40.305 Advanced Topics in Stochastic Modeling, 2020 May 2, 2020

Final examination

Duration: 75 minutes

Instructions:

1) This exam is for 75 minutes, open-book. Any interaction about the exam during the duration of exam is strictly prohibited.

2) Please make sure you have access to a device to take photos at the end of the exam.

3) Upon completion of the exam, please upload photos of the answer scripts within 10 minutes of the completion of exam. These first photos need not be of very good quality. Please take additional 10 minutes to upload readable photos (as clear as allowed by the device you have, there is no need to go searching for scanners). Please feel free to email me the photos if you encounter troubles with e-Dimension.

4) If you have questions, you may immediately join the following zoom session to ask me questions. Meeting link: https://zoom.us/j/92994871687, Meeting ID: 929 9487 1687, Password: 090896.

5) We have 30 points distributed over 4 questions. Correctly worked steps will fetch partial points even if the final answer is not correct. Good luck!

---

1. (6 points) Let $S_0 = x$ and $S_n = S_0 + X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots$ are independent and identically distributed random variables and x is a positive number. For what value of parameter c is the processs $(M_n : n \geq 0)$ defined by,

$$M_n = S_n - nc \qquad \text{for } n \geq 0,$$

a martingale?

2. (8 points) Suppose that stock price evolves according to a 3-period binomial model with the following parameters: $S_0 = 27, u = 4/3, d = 2/3$ and interest rate $r = 1/9$. Here S_0 is the initial stock price at time 0, u is the multiplicative factor by which the stock price goes up and d is the multiplicative factor by which the stock price goes down. The European "cash-or-nothing option" pays \$1 if $S_3 > 27$ and 0 otherwise. Find the values of the option V_0 at times 0.

3. (8 points) In some betting situations, the only type of wagers allowed are to choose one of the outcomes i, where $i = 1, \ldots, m$, and bet that i is the outcome of the experiment. The return from such a bet is often quoted in terms of "odds." If the odds for outcome i are o_i (often written as "o_i to 1") then a 1 unit bet will return a profit of o_i if the outcome of the experiment is i and will return a loss of -1 otherwise. Suppose the odds $o_1, \ldots, o_m$ are posted. Show that the posted odds o_i must satisfy the condition

$$\sum_{i=1}^{m} \frac{1}{1 + o_i} = 1$$

in order to be arbitrage-free (that is, no arbitrage).

4. (8 points) Suppose the value of a stock at time 0 is $S_0 = 20$ and the interest rate per unit time is $1/4$. At discrete time units $i = 1, 2$, the value of stock goes up by a multiplicative factor $1+1/(2i)$ with probability $1/2$ (or) it goes down by a factor $1-1/i$ with probability $1/2$. Let S_i denote the value of stock at time i. Find the value (at time 0) of the European call option with payoff $(S_2 - 25)^+$.