

PMATH 351: Real Analysis

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Assignments: 5, 5% each (May 24th, \dots , due at the beginning of the class)
Midterm: 20% final grade (Friday, June 27th)
Final: 55%

1 Axiom of Choice & Cardinality

1.1 Notation

- \mathbb{N} : set of natural numbers, $\{1, 2, 3, \dots\}$
- \mathbb{Z} : set of integers, $\{-3, -2, -1, 0, 1, 2, 3, \dots\}$
- \mathbb{Q} : set of rationals, $\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a, b) = 1\}$
- \mathbb{R} : set of reals
- $A \subset$ or $A \subseteq B$ (inclusion)
- $A \subsetneq B$ (proper inclusion)

Definition 1. • Let X be a set $P(X) = \{A | A \subset X\}$ is the power set of X .

- A, B sets. The union of A and B is $A \cup B = \{x | x \in A \text{ or } x \in B\}$. If $I \neq \emptyset$, $\{A_\alpha\}_{\alpha \in I}$ are sets, $A_\alpha \subseteq X, \forall \alpha$,

$$\bigcup_{\alpha \in I} A_\alpha = \{x | x \in A_\alpha \text{ for some } \alpha \in I\}$$

- Similarly for intersections
- Let $A, B \in X$, $B \setminus A = \{b \in B | b \notin A\}$. If $B = X$, $X \setminus A = A^C$ is the complement of A (in X). Note: $(A^C)^C = A, A^C = B^C \iff A = B$

Theorem 1. De Morgan's Laws:

$$1. (\bigcup_{\alpha \in I} A_\alpha)^C = \bigcap_{\alpha \in I} A_\alpha^C$$

$$\text{Proof. } x \in (\bigcup_{\alpha \in I} A_\alpha)^C \iff x \notin \bigcup_{\alpha \in I} A_\alpha \iff \forall \alpha \in I, x \notin A_\alpha \iff x \in \bigcap_{\alpha \in I} A_\alpha^C \quad \square$$

$$2. (\bigcap_{\alpha \in I} A_\alpha)^C = \bigcup_{\alpha \in I} A_\alpha^C$$

1.2 Products & Axiom of Choice

Definition 2. Let X, Y be sets. The product of X and Y is $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Let X_1, X_2, \dots, X_n be sets. The product of $\{X_1, X_2, \dots, X_n\}$ is

$$X_1 \times X_2 \cdots \times X_n = \prod_{i=1}^n X_i = \{(x_1, x_2, \dots, x_n) | x_i \in X_i, \forall i = 1, 2, \dots\}$$

An element (x_1, \dots, x_n) is called an n -tuple and x_i is called the i th coordinate.

Theorem 2. If $X_i = X, \forall i = 1, \dots, n, \prod_{i=1}^n X_i = X^n$. If X is a set, $|X|$ is the number of elements of X . If $\{X_1, \dots, X_n\}$ is a finite collection of sets

$$|\prod_{i=1}^n X_i| = \prod_{i=1}^n |X_i|$$

If $X_i = X, \forall i, |X^n| = |X|^n$

How do we define the product of an arbitrary family of sets? $(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$, then (x_1, x_2, \dots, x_n) determines a function

$$f_{(x_1, \dots, x_n)} : \{1, 2, \dots, n\} \rightarrow \bigcup_{i=1}^n X_i$$

i.e. $f_{(x_1, \dots, x_n)}(i) = x_i$

On the other hand, if we have a function

$$f : \{1, 2, 3, \dots, n\} \rightarrow \bigcup_{i=1}^n X_i$$

with $f(i) \in X_i$. We define $(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$ by

$$x_i \in X_i = f(i), \forall i = 1, \dots, n$$

$$\prod_{i=1}^n X_i = \{f : \{1, 2, \dots, n\} \rightarrow \bigcup_{i=1}^n X_i | f(i) \in X_i\}$$

Definition 3. Given a collection $\{X_\alpha\}_{\alpha \in I}$ of sets, we define

$$\prod_{\alpha \in I} X_\alpha := \{f : I \rightarrow \bigcup_{\alpha \in I} X_\alpha | f(\alpha) \in X_\alpha\}$$

Axiom 1. Zermelo's Axiom of Choice. Given a non-empty collection $\{X_\alpha\}_{\alpha \in I}$ of non-empty sets, $\prod_{\alpha \in I} X_\alpha \neq \emptyset$.

