

By the superposition principle the function

$$\boxed{u(x,\ t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x}$$

must satisfy. If we let $t = 0$, then

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

It is recognized as the half-range expansion of f in *a* sine series.

If we let $A_n = b_n$, $n = 1, 2, 3, \ldots$ thus

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\ dx}$$

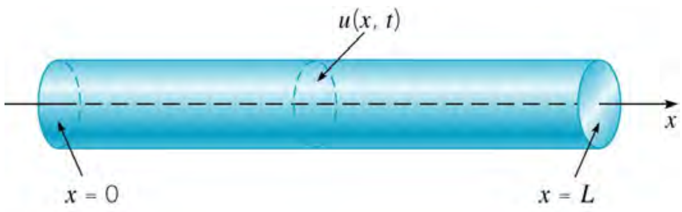
We conclude that the solution of the BVP is given by infinite series

Example : Heat Conduction Problem

Find the temperature $u(x,t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20° C throughout and whose ends are maintained at 0° C for all $t > 0$.

- This heat conduction problem has the form

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < 50, \quad t > 0$$
$$u(0,t) = 0, \quad u(50,t) = 0, \quad t > 0$$
$$u(x,0) = 20, \quad 0 < x < 50$$



- The solution to our heat conduction problem is $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi\alpha/50)^2 t} \sin(n\pi x/50)$

where

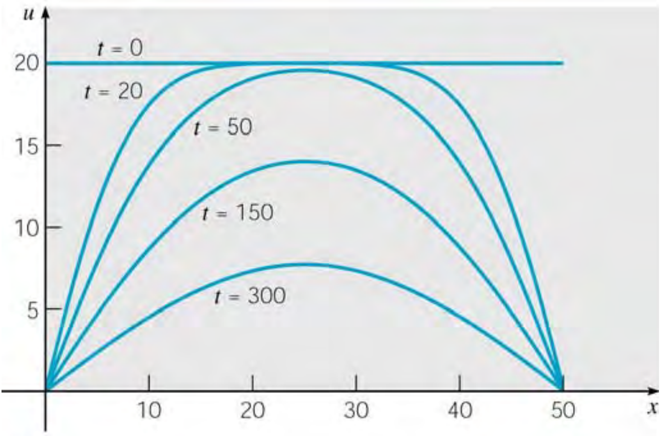
$$c_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx = \frac{2}{50}\int_0^{50} 20\sin(n\pi x/50)dx$$
$$= \frac{4}{5}\int_0^{50} \sin(n\pi x/50)dx = \frac{40}{n\pi}(1-\cos n\pi) = \begin{cases} 80/n\pi, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

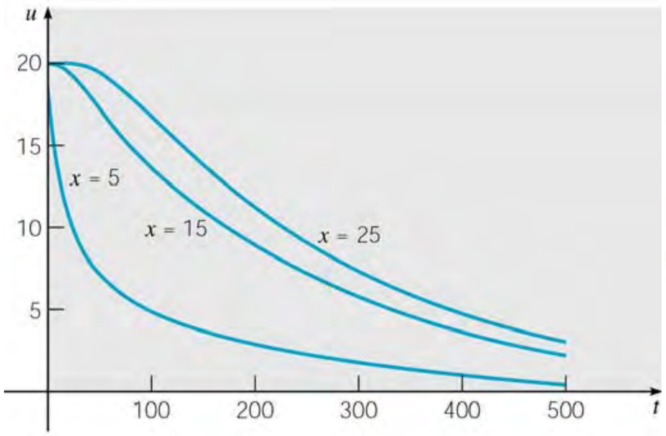
- Thus the temperature along the rod is given by

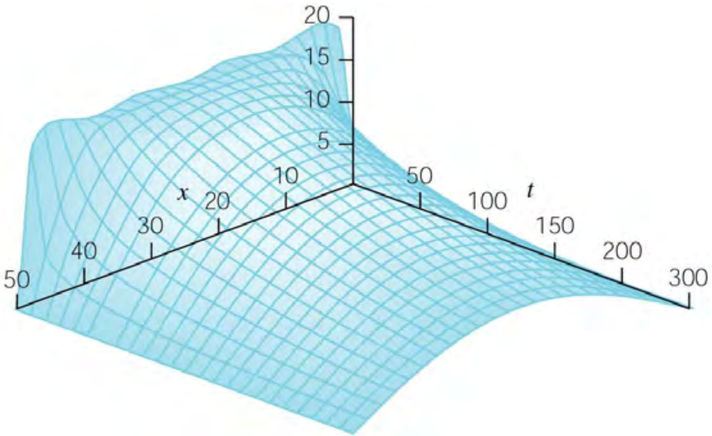
$$u(x,t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-(n\pi\alpha/50)^2 t} \sin(n\pi x/50)$$

- The negative exponential factor in each term cause the series to converge rapidly, except for small values of t or α^2.
- Therefore accurate results can usually be obtained by using only a few terms of the series.
- In order to display quantitative results, let t be measured in seconds; then α^2 has the units cm^2/sec.
- If we choose $\alpha^2 = 1$ for convenience, then the rod is of a material whose properties are somewhere between copper and aluminum.
- The graph of the temperature distribution in the bar at several times is given below on the left.
- Observe that the temperature diminishes steadily as heat in the bar is lost through the end points.

- The way in which the temperature decays at a given point is plotted in the graph below on the right, where temperature is plotted against time for a few selected points in the bar.
- A three-dimensional plot of *u* versus *x* and *t* is given below.
- Observe that we obtain the previous graphs by intersecting the surface below by planes on which either *t* or *x* is constant.
- The slight waviness in the graph below at *t* = 0 results from using only a finite number of terms in the series for *u*(*x*,*t*) and from the slow convergence of the series for *t* = 0.







EXAMPLE Insulated Bar

Find the temperature $u(x,t)$ in a metal rod of length 25 cm that is insulated on the ends as well as on the sides and whose initial temperature distribution is $u(x,0) = x$ for $0 < x < 25$.

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \quad t > 0, \tag{1}$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L. \tag{3}$$

boundary conditions are

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0, \qquad t > 0. \tag{24}$$

The problem posed by Eqs. (1), (3), and (24) can also be solved by the method of separation of variables. If we let

$$u(x,t) = X(x)T(t), \tag{25}$$

and substitute for u in Eq. (1), then it follows, that

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda, \tag{26}$$

where λ is a constant. Thus we obtain again the two ordinary differential equations

$$X'' + \lambda X = 0, \tag{27}$$

$$T' + \alpha^2 \lambda T = 0. \tag{28}$$

If we substitute for $u(x,t)$ from Eq. (25) in the boundary condition at $x = 0$, we obtain $X'(0)T(t) = 0$. We cannot permit $T(t)$ to be zero for all t, since then $u(x,t)$ would also be zero for all t. Hence we must have

$$X'(0) = 0. \tag{29}$$

Proceeding in the same way with the boundary condition at $x = L$, we find that

$$X'(L) = 0. \tag{30}$$

Thus we wish to solve Eq. (27) subject to the boundary conditions (29) and (30). It is possible to show that nontrivial solutions of this problem can exist only if λ is real. and will consider in turn the three cases $\lambda < 0, \lambda = 0$, and $\lambda > 0$.

If $\lambda < 0$, it is convenient to let $\lambda = -\mu^2$, where μ is real and positive. Then Eq. (27) becomes $X'' - \mu^2 X = 0$, and its general solution is

$$X(x) = k_1 \sinh \mu x + k_2 \cosh \mu x. \tag{31}$$

In this case the boundary conditions can be satisfied only by choosing $k_1 = k_2 = 0$.

If $\lambda = 0$, then Eq. (27) is $X'' = 0$, and therefore

$$X(x) = k_1 x + k_2. \tag{32}$$

Hence, for $\lambda = 0$, we obtain the constant solution $u(x, t) = k_2$.

Finally, if $\lambda > 0$, let $\lambda = \mu^2$, where μ is real and positive. Then Eq. (27) becomes $X'' + \mu^2 X = 0$, and consequently,

$$X(x) = k_1 \sin \mu x + k_2 \cos \mu x. \tag{33}$$

The boundary condition (29) requires that $k_1 = 0$, and the boundary condition (30) requires that $\mu = n\pi/L$ for $n = 1, 2, 3, \ldots$ but leaves k_2 arbitrary. Thus the problem (27), (29), (30) has an infinite sequence of positive eigenvalues $\lambda = n^2\pi^2/L^2$ with the corresponding eigenfunctions $X(x) = \cos(n\pi x/L)$. For these values of λ the solutions $T(t)$ of Eq. (28) are proportional to $\exp(-n^2\pi^2\alpha^2 t/L^2)$.

