Final Homework

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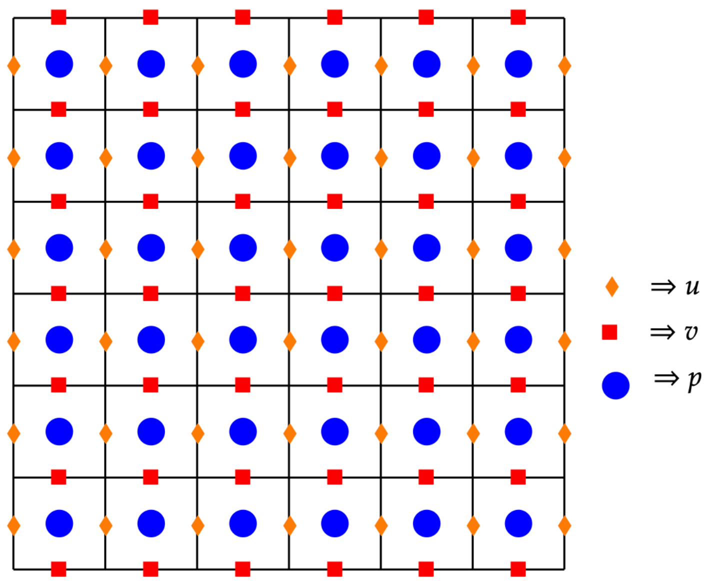
求解Stokes方程:

$$\begin{cases} -\Delta\vec{u} + \nabla p = \vec{F}, & (x,y) \in (0,1)\times(0,1) \\ \mathrm{div}\vec{u} = 0, & (x,y) \in (0,1)\times(0,1) \end{cases} \tag{1}$$

边界条件

$$\begin{aligned} \frac{\partial u}{\partial \vec{n}} &= b, y = 0, & \frac{\partial u}{\partial \vec{n}} &= t, y = 1 \\ \frac{\partial v}{\partial \vec{n}} &= l, x = 0, & \frac{\partial v}{\partial \vec{n}} &= r, x = 1 \\ u &= 0, x = 0, 1, & v &= 0, y = 0, 1 \end{aligned}$$

其中$\vec{u} = (u, v)$为速度, p为压力, $\vec{F} = (f, g)$为外力, $\vec{n}$为外法向方向



stokes方程数值形式为($h=\frac{1}{N}$):

$$-\frac{u_{i+1,j-\frac{1}{2}}-2u_{i,j-\frac{1}{2}}+u_{i-1,j-\frac{1}{2}}}{h^2}-\frac{u_{i,j+\frac{1}{2}}-2u_{i,j-\frac{1}{2}}+u_{i,j-\frac{3}{2}}}{h^2}+\frac{p_{i+\frac{1}{2},j-\frac{1}{2}}-p_{i-\frac{1}{2},j-\frac{1}{2}}}{h}=f_{i,j-\frac{1}{2}}$$

$$-\frac{v_{i-\frac{1}{2},j+1}-2v_{i-\frac{1}{2},j}+v_{i-\frac{1}{2},j-1}}{h^2}-\frac{v_{i+\frac{1}{2},j}-2v_{i-\frac{1}{2},j}+v_{i-\frac{3}{2},j}}{h^2}+\frac{p_{i-\frac{1}{2},j+\frac{1}{2}}-p_{i-\frac{1}{2},j-\frac{1}{2}}}{h}=g_{i-\frac{1}{2},j}$$

$$\frac{u_{i,j-\frac{1}{2}}-u_{i-1,j-\frac{1}{2}}}{h}+\frac{v_{i-\frac{1}{2},j}-v_{i-\frac{1}{2},j-1}}{h}=0$$

边界条件的数值形式为:

$$-\frac{u_{i+1,\frac{1}{2}}-2u_{i,\frac{1}{2}}+u_{i-1,\frac{1}{2}}}{h^2}-\frac{u_{i,\frac{3}{2}}-u_{i,\frac{1}{2}}}{h^2}+\frac{p_{i+\frac{1}{2},\frac{1}{2}}-p_{i-\frac{1}{2},\frac{1}{2}}}{h}=f_{i,\frac{1}{2}}+\frac{b_{i,0}}{h}$$

