Mini-project exercise 2: Choice II

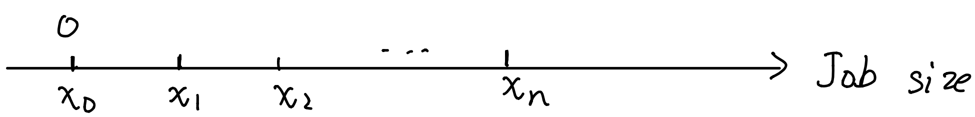
a) Recall from lecture that

$$\mathbb{E}[T_Q(k)]^{NP\text{-}Priority} = \frac{\rho \cdot \mathbb{E}[S^2]}{2\mathbb{E}[S]} \cdot \frac{1}{(1-\sum_{i=1}^{k-1}\rho_i)(1-\sum_{i=1}^{k}\rho_i)}$$

and

$$\mathbb{E}[T_Q]^{NP\text{-}Priority} = \sum_{k=1}^{n} p_k \cdot \mathbb{E}[T_Q(k)].$$

WLOG, assume in discrete case the job size range is




In continuous case $x_k \to x_{k+1}$ as $n \to \infty$.

let the pdf of serve time be $f(\cdot)$

Assign job with smaller size to a higher probability.

As $n \to \infty$, load of jobs from class 1 to k

$$= \text{load of jobs of size} < x_k$$

$$= \sum_{i=1}^{k} \rho_i = \sum_{i=1}^{k} \frac{\lambda_i}{u_i}$$

$$= \sum_{i=1}^{k} \lambda \cdot p_i \cdot \mathbb{E}[S_i] \longrightarrow \lambda \int_0^{x_k} t \cdot f(t)\,dt$$

Similarly, load of job from class 1 to k-1

$= \text{load of jobs of size} < x_{k-1}$

$$= \sum_{i=1}^{k-1} \rho_i = \sum_{i=1}^{k-1} \lambda \cdot p_i \cdot \mathbb{E}[S_i] \xrightarrow[n\to\infty]{} \lambda \int_0^{x_{k-1}} t \cdot f(t)\,dt$$

$$\xrightarrow{x_k \to x_{k-1} \text{ as } n\to\infty} \lambda \int_0^{x_k} t \cdot f(t)\,dt$$

Hence in the limit form of $\mathbb{E}[T_Q(k)]^{NP\text{-}Priority}$, we have

$$\mathbb{E}[T_Q(x)]^{SJF} = \frac{\rho\mathbb{E}[S^2]}{2\mathbb{E}[S]} \cdot \frac{1}{(1-\lambda\int_0^x t\cdot f(t)\,dt)^2}$$

$$= \frac{\rho\mathbb{E}[S^2]}{2\mathbb{E}[S]} \cdot \frac{1}{(1-\rho_x)^2}$$

In comparison with

$$\mathbb{E}[T_Q(x)]^{FCFS} = \frac{\rho\mathbb{E}[S^2]}{2\mathbb{E}[S]} \cdot \frac{1}{1-\rho}.$$

both $\mathbb{E}[T_Q(x)]^{SJF}$ and $\mathbb{E}[T_Q(x)]^{FCFS}$ are equally affected by the distribution of $S$, thus SJF does not mitigate the effect of high coefficient variation.

Note that $P_x \ll P$ for most cases if $x$ is small.
However, the square factor is significant if $x$ is large.
Thus, SJF is preferred over FCFS if most of jabs have small size.

b) We follow the same strategy in a) where $\mathbb{E}[T_Q(x)]^{PSJF}$ is derived in discrete form and take the limit to obtain the expression for continuous case.

First, we derive $\mathbb{E}[T_Q(x)]^{P\text{-}Priority}$, it is trivial to replace priority by job size to then obtain $\mathbb{E}[T_Q(x)]^{PSJF}$.