Combining all these results, we have the following fundamental solutions for the problem (1), (3), and (24):

$$
\begin{aligned}
u_0(x,t) &= c \\
u_n(x,t) &= e^{-n^2\pi^2\alpha^2 t/L^2} \cos \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots,
\end{aligned}
\tag{34}
$$

Thus, to satisfy the initial condition (3), we assume that $u(x,t)$ has the form

$$
\begin{aligned}
u(x,t) &= \frac{c_0}{2} u_0(x,t) + \sum_{n=1}^{\infty} c_n u_n(x,t) \\
&= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos \frac{n\pi x}{L}.
\end{aligned}
\tag{35}
$$

The coefficients c_n are determined by the requirement that

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L} = f(x). \tag{36}$$

Thus the unknown coefficients in Eq. (35) must be the coefficients in the Fourier cosine series of period $2L$ for f. Hence

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients $c_0, c_1, c_2, \ldots$, the series (35) provides the solution to the heat conduction problem for a rod with insulated ends, Eqs. (1), (3), and (24).

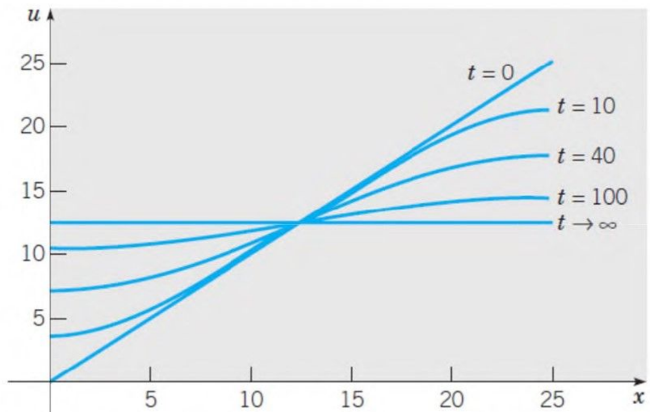


FIGURE 10.6.2 Temperature distributions at several times for the heat conduction problem

For $\alpha^2 = 1$, Figure 10.6.2 shows plots of the temperature distribution in the bar at several times. Again the convergence of the series is rapid so that only a relatively few terms are needed to generate the graphs.

The temperature in the rod satisfies the heat conduction problem (1), (3), (24) with $L = 25$. Thus, from Eq. (35), the solution is

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/625} \cos\frac{n\pi x}{25}, \tag{39}$$

where the coefficients are determined from Eq. (37). We have

$$c_0 = \frac{2}{25}\int_0^{25} x\,dx = 25 \tag{40}$$

and, for $n \geq 1$,

$$\begin{aligned} c_n &= \frac{2}{25}\int_0^{25} x\cos\frac{n\pi x}{25}\,dx \\ &= 50(\cos n\pi - 1)/(n\pi)^2 = \begin{cases} -100/(n\pi)^2, & n \text{ odd;} \\ 0, & n \text{ even.} \end{cases} \end{aligned} \tag{41}$$

Thus

$$u(x,t) = \frac{25}{2} - \frac{100}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} e^{-n^2\pi^2\alpha^2 t/625} \cos(n\pi x/25) \tag{42}$$

Homogeneous vs. Nonhomogeneous Boundary Conditions

Homogeneous Boundary Condition:

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad u(x, 0) = 0,$$

Nonhomogeneous Boundary Condition:

$$u_x(0, y) = f(y), \quad u_x(a, y) = g(y), \quad u(x, L) = u_m$$

Typically, when using separation of variables, start with the independent variable associated with homogeneous boundary conditions, to determine the value of the **separation constant**.

Example.

Find the steady-state problem and solution

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad 0 < t, \tag{8}$$

$$u(0, t) = T_0, \qquad 0 < t, \tag{9}$$

$$-\kappa \frac{\partial u}{\partial x}(a, t) = h\big(u(a, t) - T_1\big), \qquad 0 < t, \tag{10}$$

$$u(x, 0) = f(x), \qquad 0 < x < a. \tag{11}$$

When the rule given here is applied to this problem, we are led to the following equations:

$$\frac{d^2 v}{dx^2} = 0, \quad 0 < x < a,$$
$$v(0) = T_0, \quad -\kappa v'(a) = h\big(v(a) - T_1\big).$$

The solution of the differential equation is $v(x) = A + Bx$. The boundary conditions require that A and B satisfy

$$v(0) = T_0\colon\ A = T_0,$$
$$-\kappa v'(a) = h\big(v(a) - T_1\big)\colon\ -\kappa B = h(A + Ba - T_1).$$

Solving simultaneously, we find

$$A = T_0, \quad B = \frac{h(T_1 - T_0)}{\kappa + ha}.$$

Thus the steady-state solution of Eqs. (8)–(11) is

$$v(x) = T_0 + \frac{xh(T_1 - T_0)}{\kappa + ha}. \tag{12}$$

□

Transforming Nonhomogeneous BCs into Homogeneous

Problem

Consider heat flow in an insulated rod where two ends are kept at constant temperature k_1 and k_2; that is,

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant L.$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables.

- It is obvious that the solution of problem will have a steady-state solution (solution when $t = \infty$) that varies linearly (in x) between the boundary temperatures k_1 and k_2.
- It seems reasonable to think of our temperature $u(x,t)$ as the sum of two parts

$$u(x,t) = \text{steady state} + \text{transient},$$

where steady state is eventual solution for large times, while transient is a part of the solution that depends on the IC (and will go to zero).

Example: Fixed End Temperatures

we saw that the temperature $u(x, t)$ in a uniform rod with insulated material surface would be determined by the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad 0 < t, \tag{1}$$

$$u(0, t) = T_0, \qquad 0 < t, \tag{2}$$

$$u(a, t) = T_1, \qquad 0 < t, \tag{3}$$

$$u(x, 0) = f(x), \quad 0 < x < a \tag{4}$$

if the ends of the rod are held at fixed temperatures and if the initial temperature distribution is $f(x)$. we found that the steady-state temperature distribution,

$$v(x) = \lim_{t\to\infty} u(x, t),$$

satisfied the boundary value problem

$$\frac{d^2 v}{dx^2} = 0, \quad 0 < x < a, \tag{5}$$

$$v(0) = T_0, \quad v(a) = T_1. \tag{6}$$

In fact, we were able to find $v(x)$ explicitly:

$$v(x) = T_0 + (T_1 - T_0)\frac{x}{a}. \tag{7}$$

We also defined the transient temperature distribution as

$$w(x, t) = u(x, t) - v(x)$$

and determined that w satisfies the boundary value–initial value problem

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < a, \quad 0 < t, \tag{8}$$

$$w(0, t) = 0, \qquad 0 < t, \tag{9}$$

$$w(a, t) = 0, \qquad 0 < t, \tag{10}$$

$$w(x, 0) = f(x) - v(x) \equiv g(x), \qquad 0 < x < a. \tag{11}$$

Our objective is to determine the transient temperature distribution, $w(x, t)$, and — since $v(x)$ is already known — the unknown temperature will be

$$u(x, t) = v(x) + w(x, t). \tag{12}$$

The general idea of the method is to assume that the solution of the partial differential equation has the form of a product: $w(x,t) = \phi(x)T(t)$. We require that neither of the factors $\phi(x)$ and $T(t)$ be identically 0, since that would lead back to the trivial solution. Now, each of the factors depends on only one variable, so we have

$$\frac{\partial^2 w}{\partial x^2} = \phi''(x)T(t), \quad \frac{\partial w}{\partial t} = \phi(x)T'(t).$$

The partial differential equation becomes

$$\phi''(x)T(t) = \frac{1}{k}\phi(x)T'(t),$$

and on dividing through by ϕT we find

$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{kT(t)}, \quad 0 < x < a, \quad 0 < t.$$

all x in the interval $0 < x < a$ and for all $t > 0$, the common value of the two sides must be a constant, varying neither with x nor t:

$$\frac{\phi''(x)}{\phi(x)} = p, \quad \frac{T'(t)}{kT(t)} = p.$$

Now we have two ordinary differential equations for the two factor functions:

$$\phi'' - p\phi = 0, \quad T' - pkT = 0. \tag{13}$$

The two boundary conditions on w may also be stated in the product form:

$$w(0,t) = \phi(0)T(t) = 0, \quad w(a,t) = \phi(a)T(t) = 0.$$

There are two ways these equations can be satisfied for all $t > 0$. Either the function $T(t) \equiv 0$ for all t, which is forbidden, or the other factors must be zero. Therefore, we have

$$\phi(0) = 0, \quad \phi(a) = 0. \tag{14}$$

Our job now is to solve Eqs. (13) and satisfy the boundary conditions (14) while avoiding the trivial solution.