Axiom 2. Axiom of Choice: Given a non-empty set X , there exists a function $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ for every $A \subseteq X, A \neq \emptyset, f(A) \in A$

1.3 Relations and Zorn's Lemma

Definition 4. X, Y are sets A relation is a subset of $X \times Y$. We write xRy if $(x, y) \in R$.

1. Reflexive if $xRx, \forall x \in X$
2. Symmetric if $xRy \implies yRx$
3. Anti-symmetric xRy and $yRx \implies x = y$
4. Transitive if xRy and $yRz \implies xRz$

Example:

1. $x = \mathbb{R}, xRy \iff x \subseteq y$. It is reflexive, antisymmetric, transitive.
2. X set. We define a relation on $\mathcal{P}(X)$. $ARB \iff A \subseteq B$
3. R^* relation on $\mathcal{P}(x)$. $ARB \iff A \supseteq B$

Definition 5. A relation R on a set X is a partial order if it is reflexive, anti-symmetric and transitive. (X, R) is a partially order set or poset.

A partial relation R on X is a total order if $\forall x, y \in X$, either xRy or yRx . (X, R) is a totally order set or a chain.

Definition 6. (X, \leq) poset. Let $A \in X$. $x \in X$ is an upper bound for A if $a \leq x, \forall a \in A$. A is bounded above if it has an upper bound. $x \in X$ is the least upper bound (or supremum) for A if x is an upper bound and y is an upper bound, then $x \leq y$. $x = \text{lub}(A) = \text{sup}(A)$. If $x = \text{lub}(A)$ and $x \in A \implies x = \text{max}(A)$ is the maximum of A .

Axiom 3. Least Upper bound axiom for \mathbb{R} : Consider \mathbb{R} with usual order \leq . $A \subseteq \mathbb{R}, A \neq \emptyset$. If A is bounded above, the A has a least upper bound.

Example

1. $(\mathcal{P}(X), \subseteq), \{A_\alpha\}_{\alpha \in I}, A_\alpha \subseteq X, A_\alpha \neq \emptyset$. X is an upper bound for $\{A_\alpha\}_{\alpha \in I}$. \emptyset is a lower bound, $\text{lub}(\{A_\alpha\}_{\alpha \in I}) = \bigcup_{\alpha \in I} A_\alpha$, and $\text{glb}(\{A_\alpha\}_{\alpha \in I}) = \bigcap_{\alpha \in I} A_\alpha$
2. $(\mathcal{P}(X), \supseteq)$

Definition 7. (X, \leq) poset, $x \in X$ is maximal if $x \leq y$ implies $x = y$.

- (\mathbb{R}, \leq) has no maximal element
- $(\mathcal{P}(X), \leq) \implies X$ is maximal.
- $(\mathcal{P}(X), \geq) \implies \emptyset$ is maximal.

Proposition 1. Every finite, non-empty poset has a maximal element but there are poset with no maximal element.

Lemma 1. *Zorn's Lemma:* (X, \leq) non-empty poset. If every totally order subset \mathcal{C} of X has an upper bound, then (X, \leq) has a maximal element. Let \mathcal{V} be a non-zero vector space. Let $\mathcal{L} = \{A \subseteq \mathcal{V} | A \text{ is linearly independent}\}$.

Note: A basis B for \mathcal{V} is a maximal element on (\mathcal{L}, \leq) .

Theorem 3. Every non-zero vector space \mathcal{V} has a basis.

Proof. Let $\mathcal{C} = \{A_\alpha | \alpha \in I\}$ be a chain in \mathcal{L} .

Let $A = \bigcup_{\alpha \in I} A_\alpha$. Claim: A is linearly independent. Let $\{x_1, x_2, \dots, x_n\} \subseteq A$, $\{\beta_1, \beta_2, \dots, \beta_n\} \subseteq \mathbb{R}$. Then $\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n = 0$.

For each $i = 1, 2, \dots, n$, $\exists \alpha_i | x_i \in A_{\alpha_i}$.

