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1.

Let H_n be the history of evolution of M_n

$$\mathbb{E}[M_{n+1} \mid H_n] = \mathbb{E}[X_{n+1} \cdot M_n \mid H_n]$$

$$= M_n \cdot \mathbb{E}[X_{n+1} \mid H_n]$$

$$= M_n \cdot \mathbb{E}[X_{n+1}]$$

$$= M_n \cdot \int_{-\infty}^{\infty} e^{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$$

$$= M_n \cdot e^{\mu + \frac{\sigma^2}{2}}$$

For M_n to be a martingale,

$$\mathbb{E}[M_{n+1} \mid H_n] = M_n$$

$$\Rightarrow \quad e^{\mu + \frac{\sigma^2}{2}} = 1$$

$$\Rightarrow \quad \mu + \frac{\sigma^2}{2} = 0$$

2.

(a) Let $M_n = S_n^2 - n\sigma^2$ and H_n be the historical evolution of M_n.

$$\mathbb{E}[M_{n+1} \mid H_n] = \mathbb{E}[S_{n+1}^2 - (n+1)\sigma^2 \mid H_n]$$

$$= \mathbb{E}[S_{n+1}^2 \mid H_n] - (n+1)\sigma^2$$

$$= \mathbb{E}[(S_n + X_{n+1})^2 \mid H_n] - (n+1)\sigma^2$$

$$= \mathbb{E}[S_n^2 + 2X_{n+1}S_n + X_{n+1}^2 \mid H_n] - (n+1)\sigma^2$$

$$= \underbrace{\mathbb{E}[S_n^2 \mid H_n]}_{S_n^2} + 2S_n \underbrace{\mathbb{E}[X_{n+1}]}_{0} + \underbrace{\mathbb{E}[X_{n+1}^2 \mid H_n]}_{\mathbb{E}[X_{n+1}^2] = \mathbb{E}[X_{n+1}]^2 + Var(X_{n+1}) = \sigma^2} - (n+1)\sigma^2$$

$$= S_n^2 + \sigma^2 - (n+1)\sigma^2$$

$$= S_n^2 - n\sigma^2$$

Since, $\mathbb{E}[S_{n+1}^2 - (n+1)\sigma^2 \mid H_n] = S_n^2 - n\sigma^2$, $S_n^2 - n\sigma^2$ is a martingale

(b) Under condition (ii),

$$|M_{\tau \wedge n}| \leq a \quad \text{for all} \quad n \leq \tau,$$

thus by optional stopping theorem,

let $M_n = S_n^2 - n\sigma^2$

$\mathbb{E}[M_\tau] = \mathbb{E}[M_1]$

$\Rightarrow \mathbb{E}[S_\tau^2 - \tau\sigma^2] = \mathbb{E}[X_1]$

$\Rightarrow \mathbb{E}[S_\tau^2] - \sigma^2 \mathbb{E}[\tau] = 0$

Since $S_\tau \geq a$, $S_\tau^2 \geq a^2$

$\mathbb{E}[S_\tau^2] \geq \mathbb{E}[a^2] = a^2$

$\Rightarrow \sigma^2 \mathbb{E}[\tau] \geq a^2$

$\Rightarrow \mathbb{E}[\tau] \geq \dfrac{a^2}{\sigma^2}$

2.

(a) Consider buying x amounts of bet 1, y amounts of bet 2, z amounts of bet 3.

Under arbitrage opportunity, we need to win money in any three cases,

win: $2y + 0.5z - (x+y+z) > 0$

lose: $x + 2y + 1.5z - (x+y+z) > 0$

draw: $1.5x - (x+y+z) > 0$

$$\Rightarrow \begin{cases} -x+y-0.5z>0 \\ y+0.5z>0 \\ 0.5x-y-z>0 \end{cases}$$

one solution is $x=2, y=2, z=-2$.

Hence, we obtain an arbitrage opportunity by buying 2 amounts of bet 1, 2 amounts of bet 2 and -2 amounts of bet 3.

(b) Consider only buying first two options. to get an arbitrage,

$$\begin{cases} 2y-(x+y)>0 \\ x+2y-(x+y)>0 \\ 1.5x-(x+y)>0 \end{cases} \Rightarrow \begin{cases} -x+y>0 \quad - \text{①} \\ y>0 \\ 0.5x-y>0 \quad - \text{②} \end{cases}$$

From ① and ② we know that

$$(y>x) \wedge (x>2y),$$

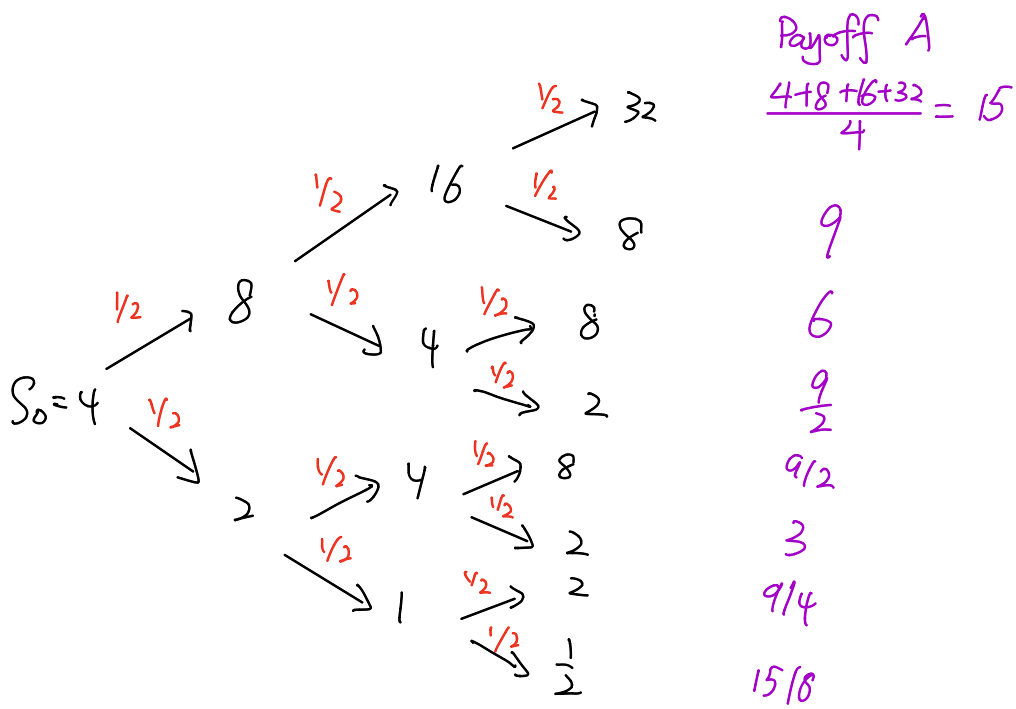
which is an contradiction thus no arbitrage.

4.

Since u, d and r remains the same through of process,

$$p^* = \frac{1+r-d}{u-d} = \frac{1+1/4-1/2}{2-1/2} = \frac{3/4}{3/2} = \frac{1}{2}$$

The stock tree



$$V_0 = \mathbb{E}[V_3] = \left[\left(\tfrac{1}{2}\right)^3 (15 + 9 + 6 + 9/2 + 9/2 + 3 + 9/4 + 15/8)\right] / (1+1/4)^3$$

$$= \frac{369}{125}$$

$$\approx 2.952$$