Case 1: If $p > 0$, the solutions of Eqs. (13) are

$$\phi(x) = c_1 \cosh\left(\sqrt{p}x\right) + c_2 \sinh\left(\sqrt{p}x\right), \quad T(t) = ce^{pkt}.$$

Now we apply the boundary conditions:

$$\phi(0) = 0: \quad c_1 = 0,$$

$$\phi(a) = 0: \quad c_2 \sinh\left(\sqrt{p}a\right) = 0.$$

Because the sinh function is 0 only when its argument is 0 — clearly not true of $\sqrt{p}a$ — we have $c_1 = c_2 = 0$ and $\phi(x) \equiv 0$, which is not acceptable.

Case 2: If we take $p = 0$, the solutions of the differential equations (13) are $\phi(x) = c_1 + c_2 x$, $T(t) = c$. The boundary conditions require

$$\phi(0) = 0: \quad c_1 = 0,$$

$$\phi(a) = 0: \quad c_2 a = 0.$$

Again we have $\phi(x) \equiv 0$.

Case 3: We now try a negative constant. Replacing p by $-\lambda^2$ in Eqs. (13) gives us the two equations

$$\phi'' + \lambda^2 \phi = 0, \quad T' + \lambda^2 kT = 0,$$

whose solutions are

$$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x), \quad T(t) = c \exp\left(-\lambda^2 kt\right).$$

If ϕ has the form given in the preceding, the boundary conditions require that $\phi(0) = c_1 = 0$, leaving $\phi(x) = c_2 \sin(\lambda x)$. Then $\phi(a) = c_2 \sin(\lambda a) = 0$.

We now have two choices: either $c_2 = 0$, making $\phi(x) \equiv 0$ for all values of x, or $\sin(\lambda a) = 0$. We reject the first possibility, for it leads to the trivial solution $w(x, t) \equiv 0$. In order for the second possibility to hold, we must have $\lambda = n\pi/a$, where $n = \pm 1, \pm 2, \pm 3, \ldots$. The negative values of n do not give any new functions, because $\sin(-\theta) = -\sin(\theta)$. Hence we allow $n = 1, 2, 3, \ldots$ only. We shall set $\lambda_n = n\pi/a$.

To review our position, we have, for each $n = 1, 2, 3, \ldots$, a function $\phi_n(x) = \sin(\lambda_n x)$ and an associated function $T_n(t) = \exp(-\lambda_n^2 kt)$. The product $w_n(x, t) = \sin(\lambda_n x) \exp(-\lambda_n^2 kt)$ has these properties:

1. $\frac{\partial^2 w_n}{\partial x^2} = -\lambda_n^2 w_n$; $\frac{\partial w_n}{\partial t} = -\lambda_n^2 k w_n$; and therefore w_n satisfies the heat equation.

2. $w_n(0,t) = \sin(0)e^{-\lambda_n^2 kt} = 0$ for any n and t; and therefore w_n satisfies the boundary condition at $x = 0$.

3. $w_n(a,t) = \sin(\lambda_n a)e^{-\lambda_n^2 kt} = 0$ for any n and t because $\lambda_n a = n\pi$ and $\sin(n\pi) = 0$. Therefore w_n satisfies the boundary condition at $x = a$.

Now we call on the Principle of Superposition in order to continue.

Principle of Superposition.
If $u_1, u_2, \ldots$ are solutions of the same linear, homogeneous equations, then so is

$$u = c_1 u_1 + c_2 u_2 + \cdots. \qquad \square$$

In fact, we have infinitely many solutions, so we need an infinite series to combine them all:

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right). \tag{15}$$

Of the four parts of the original problem, only the initial condition has not yet been satisfied. At $t = 0$, the exponentials in Eq. (15) are all unity. Thus the initial condition takes the form

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = g(x), \quad 0 < x < a. \tag{16}$$

We immediately recognize a problem in Fourier series, which is solved by choosing the constants b_n according to the formula

$$b_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right) dx. \tag{17}$$

Once the transient temperature has been determined, we find the original unknown $u(x,t)$ as the sum of the transient and the steady-state solutions,

$$u(x,t) = v(x) + w(x,t).$$

Example.

Suppose the original problem to be

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad 0 < t,$$

$$u(0,t) = T_0, \quad 0 < t,$$

$$u(a,t) = T_1, \quad 0 < t,$$

$$u(x,0) = 0, \quad 0 < x < a.$$

The steady-state solution is $\quad v(x) = T_0 + (T_1 - T_0)\dfrac{x}{a}.$

The transient temperature, $w(x,t) = u(x,t) - v(x)$, satisfies

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < a, \quad 0 < t,$$

$$w(0,t) = 0, \qquad 0 < t,$$

$$w(a,t) = 0, \qquad 0 < t,$$

$$w(x,0) = -T_0 - (T_1 - T_0)\frac{x}{a} \equiv g(x), \quad 0 < x < a.$$

According to the preceding calculations, w has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right) \tag{18}$$

and the initial condition is

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) = g(x), \quad 0 < x < a.$$

The coefficients b_n are given by

$$b_n = \frac{2}{a}\int_0^a \left[-T_0 - (T_1 - T_0)\frac{x}{a}\right] \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2T_0}{a} \frac{\cos(n\pi x/a)}{(n\pi/a)}\bigg|_0^a - \frac{2}{a^2}(T_1 - T_0)\frac{\sin(n\pi x/a) - (n\pi x/a)\cos(n\pi x/a)}{(n\pi/a)^2}\bigg|_0^a$$

$$= -\frac{2T_0}{n\pi}\big(1-(-1)^n\big) + \frac{2(T_1-T_0)}{n\pi}(-1)^n$$

$$b_n = \frac{-2}{n\pi}\big(T_0 - T_1(-1)^n\big).$$

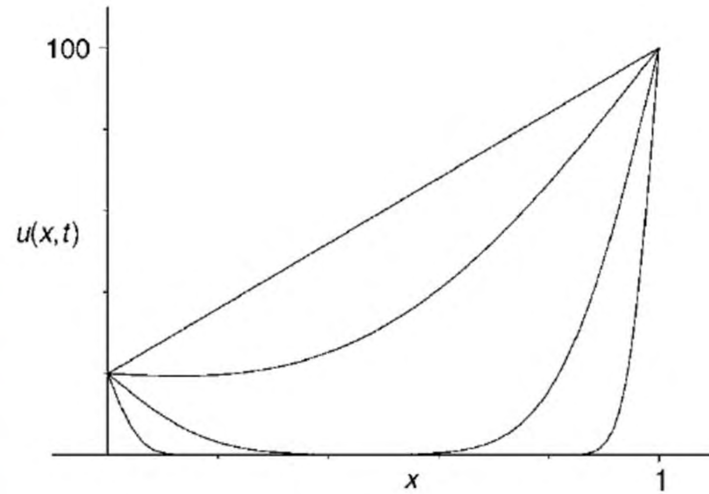
Now the complete solution (see Fig. 3) is

$$u(x,t) = w(x,t) + T_0 + (T_1 - T_0)\frac{x}{a},$$

where $$w(x,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{T_0 - T_1(-1)^n}{n} \sin(\lambda_n x)\exp\big(-\lambda_n^2 kt\big). \qquad (19)$$

We can discover certain features of $u(x,t)$ by examining the solution. First, $u(x,0)$ really is zero ($0 < x < a$) because the Fourier series converges to $-v(x)$ at $t=0$. Second, when t is positive but very small, the series for $w(x,t)$ will almost equal $-T_0 - (T_1 - T_0)x/a$. But at $x=0$ and $x=a$, the series adds up to zero (and $w(x,t)$ is a continuous function of x); thus $u(x,t)$ satisfies the boundary conditions. Third, when t is large, $\exp(-\lambda_1^2 kt)$ is small, and the other exponentials are still smaller. Then $w(x,t)$ may be well approximated by the first term (or first few terms) of the series. Finally, as $t \to \infty$, $w(x,t)$ disappears completely.