Assume, $A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \dots \subseteq A_{\alpha_n}$ (\mathcal{C} is a chain, change name of index if needed). Therefore, $\{x_1, x_2, \dots, x_n\} \subseteq A_{\alpha_n}$ and A_{α_n} is linearly independent. Hence, $\{x_1, x_2, \dots, x_n\}$ is linearly independent. Lastly, $\beta_i = 0, \forall i$. Then A is linearly independent. A is an upper bound for \mathcal{C} on \mathcal{L} . By Zorn's lemma, \mathcal{L} has a maximal element. \square

Definition 8. A poset (X, \leq) is well-ordered, if every non-empty subset A has a least element.

Examples

- (\mathbb{N}, \leq) is well-ordered.
- $\mathbb{Q} = \{\frac{n}{m} | n \in \mathbb{Z}, m \in \mathbb{N}, \gcd(n, m) = 1\}$.

We can construct a well-order on \mathbb{Q} . $\phi : \mathbb{Q} \rightarrow \mathbb{N}$ by $\phi(\frac{n}{m}) = \begin{cases} 2^n 3^m & n > 0 \\ 1 & n = 0 \\ 5^{-n} 7^m & n < 0 \end{cases}$. ϕ is

1-to-1. $\frac{n}{m} \leq \frac{p}{q} \iff \phi(\frac{n}{m}) \leq \phi(\frac{p}{q})$
 (\mathbb{Q}, \leq) is well-ordered.

Axiom 4. *Well-ordering principle:* Given any set $X \neq \emptyset$, there exists a partial order \leq such that (X, \leq) is well-ordered.

Theorem 4. *TFAE:*

1. Axiom of Choice
2. Zorn's lemma
3. Well-ordering principle

1.4 Equivalence Relations & Cardinality

Definition 9. A relation \sim on a set X is an equivalence relation if

1. Reflexive
2. Symmetric
3. Transitive

Given $x \in X$, let $[x] = \{y \in X | x \sim y\}$ be the equivalence class of x .

Proposition 2. Let \sim be an equivalence relation on X

1. $[x] \neq \emptyset, \forall x \in X$
2. For each $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.
3. $X = \bigcup_{x \in X} [x]$

Definition 10. If X is a set, a partition of X is a collection $\mathcal{P} = \{A_\alpha \subseteq X | \alpha \in I\}$.

1. $A_\alpha \neq \emptyset, \forall \alpha$.
2. If $\beta \neq \alpha \implies A_\alpha \cap A_\beta = \emptyset$
3. $X = \bigcup_{\alpha \in I} A_\alpha$.

Note:

Given \sim on $X \implies \sim$ induces a partition on X . Given a partition on X ($\mathcal{P} = \{A_\alpha | \alpha \in I\}$) we define an equivalence relation on X :

$$x \sim y \iff x, y \in A_\alpha, \text{ for some } \alpha$$

Example: Define \sim on $\mathcal{P}(X)$ by $A \sim B \iff \exists$ a 1-to-1 and onto function $f : A \rightarrow B$. \sim is an equivalence relation.

Definition 11. Two sets X and Y are equivalent if there exists a 1-to-1 and onto function $f : X \rightarrow Y$. In this case, we write $X \sim Y$. We say that X and Y have the same cardinality, $|X| = |Y|$.

A set X is finite if $X = \{1, 2, \dots, n\}$ or $X \sim \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, $|X| = n$. Otherwise, X is infinite.

Can X be equivalent to both $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\}$, with $n \neq m$? If $X \sim \{1, 2, \dots, n\}$ and $X \sim \{1, 2, \dots, m\} \implies \{1, 2, \dots, n\} \sim \{1, 2, \dots, m\}$.

Proposition 3. The set $\{1, 2, \dots, m\}$ is not equivalent to any proper subset of itself.

Proof. Induction on m

$m = 1$: The only proper subset of $\{1\}$ is \emptyset . and $\{1\} \sim \emptyset$.

$m = k$ Statement holds for $\{1, 2, \dots, k\}$. Assume $\exists S \subsetneq \{1, 2, \dots, k, k+1\}$ and $f : \{1, 2, \dots, k+1\} \rightarrow S$, 1-to-1 and onto.

