Integration

Indefinite Integrals

$$\int x\,dx = \frac{x^2}{2} + c$$

Indefinite Integrals of a function are functions that differ from each other by a constant.

Definite Integrals

$$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

Definite Integrals are numbers.

Upper and Lower Sums

The interval is divided into subintervals.

$$Partition\ P = \{a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b\}$$

$$Define$$

$$m_i = \min\{f(x) : x_i \le x \le x_{i+1}\}$$

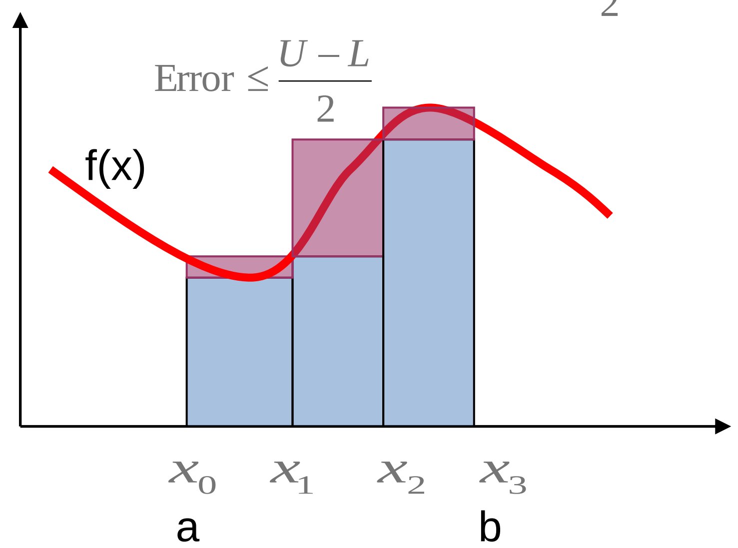
$$M_i = \max\{f(x) : x_i \le x \le x_{i+1}\}$$

$$Lower\ sum \quad L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

$$Upper\ sum \quad U(f,P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$

$$\text{Estimate of the integral} = \frac{L+U}{2}$$

$$\text{Error} \le \frac{U-L}{2}$$



Example: $\int_0^1 x^2 \, dx \qquad Partition: \quad P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$

$$n = 4 \text{ (four equal intervals)} \quad m_0 = 0, \qquad m_1 = \frac{1}{16}, \qquad m_2 = \frac{1}{4}, \qquad m_3 = \frac{9}{16}$$

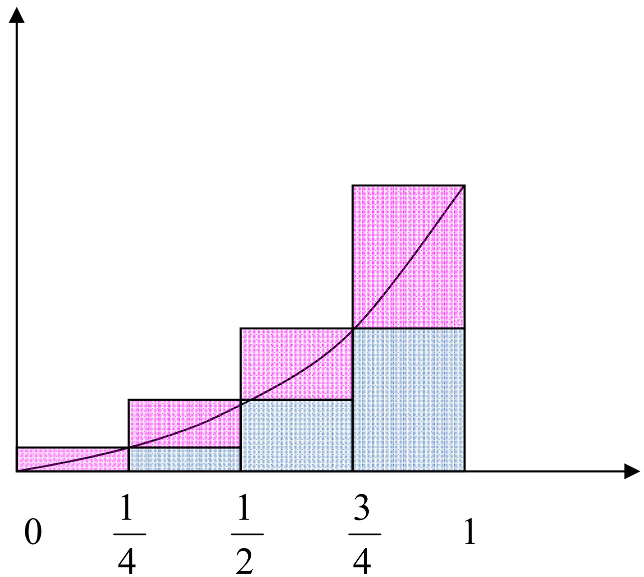
$$M_0 = \frac{1}{16}, \quad M_1 = \frac{1}{4}, \quad M_2 = \frac{9}{16}, \quad M_3 = 1 \quad \Rightarrow \quad x_{i+1} - x_i = \frac{1}{4} \qquad for \quad i = 0,1,2,3$$

$$Lower\ sum \quad L(f,P) = \sum_{i=0}^{n-1} m_i \left(x_{i+1} - x_i\right)$$

$$L(f,P) = \frac{1}{4}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] = \frac{14}{64}$$

$$Upper\ sum \quad U(f,P) = \sum_{i=0}^{n-1} M_i \left(x_{i+1} - x_i\right)$$

$$U(f,P) = \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right] = \frac{30}{64}$$



$$\text{Estimate of the integral} = \frac{1}{2}\left(\frac{30}{64} + \frac{14}{64}\right) = \frac{11}{32}$$

$$Error < \frac{1}{2}\left(\frac{30}{64} - \frac{14}{64}\right) = \frac{1}{8}$$

Estimates based on Upper and Lower Sums are easy to obtain for **monotonic** functions. For non-monotonic functions, finding maximum and minimum of the function can be difficult and other methods can be more attractive.

NUMERICAL INTEGRATION

$$\int_a^b f(x)dx =$$ area under the curve *f*(*x*) between $x = a$ to $x = b$.

In many cases a mathematical expression for *f*(*x*) is unknown and in some cases even if *f*(*x*) is known its complex form makes it difficult to perform the integration.

The Trapezoidal Rule

- The *Trapezoidal rule* is the first of the Newton-Cotes closed integration formulas, corresponding to the case where the polynomial is first order:

$$I = \int_a^b f(x)dx \cong \int_a^b P_1(x)dx \quad \Rightarrow \quad P_1(x) = a_0 + a_1 x$$

- The area under this first order polynomial is an estimate of the integral of *f*(*x*) between the limits of *a* and *b*:

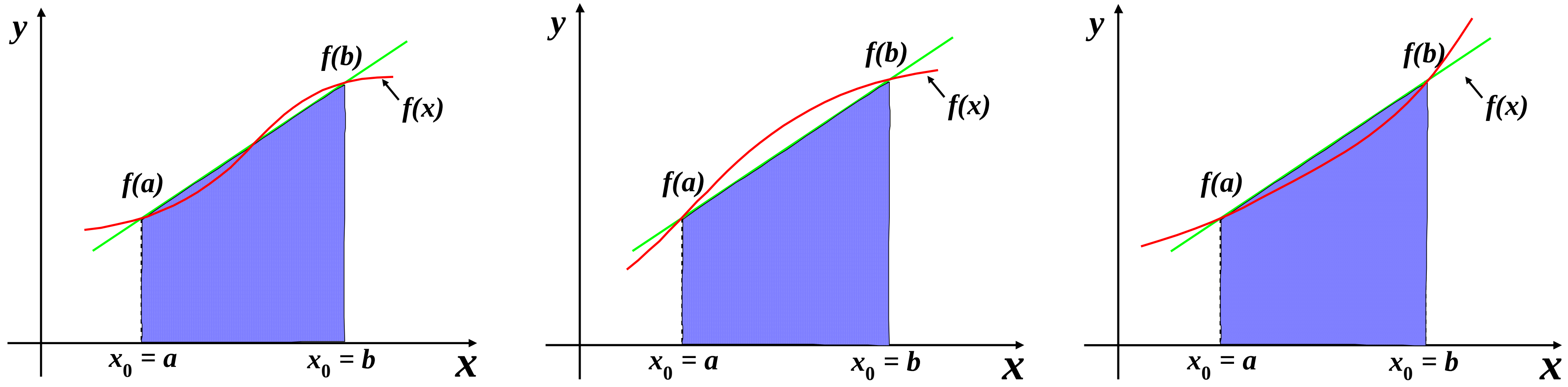
$$I = (b-a)\frac{f(a)+f(b)}{2}$$ Trapezoidal rule

Error of the Trapezoidal Rule

- When we employ the integral under a straight line segment to approximate the integral under a curve, error may be substantial:

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

where $a < \xi < b$ and ξ lies somewhere in the interval from *a* to *b*.



Prove

Using the Lagrange formulation, we find that the interpolating polynomial with error term is

$$f(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} + \frac{(x - x_0)(x - x_1)}{2!} f''(c_x) = P(x) + E(x).$$

Integrating both sides on the interval of interest $[x_0, x_1]$ yields

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} P(x)\,dx + \int_{x_0}^{x_1} E(x)\,dx.$$

Computing the first integral gives

$$\begin{aligned}\int_{x_0}^{x_1} P(x)\,dx &= y_0 \int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1}\,dx + y_1 \int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0}\,dx \\ &= y_0 \frac{h}{2} + y_1 \frac{h}{2} = h \frac{y_0 + y_1}{2},\end{aligned}$$

where we have defined $h = x_1 - x_0$ to be the interval length

For example, substituting $w = -x + x_1$ into the first integral

$$\int_{x_0}^{x_1} \frac{x - x_1}{x_0 - x_1}\,dx = \int_h^0 \frac{-w}{-h}(-dw) = \int_0^h \frac{w}{h}\,dw = \frac{h}{2},$$

and the second integral, after substituting $w = x - x_0$, is

$$\int_{x_0}^{x_1} \frac{x - x_0}{x_1 - x_0}\, dx = \int_0^h \frac{w}{h}\, dw = \frac{h}{2}.$$

The error term is

$$\begin{aligned}
\int_{x_0}^{x_1} E(x)\, dx &= \frac{1}{2!}\int_{x_0}^{x_1} (x - x_0)(x - x_1) f''(c(x))\, dx \\
&= \frac{f''(c)}{2}\int_{x_0}^{x_1} (x - x_0)(x - x_1)\, dx \\
&= \frac{f''(c)}{2}\int_0^h u(u - h)\, du \\
&= -\frac{h^3}{12} f''(c),
\end{aligned}$$

Trapezoid Rule

$$\boxed{\int_{x_0}^{x_1} f(x)\, dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12} f''(c),}$$

where $h = x_1 - x_0$ and c is between x_0 and x_1.

