Title : **On complete generalized Fibonacci sequences** **M2_12_01**

Field : Mathematics and Statistics

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Abstract

Generalized Fibonacci sequence $(A_n)_{n\in\mathbb{N}}$ is a sequence defined by a recurrence relation $A_n = aA_{n-1} + bA_{n-2}$ when A_1, A_2, a, b are non-negative integers. In this project, we study all pairs of (a, b) such that $(A_n)_{n\in\mathbb{N}}$ is a complete sequence. Moreover, we provide conditions in which each positive integer can be uniquely written as a linear combinations of complete generalized Fibonacci sequences.

Keywords : Complete sequence, Zeckendorf representation, Fibonacci sequences

Introduction

Fibonacci sequence $(F_n)_{n\in\mathbb{N}\cup\{0\}}$ is a sequence determined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ Where $F_0 = 0$ and $F_1 = 1$. It was shown in [5] that Fibonacci sequence is a complete sequence such that each positive integer m can be uniquely written as $m = \sum_{i=2}^{\infty} \alpha_i F_i$, $\alpha_i \in \{0,1\}$ under the condition $\alpha_i\alpha_{i+1} = 0$. This linear combination is called Zeckendorf representation [8], which has been used in coding [1,4] and gaming algorithm [2,6,7]. Later in 1969, J.L. Brown [3] showed the condition that uniquely represented each positive integer as a summation of elements in Lucas sequence. He found that if m is a non-negative integer, then there exists a unique sequence $(\alpha_n)_{n\in\{0\}\cup\mathbb{N}}$ in $\{0,1\}$ such that $m = \sum_0^{\infty} \alpha_i L_i$ where $\alpha_i\alpha_{i+1} = 0$ for all $i \geq 0$ and $\alpha_0\alpha_2 = 0$.

A generalize Fibonacci sequence $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ is a sequence determined by the recurrence relation $A_n = aA_{n-1} + bA_{n-2}$ for all $n \geq 2$ where A_1, A_2, a, b are non-negative integers. In general the sequence $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ is not a complete sequence. For example if $A_1 = A_2 = 1$ and $(a, b) = (0,4)$, then $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ is not a complete sequence. In this project, we are interested in finding the conditions on (a, b) that make $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ complete. Moreover, we will find a condition for unique writing each positive integer as a summation of elements in a complete generalized Fibonacci sequence.

Methodology

1.Study sequence theory.

2.Study related information.

3.Find all pairs (a,b) such that the generalized Fibonacci sequences $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ are complete.

4.Find conditions that each positive integer can be uniquely written as a linear combination of elements in complete generalized Fibonacci sequences.

5.Write the project report.

6.Check and edit.

Results, Discussion and Conclusion

(a,b)	conditions
$(0,1)$	$\alpha_i = 0$ for all $i > n$
$(0,2)$	$\alpha_{2i-1} = 0$ for all $i \in \mathbb{N}$
$(0,3)$	$\alpha_{2i-1} \leq \alpha_i$ for all $i \in \mathbb{N}$
$(1,0)$	$\alpha_i = 0$ for all $i > n$
$(1,1)$	$\alpha_1 = 0, \alpha_i\alpha_{i+1} = 0$ for all $i \in \mathbb{N}$
$(1,2)$	$\min\{k \mid \alpha_k = 1\}$ are odd numbers
$(2,0)$	$\alpha_1 = 0$

All pairs of (a,b) such that $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ are complete and conditions in which each positive integer can be uniquely written as a linear combination of elements in complete generalized Fibonacci sequences, when $A_1 = 1,\ A_2 = 1$.

(a,b)	conditions
$(0,2)$	$\alpha_{2i} = 0$ for all $i \in \mathbb{N}$
$(0,3)$	$\alpha_i\alpha_{i+1} = 0$ for all i are odd numbers
$(0,4)$	$-$
$(1,0)$	$\alpha_{2i+1} \leq \alpha_i$ for all $i \in \mathbb{N} - \{1\}$
$(1,1)$	$\alpha_i\alpha_{i+1} = 0$ for all $i \in \mathbb{N}$
$(1,2)$	$-$
$(2,0)$	$-$

All pairs of (a,b) such that $(A_n)_{n\in\mathbb{N}\cup\{0\}}$ are complete and conditions in which each positive integer can be uniquely written as a linear combination of elements in complete generalized Fibonacci sequences, when $A_1 = 1,\ A_2 = 2$.

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