Consider a job of priority $k$, then

$\mathbb{E}[T(k)]^{P\text{-}Priority}$

$= \mathbb{E}[$ mean service time of priority $k]$

$+ \mathbb{E}[$ the total service time of job of priority 1 to $k-1$ before the service of priority $k$ is completed$]$

$+ \mathbb{E}[$ the expected waiting time of jobs of 1 to $k$ before our job arrives$]$

$$= \mathbb{E}[S_k] + \mathbb{E}[T(k)]^{P\text{-}Priority} \cdot \sum_{i=1}^{k-1} \rho_i$$

$$+ \underbrace{\mathbb{E}[T_Q(k) \text{ of system with any policy}]}_{\substack{\text{All policy have the} \\ \text{Same remaing work}}}$$

WLOG, FCFS

$$= \mathbb{E}[S_k] + \mathbb{E}[T(k)]^{P\text{-Priority}} \cdot \sum_{i=1}^{k-1} \rho_i + \frac{\sum_{i=1}^{k} \rho_i \cdot \frac{\mathbb{E}[S_i^2]}{2\mathbb{E}[S_i]}}{1 - \sum_{i=1}^{k} \rho_i}$$

Hence

$$\mathbb{E}[T(k)]^{P\text{-}P} = \mathbb{E}[S_k] + \frac{\sum_{i=1}^{k} \rho_i \cdot \frac{\mathbb{E}[S_i^2]}{2\mathbb{E}[S_i]}}{1 - \sum_{i=1}^{k} \rho_i} + \mathbb{E}[T(k)^{P\text{-}P}] \cdot \sum_{i=1}^{k-1} \rho_i$$

$$\Rightarrow \left(1 - \sum_{i=1}^{k-1} \rho_i\right) \mathbb{E}[T(k)]^{P\text{-}P} = \mathbb{E}[S_k] + \frac{\sum_{i=1}^{k} \rho_i \frac{\mathbb{E}[S_i^2]}{2\mathbb{E}[S_i]}}{1 - \sum_{i=1}^{k} \rho_i}$$

$$\Rightarrow \mathbb{E}[T(k)]^{P\text{-}P} = \frac{\mathbb{E}[S_k]}{1 - \sum_{i=1}^{k-1} \rho_i} + \frac{\sum_{i=1}^{k} \rho_i \cdot \frac{\mathbb{E}[S_i^2]}{2\mathbb{E}[S_i]}}{\left(1 - \sum_{i=1}^{k} \rho_i\right)\left(1 - \sum_{i=1}^{k} \rho_{i-1}\right)}$$

$$= \frac{\mathbb{E}[S_k]}{1 - \sum_{i=1}^{k-1} \rho_i} + \frac{\frac{\lambda}{2} \sum_{i=1}^{k} \mathbb{E}[S_i^2] \cdot p_i}{\left(1 - \sum_{i=1}^{k} \rho_i\right)\left(1 - \sum_{i=1}^{k} \rho_{i-1}\right)}$$

c)

Assign job with smallest size to higher priority and take the limit as we did in (a), we have

$$\mathbb{E}[T(x)]^{PSJF} = \frac{x}{1 - \rho_x} + \frac{\frac{\lambda}{2} \int_0^{x} t^2 f(t)\, dt}{(1 - \rho_x)^2}$$

PSJF tends to perform better as they are less sensitive to the job size distribution.

Note that $\mathbb{E}[T(x)]^{PSJF}$ solely depends on job with size up to $x$, it only suffers the variability of the job up to $x$ instead of the entire distribution compared to Part a).

Hence under circumstance when the job distribution suffers high-variability, PSJF will outperforms SJF.

# Appendix

$$\mathbb{E}[T_Q(x)]^{PSJF} = \frac{1}{1-\rho_x} + \frac{\frac{\lambda}{2}\int_0^x t^2 f(t)\,dx}{(1-\rho_x)^2}$$

$$= \frac{1-\rho_x + \frac{\lambda}{2}\int_0^x t^2 f(t)\,dx}{(1-\rho_x)^2}$$

$$\mathbb{E}[T_Q(x)]^{SJF} = \frac{\rho\,\mathbb{E}[S^2]}{2\mathbb{E}[S]} \cdot \frac{1}{(1-\rho_x)^2}$$

$$= \frac{\frac{\lambda}{2}\mathbb{E}[S^2]\,\cdot}{(1-\rho_x)^2}$$

As $x \to x_n$ (The maximum job size),

$\rho_x \to \rho$

$\int_0^x t^2 f(t)\,dx \to \mathbb{E}[S^2]$,

thus

$$\mathbb{E}[T_Q(x)]^{PSJF} \xrightarrow{x \to x_{max}} \frac{1-\rho + \frac{\lambda}{2}\mathbb{E}[S^2]}{(1-\rho_x)^2} \xrightarrow{\rho \to 1} \frac{\frac{\lambda}{2}\mathbb{E}[S^2]}{(1-\rho_x)^2}$$

$$= \mathbb{E}[T_Q(x)]^{SJF}$$

Hence, as $\rho \to 1$, PSJF consistently outperforms SJF for all job size $x$.