$$-\frac{u_{i+1,N-\frac{1}{2}}-2u_{i,N-\frac{1}{2}}+u_{i-1,N-\frac{1}{2}}}{h^2}+\frac{u_{i,N-\frac{1}{2}}-u_{i,N-\frac{3}{2}}}{h^2}+\frac{p_{i+\frac{1}{2},N-\frac{1}{2}}-p_{i-\frac{1}{2},N-\frac{1}{2}}}{h}=f_{i,N-\frac{1}{2}}+\frac{t_{i,N}}{h}$$

$$-\frac{v_{\frac{1}{2},j+1}-2v_{\frac{1}{2},j}+v_{\frac{1}{2},j-1}}{h^2}-\frac{v_{\frac{3}{2},j}-v_{\frac{1}{2},j}}{h^2}+\frac{p_{\frac{1}{2},j+\frac{1}{2}}-p_{\frac{1}{2},j-\frac{1}{2}}}{h}=g_{\frac{1}{2},j}+\frac{l_{0,j}}{h}$$

$$-\frac{v_{N-\frac{1}{2},j+1}-2v_{N-\frac{1}{2},j}+v_{N-\frac{1}{2},j-1}}{h^2}+\frac{v_{N-\frac{1}{2},j}-v_{N-\frac{3}{2},j}}{h^2}+\frac{p_{N-\frac{1}{2},j+\frac{1}{2}}-p_{N-\frac{1}{2},j-\frac{1}{2}}}{h}=g_{N-\frac{1}{2},j}+\frac{r_{N,j}}{h}$$

$$u_{0,j-\frac{1}{2}}=u_{N,j-\frac{1}{2}}=v_{i-\frac{1}{2},0}=v_{i-\frac{1}{2},N}=0$$

将第k次迭代的$u_{i,j-\frac{1}{2}}$记为$u^k_{i,j}$, $v_{i-\frac{1}{2},j}$记为$v^k_{i,j}$, $p_{i-\frac{1}{2},j-\frac{1}{2}}$记为$p^k_{i,j}$
则我们需要解线性方程:

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}\begin{bmatrix} U \\ P \end{bmatrix}=\begin{bmatrix} F \\ 0 \end{bmatrix}$$

1 DGS+Vcycle

1.1 Algorithm

1.1.1 单次DGS迭代

Gauss迭代更新速度分量

stokes方程的Gauss迭代可以写成:

$$-\frac{u^k_{i+1,j}-2u^{k+1}_{i,j}+u^{k+1}_{i-1,j}}{h^2}-\frac{u^k_{i,j+1}-2u^{k+1}_{i,j}+u^{k+1}_{i,j-1}}{h^2}+\frac{p^k_{i+1,j}-p^k_{i,j}}{h}=f_{i,j-\frac{1}{2}}$$

$$-\frac{v^k_{i,j+1}-2v^{k+1}_{i,j}+v^{k+1}_{i,j-1}}{h^2}-\frac{v^k_{i+1,j}-2v^{k+1}_{i,j}+v^{k+1}_{i-1,j}}{h^2}+\frac{p^k_{i,j+1}-p^k_{i,j}}{h}=g_{i-\frac{1}{2},j}$$

边界条件的Gauss迭代可以写成:

$$
\begin{gathered}
-\frac{u_{i+1,1}^{k}-2u_{i,1}^{k+1}+u_{i-1,1}^{k+1}}{h^2}-\frac{u_{i,2}^{k}-u_{i,1}^{k}}{h^2}+\frac{p_{i+1,1}^{k}-p_{i,1}^{k}}{h}=f_{i,\frac{1}{2}}+\frac{b_{i,0}}{h}\\
-\frac{u_{i+1,N}^{k}-2u_{i,N}^{k+1}+u_{i-1,N}^{k+1}}{h^2}+\frac{u_{i,N}^{k+1}-u_{i,N-1}^{k+1}}{h^2}+\frac{p_{i+1,N}^{k}-p_{i,N}^{k}}{h}=f_{i,N-\frac{1}{2}}+\frac{t_{i,N}}{h}\\
-\frac{v_{1,j+1}^{k}-2v_{1,j}^{k+1}+v_{1,j-1}^{k+1}}{h^2}-\frac{v_{2,j}^{k}-v_{1,j}^{k+1}}{h^2}+\frac{p_{1,j+1}^{k}-p_{1,j}^{k}}{h}=g_{\frac{1}{2},j}+\frac{l_{0,j}}{h}\\
-\frac{v_{N,j+1}^{k}-2v_{N,j}^{k+1}+v_{N,j-1}^{k+1}}{h^2}+\frac{v_{N,j}^{k+1}-v_{N-1,j}^{k+1}}{h^2}+\frac{p_{N,j+1}^{k}-p_{N,j}^{k}}{h}=g_{N-\frac{1}{2},j}+\frac{r_{N,j}}{h}\\
u_{0,j}^{k+1}=u_{N,j}^{k+1}=v_{i,0}^{k+1}=v_{i,N}^{k+1}=0
\end{gathered}
$$