Two cases:

1. If $k+1 \in S \implies f_{\{1,2,\dots,k\}} : \{1, 2, \dots, k\} \rightarrow S \setminus \{f(k+1)\} \subsetneq \{1, 2, \dots, k\}$. This is impossible.

2. If $k+1 \in S$, $f(k+1) = k+1$, then $f_{\{1,2,\dots,k\}} : \{1, 2, \dots, k\} \rightarrow S \setminus \{k+1\} \subsetneq \{1, 2, \dots, k\}$. This is impossible.

If $f(k+1) = j$ and $f(i) = k+1$. Define $f^* : \{1, 2, \dots, k+1\} \rightarrow S$, $f^*(l) = \begin{cases} k+1 & l = k+1 \\ j & l = i \\ f(l) & \text{otherwise} \end{cases}$. This is impossible

□

Corollary 1. *If X is finite, then X is not equivalent to any proper subset of itself.*

Example:

$f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{1\} = n \rightarrow n+1$ is 1-to-1 and onto. Hence $\mathbb{N} \sim \mathbb{N} \setminus \{1\}$.

Definition 12. *A set X is countable if X is finite or $X \sim \mathbb{N}$. Otherwise, uncountable. X is countable infinite if $X \sim \mathbb{N}$, $|X| = |\mathbb{N}| = \aleph_0$*

Proposition 4. Every infinite set contains a countable infinite subsets.

Proof. By Axiom of Choice, $\exists f : \mathcal{P}(X) \setminus \emptyset \rightarrow X$, $f(A) \in A$. $x_1 = f(X)$ and $x_2 = f(X \setminus \{x_1\}) \dots x_{n+1} = f(X \setminus \{x_1, x_2, \dots, x_n\})$

$A = \{x_1, x_2, \dots, x_{n+1}, \dots\} \subseteq X$ is countable infinite. □

Corollary 2. *A set X is infinite if and only if it is equivalent to a proper subset of itself.*

Theorem 5. (Cantor-Schroeder-Berstein) (CSB) Assume that $A_2 \subseteq A_1 \subseteq A_0$. If $A_2 \sim A_0$, then $A_1 \sim A_0$.

Corollary 3. Assume $A_1 \subseteq A$ and $B_1 \subseteq B$. If $A \sim B_1$ and $B \sim A_1$, then $A \sim B$. $f : A \rightarrow B_1$ is 1-to-1 and onto and $g : B \rightarrow A_1$ is 1-to-1 and onto. $A_2 = g(f(A)) = g(B) \subseteq A_1 \subseteq A$ and $g \circ f$ is 1-to-1 and onto on A_2 . Hence $A_2 \sim A \xrightarrow{CSB} A_1 \sim A$ and $A_1 \sim B$. Hence $A \sim B$.

Corollary 4. *An infinite set X is countable infinite if and only if there exists a 1-to-1 function $f : X \rightarrow \mathbb{N}$.*

Proposition 5. Assume there exists $g : X \rightarrow Y$ onto. Then there exists a 1-to-1 function $f : Y \rightarrow X$.

Proof. By axiom of choice, $\exists h : \mathcal{P}(x) \setminus \emptyset \rightarrow X, h(A) \in A, A \neq \emptyset, A \subseteq X$. $\forall y \in Y$, define $f(y) = h(g^{-1}(\{y\})) \in X$. $f : Y \rightarrow X$. Check f is 1-to-1. \square

Corollary 5. X, Y sets. *TFAE*

1. $\exists f : X \rightarrow Y$, 1-to-1
2. $\exists g : Y \rightarrow X$, is onto
3. $|Y| \succeq |X|$

Theorem 6. $[0, 1]$ is uncountable.

Proof. Assume $[0, 1]$ is countable

$$[0, 1] = \{a_1, a_2, \dots, a_n, \dots\}$$

each real number has a unique decimal expansion if we don't allow $.99\bar{9}$ (∞ times 9)

$$a_1 = 0.a_{11}a_{12}a_{13} \dots$$

$$a_2 = 0.a_{21}a_{22}a_{23} \dots$$

$$a_3 = 0.a_{31}a_{32}a_{33} \dots$$

$$\vdots$$

Let $b \in [0, 1)$, $b = 0.b_1b_2 \dots$ where $b_n := \begin{cases} 1 & a_{nn} \neq 1 \\ 2 & a_{nn} = 1 \end{cases}$ Well, $b \neq a_n, \forall n$. It is impossible.

