1.

For M_n to be a martingale

$$\mathbb{E}[M_{n+1} \mid H_n] = M_n$$

where $H_n = \{M_0, X_1, \dots, X_n\}$.

$$\mathbb{E}[M_{n+1} \mid H_n]$$

$$= \mathbb{E}[S_{n+1} - (n+1)c \mid H_n]$$

$$= \mathbb{E}[S_{n+1} \mid H_n] - (n+1)C$$

$$= \mathbb{E}[S_n + X_{n+1} \mid H_n] - (n+1)c$$

$$= \mathbb{E}[X_{n+1}] + \mathbb{E}[S_n \mid H_n] - (n+1)c$$

$$= \mathbb{E}[X_{n+1}] + (S_n - nc) - c$$

$$= \mathbb{E}[X_{n+1}] + M_n - c$$

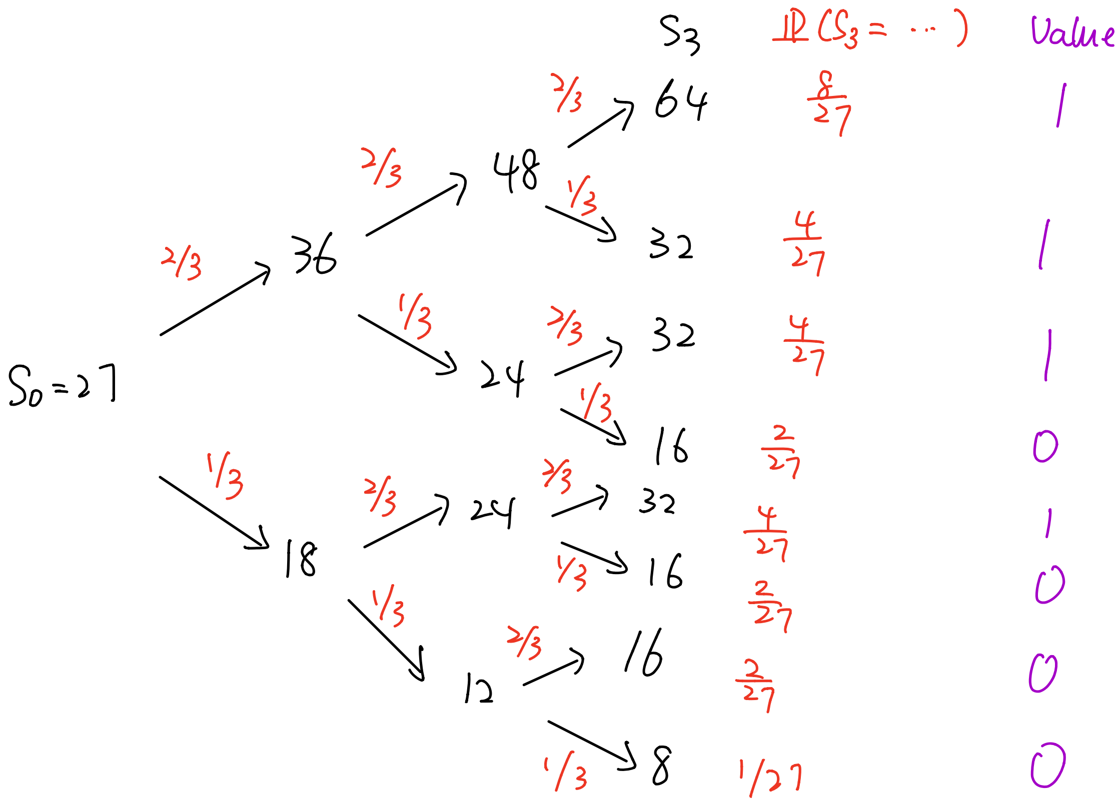
Thus $\mathbb{E}[X_{n+1}] + M_n - C = M_n$

$$\Rightarrow C = \mathbb{E}[X_{n+1}] = \mathbb{E}[X_1$$

Hence M_n is a martingale if $c = \mathbb{E}[X]$.