The Multiple Application Trapezoidal Rule

- One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from *a* to *b* into a number of segments and apply the method to each segment.
- The areas of individual segments can then be added to yield the integral for the entire interval.

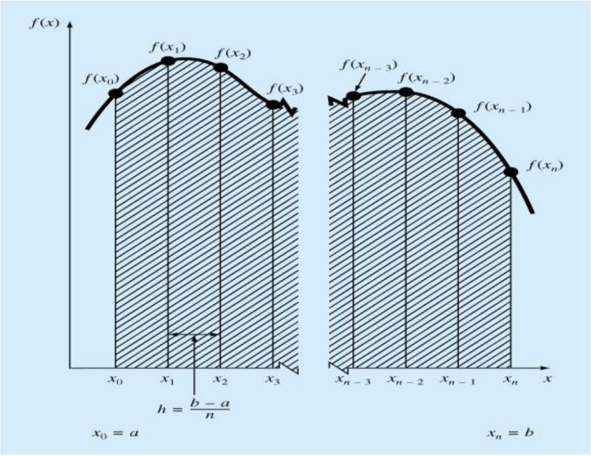
$$h=\frac{b-a}{n} \;\; ; \; a=x_0,\; b=x_n \;\Rightarrow I=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\cdots+\int_{x_{n-1}}^{x_n} f(x)dx$$

Substituting the trapezoidal rule for each integral yields:

$$I=h\frac{f(x_0)+f(x_1)}{2}+h\frac{f(x_1)+f(x_2)}{2}+\cdots+h\frac{f(x_{n-1})+f(x_n)}{2}$$

We can group terms to express a general form

$$I \cong \underbrace{(b-a)}_{\text{width}}\underbrace{\frac{f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)}{2n}}_{\text{average height}}$$



- The average height represents a weighted average of the function values
- The interior points are given twice the weight of the two end points
- An error for multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment:

$$E_t = \sum_{i=1}^{n} E_i = \frac{h^3}{12} \sum_{i=1}^{n} f''(\zeta_i) = \frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The **term** $\dfrac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the

$f''(x),\ a < x < b$ Thus, if the number of segments is doubled, the

truncation error will be quartered.

Hence:

$$E_a = -\frac{(b-a)^3}{12n^2} \bar{f}''$$

Example: Find $\int_0^{\pi} \sin(x)dx$ by dividing the interval into 20 subintervals.

Solution: $n = 20$ and $h = \dfrac{b-a}{n} = \dfrac{\pi}{20}$

$$x_k = a + kh = \frac{k\pi}{20}, \quad k = 0, 1, 2, \ldots\ldots, 20$$

$$\int_0^{\pi} \sin(x)dx \approx \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right]$$

$$= \frac{\pi}{40}\left[\sin(0) + 2\sum_{k=1}^{19}\sin\left(\frac{k\pi}{20}\right) + \sin(\pi)\right]$$

$$= 1.995886$$

Example of Trapezoidal Rule: The vertical distance covered by a rocket from t=8 to t=30 seconds is given by:

$$x = \int_{8}^{30} \left(2000 ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use single segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution: a) $I \approx (b-a)\left[\frac{f(a)+f(b)}{2} \right]$ $\quad a = 8 \;,\; b = 30$

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000 \ln \left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s}$$

$$f(30) = 2000 \ln \left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s}$$

$$I = (30-8)\left[\frac{177.27+901.67}{2}\right] = 11868 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000 ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt = 11061 \text{ m}$$

$$E_t = True\ Value - Approximate\ Value = 11061 - 11868 = -807\ m$$

c) The absolute relative true error $|\in_t|$, would be

$$|\in_t| = \left|\frac{11061-11868}{11061}\right| \times 100 = 7.2959\%$$

a) Use two-segment Trapezoidal rule to find the distance covered.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution: a) The solution using 2-segment Trapezoidal rule is

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$n = 2 \qquad a = 8 \qquad b = 30$$

$$h = \frac{b-a}{n} = \frac{30-8}{2} = 11$$

Then: $$I = \frac{30-8}{2(2)}\left[f(8) + 2\left\{\sum_{i=1}^{2-1} f(a+ih)\right\} + f(30)\right]$$

$$= \frac{22}{4}\left[f(8) + 2f(19) + f(30)\right]$$

$$= \frac{22}{4}\left[177.27 + 2(484.75) + 901.67\right] \quad = 11266 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt \quad = 11061 \text{ m}$$

so the true error is

$$E_t = True\ Value - Approximate\ Value = 11061 - 11266$$

c) The absolute relative true error $\left|\in_t\right|$, would be

$$\left|\in_t\right| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 \quad = \left|\frac{11061-11266}{11061}\right| \times 100 \quad = 1.8534\%$$

The Table gives the values obtained using multiple segment Trapezoidal rule for:

$$x = \int_8^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Table : Multiple Segment Trapezoidal Rule Values

Example: Use Multiple Segment Trapezoidal Rule to find the area under the curve

$$f(x) = \frac{300x}{1+e^x} \quad \text{from } x = 0 \text{ to } x = 10$$

Solution: Using two segments, we get $h = \frac{10-0}{2} = 5$

$$f(0) = \frac{300(0)}{1+e^0} = 0 \qquad f(5) = \frac{300(5)}{1+e^5} = 10.039 \qquad f(10) = \frac{300(10)}{1+e^{10}} = 0.136$$

Then: $$I = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$= \frac{10-0}{2(2)}\left[f(0) + 2\left\{ \sum_{i=1}^{2-1} f(5) \right\} + f(10) \right] \quad = \frac{10}{4}\left[f(0) + 2f(5) + f(10) \right]$$

$$= \frac{10}{4}\left[0 + 2(10.039) + 0.136 \right] \quad = 50.535$$

So what is the true value of this integral? $\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59$

Making the absolute relative true error:

$$\left|\in_t\right| = \left| \frac{246.59 - 50.535}{246.59} \right| \times 100\%$$

$$= 79.506\%$$

n	Approximate Value	E_t	$\left\|\in_t\right\|$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Table : Values obtained using Multiple Segment Trapezoidal Rule

Simpson's Rules

- More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points. The formulas that result from taking the integrals under such polynomials are called *Simpson's rules*.

Simpson's 1/3 Rule

- Results when a second-order interpolating polynomial is used.

$$I = \int_a^b f(x)dx \approx \int_a^b P_2(x)dx \quad \Longrightarrow \quad P_2(x) = a_0 + a_1 x + a_2 x^2$$

$$\int_a^b P_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

Hence $$\int_a^b P_2(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \qquad h = \frac{b-a}{2}$$

Because the above form has 1/3 in its formula, it is called Simpson's 1/3rd Rule.

- **General formula**

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{average height}}$$

Single segment application of Simpson's 1/3 rule has a truncation error of:

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \qquad a < \xi < b$$

Simpson's 1/3 rule is more accurate than trapezoidal rule, which is

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 \qquad a < \xi < b$$

Prove

the sum of the interpolating parabola and the interpolation error:

$$f(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{(x - x_0)(x - x_1)(x - x_2)}{3!} f'''(c_x)$$

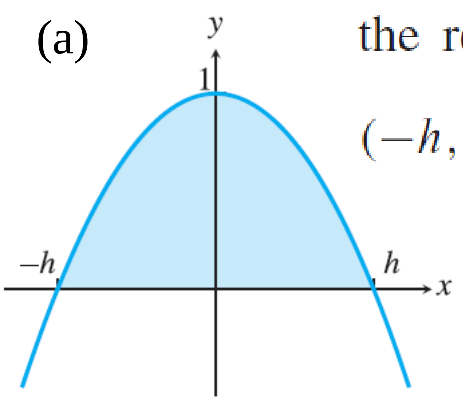
$$= P(x) + E(x).$$

Integrating gives

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} P(x)\,dx + \int_{x_0}^{x_2} E(x)\,dx, \qquad (1)$$