Figure 3 The solution of the example with T_1= 100 and T_0= 20. The function $u(x, t)$ is graphed as a function of x for four values of t, chosen so that the dimensionless time kt/a^2 has the values 0.001, 0.01, 0.1, and 1.



Example: Different Boundary Conditions

In many important cases, boundary conditions at the two endpoints will be different kinds. In this section we shall solve the problem of finding the temperature in a rod having one end insulated and the other held at a constant temperature. The boundary value–initial value problem satisfied by the temperature in the rod is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < a, \quad 0 < t, \tag{1}$$

$$u(0, t) = T_0, \qquad 0 < t, \tag{2}$$

$$\frac{\partial u}{\partial x}(a, t) = 0, \qquad 0 < t, \tag{3}$$

$$u(x, 0) = f(x), \quad 0 < x < a. \tag{4}$$

It is easy to verify that the steady-state solution of this problem is $v(x) = T_0$. Using this information, we can find the boundary value–initial value problem satisfied by the transient temperature $w(x, t) = u(x, t) - T_0$:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \qquad 0 < x < a, \quad 0 < t, \tag{5}$$

$$w(0,t)=0, \quad \frac{\partial w}{\partial x}(a,t)=0, \quad 0<t, \tag{6}$$

$$w(x,0)=f(x)-T_0=g(x), \quad 0<x<a. \tag{7}$$

Since this problem is homogeneous, we can attack it by the method of separation of variables. The assumption that $w(x,t)$ has the form of a product, $w(x,t)=\phi(x)T(t)$, and insertion of w in that form into the partial differential equation (5) lead, as before, to

$$\frac{\phi''(x)}{\phi(x)}=\frac{T'(t)}{kT(t)}=\text{constant}. \tag{8}$$

The boundary conditions take the form

$$\phi(0)T(t)=0, \quad 0<t, \tag{9}$$

$$\phi'(a)T(t)=0, \quad 0<t. \tag{10}$$

As before, we conclude that $\phi(0)$ and $\phi'(a)$ should both be zero:

$$\phi(0)=0, \quad \phi'(a)=0. \tag{11}$$

By trial and error we find that a positive or zero separation constant in Eq. (8) forces $\phi(x) \equiv 0$. Thus we take the constant to be $-\lambda^2$. The separated equations are

$$\phi'' + \lambda^2\phi = 0, \qquad 0 < x < a, \tag{12}$$

$$T' + \lambda^2 kT = 0, \qquad 0 < t. \tag{13}$$

Now, the general solution of the differential equation (12) is

$$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

The boundary condition, $\phi(0) = 0$, requires that $c_1 = 0$, leaving

$$\phi(x) = c_2 \sin(\lambda x).$$

The boundary condition at $x = a$ now takes the form

$$\phi'(a) = c_2\lambda \cos(\lambda a) = 0.$$

The three choices are: $c_2 = 0$, which gives the trivial solution; $\lambda = 0$, which should be investigated separately (Exercise 2), and $\cos(\lambda a) = 0$. The third alternative — the only acceptable one — requires that λa be an odd multiple of $\pi/2$, which we may express as

$$\lambda_n = \frac{(2n-1)\pi}{2a}, \quad n = 1, 2, \ldots. \tag{14}$$

Thus, we have found that the eigenvalue problem consisting of Eqs. (11) and (12) has the solution

$$\lambda_n = \frac{(2n-1)\pi}{2a}, \quad \phi_n(x) = \sin(\lambda_n x), \quad n = 1, 2, 3, \ldots. \tag{15}$$

With the eigenfunctions and eigenvalues now in hand, we return to the differential equation (13), whose solution is

$$T_n(t) = \exp\left(-\lambda_n^2 kt\right).$$

As in previous cases, we assemble the general solution of the homogeneous problem expressed in Eqs. (5)–(7) by forming a general linear combination of our product solutions,

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right). \tag{16}$$

The choice of the coefficients, b_n, must be made so as to satisfy the initial condition, Eq. (8). Using the form of w given by Eq. (16), we find that the initial condition is

$$w(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2a}\right) = g(x), \quad 0 < x < a. \tag{17}$$

A routine Fourier sine series for the interval $0 < x < a$ would involve the functions $\sin(n\pi x/a)$, rather than the functions we have. By one of several means (Exercises 10–12), it may be shown that the series in Eq. (17) represents the function $g(x)$, provided that g is sectionally smooth and that we choose the coefficients by the formula

$$b_n = \frac{2}{a} \int_0^a g(x) \sin\left(\frac{(2n-1)\pi x}{2a}\right) dx. \tag{18}$$

Now the original problem is completely solved. The solution is

$$u(x, t) = T_0 + \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp\left(-\lambda_n^2 kt\right). \tag{19}$$

It should be noted carefully that the T_0 term in Eq. (19) is the steady-state solution in this case; it is not part of the separation-of-variables solution.

Orthogonal Series Expansions

Example The temperature in a rod of unit length is determined from

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0$$

$$u(0,t) = 0, \qquad u_x(1,t) = -hu(1,t),\ h > 0\,, t > 0$$

$$u(x,0) = 1, \qquad 0 < x < 1$$

solve for $u(x, t)$.

Solution

If we let $u(x, t) = X(x)T(t)$ and $-\lambda$ as the separation constant, we have

$$X'' + \lambda X = 0$$

$$T' + k\lambda T = 0$$

$$X(0) = 0, \ \ X'(1) = -hX(1)$$

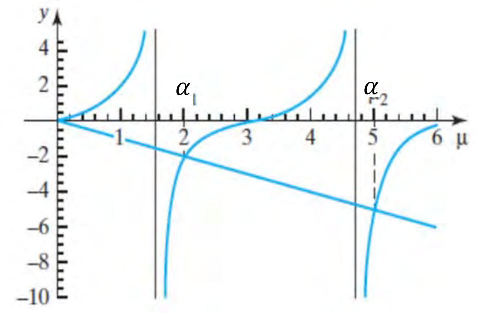
These comprise the regular Sturm-Liouville problem

$$X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(1) + hX(1) = 0$$

This possesses nontrivial solutions only for $\lambda = \alpha^2 > 0$, $\alpha > 0$.

The general solution is $X = c_1 \cos \alpha x + c_2 \sin \alpha x$. $X(0) = 0$ implies $c_1 = 0$.

Applying the second condition to $X = c_2 \sin \alpha x$ implies



$$\alpha \cos \alpha + h \sin \alpha = 0 \quad \text{or} \quad \tan \alpha = -\frac{\alpha}{h}$$

Because the graph of $y = \tan x$ and $y = -x/h$, $h > 0$, have an infinite number of points of intersections for $x > 0$, this equation has an infinite number of roots. If the consecutive positive roots are denoted by α_n, $n = 1, 2, \ldots$, then the eigenvalues $\lambda_n = \alpha_n{}^2$ and the corresponding eigenfunctions $X(x) = c_2 \sin \alpha_n x$, $n = 1, 2, \ldots$. The solution is

$$T(t) = c_3 e^{-k\alpha_n^2 t}, \text{ and so } \quad u_n = XT = A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x}$$

Now at t = 0, $u(x, 0) = 1$, $0 < x < 1$, so that

the sine function is not an integer multiple of $\pi x/L$

$$1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$$

It is an expansion of $u(x, 0) = 1$ in terms of the orthogonal functions arising from the Sturm-Liouville problem. The set $\{\sin \alpha_n x\}$ is orthogonal w.r.t. the weight function $p(x) = 1$. we have

$$A_n = \frac{\int_0^1 \sin \alpha_n x \, dx}{\int_0^1 \sin^2 \alpha_n x \, dx}$$

We found that

$$\int_0^1 \sin^2 \alpha_n x \, dx = \frac{1}{2}\int_0^1 [1 - \cos 2\alpha_n x] \, dx = \frac{1}{2}\left[1 - \frac{1}{2\alpha_n} \sin 2\alpha_n \right]$$

Using $\sin 2\alpha_n = 2 \sin \alpha_n \cos \alpha_n$ and $\alpha_n \cos \alpha_n = -h \sin \alpha_n$, This becomes

$$\int_0^1 \sin^2 \alpha_n x dx = \frac{1}{2h}(h + \cos^2 \alpha_n),$$

$$\int_0^1 \sin \alpha_n x dx = -\frac{1}{\alpha_n} \cos \alpha_n x \Big|_0^1 = \frac{1}{\alpha_n}(1 - \cos \alpha_n)$$

Thus becomes

$$A_n = \frac{2h(1 - \cos \alpha_n)}{\alpha_n (h + \cos^2 \alpha_n)}$$

Finally,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2h(1-\cos \alpha_n)}{\alpha_n (h+\cos^2 \alpha_n)} e^{-k\alpha_n{}^2 t} \sin \alpha_n x$$

Example Nonhomogeneous Problems

A typical nonhomogeneous BVP for the heat equation is

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \qquad t > 0$$

$$u(0,t) = u_0(t), \qquad u(L,t) = u_1(t) \quad , \qquad t > 0$$

$$u(x,\ 0) = f(x), \qquad 0 < x < L$$

When heat is generated at a constant rate *r* within a rod, the heat equation takes the form

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}$$

The equation is shown not to be separable.