Then $[0, 1]$ is uncountable. \square

Corollary 6. \mathbb{R} is uncountable. $\mathbb{R} \sim (0, 1)$. Note $|\mathbb{R}| = c$.

Theorem 7. Comparability theorem for cardinals: Given X, Y sets, either $|X| \preceq |Y|$ or $|Y| \preceq |X|$.

1.5 Cardinal Arithmetic

1.5.1 Sums of Cardinals

Definition 13. Let X, Y be disjoint sets, then

$$|X| + |Y| = |X \cup Y|$$

Examples

1. $X = \{1, 3, 5, \dots\}, Y = \{2, 4, 6, \dots\}$. $|X| + |Y| = \aleph_0 + \aleph_0 = \aleph_0$.

Theorem 8. If X is infinite, then

$$|X| + |Y| = \max\{|X|, |Y|\}$$

In particular,

$$|X| + |X| = |X|$$

X_1, \dots, X_n countable sets. Then $|\bigcup_{i=1}^n X_i| = \aleph_0$.

Theorem 9. $\{X_i\}_{i=1}^\infty$ countable collection of countable sets, then $X = \bigcup_{i=1}^\infty X_i$ is countable.

Note: we can assume $X_i \cap X_j = \emptyset$ if $i \neq j$. Otherwise, let $E_1 = X_1, E_2 = X_2 \setminus X_1, \dots, E_n = X_n \setminus \bigcup_{i=1}^{n-1} X_i$. Assume $\{X_i\}_{i=1}^\infty$ is pairwise disjoint if $X_i \neq \emptyset$, let $X_i = \{x_{i1}, x_{i2}, \dots\}$ countable. Let $f : X = \bigcup_{i=1}^\infty X_i \rightarrow \mathbb{N}$ 1-to-1 such that $f(x_{ij}) = 2^i 3^j$.

1.5.2 Product of cardinals

Let X, Y be two sets

$$|X| \cdot |Y| = |X \times Y|$$

Theorem 10. If X is infinite and $Y \neq \emptyset$, then

$$|X| \cdot |Y| = \max\{|X|, |Y|\}$$

In particular,

$$|X| \cdot |X| = |X|$$

1.5.3 Exponentiation of Cardinals

Recall: Given a collection $\{Y_x\}_{x \in X}$ of non-empty sets, we defined

$$\prod_{x \in X} Y_x = \{f : X \rightarrow \bigcup_{x \in X} Y_x \mid f(x) \in Y_x\}$$

If $\forall x \in X, Y_x = Y$ for some set Y , $Y^X = \prod_{x \in X} Y_x = \prod_{x \in X} Y = \{f : X \rightarrow Y\}$.

Definition 14. Let X, Y non empty sets, we define

$$|Y|^{|X|} = |Y^X|$$

Theorem 11. X, Y, Z non-empty sets.

1. $|Y|^{|X|}|Y|^{|Z|} = |Y|^{|X|+|Z|}$
2. $(|Y|^{|X|})^{|Z|} = |Y|^{|X|+|Z|}$

Example ($2^{\aleph_0} = c$) $2^{\aleph_0} = |\{0, 1\}^{\mathbb{N}}| = |\{\{a_n\}_{n \in \mathbb{N}} | a_n = 0 \text{ or } a_n = 1\}|$.

$2^{\aleph_0} \preceq c$: $f: \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$ is 1-to-1 such that $\{a_n\} \rightarrow \sum_{n=1}^{\infty} \frac{a_n}{3^n}$.

$2^{\aleph_0} \succeq c$: $g: [0, 1] \rightarrow \{0, 1\}^{\mathbb{N}}$ is 1-to-1. $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \rightarrow \{a_n\}$.

Hence done.

Given a set X , we want to find $|\mathcal{P}(X)| = 2^{|X|}$.

