1. Solution

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Let $U_i$ be the time taken by each repair.
Let $V_i$ be the time duration between successive failure.

① Identify cycle:

$$X = \min(U, V), \quad U_i \overset{i.i.d.}{\sim} Exp(u), \quad V_i \overset{i.i.d.}{\sim} Exp(\lambda)$$

$$\begin{aligned}\mathbb{P}(X \le x) &= 1 - \mathbb{P}(X > x) \\ &= 1 - \mathbb{P}(\min(U, V) > x) \\ &= 1 - \mathbb{P}(U > x) \cdot \mathbb{P}(V > x) \\ &= 1 - e^{-(\lambda+u)x}\end{aligned}$$

$$f_X(x) = \frac{d}{dx}\mathbb{P}(X \le x) = (\lambda+u)e^{-(\lambda+u)x} \Rightarrow X \sim Exp(\lambda+u)$$

$$\mathbb{E}[X] = \frac{1}{\lambda+u}.$$

② Identify reward

$$R_i = \mathbb{1}\{U_i \le V_i\}$$

$$\begin{aligned}\mathbb{E}[R_i] = \mathbb{P}(U \le V) &= \int_0^\infty \int_0^y f_{U,V}(x,y)\, dx\, y \\ &= \int_0^\infty \int_0^v u e^{-ux} \cdot \lambda e^{-\lambda y}\, dx\, dy\end{aligned}$$

$$= \int_0^\infty \left[ -e^{-\mu x} \right]_0^y \lambda e^{-\lambda y} dy$$

$$= \int_0^\infty (1 - e^{-\mu y}) \lambda e^{-\lambda y} dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} - \lambda e^{-(\mu+\lambda) y} dy$$

$$= \left[ -e^{-\lambda y} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda) y} \right]_0^\infty$$

$$= 1 - \frac{\lambda}{\mu+\lambda} = \frac{\mu}{\mu+\lambda}$$

By Renewal- Reward theorem,

Long-run rate of machines are fully repaired

$$= \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]} = \frac{\mu/(\mu+\lambda)}{1/(\mu+\lambda)} = \mu$$

Long- run fraction of machines are fully done

$$= \mathbb{P}(U_i \leq V_i) = \frac{\mu}{\mu+\lambda}.$$

## 2. Solution

If $s < t$,

$$\mathbb{P}(S_1 \le s \mid N(t)=1)$$

$$= \frac{\mathbb{P}(S_1 \le s,\ N(t)=1)}{\mathbb{P}(N(t)=1)}$$

Note $N(t) \sim \text{Poiss}(\lambda t)$

$$= \frac{\mathbb{P}(N(s-0)=1,\ N(t-s)=0)}{\mathbb{P}(N(t)=1)}$$

$$= \frac{\mathbb{P}(N(s)=1)\cdot \mathbb{P}(N(t-s)=0)}{\mathbb{P}(N(t)=1)}$$

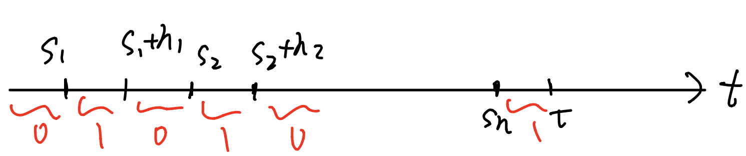
$$= \frac{(\lambda s)^1 e^{-\lambda s}}{1!} \cdot \frac{(\lambda(t-s))^0 e^{-\lambda(t-s)}}{0!} \Big/ \frac{(\lambda t)^1 e^{-\lambda t}}{1!}$$

$$= \frac{\lambda s e^{-\lambda t}}{\lambda t e^{-\lambda t}} = \frac{s}{t}$$

If $s \ge t$, $\mathbb{P}(S_1 \le s \mid N(t)=1) = 1$.