Change of Dependent variables

$u = v + \psi$, ψ is a function to be determined.

Time Independent PDE and BCs

First consider the heat source *F* and the boundary conditions are time-independent:

$$\boxed{\begin{aligned} & k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \qquad t > 0 \\ & u(0,t) = u_0, \qquad u(L,t) = u_1, t > 0 \\ & u(x,\ 0) = f(x), \qquad 0 < x < L \end{aligned}}$$

where u_0 and u_1 denotes constants. If we let $u(x, t) = v(x, t) + \psi(x)$, The equation can be decomposed into two problems:

Problem 1: $\{k\psi'' + F(x) = 0, \psi(0) = u_0, \psi(L) = u_1$

Problem 2: $\begin{cases} k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t} \\ v(0,t) = 0, v(L,t) = 0 \\ v(x,0) = f(x) - \psi(x) \end{cases}$

Example Solve $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$ subject to $u(0,t)=0, u(1,t)=u_0, t>0$

$u(x,0)=f(x), 0<x<1$

Solution

If we let $u(x, t) = v(x, t) + \psi(x)$, then

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x^2}+\psi'', \qquad \frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}$$

since $\psi_t = 0$. Substituting gives

$$k\frac{\partial^2 v}{\partial x^2}+k\psi''+r=\frac{\partial v}{\partial t}$$

The equation is reduced to a homogeneous PDE if we demand that ψ be a function satisfying the ODE

$$k\psi''+r=0 \quad \text{or} \quad \psi''=-\frac{r}{k}$$

Thus we have

$$\psi(x)=-\frac{r}{2k}x^2+c_1x+c_2$$

Furthermore, $u(0,t) = v(0,t) + \psi(0) = 0$

$$u(1,t) = v(1,t) + \psi(1) = u_0$$

We have $v(0, t) = 0$ and $v(1, t) = 0$, provided we choose $\psi(0) = 0$ and $\psi(1) = u_0$

Applying these conditions implies $c_2 = 0$, $c_1 = r/2k + u_0$.

Thus $$\boxed{\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x}$$

Finally the initial condition $u(x,0) = v(x, 0) + \psi(x)$ implies $v(x,0) = u(x, 0) - \psi(x) = f(x) - \psi(x)$.

We have the new homogeneous BVP:

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \qquad 0 < x < 1, \qquad t > 0$$

$$v(0,t) = 0, \qquad v(1,t) = 0, \qquad t > 0$$

$$v(x,0) = f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x, \qquad 0 < x < 1$$

In the usual manner we find

$$v(x,t) = \sum_{n=1}^{\infty} \left\{ A_n e^{-kn^2\pi^2 t} \sin n\,\pi x \right\}$$

With the initial condition $v(x,0)$, we have

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x \right] \sin n\,\pi x dx$$

A solution of the original problem is

$$u(x,\ t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)\ x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\,\pi x$$

Observe that

$$u(x,t) \to \psi(x) \text{ as } t \to \infty: \mathit{steady - state\ solution}$$

$$v(x,t) \to 0 \text{ as } t \to \infty: \mathit{transient\ solution}$$

Semi-Infinite Rod

Frequently, however, it is justifiable and useful to assume that an object is infinite in length. Thus, if the rod we have been studying is very long, we may treat it as *semi-infinite*—that is, as extending from 0 to ∞. If properties are uniform and there is no "generation," the partial differential equation governing the temperature $u(x, t)$ remains

Example :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x, \quad 0 < t.$$

Let us suppose that at $x = 0$ the temperature is held constant, say, $u(0, t) = 0$ in some temperature scale. In the absence of another boundary, there is no other boundary condition. However, it is desirable that $u(x, t)$ remain finite — less than some fixed bound — as $x \to \infty$.

Thus, our mathematical model is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \quad 0 < t, \tag{1}$$

$$u(0, t) = 0, \qquad 0 < t, \tag{2}$$

$$u(x, t) \quad \text{bounded} \quad \text{as } x \to \infty, \tag{3}$$

$$u(x, 0) = f(x), \qquad 0 < x. \tag{4}$$

The heat equation (1) and the boundary condition (2) are homogeneous. The boundedness condition (3) is also homogeneous in an important way: A (finite) sum of bounded functions is bounded. Thus, we can attack Eqs. (1)–(3) by separation of variables. Assume that $u(x, t) = \phi(x)T(t)$, so the partial differential equation can be separated into two ordinary equations as usual:

$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{kT(t)} = \text{const.} \tag{5}$$

Thus, we must choose a negative separation constant, $-\lambda^2$. The differential equation, together with the boundary and boundedness conditions, forms a *singular* eigenvalue problem (singular because of the semi-infinite interval),

$$\phi'' + \lambda^2\phi = 0, \quad 0 < x, \tag{6}$$

$$\phi(0) = 0, \qquad \phi(x) \quad \text{bounded} \quad \text{as } x \to \infty. \tag{7}$$

The general solution of the differential equation is

$$\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x),$$

which is bounded for any choice of the constants and for any value of λ. The boundedness condition told us to use a negative constant in Eq. (5) and now contributes nothing further.

Applying the boundary condition at $x = 0$ shows that $c_1 = 0$, leaving $\phi(x) = c_2 \sin(\lambda x)$.

$$\phi(x; \lambda) = \sin(\lambda x), \quad \lambda > 0. \tag{8}$$

The solution of Eq. (5) for $T(t)$, with constant $-\lambda^2$, is

$$T(t) = \exp(-\lambda^2 kt).$$

For any value of λ^2, the function

$$u(x, t; \lambda) = \sin(\lambda x) \exp(-\lambda^2 kt)$$

satisfies Eqs. (1)–(3). Equation (1) and the boundary condition Eq. (2) are homogeneous, and Eq. (3) is homogeneous in effect; therefore any linear combination of solutions is a solution.

Thus u should have the form

$$u(x,t) = \int_0^\infty B(\lambda)\sin(\lambda x)\exp\left(-\lambda^2 kt\right)d\lambda. \tag{9}$$

The initial condition will be satisfied if $B(\lambda)$ is chosen to make

$$u(x,0) = \int_0^\infty B(\lambda)\sin(\lambda x)\,d\lambda = f(x), \quad 0 < x.$$

We recognize this as a Fourier integral; $B(\lambda)$ is to be chosen as

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)\,dx. \tag{10}$$

If $B(\lambda)$ exists, then Eq. (9) is the solution of the problem. Notice that when $t > 0$, the exponential function makes the improper integral in Eq. (9) converge very rapidly.

Some care must be taken in the interpretation of our solution. If the rod really is finite (say, length L) the expression in Eq. (9) is, of course, meaningless for x greater than L. The presence of a boundary condition at $x = L$ would influence temperatures nearby, so Eq. (9) can be considered a valid approximation only for $x \ll L$.