To develop the Newton–Cotes formulas,

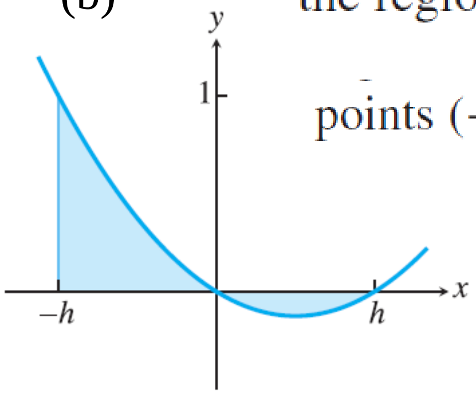
(a)



the region under the parabola $P(x)$ interpolating the data points $(-h, 0)$, $(0, 1)$, and $(h, 0)$, which has area

$$\int_{-h}^{h} P(x)\,dx = x - \frac{x^3}{3h^2} = \frac{4}{3}h. \qquad ; \quad P(x) = 1 - (\frac{x}{h})^2$$

(b) the region between the x-axis and the parabola interpolating the data points $(-h, 1)$, $(0, 0)$, and $(h, 0)$, with net positive area



$$\int_{-h}^{h} P(x)\,dx = \frac{1}{3}h. \qquad ; \; P(x) = \frac{1}{2}[(\frac{x}{h})^2 - \frac{x}{h}]$$

From (1) We have set $h = x_2 - x_1 = x_1 - x_0$

$$\int_{x_0}^{x_2} P(x)\,dx = y_0 \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)\,dx}{(x_0-x_1)(x_0-x_2)} + y_1 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)\,dx}{(x_1-x_0)(x_1-x_2)}$$

$$+ y_2 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)\,dx}{(x_2-x_0)(x_2-x_1)}$$

used (a) for the middle integral and (b) for the first and third. So we have

$$\int_{x_0}^{x_2} P(x)\,dx = y_0\frac{h}{3} + y_1\frac{4h}{3} + y_2\frac{h}{3}.$$

The error term can be computed

$$\int_{x_0}^{x_2} E(x)\,dx = -\frac{h^5}{90} f^{(iv)}(c)$$

for some c in the interval $[x_0, x_2]$, provided that $f^{(iv)}$ exists and is continuous.

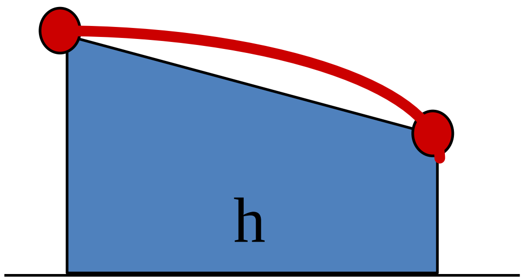
Simpson's Rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} f^{(iv)}(c),$$

where $h = x_2 - x_1 = x_1 - x_0$ and c is between x_0 and x_2.

Numerical Integration

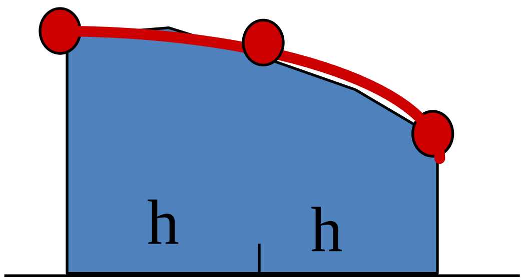
Trapezoidal Rule



$$I = (b-a)\frac{f(a)+f(b)}{2}$$

$$I = h\frac{f(a)+f(b)}{2}$$

Simpson's Rule (simple)



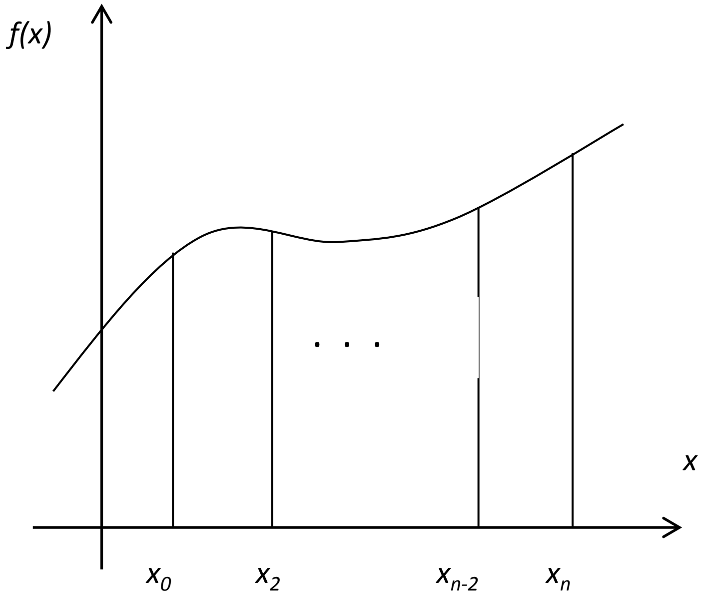
$$I = \frac{(b-a)}{2}\,\frac{f(x_0)+4f(x_1)+f(x_2)}{3}$$

$$I = h\frac{f(x_0)+4f(x_1)+f(x_2)}{3}$$

The Multiple-Application Simpson's 1/3 Rule

- Just as the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width.
- Yields accurate results and considered superior to trapezoidal rule for most applications.
- However, it is limited to cases where values are equi-spaced.
- Further, it is limited to situations where there are an even number of segments and odd number of points.
- Just like in multiple segment Trapezoidal Rule, one can subdivide the interval [a, b] into n segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval [*a*, *b*] into equal segments, hence the segment width

$$h = \frac{b-a}{n} \qquad x_0 = a \qquad x_n = b$$



$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx = (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + \ldots + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since $x_i - x_{i-2} = 2h \qquad i = 2, 4, \ldots, n$

Then $\int_a^b f(x)dx = 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + \ldots + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + \ldots + f(x_{n-1})\} + \ldots\right.$$

$$\left.\ldots + 2\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\} + f(x_n)\}\right]$$

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\int_a^b f(x)dx = \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$I \cong \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{f(x_0) + 4\sum\limits_{i=1,3,5..}^{n-1} f(x_i) + 2\sum\limits_{j=2,4,6..}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{average height}}$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

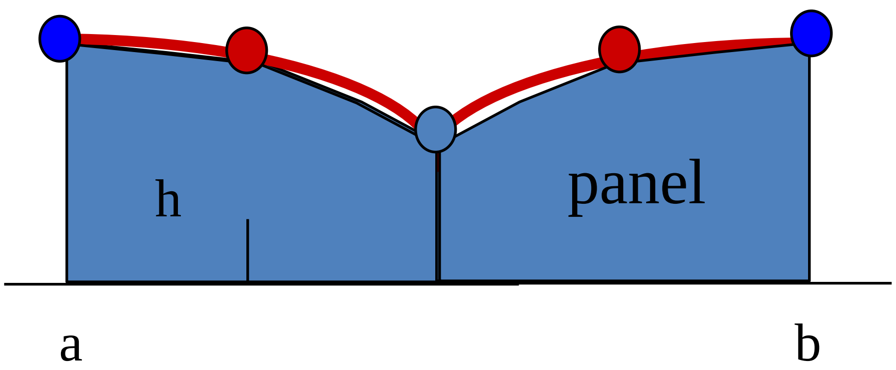
$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^4}\frac{\sum\limits_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\dfrac{\sum\limits_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x),\ a < x < b$

Hence

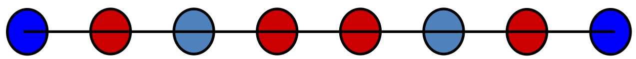
$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

Repeated Simpson's 1/3 rule



weights:

1 4 2 4 … 4 2 4 1



h

k panels

2k intervals

h ?

(k=2) $$I_h = h\frac{f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)}{3}$$

(general) $$I_h = \frac{h}{3}\left[f(x_1)+4\sum_{i=2,4,\dots}^{2k} f(x_i)+2\sum_{i=3,5,\dots}^{2k-1} f(x_i)+f(x_{2k+1})\right]$$

EXAMPLE The table below gives the velocity v of a moving particle at time t seconds.

t	0	2	4	6	8	10	12
v	4	6	16	34	60	94	136

Find the distance covered by the particle in 12 seconds and also the acceleration at $t = 2$ seconds.

To get S, we have to integrate v. $S = \int_0^{12} v\,dt$ (using simpson's one-third rule)

$$= \frac{2}{3}[(4+136)+2(16+60)+4(6+34+94)]$$

$$= \frac{2}{3}(140+152+536) = 552 \text{ m}$$

Acceleration $a = \left(\frac{dv}{dt}\right)_{t=2}$

$\therefore$ Hence, we require differentiation.

Now, we form difference table.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$
0	4			
		2		
2	6		8	
		10		0
4	16		8	
		18		0
6	34		8	
		26		0
8	60		8	
		34		0
10	94		8	
		42		
12	136			

$$\left(\frac{dv}{dt}\right)_{t=2} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right] \qquad (\text{taking } v_0 = 6)$$

$$= \frac{1}{2}\left(10 - \frac{1}{2}8\right) = 3 \text{ m/s}^2$$

Example: Find $\int_0^{\pi} \sin(x)dx$ by dividing the interval into 20 subintervals.