对每个内部单元(i,j)计算散度残量

$$
r_{i,j}=-\frac{u_{i,j}^{k+1}-u_{i-1,j}^{k+1}}{h}-\frac{v_{i,j}^{k+1}-v_{i,j-1}^{k+1}}{h}
$$

令$delta=r_{i,j}h/4$

更新内部单元速度

$$
\begin{gathered}
u_{i-1,j}^{k+2}=u_{i-1,j}^{k+1}-\delta,u_{i,j}^{k+2}=u_{i,j}^{k+1}+\delta\\
v_{i,j-1}^{k+2}=v_{i,j-1}^{k+1}-\delta,v_{i,j}^{k+2}=v_{i,j}^{k+1}+\delta
\end{gathered}
$$

更新内部单元压力

$$
\begin{aligned}
&p_{i,j}^{k+2}=p_{i,j}^{k}+r_{i,j},\\
&p_{i+1,j}^{k+2}=p_{i+1,j}^{k}-r_{i,j}/4,p_{i-1,j}^{k+2}=p_{i-1,j}^{k}-r_{i,j}/4,\\
&p_{i,j+1}^{k+2}=p_{i,j+1}^{k}-r_{i,j}/4,p_{i,j-1}^{k+2}=p_{i,j-1}^{k}-r_{i,j}/4,
\end{aligned}
$$

更新边界单元和顶点单元

与前面类似, 只是将分子的4换成3和2

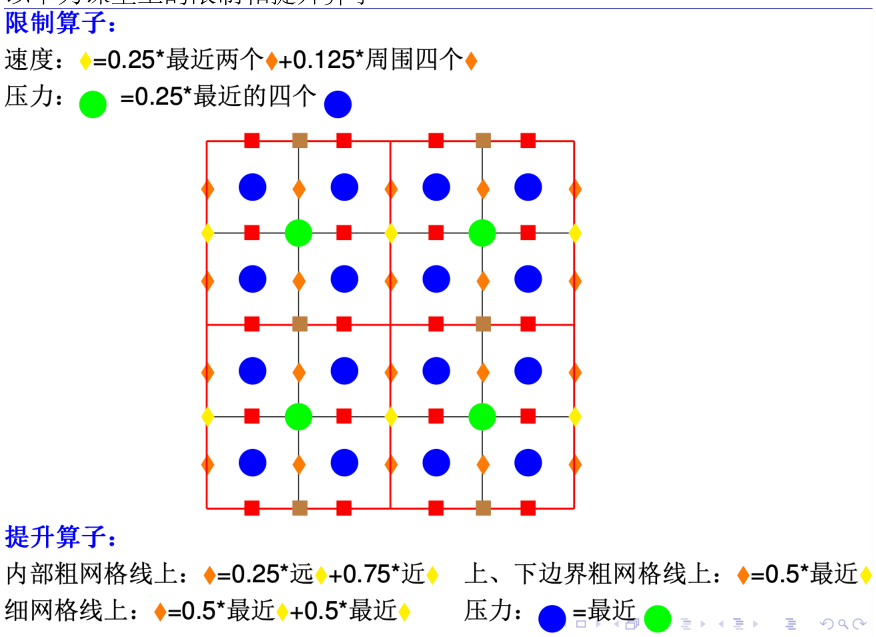
1.1.2 Vcycle

Vcycle算法框架如下 (求解$Ax=b$):
1) 每一轮开始时, 如果是最上层循环, 以上一轮得到的x_0作为初始值, 否则用0作为初始值
2) 先做ν_1次DGS得到x_0
3) 再计算当今误差$b_0=b-Ax_0$, 对其做一次限制算子降低维数得到b_1
4) 重复Vcycle, 近似求解$Ax_1=b_1$
5) 对求解到的x_1做一次提升算子得到x_2