Infinite Rod If we wish to study heat conduction in the center of a very long rod, we may assume that it extends from $-\infty$ to ∞. Then there are no boundary conditions, and the problem to be solved is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \quad 0 < t, \tag{1}$$

$$u(x, 0) = f(x), \qquad -\infty < x < \infty, \tag{2}$$

$$|u(x, t)| \quad \text{bounded} \quad \text{as } x \to \pm\infty. \tag{3}$$

Using the same techniques as before, we look for solutions in the form $u(x, t) = \phi(x)T(t)$ so that the heat equation (1) becomes

$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{T(t)} = \text{constant}.$$

As in the previous section, the constant must be nonpositive (say, $-\lambda^2$) in order for the solutions to be bounded. Thus, we have the singular eigenvalue problem

$$\phi'' + \lambda^2\phi = 0, \quad -\infty < x < \infty,$$

$$\phi(x) \quad \text{bounded} \quad \text{as } x \to \pm\infty.$$

It is easy to see that *every* solution of $\phi''/\phi = -\lambda^2$ is bounded. Thus, our factors $\phi(x)$ and $T(t)$ are

$$\begin{aligned}\phi(x;\lambda) &= A\cos(\lambda x) + B\sin(\lambda x),\\ T(t;\lambda) &= \exp\left(-\lambda^2 kt\right).\end{aligned}$$

We combine the solutions $\phi(x)T(t)$ in the form of an integral to obtain

$$u(x,t) = \int_0^\infty \big(A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\big)\exp\left(-\lambda^2 kt\right)d\lambda. \tag{4}$$

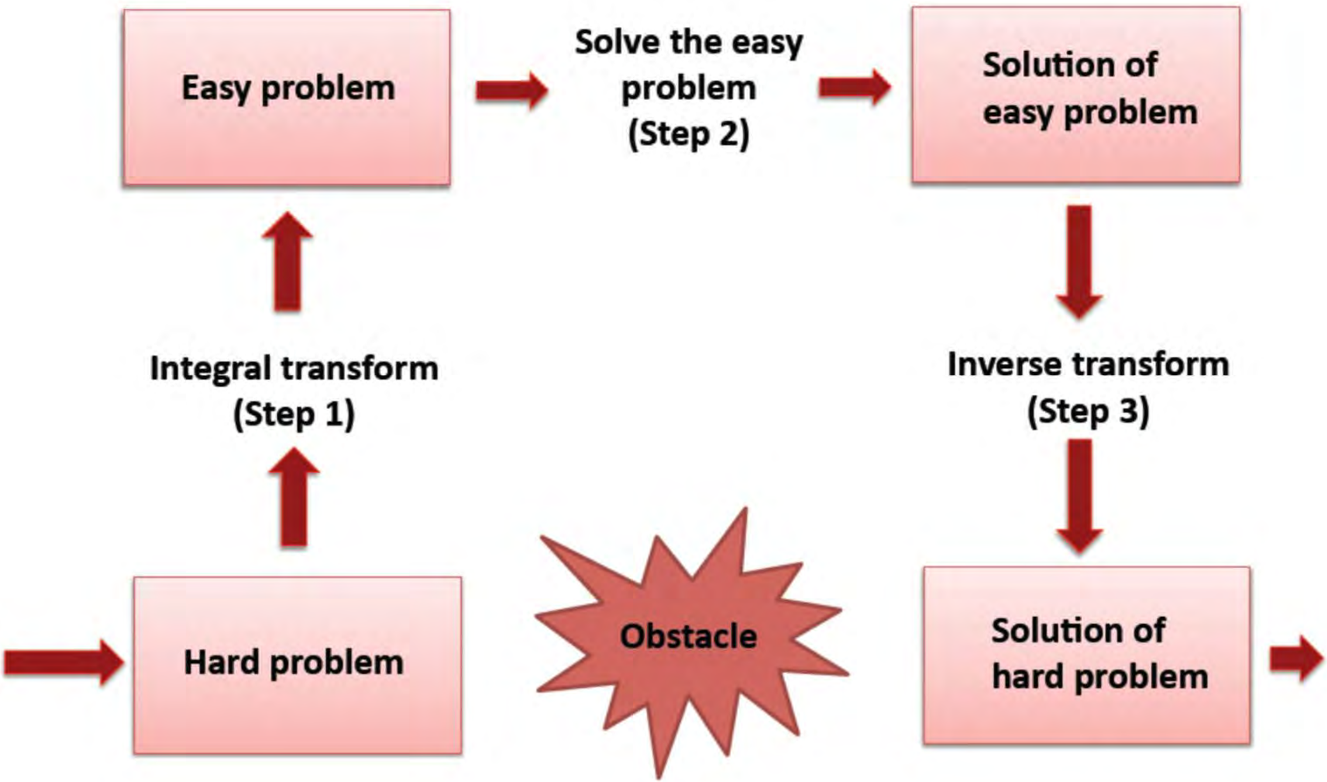
At time $t = 0$, the exponential factor becomes 1, and the initial condition is

$$\int_0^\infty \big(A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\big)\,d\lambda = f(x), \quad -\infty < x < \infty.$$

As this is clearly a Fourier integral problem, we must choose $A(\lambda)$ and $B(\lambda)$ to be the Fourier integral coefficient functions,

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx, \quad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx. \tag{5}$$

General philosophy of transforms



Laplace transforms:

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)e^{-st}dt \quad \text{(Laplace transform)}$$

$$\mathcal{L}^{-1}[F] = f(t) = \int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds \quad \text{(inverse Laplace transform)}$$

Remarks

- The Laplace transform has one major advantage over the Fourier transform in that the damping factor e^{-st} in the integrand allows us to transform a wider class of functions.

- The factor $e^{i\xi x}$ in the Fourier transform doesn't do any damping, since its absolute value is one.

Sufficient Conditions to Insure the Existence of a Laplace Transform

Laplace transform

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
4.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
5.	t^n ($n =$ positive integer)	$\frac{n!}{s^{n+1}}, \quad s > 0$

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
6.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
7.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
9.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
10.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
11.	$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$
12.	$H(t-a)f(t-a)$	$e^{-as}F(s)$
13.	$e^{at}f(t)$	$F(s-a)$
14.	$f^{(n)}(t)$ (nth derivative)	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
15.	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
16.	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
17.	$\operatorname{erf}(t/2a)$	$\frac{1}{s}e^{a^2s^2}\operatorname{erfc}(as)$
18.	$\operatorname{erfc}(a/(2\sqrt{t}))$	$\frac{1}{s}e^{-a\sqrt{s}}$
19.	$\delta(t-a)$	e^{-sa}
20.	$\frac{1}{\sqrt{\pi t}} - ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$	$\frac{1}{\sqrt{s}+a}$

Heat Conduction in a Semi Infinite Medium

Consider a deep container of liquid that is insulated on the sides. Suppose the liquid has an initial temperature of u_0 and the temperature of the air above the liquid is zero (some reference temperature). Our goal is to find the temperature of the liquid at various depths of the container at different values of time.

Problem

To find the function $u(x,t)$ that satisfies

$$\begin{array}{rlr} \text{PDE:} & u_t = u_{xx}, & 0 < x < \infty, \quad 0 < t < \infty \\ \text{BC:} & u_x(0,t) - u(0,t) = 0, & 0 < t < \infty \\ \text{IC:} & u(x,0) = u_0, & 0 < x < \infty \end{array}$$

Step 1. (Transformation)

- We take the Laplace transform with respect to t-variable. We transform the PDE and the BC - not the IC! As a result we get and ODE in x

$$\begin{aligned} &\text{ODE:} \quad sU(x) - u_0 = \frac{d^2}{dx^2}U(x), \qquad 0 < x < \infty \\ &\text{BC:} \qquad \frac{d}{dx}U(0) = U(0) \end{aligned} \tag{8.2}$$

Step 2. (Solving the BVP for ODE)

- The first equation in (8.2) is a second-order ODE with one BC at $x = 0$.
- For physical reasons, we really have a second, implied BC that says $U(x)$ is bounded.
- To solve (8.2), we first find the general solution (homogeneous + a particular solution), which is

$$U(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{u_0}{s}$$

- Note that $c_1 = 0$ or else the temperature will go to infinity as x gets large.
- Finding c_2 from the BC at $x = 0$ provides

$$U(x) = -u_0 \left\{ \frac{e^{-\sqrt{s}x}}{s(\sqrt{s}+1)} \right\} + \frac{u_0}{s}$$

Step 3. (Inverse transform)

- The last step is to find the inverse transform of $U(s)$; that is,

$$u(x,t) = \mathcal{L}^{-1}\left[U(x,s)\right].$$

Using the tables we see that

$$\boxed{u(x,t) = u_0 - u_0 \left[\operatorname{erfc}\left(x/(2\sqrt{t})\right) - \operatorname{erfc}\left(\sqrt{t} + x/(2\sqrt{t})\right)e^{x+t}\right]}.$$

EXAMPLE : A Solution in Terms of erf (x)

Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \ t > 0$$

subject to

$$u(0, t) = 0, \quad u(1, t) = u_0, \ t > 0$$

$$u(x, 0) = 0, \quad 0 < x < 1.$$

SOLUTION From (1) and (3) and the given initial condition,

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{L}\left\{\frac{\partial u}{\partial t}\right\}$$

becomes

$$\frac{d^2 U}{dx^2} - sU = 0. \tag{7}$$

The transforms of the boundary conditions are

$$U(0, s) = 0 \quad \text{and} \quad U(1, s) = \frac{u_0}{s}. \tag{8}$$

Since we are concerned with a finite interval on the x-axis, we choose to write the general solution of (7) as

$$U(x, s) = c_1 \cosh(\sqrt{s}x) + c_2 \sinh(\sqrt{s}x).$$

Applying the two boundary conditions in (8) yields, respectively, $c_1 = 0$ and $c_2 = u_0/(s \sinh \sqrt{s})$. Thus

$$U(x, s) = u_0 \frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}}.$$

Now the inverse transform of the latter function cannot be found in most tables. However, by writing

$$\frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}} = \frac{e^{\sqrt{s}x} - e^{-\sqrt{s}x}}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})} = \frac{e^{(x-1)\sqrt{s}} - e^{-(x+1)\sqrt{s}}}{s(1 - e^{-2\sqrt{s}})}$$

and using the geometric series

$$\frac{1}{1 - e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}}$$

we find

$$\frac{\sinh(\sqrt{s}x)}{s \sinh \sqrt{s}} = \sum_{n=0}^{\infty} \left[\frac{e^{-(2n+1-x)\sqrt{s}}}{s} - \frac{e^{-(2n+1+x)\sqrt{s}}}{s} \right].$$

If we assume that the inverse Laplace transform can be done term by term, it follows from entry 3 of Table that

$$u(x, t) = u_0\,\mathscr{L}^{-1}\left\{\frac{\sinh(\sqrt{s}x)}{s\sinh\sqrt{s}}\right\}$$

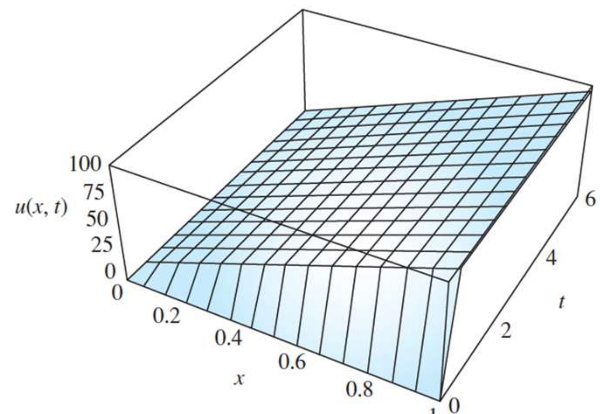
$$= u_0\sum_{n=0}^{\infty}\left[\mathscr{L}^{-1}\left\{\frac{e^{-(2n+1-x)\sqrt{s}}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-(2n+1+x)\sqrt{s}}}{s}\right\}\right]$$

$$= u_0\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]. \qquad (9)$$

The solution (9) can be rewritten in terms of the error function using $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$:

$$u(x, t) = u_0\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]. \qquad (10) \equiv$$

The rectangular region $0 < x < 1$, $0 < t < 6$ defined by the partial sum $S_{10}(x, t)$ of the solution (10).



(a) $u_0 = 100$

Example Laplace transform

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \quad 0 < t,$$

$$u(0, t) = 1, \quad u(1, t) = 1, \quad 0 < t,$$

$$u(x, 0) = 1 + \sin(\pi x), \qquad 0 < x < 1.$$

The partial differential equation *and the boundary conditions* (that is, everything that is valid for $t > 0$) are transformed, while the initial condition is incorporated by the transform

$$\frac{d^2 U}{dx^2} = sU - \big(1 + \sin(\pi x)\big), \quad 0 < x < 1,$$

$$U(0, s) = \frac{1}{s}, \quad U(1, s) = \frac{1}{s}.$$

This boundary value problem is solved to obtain

$$U(x, s) = \frac{1}{s} + \frac{\sin(\pi x)}{s + \pi^2}.$$

We direct our attention now to U as a function of s. Because $\sin(\pi x)$ is a constant with respect to s, tables may be used to find

$$u(x, t) = 1 + \sin(\pi x) \exp\big(-\pi^2 t\big). \qquad \square$$

Fourier Transform

If we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in x.

- The transform and its inverse together form what is called a transform pair.

Sine transforms

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\sin(\omega t)dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\sin(\omega t)d\omega & \text{(inverse sine transform)} \end{cases}$$

Cosine transforms

$$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(t)\cos(\omega t)dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega)\cos(\omega t)d\omega & \text{(inverse cosine transform)} \end{cases}$$

Sine and Cosine transforms of derivatives

$$\mathscr{F}_s\{f'(x)\} = -\alpha\mathscr{F}_c\{f(x)\}$$

$$\mathscr{F}_c\{f'(x)\} = \alpha\mathscr{F}_s\{f(x)\} - f(0).$$

$$\mathscr{F}_s\{f''(x)\} = -\alpha^2F(\alpha) + \boxed{\alpha f(0).}$$

$$\mathscr{F}_c\{f''(x)\} = -\alpha^2F(\alpha) - \boxed{f'(0).}$$

$$\mathscr{F}\{f'(x)\} = -i\alpha F(\alpha).$$

$$\mathscr{F}\{f''(x)\} = (-i\alpha)^2\,\mathscr{F}\{f(x)\} = -\alpha^2F(\alpha).$$

$$\mathscr{F}\{f^{(n)}(x)\} = (-i\alpha)^n\,\mathscr{F}\{f(x)\} = (-i\alpha)^nF(\alpha),$$

Solution of an Infinite-Diffusion Problem via the Sine Transform

We now show how the sine transform can solve an important IBVP (the infinite diffusion problem).

Problem

To find the function $u(x,t)$ that satisfies

$$\text{PDE:}\quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC:}\quad u(0,t) = A, \qquad 0 < t < \infty$$

$$\text{IC:}\quad u(x,0) = 0, \qquad 0 \leqslant x \leqslant \infty$$

Exponential Fourier transforms:

$$\mathcal{F}[f] \equiv F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x) e^{-i\xi x} dx \right]$$ (Fourier transform - FT)

$$\mathcal{F}^{-1}[F] \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(\xi) e^{i\xi x} d\xi \right]$$ (inverse FT)

Transformation of partial derivatives:

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_x(x,t) e^{-i\xi x} dx = i\xi \mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{xx}(x,t) e^{-i\xi x} dx = -\xi^2 \mathcal{F}[u]$$

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-i\xi x} dx = \frac{\partial}{\partial t} \mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{tt}(x,t) e^{-i\xi x} dx = \frac{\partial^2}{\partial t^2} \mathcal{F}[u]$$

Exponential Fourier transform

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
1.	$f^{(n)}(x)$ (n^{th} derivative)	$(i\omega)^n F(\omega)$
2.	$f(ax),\ a > 0$	$\frac{1}{a} F\left(\frac{\omega}{a}\right)$
3.	$f(x-a)$	$e^{-ia\omega} F(\omega)$
4.	$e^{-a^2 x^2}$	$\frac{1}{a\sqrt{2}} e^{-\omega^2/(4a^2)}$
5.	$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}$

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
6.	$\begin{cases} 1, & \|x\| < a \\ 0, & \|x\| > a \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(a\omega)}{\omega}$
7.	$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}} e^{-ia\omega}$
8.	$(1+x^2)^{-1}$	$\sqrt{\frac{\pi}{2}} e^{-\|\omega\|}$
9.	$xe^{-a\|x\|},\ a > 0$	$-2\sqrt{\frac{2}{\pi}} \frac{ia\omega}{(\omega^2+a^2)^2}$

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
10.	$H(x+a) - H(x-a)$	$\sqrt{\frac{2}{\pi}} \frac{\sin(a\omega)}{\omega}$
11.	$\frac{a}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} e^{-a\|\omega\|}$
12.	$\frac{2ax}{(x^2+a^2)^2}$	$-i\sqrt{\frac{\pi}{2}} \omega e^{-a\|\omega\|}$
13.	$\begin{cases} \cos(ax), & \|x\| < \pi/(2a) \\ 0, & \|x\| > \pi/(2a) \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2-\omega^2} \cos(\pi\omega/(2a))$