Solution: $n = 20$ and $h = \frac{b-a}{n} = \frac{\pi}{20}$

$$x_k = a + kh = \frac{k\pi}{20}, \quad k = 0, 1, 2, \ldots\ldots, 20$$

$$\int_0^{\pi} \sin(x)dx \approx \frac{\pi}{60}\left[\sin(0) + 2\sum_{k=1}^{9} \sin\left(\frac{2k\pi}{20}\right) + 4\sum_{k=1}^{10} \sin\left(\frac{(2k-1)\pi}{20}\right) + \sin(\pi)\right]$$

$$= 2.000006$$

Example: The distance covered by a rocket from t=8 to t=30 is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x
b) Find the true error E_t
c) Find the absolute relative true error $|\epsilon_t|$

Solution: a) $x = \int_{8}^{30} f(t)dt = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$

$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30) \right]$$

$$= \left(\frac{22}{6}\right)\left[177.2667 + 4(484.7455) + 901.6740 \right]$$

$$= 11065.72 \text{ m}$$

b) The exact value of the above integral is $x = 11061.34 \text{ m}$

True Error $E_t = 11061.34 - 11065.72 = -4.38 \text{ m}$

c) Absolute relative true error

$$\left|\in_t\right| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\% = 0.0396\%$$

a) Use 4 segment Simpson's 1/3rd Rule to find the approximate value of *x*.
b) Find the true error E_t for part (a).
c) Find the absolute relative true error $\left|\in_a\right|$ for part (a).

Solution: a) Using n segment Simpson's 1/3rd Rule $h = \frac{30-8}{4} = 5.5$

So $f(t_0) = f(8)$ $\quad f(t_1) = f(8+5.5) = f(13.5)$

$f(t_2) = f(13.5+5.5) = f(19)$ $\quad f(t_3) = f(19+5.5) = f(24.5)$

$f(t_4) = f(30)$

$$I = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n)\right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(t_i) + f(30)\right]$$

$$I = \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30)\right]$$

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30)\right]$$

$$= \frac{11}{6}\left[177.2667 + 4(320.2469) + 4(676.0501) + 2(484.7455) + 901.6740\right]$$

$$I = 11061.64 \text{ m}$$

b) In this case, the true error is

$$E_t = 11061.34 - 11061.64 = -0.30 \text{ m}$$

c) The absolute relative true error

$$\left|\in_t\right| = \left|\frac{11061.34 - 11061.64}{11061.34}\right| \times 100\%$$

$$= 0.0027\%$$

n	Approximate Value	E_t	$\lvert\epsilon_t\rvert$
2	11065.72	4.38	0.0396%
4	11061.64	0.30	0.0027%
6	11061.40	0.06	0.0005%
8	11061.35	0.01	0.0001%
10	11061.34	0.00	0.0000%

Table : Values of Simpson's 1/3rd Rule with multiple segments

Simpson’s 3/8 Rule

- An odd-segment-even-point formula is used where the function is a third order polynomial.

$$I = \int_a^b f(x)dx \cong \int_a^b P_3(x)dx$$

where $\quad P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

- The formula is

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

where $\quad h = \dfrac{(b-a)}{3}$

- The true errors (E_{ts} for single) and (E_{tc} for composite) are

$$E_{ts} = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

and

$$E_{tc} = -\frac{3(b-a)^5}{80} \bar{f}^{(4)}$$

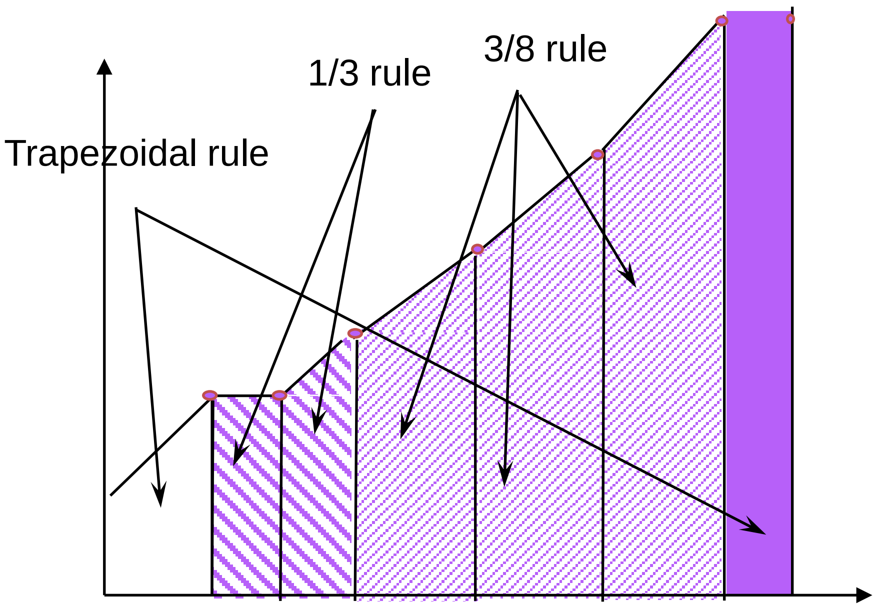
More accurate

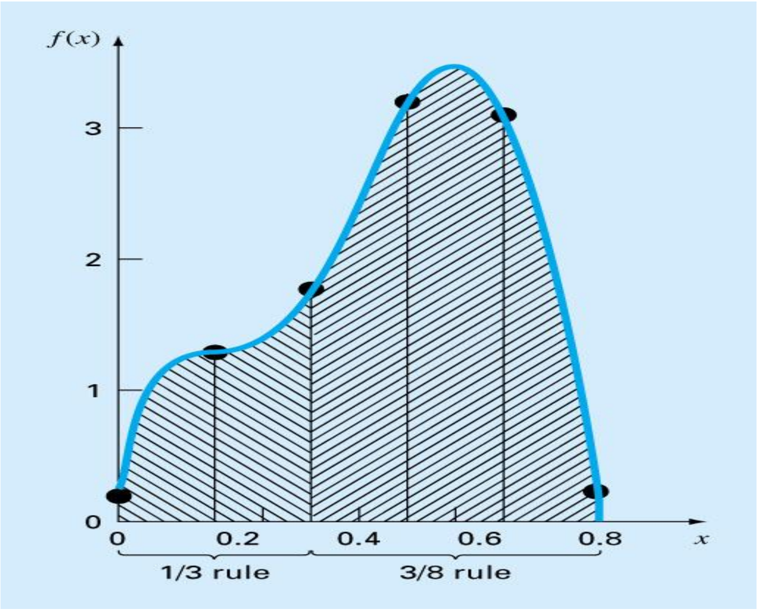
Integration of Unequal Segments

- Experimental and field study data is often unevenly spaced
- In previous equations we grouped the term (i.e. h_i) which represented segment width.

$$I \cong (b-a)\frac{f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)}{2n} \cong h\frac{f(x_0)+f(x_1)}{2}+h\frac{f(x_1)+f(x_2)}{2}+\cdots h\frac{f(x_{n-1})+f(x_n)}{2}$$

- We should also consider alternately using higher order equations if we can find data in consecutively even segments

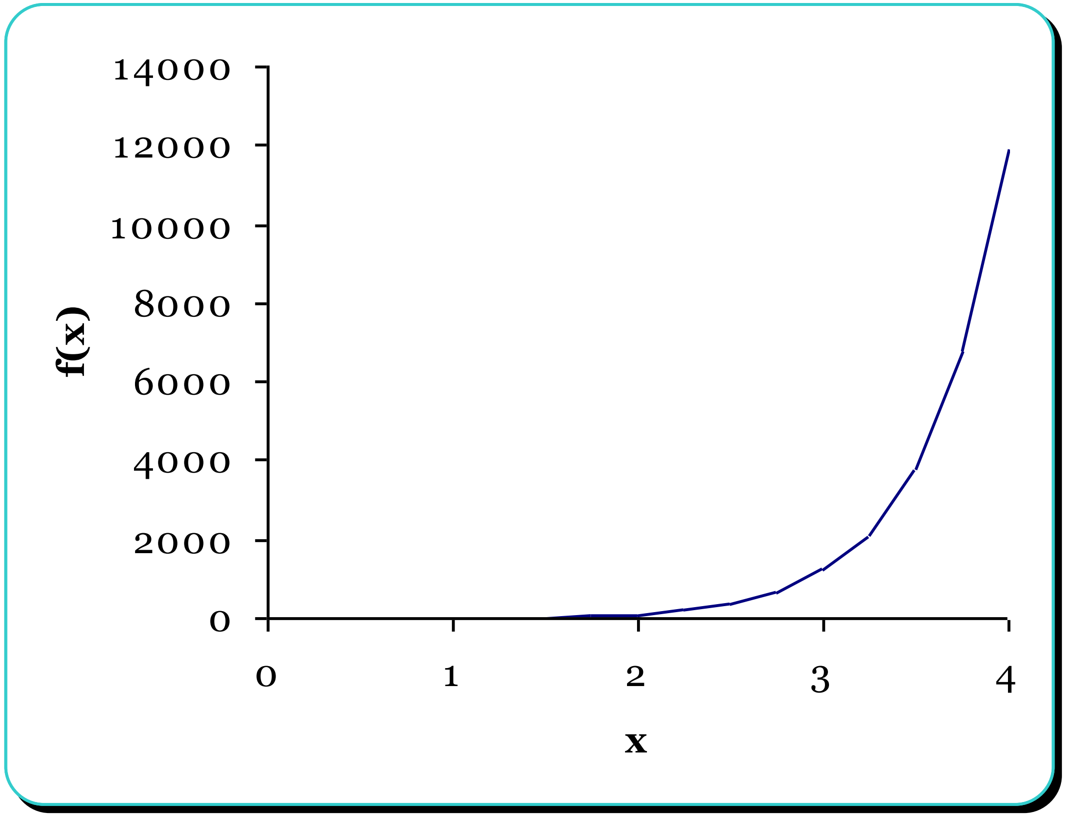




Exercise:

Integrate the following using the trapezoidal rule, Simpson's 1/3 Rule, a multiple application of the trapezoidal rule with n=2 and Simpson's 3/8 Rule. Compare results with the analytical solution.

$$\int_0^4 xe^{2x}\,dx$$



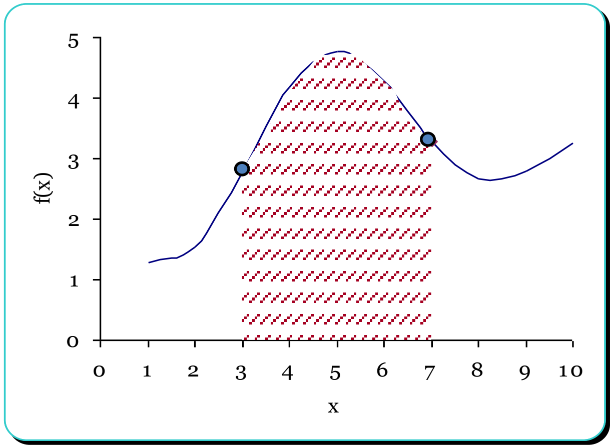
Newton-Cotes Integration Formulas

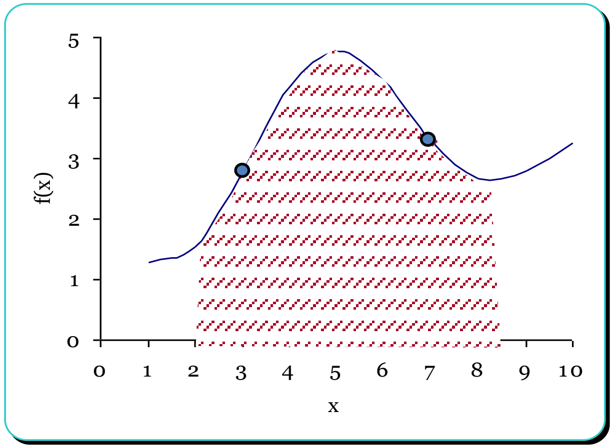
- The *Newton-Cotes formulas* are the most common numerical integration schemes.
- They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

$$I = \int_a^b f(x)dx \cong \int_a^b P_n(x)dx$$

$$P_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

  - **Closed form** - data is at the beginning and end of the limits of integration.
  - **Open form** - integration limits extend beyond the range of data.





Newton-Cotes formulas

Table Newton-Cotes Formulas

n	Formula
0	Rectangle rule
1	Trapezoid rule
2	Simpson's 1/3 rule
3	Simpson's 3/8 rule

Newton-Cotes formulas can be expressed in the general form:

$$\boxed{I = \int_a^b f(x)\,dx = n\beta h(\alpha_0 f_0 + \alpha_1 f_1 + \cdots) + \text{Error}}$$

Table Newton-Cotes Formulas

n	β	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	Local Error
1	1/2	1	1							$-1/12 f^{(2)} h^3$
2	1/6	1	4	1						$-1/90 f^{(4)} h^5$
3	1/8	1	3	3	1					$-3/80 f^{(4)} h^5$
4	1/90	7	32	12	32	7				$-8/945 f^{(6)} h^7$
5	1/288	19	75	50	50	75	19			$-275/12096 f^{(6)} h^7$
6	1/840	41	216	27	272	27	216	41		$-9/1400 f^{(8)} h^9$
7	1/17280	751	3577	1323	2989	2989	1323	3577	751	$-8183/518400 f^{(8)} h^9$

Richardson Extrapolation

- Two ways to improve derivative estimates
  - decrease step size
  - use a higher order formula that employs more points
- Third approach, based on Richardson extrapolation, uses two derivatives estimates to compute a third, more accurate approximation

Richardson's method

$$F_1[h] = \frac{r^n F[h] - F[rh]}{r^n - 1}, \qquad \text{where } rh = h_{\text{larger}}$$

- Make two separate estimates using step sizes of h_1 and h_2 .

$$I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

- Recall the error of the multiple-application of the trapezoidal rule

$$E = -\frac{b-a}{12} h^2 \bar{f}''$$

- Assume that $\bar{f}''$ is constant regardless of the step size

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2}$$

Substitute $E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2$ into $I(h_1) + E(h_1) = I(h_2) + E(h_2)$

Thus we have developed an estimate of the truncation error in terms of the integral estimates and their step sizes.

$$E(h_2) = \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^2}$$

This estimate can then be substituted into $I = I(h_2) + E(h_2)$ to yield an improved estimate of the integral:

Richardson General formula

$$\boxed{I \cong I(h_2) + \frac{I(h_2) - I(h_1)}{\left(\frac{h_1}{h_2}\right)^2 - 1}}$$

Richardson for trapezoidal

What is the equation for the special case where the interval is halved?

$$\text{for} \quad \left(\frac{h_1}{h_2}\right) = 2 \quad \Rightarrow \quad \boxed{I \cong I(h_2) + \left[\frac{1}{2^2 - 1}\right]\left[I(h_2) - I(h_1)\right] = \frac{4I_2 - I_1}{3}}$$

Richardson for Simpson

- Do calculation with **h** and with **h/2**
- Express remainder term as a power of h
- Try to extrapolate to infinitely small h

In case of repeated Simpson's rule

$$I_1 \cong I + Ch^4$$

$$I_2 \cong I + C\frac{h^4}{2^4}$$

$$\Rightarrow \quad 2^4 I_2 \cong 2^4 I + I_1 - I$$

$$2^4 I_2 - I_1 \cong (2^4 - 1)I$$

$$\Rightarrow \quad \boxed{I \cong \frac{16 I_2 - I_1}{15}}$$

Richardson calculated in ordering number

$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$
1: $N_1(h) \equiv N(h)$			
2: $N_1(\frac{h}{2}) \equiv N(\frac{h}{2})$	**3:** $N_2(h)$		
4: $N_1(\frac{h}{4}) \equiv N(\frac{h}{4})$	**5:** $N_2(\frac{h}{2})$	**6:** $N_3(h)$	
7: $N_1(\frac{h}{8}) \equiv N(\frac{h}{8})$	**8:** $N_2(\frac{h}{4})$	**9:** $N_3(\frac{h}{2})$	**10:** $N_4(h)$

Example of Trapeziodal improved by Richardson Extrapolation

$N_1(0.2) = 22.414160$

$N_1(0.1) = 22.228786 \qquad N_2(0.2) = N_1(0.1) + \dfrac{N_1(0.1) - N_1(0.2)}{3} = 22.166995$

$$F_1[h] = \frac{r^n F[h] - F[rh]}{r^n - 1}, \qquad \text{where } rh = h_{\text{larger}}$$

$N_1(0.05) = 22.182564 \qquad N_2(0.1) = N_1(0.05) + \dfrac{N_1(0.05) - N_1(0.1)}{3} = 22.167157 \qquad N_3(0.2) = N_2(0.1) + \dfrac{N_2(0.1) - N_2(0.2)}{15} = 22.167168$

Example: The vertical distance covered by a rocket from 8 to 30 seconds is given by

$$x = \int_8^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t \right) dt$$

a) Use the 2-segment and 4-segment Trapezoidal rule results given in the Table to find the distance by Richardson's rule.
b) Find the true error E_t for part (a).
c) Find the absolute relative true error $|\in_a|$ for part (a).