2. $S_0 = 27, \quad u = \frac{4}{3}, \quad d = \frac{2}{3}, \quad r = \frac{1}{9}$

$$p^* = \frac{1+r-d}{u-d} = \frac{1+\frac{1}{9}-\frac{2}{3}}{\frac{4}{3}-\frac{2}{3}} = \frac{2}{3}$$



$$\mathbb{E}[V_0] = \frac{1}{(1+\frac{1}{9})^3} \cdot \left(1 \cdot \frac{8}{27} + 1 \cdot \frac{4}{27} + 1 \cdot \frac{4}{27} + 0 \cdot \frac{2}{27} + 1 \cdot \frac{4}{27} + 0 \cdot \frac{2}{27} + 0 \cdot \frac{2}{27} + 0 \cdot \frac{1}{27} \right)$$

$$= \frac{27}{50}$$

Hence $V_0 = \mathbb{E}[V_0] = \frac{27}{50}$.

3.

Let the reward be $r_i(j)$ where i is the predicted outcome and j is the actual outcome.

As such,

$$r_i(j) = O_i \cdot \mathbb{I}\{j=i\} - \mathbb{I}\{j \neq i\}$$

Given that $\{O_1, \ldots, O_m\}$ is posted, in order for the experiment to be arbitrage-free, by Theorem 1 under session 2 of week 13, there must exists a probability vector $\vec{p} = (p_1, \ldots, p_m)^T$ such that

$$\sum_{j=1}^{n} r_i(j) \cdot p_j = \mathbb{E}[r_i(j)] = 0$$

Note that $\mathbb{E}[r_i(j)] = O_i \cdot p_i - (1-p_i)$

Thus,

$$O_i \cdot p_i - 1 + p_i = 0$$

$$\Rightarrow p_i = \frac{1}{1+O_i}$$

As all p_i must sum to 1 to form a probability vector,

$$\sum_{i=1}^{m} p_i = 1$$

$$\Rightarrow \quad \sum_{i=1}^{m} \frac{1}{1+o_i} = 1.$$

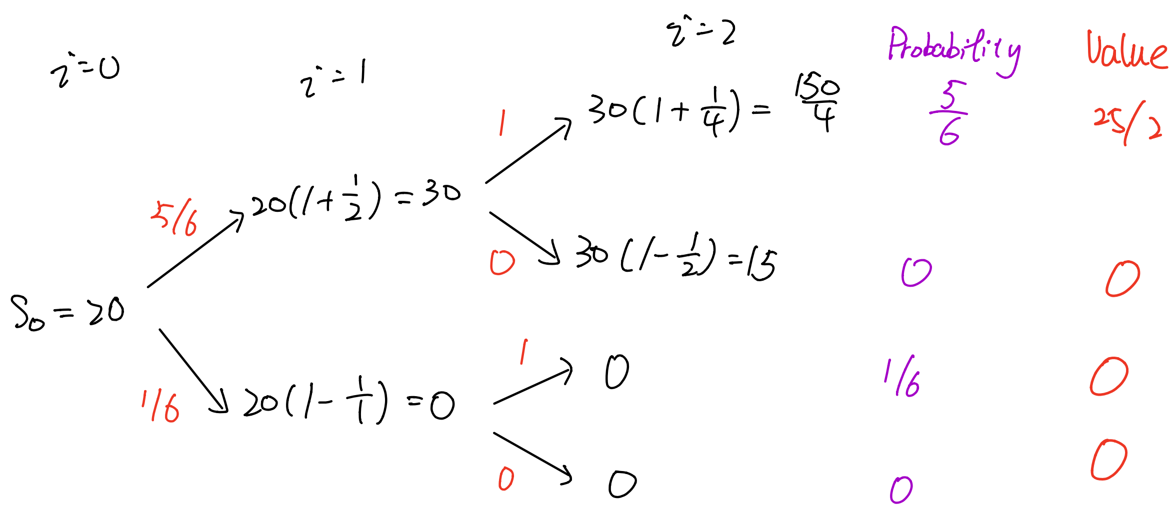
$\blacksquare$

4.

$$p^* = \frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\left(1-\frac{1}{i}\right)}{1+\frac{1}{2i}-\left(1-\frac{1}{i}\right)} = \frac{\frac{1}{4}+\frac{1}{i}}{\frac{3}{2i}} = \frac{4+i}{6}$$

At $i=1$, $p(\text{up}) = \frac{5}{6}$, $p(\text{down}) = \frac{1}{6}$

At $i=2$, $p(\text{up}) = 1$, $p(\text{down}) = 0$



$$V_0 = \mathbb{E}[V_0] = \frac{1}{\left(1+\frac{1}{4}\right)^2}\left[\frac{5}{6}\cdot\frac{25}{2}\right]$$

$$= \frac{20}{3}$$

Hence $V_0 = \frac{20}{3}$.