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
14.	$\begin{cases} 1-\|x\|, & \|x\| < 1 \\ 0, & \|x\| > 1 \end{cases}$	$2\sqrt{\frac{2}{\pi}} \left[\frac{\sin(\omega/2)}{\omega}\right]^2$
15.	$\cos(ax)$	$\sqrt{\frac{\pi}{2}} [\delta(\omega+a) + \delta(\omega-a)]$
16.	$\sin(ax)$	$i\sqrt{\frac{\pi}{2}} [\delta(\omega+a) + \delta(\omega-a)]$

Example the heat equation for a semi-infinite rod

$$
\begin{aligned}
u_t &= u_{xx}, \quad x > 0, t > 0, \\
u(x,0) &= f(x), x > 0, \\
u(0,t) &= 0, t > 0, \\
\lim_{x\to\infty} u(x,t) &= 0.
\end{aligned}
$$

We choose to transform x and, since we're given $\boxed{u(0,t)}$ (as opposed to $u_x(0,t)$), we use the sine transform. Specifically, we define

$$
\begin{aligned}
U(\alpha,t) &= \mathcal{F}_s[u(x,t)] \\
&= \frac{2}{\pi}\int_0^\infty u(x,t)\sin\alpha x \; dx, \text{ for each } t.
\end{aligned}
$$

As with Laplace transforms, we will need the property

$$
\frac{\partial}{\partial t}U(\alpha,t) = \frac{2}{\pi}\int_0^\infty u_t(x,t)\sin\alpha x \; dx.
$$

Then, the transformed PDE is

$$
U_t = -\alpha^2 U + \quad u(0,t)\alpha
$$

or

$$U_t + \alpha^2 U = 0,$$

with solution

$$U(\alpha, t) = e^{-\alpha^2 t} G(\alpha), G \text{ arbitrary.}$$

Then, the initial condition gives

$$U(\alpha, 0) = F_s(\alpha)$$

which implies that

$$U(\alpha, t) = e^{-\alpha^2 t} F_s(\alpha).$$

Therefore, we may transform back:

$$\begin{aligned} u(x,t) &= \int_0^\infty e^{-\alpha^2 t} F_s(\alpha) \sin \alpha x \; d\alpha \\ &= \int_0^\infty e^{-\alpha^2 t} \frac{2}{\pi} \int_0^\infty f(z) \sin \alpha z \; dz \sin \alpha x \; d\alpha. \end{aligned}$$

This is our not-so-satisfying solution. However, let's switch the order of integration (if it's actually allowed! More, later.):

EXAMPLE : Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $-\infty < x < \infty$, $t > 0$, subject to

$$u(x, 0) = f(x) \qquad \text{where} \qquad f(x) = \begin{cases} u_0, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

SOLUTION The problem can be interpreted as finding the temperature $u(x, t)$ in an infinite rod. Because the domain of x is the infinite interval $(-\infty, \infty)$ we use the Fourier transform (5) and define the transform of $u(x, t)$ to be

$$\mathscr{F}\{u(x, t)\} = \int_{-\infty}^{\infty} u(x, t)\, e^{i\alpha x}\, dx = U(\alpha, t).$$

If we write

$$\mathscr{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\alpha^2 U(\alpha, t) \qquad \text{and} \qquad \mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\mathscr{F}\{u(x, t)\} = \frac{dU}{dt},$$

then the Fourier transform of the partial differential equation,

$$\mathscr{F}\left\{k\,\frac{\partial^2 u}{\partial x^2}\right\} = \mathscr{F}\left\{\frac{\partial u}{\partial t}\right\},$$

becomes the ordinary differential equation

$$-k\alpha^2 U(\alpha, t) = \frac{dU}{dt} \qquad \text{or} \qquad \frac{dU}{dt} + k\alpha^2 U(\alpha, t) = 0.$$

Solving the last equation by the method of Section 2.3 gives $\boxed{U(\alpha, t) = ce^{-k\alpha^2 t}.}$ The initial temperature $u(x, 0) = f(x)$ in the rod is shown in **FIGURE** and its Fourier transform is

$$\mathscr{F}\{u(x, 0)\} = U(\alpha, 0) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx = \int_{-1}^{1} u_0 e^{i\alpha x}\, dx = u_0 \frac{e^{i\alpha} - e^{-i\alpha}}{i\alpha}.$$

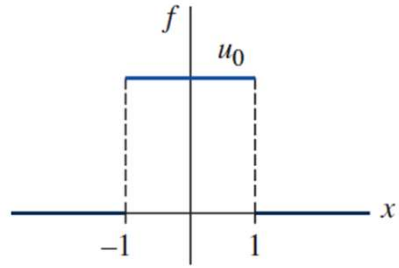
By Euler's formula

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$
$$e^{-i\alpha} = \cos\alpha - i\sin\alpha.$$

Subtracting these two results and solving for $\sin\alpha$ gives $\sin\alpha = \dfrac{e^{i\alpha} - e^{-i\alpha}}{2i}$. Hence we can rewrite the transform of the initial condition as $U(\alpha, 0) = 2u_0 \dfrac{\sin\alpha}{\alpha}$. Applying this condition to the solution $U(\alpha, t) = ce^{-k\alpha^2 t}$ gives $U(\alpha, 0) = c = 2u_0 \dfrac{\sin\alpha}{\alpha}$ and so

$$\boxed{U(\alpha, t) = 2u_0 \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t}.}$$

It then follows from the inverse Fourier transform (6) that



$$\boxed{u(x, t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha.}$$

This integral can be simplified somewhat by using Euler's formula again as $e^{-i\alpha x} = \cos \alpha x - i \sin \alpha x$ and noting that

$$\int_{-\infty}^{\infty} i\, \frac{\sin \alpha}{\alpha} e^{-k\alpha^2 t} \sin \alpha x\, d\alpha = 0$$

because the integrand is an odd function of α. Hence we finally have the solution

$$u(x, t) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} e^{-k\alpha^2 t}\, d\alpha. \tag{15}$$

Two Useful Fourier Transforms

$$\boxed{\begin{aligned} \mathscr{F}_s\{e^{-bx}\} &= \int_0^{\infty} e^{-bx} \sin \alpha x\, dx = \frac{\alpha}{b^2 + \alpha^2}, \\ \mathscr{F}_c\{e^{-bx}\} &= \int_0^{\infty} e^{-bx} \cos \alpha x\, dx = \frac{b}{b^2 + \alpha^2}. \end{aligned}}$$

((sinx/x)*exp(-x^2))*sin5x

Extended Keyboard Upload

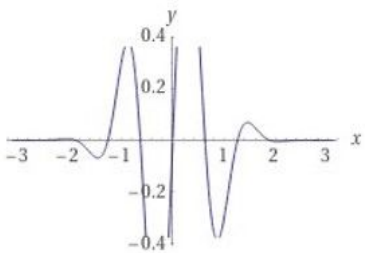
Input:

$$\left(\frac{\sin(x)}{x}\exp(-x^2)\right)\sin(5\,x)$$

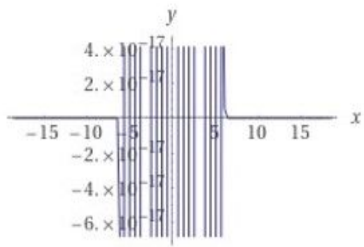
Exact result:

$$\frac{e^{-x^2}\sin(x)\sin(5\,x)}{x}$$

Plots:



(x from -3.1 to 3.1)



(x from -18.5 to 18.5)

((sinx/x)*exp(-x^2))*cos5x

Extended Keyboard Upload

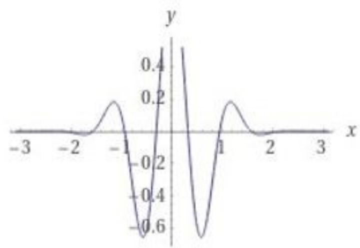
Input:

$$\left(\frac{\sin(x)}{x}\exp(-x^2)\right)\cos(5\,x)$$

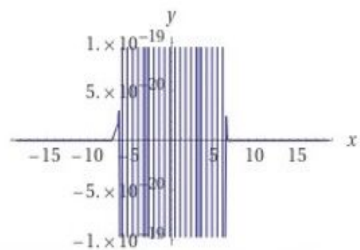
Exact result:

$$\frac{e^{-x^2}\sin(x)\cos(5\,x)}{x}$$

Plots:



(x from -3.1 to 3.1)



(x from -18.5 to 18.5)