Solution: a) $I_2 = 11266m \quad I_4 = 11113m$

Using Richardson's extrapolation formula for Trapezoidal rule

$$I \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$ and choosing *n=2*

$$I \approx I_4 + \frac{I_4 - I_2}{3} = 11113 + \frac{11113 - 11266}{3} = 11062 \text{ m}$$

b) The exact value of the above integral is $x = 11061 \text{ m}$

Hence $E_t = True\ Value - Approximate\ Value$

$$= 11061 - 11062 = -1 \text{ m}$$

c) The absolute relative true error $|\in_t|$ would then be

$$|\in_t| = \left| \frac{11061 - 11062}{11061} \right| \times 100 = 0.00904\%$$

The table shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

The values obtained using Richardson's extrapolation formula for Trapezoidal rule

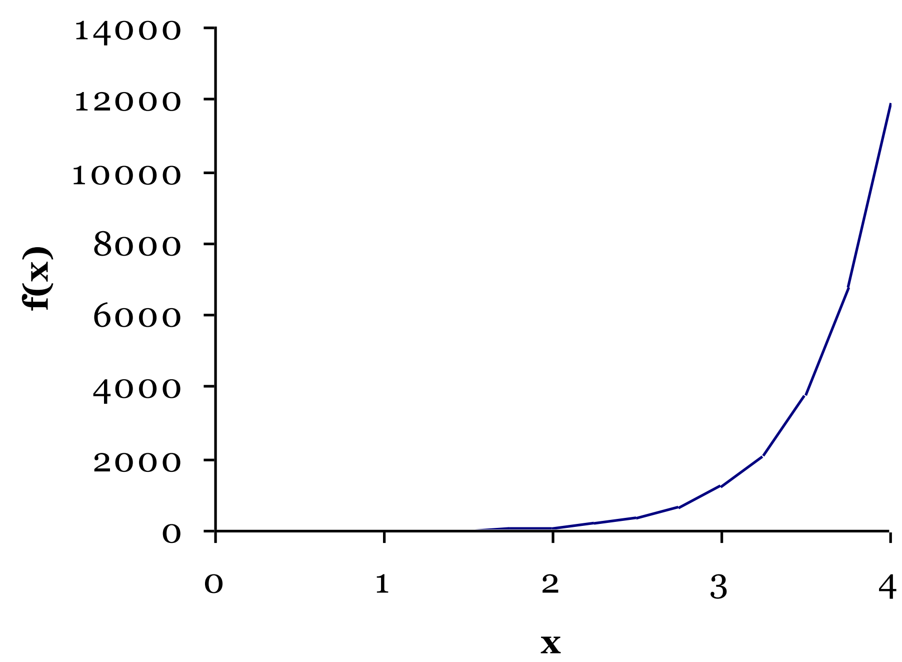
n	Trapezoidal Rule	$\left\|\in_t\right\|$ for Trapezoidal Rule	Richardson's Extrapolation	$\left\|\in_t\right\|$ for Richardson's Extrapolation
1	11868	7.296	--	--
2	11266	1.854	11065	0.03616
4	11113	0.4655	11062	0.009041
8	11074	0.1165	11061	0.0000

Table : Richardson's Extrapolation Values

Exercise

Use Richardson's extrapolation to evaluate:

$$\int_0^4 xe^{2x}dx$$



Integration of Equations

- Functions to be integrated numerically are in two forms:
  - *A table of values*. We are limited by the number of points that are given.
  - *A function*. We can generate as many values of $f(x)$ as needed to attain acceptable accuracy.
- Two techniques that are designed to analyze functions:
  - *Romberg integration*
  - *Gauss quadrature*

What is the Romberg Rule?

- Romberg Integration is an extrapolation formula of the Trapezoidal Rule for integration. It provides a better approximation of the integral by reducing the True Error.

Romberg integration is the same as Richardson's extrapolation formula as given previously. However, Romberg used a recursive algorithm for the extrapolation. Recall

$$I \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

This can alternately be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3}$$

Determine another integral value with further halving the step size (doubling the number of segments)

$$(I_{4n})_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3}$$

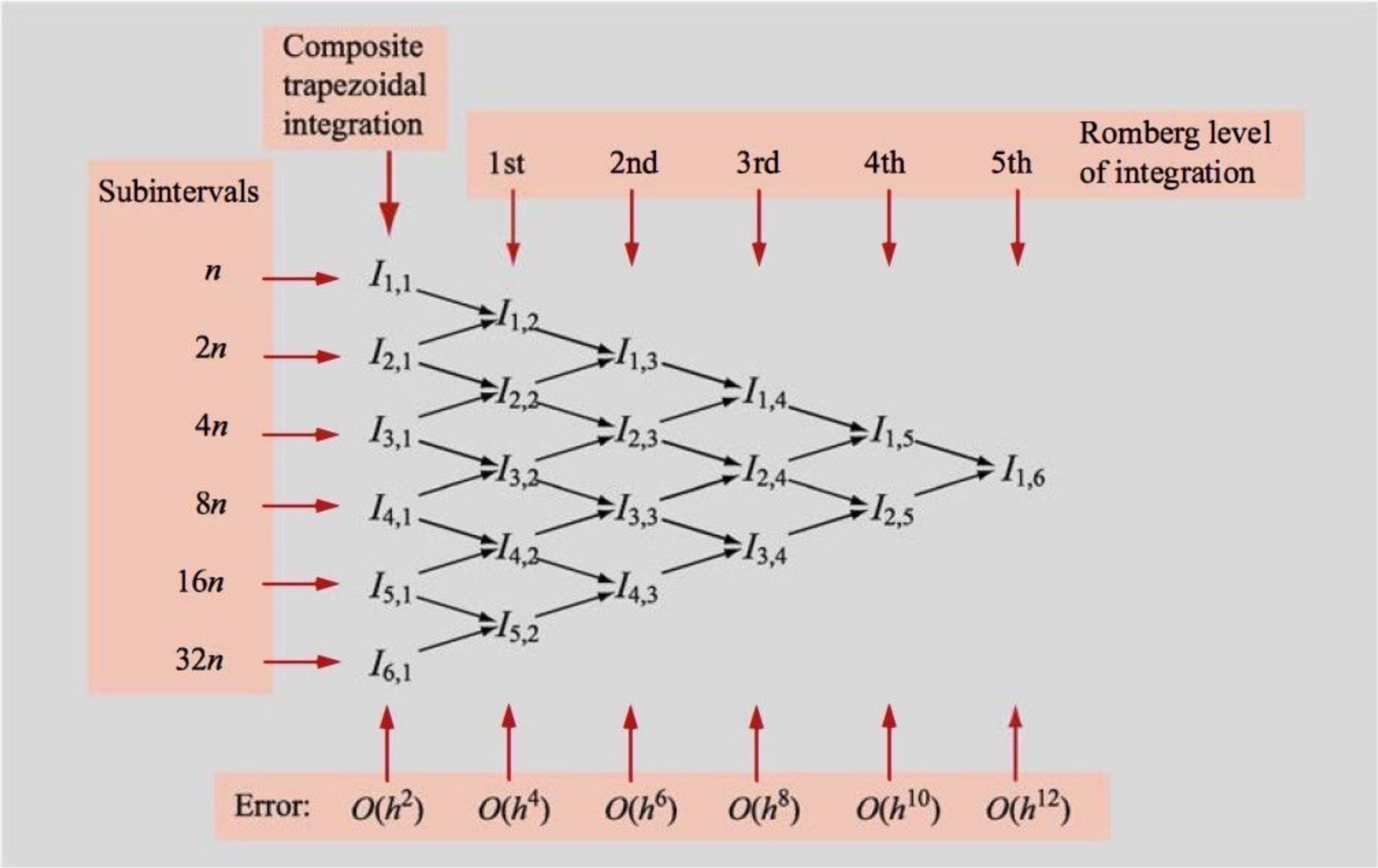
It follows from the two previous expressions that the true value can be written as

$$I \approx (I_{4n})_R + \frac{(I_{4n})_R - (I_{2n})_R}{4^{3-1} - 1} = I_{4n} + \frac{(I_{4n})_R - (I_{2n})_R}{15}$$

A general expression for Romberg integration can be written as

$$\boxed{I_{k,j} = I_{k,j-1} + \frac{I_{k,j-1} - I_{k-1,j-1}}{4^{j-1} - 1} \quad ; \quad k, j \geq 2}$$

Romberg integration method



The index j represents the order of extrapolation. $j = 1$ represents the values obtained from the Trapezoidal rule, $j = 2$ represents values obtaining $O(h^2)$. The index k is the improved step size of the integral.