Hence,

$$\mathbb{P}(S_1 \le s \mid N(t)=1) = \max\left(\min\left(\frac{s}{t}, 1\right), 0\right)$$



3. Solution

a) $\mathbb{P}\left\{ s_i \le S_i \le s_i + h_i ,\ i = 1, \dots, n \mid N(t) = n \right\}$

$$= \frac{\prod_{i=1}^{n+1} \mathbb{P}\left\{ N(s_i - s_{i-1} - h_{i-1}) = 0 \right\} \cdot \prod_{i=1}^{n} N(h_n) = 1 \}}{\mathbb{P}(N(t) = n\}}$$

$$= \frac{\prod_{i=1}^{n+1} e^{-\lambda(s_i - s_{i-1} - h_{i-1})} \cdot \prod_{i=1}^{n} \lambda h_i \cdot e^{-\lambda h_i}}{(\lambda t)^n \cdot e^{-\lambda t} / n!}$$

$$= \frac{\lambda^n \prod_{i=1}^{n} h_i \cdot \prod_{i=1}^{n+1} e^{-\lambda(s_i - s_{i-1})}}{\lambda^n t^n \cdot e^{-\lambda t} / n!} = \frac{\lambda^n \cdot \prod_{i=1}^{n} h_i \cdot e^{-\lambda t}}{\lambda^n \cdot t^n \cdot e^{-\lambda t} / n!}$$

$$= \frac{n!}{t^n} \cdot \prod_{i=1}^{n} h_i$$

b) From a),

$$\lim_{\substack{h_i \to 0 \\ i = 1 \dots n}} \mathbb{P}\left\{ s_i \le S_i \le s_i + h_i \mid N(t) = n \right\}$$

$$= \lim_{\substack{h_i \to 0 \\ i = 1 \dots n}} f_{S_1, S_2, \dots S_n}(s_1, s_2, \dots, s_n) \cdot \prod_{i=1}^{n} h_i$$

Canceling $\prod_{i=1}^{n} h_i$ on both sides yields

$$f_{S_1, S_2, \ldots S_n}(\underbrace{s_1, s_2, \cdots s_n}_{0 < s_1 < \cdots < s_n}) = n!\, t^{-n}$$ , which is exactly

he distribution of order statistics of $n$ i.i.d. r.v.s.

4. Solution

Let $S$ be the number of messages stored in buffer before clearing.

① Identify cycle

$$X_i = T$$

② Identify reward

$$R_i = K + \left[(T-X_1) + (T-X_2) + \cdots + (T-X_{N(T)})\right] \cdot h$$

$$\mathbb{E}[R_i] = K + h \cdot \mathbb{E}\left[N(T)\cdot T - \sum_{i=1}^{N(T)} X_i\right]$$

$$= K + h \cdot \mathbb{E}\left[\mathbb{E}\left[N(T)\cdot T - \sum_{i=1}^{N(T)} X_i \,\middle|\, N(T)\right]\right]$$ law of total expectation

$$= K + h \cdot \mathbb{E}\left[N(T)\cdot T - N(T)\cdot \mathbb{E}[X_i]\right]$$ $X_i \mid N(t) = t \sim \mathrm{Unif}(0,t)$

$$= K + h \cdot \mathbb{E}\left[T \cdot N(T) - \frac{T}{2} N(T)\right]$$

$$= K + h \cdot \frac{T}{2} \cdot \mathbb{E}[N(T)]$$

$$= K + h \cdot \frac{T}{2} \cdot \lambda T$$

$$= \frac{h\lambda}{2} T^2 + K$$

Let $g(T)$ be the long-run average cost per unit time,

by renewel-reward theorem

$$g(T) = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]} = \frac{h\lambda}{2} T + \frac{K}{T},$$

$$g'(T) = \frac{h\lambda}{2} - \frac{K}{T^2} = 0 \Rightarrow T = \sqrt{\frac{2K}{h\lambda}}$$

Hence the $T = \sqrt{\frac{2K}{h\lambda}}$ gives the optimal value.