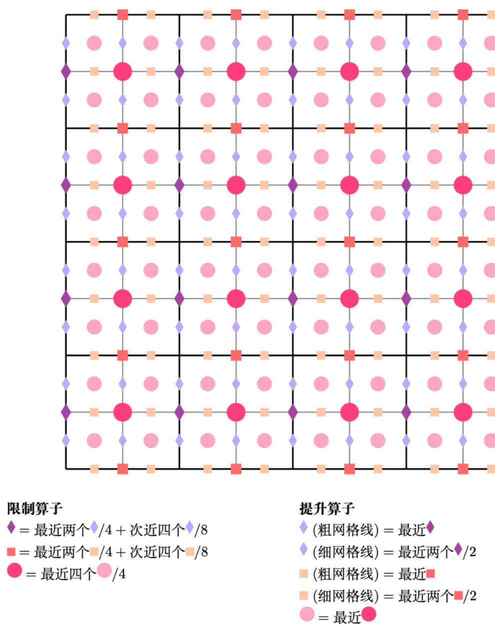
6) 更新$x_0 < -x_0 + x_2$ 7) 再做ν_2次DGS得到x_0

8) 回溯

以下为课堂上的限制和提升算子:



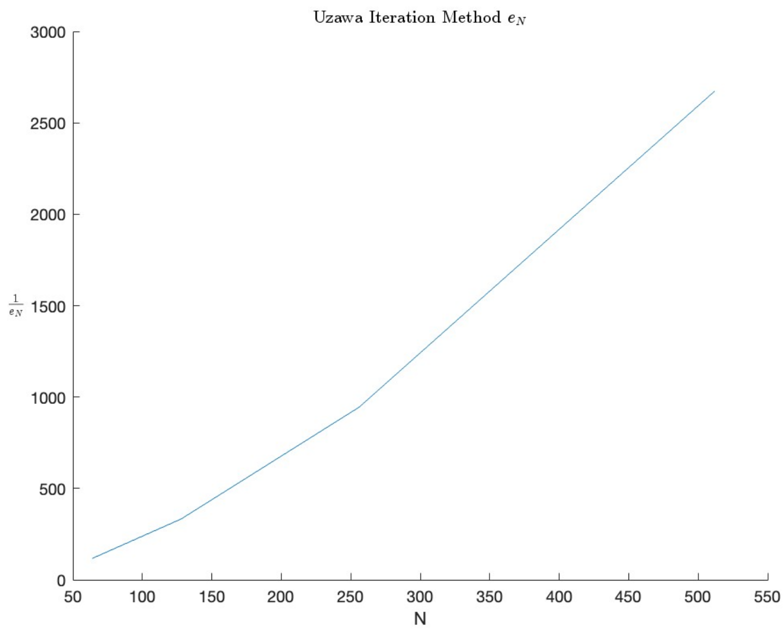
以下为作业pdf中新加上的限制和提升算子:



1.2 Implementation

具体函数见code里的README.md

1.3 Result



N	time	e_N	steps
64	0.14275	0.0085233	6
128	0.12871	0.0030007	6
256	0.57573	0.0010587	7
512	2.4236	0.00037395	7
1024	18.5366	0.00013215	7
2048	94.1432	4.6709e-05	7

由图易见$e_N = O(\frac{1}{N})$

2 Uzawa Iteration Method + CG

2.1 Algorithm

算法框架如下:
1) 每一轮开始时, 用共轭梯度法求解$AU_{k+1} = F - BP_k$
2) 然后对压力进行更新$P_{k+1} = P_k + \alpha(B^T U_{k+1})$
3) 判断误差是否小于允许的值, 从而判断是否回到第一步