General Richardson in the table

$$R(n,1) = \frac{1}{2} R(n-1,1) + h \sum_{i=0}^{n-1} f(a + (2i+1)h)$$

$R_{1,1}$					
$R_{2,1}$	$R_{2,2}$				
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

Example: Approximating $\int_0^{\pi} \sin x \, dx$. by the Romberg table

0					
1.57079633	2.09439511				
1.89611890	2.00455976	1.99857073			
1.97423160	2.00026917	1.99998313	2.00000555		
1.99357034	2.00001659	1.99999975	2.00000001	1.99999999	
1.99839336	2.00000103	2.00000000	2.00000000	2.00000000	2.00000000

Example: Approximating $\int_0^1 x^2 dx$ by the Romberg table

$$h = 1,\ \ R(1,1) = \frac{b-a}{2}\left[f(a) + f(b)\right] = \frac{1}{2}[0+1] = 0.5$$

0.5	
3/8	1/3

$$h = \frac{1}{2},\ R(2,1) = \frac{1}{2}R(1,1) + h(f(a+h)) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

$$R(n,m) = \frac{1}{4^{m-1}-1}\left[4^{m-1} \times R(n,m-1) - R(n-1,m-1)\right] \quad for \quad n \geq 2, \quad m \geq 2$$

$$R(2,2) = \frac{1}{4^1-1}\left[4 \times R(2,1) - R(1,1)\right] = \frac{1}{3}\left[4 \times \frac{3}{8} - \frac{1}{2}\right] = \frac{1}{3}$$

$$h = \frac{1}{4},\ R(3,1) = \frac{1}{2}R(2,1) + h(f(a+h) + f(a+3h)) = \frac{1}{2}\left(\frac{3}{8}\right) + \frac{1}{4}\left(\frac{1}{16} + \frac{9}{16}\right) = \frac{11}{32}$$

$$R(n,m) = \frac{1}{4^{m-1}-1}\left[4^{m-1} \times R(n,m-1) - R(n-1,m-1)\right]$$

$$R(3,2) = \frac{1}{3}\left[4 \times R(3,1) - R(2,1)\right] = \frac{1}{3}\left[4 \times \frac{11}{32} - \frac{3}{8}\right] = \frac{1}{3}$$

$$R(3,3) = \frac{1}{4^2-1}\left[4^2 \times R(3,2) - R(2,2)\right] = \frac{1}{15}\left[\frac{16}{3} - \frac{1}{3}\right] = \frac{1}{3}$$

0.5		
3/8	1/3	
11/32	1/3	1/3

STOP if $\left|R(n,n) - R(n,n-1)\right| \leq \varepsilon$ or After a given number of steps

Example: The vertical distance covered by a rocket from 8 to 30 seconds is given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Romberg's rule to find from the Table of the 1, 2, 4, and 8-segment Trapezoidal rule. From the Table, the values from original Trapezoidal rule are

$$I_{1,1} = 11868 \qquad I_{2,1} = 11266 \qquad I_{3,1} = 11113 \qquad I_{4,1} = 11074$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values

$$I_{2,2} = I_{2,1} + \frac{I_{2,1} - I_{1,1}}{3} = 11266 + \frac{11266 - 11868}{3} = 11065$$

$$I_{3,2} = I_{3,1} + \frac{I_{3,1} - I_{2,1}}{3} = 11113 + \frac{11113 - 11266}{3} = 11062$$

$$I_{4,2} = I_{4,1} + \frac{I_{4,1} - I_{3,1}}{3} = 11074 + \frac{11074 - 11113}{3} = 11061$$

For the second order extrapolation values,

$$I_{3,3} = I_{3,2} + \frac{I_{3,2} - I_{2,2}}{15} = 11062 + \frac{11062 - 11065}{15} = 11062$$

$$I_{4,3} = I_{4,2} + \frac{I_{4,2} - I_{3,2}}{15} = 11061 + \frac{11061 - 11062}{15} = 11061$$

For the third order extrapolation values

$$I_{4,4} = I_{4,3} + \frac{I_{4,3} - I_{3,3}}{63} = 11061 + \frac{11061 - 11062}{63} = 11061 \text{ m}$$

		First Order	*Second Order*	*Third Order*
1-segment	11868			
		11065		
2-segment	1126		11062	
		11062		11061
4-segment	11113		11061	
		11061		
8-segment	11074			

Table : Improved estimates of the integral value using Romberg Integration

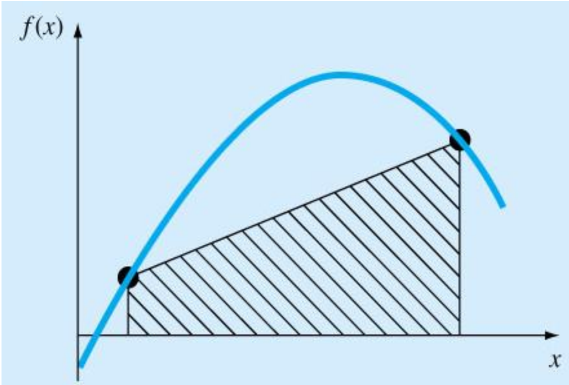
Gauss Quadrature

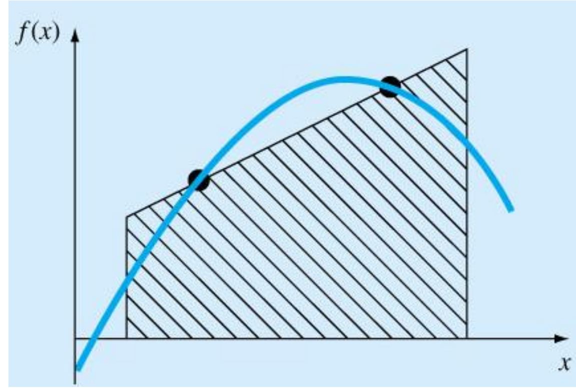
- **Gauss quadrature** implements a strategy of positioning any two points on a curve to define a straight line that would balance the positive and negative errors. Hence the area evaluated under this straight line provides an improved estimate of the integral.
- Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

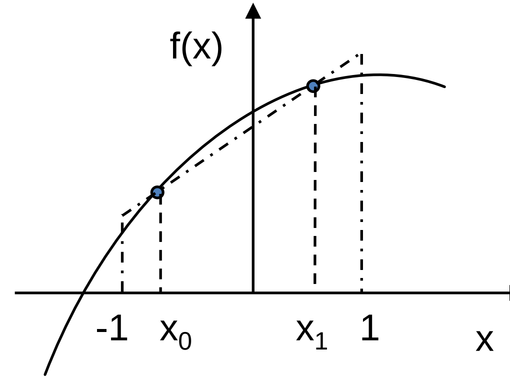
$$I = \int_a^b f(x)dx = (b-a)\frac{f(a)+f(b)}{2} \approx c_1 f(a) + c_2 f(b)$$

- The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as *a* and *b* but as unknowns x_1 and x_2.
- The two cases that should be evaluated exactly by the trapezoidal rule, *y* is constant and a straight line. Thus, the following equalities should hold.

From $$I \cong c_0 f(a) + c_1 f(b)$$







- Previously, we assumed that the equation fit the integrals of a constant and linear function. Extend the reasoning by assuming that it also fits the integral of a parabolic and a cubic function.

Considering two points between a =-1 and b = 1

For y=1 $\Longrightarrow$ $$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} 1dx = 2$$

For y=x $\Longrightarrow$ $$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x dx = 0$$

For y=x^2 $\Longrightarrow$ $$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^2 dx = 2/3$$

For y=x^3 $\Longrightarrow$ $$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^3 dx = 0$$

- We now have four equations and four unknowns c_0, c_1, x_0 and x_1 What equations are you solving?

Solve these equations simultaneously

$$c_0 \cdot 1 + c_1 \cdot 1 = 2$$

$$c_0 x_0 + c_1 x_1 = 0$$

$$c_0 x_0^2 + c_1 x_1^2 = 2/3$$

$$c_0 x_0^3 + c_1 x_1^3 = 0$$

This results in the following $c_0 = c_1 = 1, \quad x_0 = \frac{-1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}$

$$I \cong f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

- The interesting result is that the integral can be estimated by the simple addition of the function values at $\frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$

What if we aren't integrating from –1 to 1?

- A simple change in variables can be use to translate other limits.
- Assume that the new variable x_d is related to the original variable x in a linear fashion. $x = a_0 + a_1 x_d$
- Let the lower limit $x = a$ correspond to $x_d = -1$ and the upper limit $x = b$ correspond to $x_d = 1$

$a = a_0 + a_1(-1)$, $b = a_0 + a_1(1)$

Solve these equations simultaneously $a_0 = \frac{b+a}{2}$, $a_1 = \frac{b-a}{2}$

$$x = a_0 + a_1 x_d = \frac{(b+a)+(b-a)x_d}{2} \Longrightarrow dx = \frac{b-a}{2} dx_d$$

These equations are substituted for *x* and *dx* respectively. Let's do an example to appreciate the theory behind this numerical method.

Argumented Gauss Quadrature Formulas

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$? The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$. Substituting the values of *x* and *dx* into the integral gives us:

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\frac{b-a}{2}dt$$

Weighing Factors for *n-point* Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for *n-point* Gauss Quadrature Rule are given for integrals shown in the Table.

