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1.

a) Let S be the overall service time

Then S is hyperexponential

$$S = \begin{cases} S_R \text{ w.p. } \dfrac{\lambda_R}{\lambda_R+\lambda_B} & \text{where } \mathbb{E}[S_R] = 0.5 \quad Var(S_R) = 1 \\ S_B \text{ w.p. } \dfrac{\lambda_B}{\lambda_B+\lambda_R} & \text{where } \mathbb{E}[S_B] = 1, \quad Var(S_R) = 1 \end{cases}$$

Let $\lambda = \lambda_R + \lambda_B$

$$\mathbb{E}[S] = \frac{\lambda_R}{\lambda}\mathbb{E}[S_R] + \frac{\lambda_B}{\lambda}\mathbb{E}[S_B]$$

$$= \frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times 1 = \frac{7}{8}$$

$$\mathbb{E}[S^2] = \frac{\lambda_R}{\lambda}\mathbb{E}[S_R^2] + \frac{\lambda_B}{\lambda}\mathbb{E}[S_B^2]$$

$$= \frac{\lambda_R}{\lambda}\left(Var(S_R) + \mathbb{E}[S_R]^2\right) + \frac{\lambda_B}{\lambda}\left(Var(S_B) + \mathbb{E}[S_B]^2\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{4}\right) + \frac{3}{4}(1+1)$$

$$= \frac{29}{16}$$

$$Var(S) = \mathbb{E}[S^2] - \mathbb{E}[S]^2 = \frac{67}{64}$$

$$\rho_R = \frac{\lambda_R}{\mu_R} = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}, \quad \rho_B = \frac{\lambda_B}{\mu_B} = \frac{3}{5} \times 1 = \frac{3}{5}$$

$$\Rightarrow \rho = \rho_R + \rho_B = \frac{7}{10}$$

Use P-K formula

$$\mathbb{E}[T_Q] = \frac{\rho}{1-\rho} \cdot \frac{\mathbb{E}[S]}{2} (C_S^2+1) = \frac{29}{12}$$

Hence

$$\mathbb{E}[T_R] = \mathbb{E}[T_Q] + \mathbb{E}[S_R] = \frac{29}{12} + 0.5 = \frac{35}{12}$$

$$\mathbb{E}[T_B] = \mathbb{E}[T_Q] + \mathbb{E}[S_B] = \frac{29}{12} + 1 = \frac{41}{12}$$

(b) In this case, it is a priority queue,

Red job has priority 1, blue job has priority 2.

$$\mathbb{E}[S_e] = \frac{\mathbb{E}[S^2]}{2\mathbb{E}[S]} = \frac{29/16}{2 \times 7/8} = \frac{29}{28}$$

$$\rho = \rho_R + \rho_B = \frac{7}{10}$$

$$\mathbb{E}[T_Q(R)] = \frac{\frac{7}{10} \cdot \frac{29}{28}}{1 - \frac{1}{5} \cdot \frac{1}{2}} = \frac{29}{36}$$

$$\mathbb{E}[T_Q(B)] = \frac{\frac{7}{10} \cdot \frac{29}{28}}{(1 - \frac{1}{5} \cdot \frac{1}{2})(1 - \frac{1}{5} \cdot \frac{1}{2} - \frac{3}{5})} = \frac{145}{54}$$

Hence

$$\mathbb{E}[T_R] = \mathbb{E}[T_Q] + \mathbb{E}[S_R] = \frac{29}{36} + 0.5 = \frac{47}{36}$$

$$\mathbb{E}[T_B] = \mathbb{E}[T_Q] + \mathbb{E}[S_B] = \frac{145}{54} + 1 = \frac{199}{54}$$

2. Say the arrival rate $\sim Pois(\lambda)$

service time $\sim Exp(\mu)$

The system have regenerative point iff $\rho = \frac{\lambda}{\mu} < 1$.

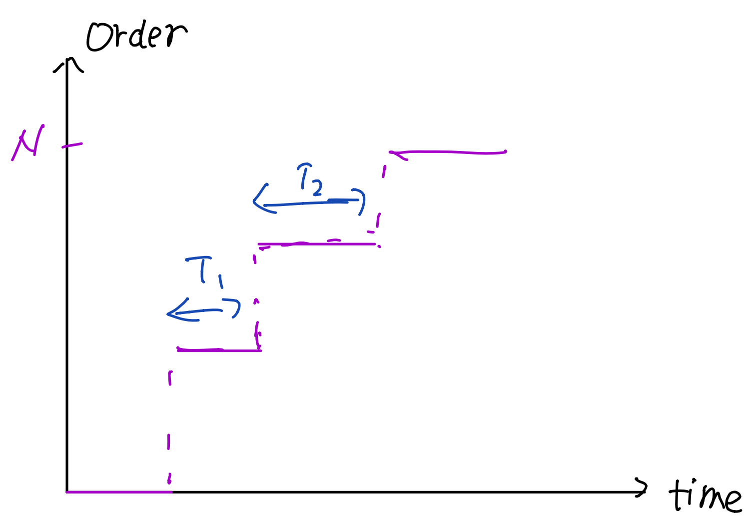
① The set of regenerative point is the set of time when an arriving customer find the the system empty.

② The set of regenerative point is the set of time when the service is completed without any customer left in the system.

3. My student ID: 1003040

Take interarrival time $E[T_i] = \mu = 6$

setup cost $c = 4$



Let the long run cost be $f(N)$.

By Renewal - Reward theorem

The length of each cycle,

$$X_i = \mathbb{E}[S_N] = \mathbb{E}\left[\sum_{i=1}^{N} T_i\right] = N\mu$$

The cost of each cycle,

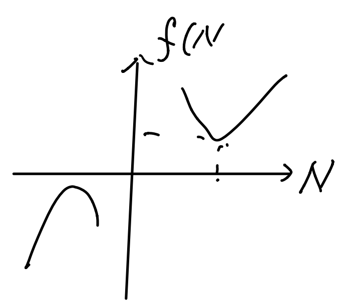
$$R_i = \mathbb{E}[T_1 + 2T_1 + \cdots + (N-1)T_{N-1}] \quad \mathbb{E}[T_i] = \mu$$

$$= \mu \cdot \frac{N}{2}(N-1)$$

$$f(N) = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]} = \frac{\mu \cdot \frac{N}{2}(N-1) + C}{N\mu}$$

$$= \frac{1}{2}(N-1) + \frac{C}{N\mu}$$

Substitute numeric value



$$f(N) = \frac{1}{2}(N-1) + \frac{4}{6N}$$

$$f'(N) = \frac{1}{2} - \frac{2}{3N^2} = 0 \Rightarrow N = \pm\frac{2}{\sqrt{3}}$$

Hence $N^* = \frac{2}{\sqrt{3}}$ minimizes the cost (N can be take as the neighouring integer whichever minimizes the cost)

In my case $N^* = 1$ is better.

4.

Yes. Consider a renewal process that observed it going backwards in. In doing so, we observe a counting process where the time between successive customer departure are i.i.d. Hence, we again observe a renewal process having the same probability structure as before. By symmetry, the point where service is completed without any customer left is equivalent to the point where arriving customer finds the system empty. Thus it is a valid regenerative point by imaging time goes backward.