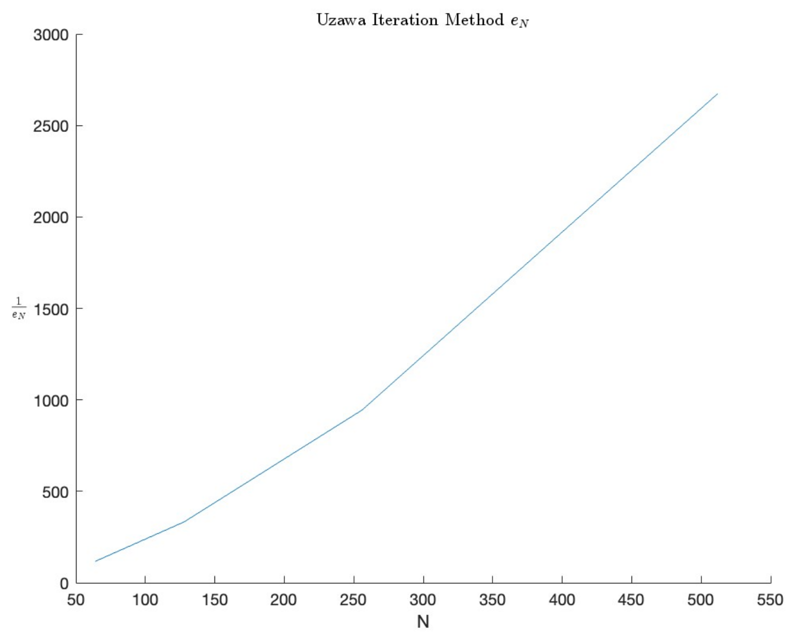
上面α选取为

$$\alpha_\star = \frac{2}{\lambda_{min}B^TA^{-1}B + \lambda_{max}B^TA^{-1}B}$$

2.2 Implementation

具体函数见code里的README.md

2.3 Result



N	time	e_N
64	0.13661	0.0085233
128	0.47766	0.0030007
256	4.7065	0.0010587
512	27.9452	0.00037395

由图易见$e_N = O(\frac{1}{N})$

3 Inexact Uzawa Iteration Method + Vcycle

3.1 Algorithm

算法框架如下:
1) 每一轮开始时, 用预条件共轭梯度法近似求解$AU_{k+1} = F - BP_k$, 使用$M =$

A作为预条件矩阵,求解预条件方程的时候使用Vcycle方法求解
2) 然后对压力进行更新$P_{k+1} = P_k + \alpha(B^T U_{k+1})$
3) 判断误差是否小于允许的值,从而判断是否回到第一步
上面α选取为

$$\alpha_\star = \frac{2}{\lambda_{min} B^T A^{-1} B + \lambda_{max} B^T A^{-1} B}$$

近似求解$AU_{k+1} = F - BP_k$得到的$\hat{U}_{k+1}$需要满足若定义

$$\delta_k = A\hat{U}_{k+1} - F + BP_k$$

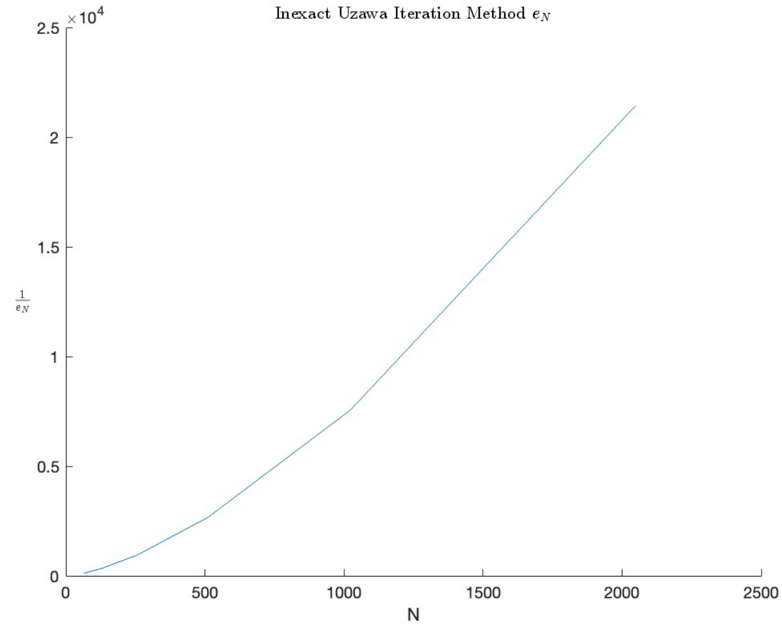
则

$$||\delta_k||_2 \leq \tau ||B^T \hat{U}_k||$$

3.2 Implementation

具体函数见code里的README.md

3.3 Result



N	time	e_N	steps
64	0.065143	0.0085233	4
128	0.25165	0.0030007	4
256	0.99247	0.0010587	4
512	4.0304	0.00037395	4
1024	24.4011	0.00013212	4
2048	129.3766	4.6674e-05	4

由图易见$e_N = O(\frac{1}{N})$