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

Points	Weighting Factors	Function Arguments
2	c_1 = 1.000000000 c_2 = 1.000000000	x_1 = -0.577350269 x_2 = 0.577350269
3	c_1 = 0.555555556 c_2 = 0.888888889 c_3 = 0.555555556	x_1 = -0.774596669 x_2 = 0.000000000 x_3 = 0.774596669
4	c_1 = 0.347854845 c_2 = 0.652145155 c_3 = 0.652145155 c_4 = 0.347854845	x_1 = -0.861136312 x_2 = -0.339981044 x_3 = 0.339981044 x_4 = 0.861136312

Points	Weighting Factors	Function Arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.238619186 x_4 = 0.238619186 x_5 = 0.661209386 x_6 = 0.932469514

Table : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments	Truncation Error
1	$c_0 = 2$	$x_0 = 0.0$	$\cong f^{(2)}(\xi)$
2	$c_0 = 1$ $c_1 = 1$	$x_0 = -1/\sqrt{3}$ $x_1 = 1/\sqrt{3}$	$\cong f^{(4)}(\xi)$
3	$c_0 = 5/9$ $c_1 = 8/9$ $c_2 = 5/9$	$x_0 = -\sqrt{3/5}$ $x_1 = 0.0$ $x_2 = \sqrt{3/5}$	$\cong f^{(6)}(\xi)$
4	$c_0 = (18 - \sqrt{30})/36$ $c_1 = (18 + \sqrt{30})/36$ $c_2 = (18 + \sqrt{30})/36$ $c_3 = (18 - \sqrt{30})/36$	$x_0 = -\sqrt{525 + 70\sqrt{30}}/35$ $x_1 = -\sqrt{525 - 70\sqrt{30}}/35$ $x_2 = \sqrt{525 - 70\sqrt{30}}/35$ $x_3 = \sqrt{525 + 70\sqrt{30}}/35$	$\cong f^{(8)}(\xi)$
5	$c_0 = (322 - 13\sqrt{70})/900$ $c_1 = (322 + 13\sqrt{70})/900$ $c_2 = 128/225$ $c_3 = (322 + 13\sqrt{70})/900$ $c_4 = (322 - 13\sqrt{70})/900$	$x_0 = -\sqrt{245 + 14\sqrt{70}}/21$ $x_1 = -\sqrt{245 - 14\sqrt{70}}/21$ $x_2 = 0.0$ $x_3 = \sqrt{245 - 14\sqrt{70}}/21$ $x_4 = \sqrt{245 + 14\sqrt{70}}/21$	$\cong f^{(10)}(\xi)$
6	$c_0 = 0.171324492379170$ $c_1 = 0.360761573048139$ $c_2 = 0.467913934572691$ $c_3 = 0.467913934572691$ $c_4 = 0.360761573048131$ $c_5 = 0.171324492379170$	$x_0 = -0.932469514203152$ $x_1 = -0.661209386466265$ $x_2 = -0.238619186083197$ $x_3 = 0.238619186083197$ $x_4 = 0.661209386466265$ $x_5 = 0.932469514203152$	$\cong f^{(12)}(\xi)$

Example: For an integral $\int_a^b f(x)dx$ derive the one-point Gaussian Quadrature Rule.

Solution: The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f\left(x_1\right) \quad ; \quad f(x) = a_0 + a_1 x.$$

The two unknowns x_1, and c_1 are found by assuming that the formula gives exact results for integrating a general first order polynomial,

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x\right)dx = \left[a_0 x + a_1 \frac{x^2}{2}\right]_a^b = a_0\left(b-a\right) + a_1\left(\frac{b^2 - a^2}{2}\right)$$

It follows that

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1 x_1\right)$$

Equating Equations, the two previous two expressions yield

$$a_0\left(b-a\right) + a_1\left(\frac{b^2 - a^2}{2}\right) = c_1\left(a_0 + a_1 x_1\right) = a_0\left(c_1\right) + a_1\left(c_1 x_1\right)$$

Since the constants a_0, and a_1 are arbitrary $c_1 = b - a$ and $x_1 = \frac{b+a}{2}$

Hence **One-Point Gaussian Quadrature Rule**

$$\int_a^b f(x)dx \approx c_1 f(x_1) = (b-a)\ f\left(\frac{b+a}{2}\right)$$

Example: Form the integral, derive the one-point Gaussian Quadrature Rule. Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from t=8 to t=30 as given by

$$x = \int_8^{30}\left(2000\,ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

a) Find the true error E_t for part (a).
b) Find the absolute relative true error $\left|\in_a\right|$ for part (a).

Solution:

First, change the limits of integration from [8,30] to [-1,1] by previous relations as follows.

$$\int_8^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x+\frac{30+8}{2}\right)dx = 11\int_{-1}^{1} f(11x+19)dx$$

Next, get weighting factors and function argument values from the Table for the two point rule

$$c_1 = 1.000000000 \qquad c_2 = 1.000000000$$

$$x_1 = -0.577350269 \qquad x_2 = 0.577350269$$

Now we can use the Gauss Quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19) + 11c_2 f(11x_2+19) = 11f(11(-0.5773503)+19) + 11f(11(0.5773503)+19)$$

$$= 11f(12.64915) + 11f(25.35085) = 11(296.8317) + 11(708.4811) = 11058.44 \text{ m}$$

since

$$f(12.64915) = 2000\ln\left[\frac{140000}{140000-2100(12.64915)}\right] - 9.8(12.64915) = 296.8317$$

$$f(25.35085) = 2000\ln\left[\frac{140000}{140000-2100(25.35085)}\right] - 9.8(25.35085) = 708.4811$$

a) The true error E_t is

$$E_t = True\ Value - Approximate\ Value = 11061.34 - 11058.44 = 2.9000 \text{ m}$$

b) The absolute relative true error $|\in_t|$ is (Exact value = 11061.34m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100\% = 0.0262\%$$

EXAMPLE By the Gaussian formulae, evaluate $\int_2^3 \frac{dt}{1+t}$.

Solution

Let $t = \frac{b-a}{2}x + \frac{(b+a)}{2} = \frac{x}{2} + \frac{5}{2}$, $dt = \frac{1}{2}dx$

$$\int_2^3 \frac{dt}{1+t} = \frac{1}{2}\int_{-1}^{1} \frac{2dx}{7+x} = \int_{-1}^{1} \frac{dx}{7+x}$$

By two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{7-\frac{1}{\sqrt{3}}} + \frac{1}{7+\frac{1}{\sqrt{3}}}$$

$$= \frac{42}{149} = 0.28188$$

EXAMPLE Apply Gauss two-point quadrature formula to evaluate

$$\text{(i)} \int_{-1}^{1} \frac{1}{1+x^2}\,dx \text{ and (ii)} \int_{0}^{1} \frac{dx}{1+x^2}$$

Solution

(i) Here, $f(x) = \dfrac{1}{1+x^2} \Rightarrow f\left(\dfrac{1}{\sqrt{3}}\right) = f\left(\dfrac{-1}{\sqrt{3}}\right) = \dfrac{3}{4}$

Hence, $$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 1.5$$

But $$\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2(\tan^{-1} x)_0^1 = 1.5708$$

Here, the error due to two-point formula is 0.0708.

(ii) $$\int_{0}^{1} \frac{dx}{1+x^2} = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+x^2} = 0.75$$

EXAMPLE Using the Gaussian three-point formula, evaluate

(i) $\int_{-1}^{1} (3x^2 + 5x^4)\, dx$, and (ii) $\int_{0}^{1} (3x^2 + 5x^4)\, dx$ Also compare with exact value.

Solution

Let $$f(x) = 3x^2 + 5x^4$$

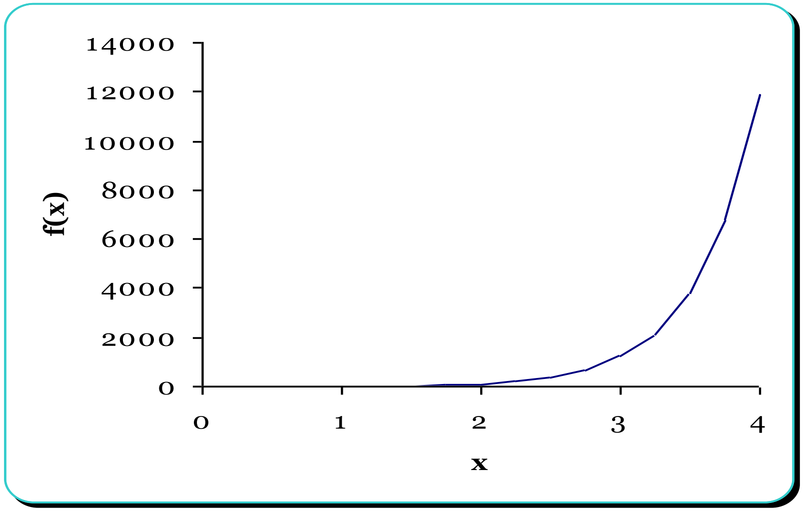
$\therefore$ $$f(0) = 0 \qquad f(-\sqrt{0.6}) = f(\sqrt{0.6}) = 3.6$$

$$\int_{-1}^{1} f(x)\, dx = \frac{5}{9}[f(-\sqrt{0.6}) + f(\sqrt{0.6})] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}(3.6 + 3.6) + 0 = 4$$

Exact value of the integral in also 4 by direct integration.

(ii) $$\int_{0}^{1} (3x^2 + 5x^4)\, dx = \frac{1}{2}\int_{-1}^{1} (3x^2 + 5x^4)\, dx = \frac{1}{2} \times 4 = 2$$



Exercise: Estimate the following using two-point Gauss Legendre:

$$\int_0^4 x e^{2x} dx$$

Improper Integrals

- Improper integrals is produced by changing of variable that transforms the infinite range into the finite one,

$$\int_a^{\infty} f(x)dx = \int_0^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt \Leftarrow t = \frac{1}{x}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

where *A* is a sufficiently large value so that the function has to approach zero asymptotically at least as fast as $1/x^2$.