Let $A \subseteq X$, $\chi_A: X \rightarrow \{0, 1\}$, such that $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. This is called character-

istics function of A . $\chi_A \in \{0, 1\}^X$. If $f \in \{0, 1\}^X$, $A = \{x \in X | f(x) = 1\}$. Hence $\chi_A = f$. Let $\Gamma: \mathcal{P}(X) \rightarrow \{0, 1\}^X$. Hence Γ is a bijection. Therefore $|\mathcal{P}(X)| = 2^{|X|}$.

Theorem 12. $|\mathcal{P}(X)| > |X|$ for any $X \neq \emptyset$ (Russel's Paradox)

It is enough to show that there is no onto function $X \rightarrow \mathcal{P}(X)$. Assume to the contrary: there exists $f: X \rightarrow \mathcal{P}(X)$ onto.

$A = \{x \in X | x \notin f(x)\}$. $\exists x_0 \in X | f(x_0) = A$. If $x_0 \in A$: $\implies x_0 \notin f(x_0) = A$. Impossible. If $x_0 \notin A$: $\implies x_0 \in f(x_0) = A$. OK

2 Metric spaces

Definition 15. Let $X \neq \emptyset$. A metric on X is a function $d: X \times X \rightarrow \mathbb{R}$.

1. $d(x, y) \geq 0, \forall x, y \in X$. $d(x, y) = 0 \implies x = y$.
2. $d(x, y) = d(y, x), \forall x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

(X, d) is a metric space.

Examples

1. $X = \mathbb{R}$ $d(x, y) = |x - y|$ "usual metric on \mathbb{R} "
2. X any non-empty set $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ "discrete metric"

3. $X = \mathbb{R}^n$. $d_2((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. d_2 verifies 1), 2). This is called “Euclidean Metric”.

Definition 16. Let V be a vector space. A norm on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

1. $\|x\| \geq 0, \forall x \in V$. $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, \forall x \in V$.
3. $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$

$(V, \|\cdot\|)$ is normed vector space.

Remark: $(V, \|\cdot\|)$ normed vector space. $\|\cdot\|$ induces a metric on V . $d_{\|\cdot\|}(x, y) = \|x - y\|$

1. $d_{\|\cdot\|}(x, y) = \|x - y\| \geq 0, \forall x, y \in V$. $\|x - y\| = 0 \implies x = y$.
2. $d_{\|\cdot\|}(x, y) = \|x - y\| = |-1| \|y - x\| = d_{\|\cdot\|}(y, x)$
3. $d_{\|\cdot\|}(x, y) = \|x - y\| \leq \|x - z\| + \|z - y\|$

Examples

1. $X = \mathbb{R}^n$, $\|(x_1, \dots, x_n)\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. $d_{\|\cdot\|_2} = d_2$. This is a 2-norm or Euclidean norm.
2. $X = \mathbb{R}^n$, $1 < p < \infty$. $\|(x_1, x_2, \dots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ This is called p-norm.
3. $X = \mathbb{R}^n$, $\|(x_1, \dots, x_n)\|_\infty = \max\{|x_i|\}$. This is called ∞ -norm.
4. $\|(x_1, \dots, x_n)\|_1 = \sum_{i=1}^n |x_i|$. This is called 1-norm.

Remark: Let $p, 1 < p < \infty$, and $q, \frac{1}{p} + \frac{1}{q} = 1$. Then $1 + \frac{p}{q} = p \implies \frac{p}{q} = p - 1 \implies \frac{p}{p-1} = q \implies \frac{q}{p} = q - 1 \implies \frac{1}{p-1} = \frac{q}{p} = q - 1$.

Lemma 2. Let $\alpha, \beta > 0, 1 < p < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$, then $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ (Young's inequality)

$$u = t^{p-1} \implies t = u^{\frac{1}{p-1}} = u^{q-1}. \alpha\beta \leq \int_0^\alpha t^{p-1} dt + \int_0^\beta u^{q-1} du = \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

Theorem 13. Hölder's Inequality: Let (a_1, \dots, a_n) and $(b_1, \dots, b_n) \in \mathbb{R}^n$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}$$

Proof. Assume $a \neq 0 \neq b$.

Note: $\alpha, \beta > 0$,

$$\begin{aligned}\sum_{i=1}^n |(\alpha a_i)(\beta b_i)| &= \alpha\beta \sum_{i=1}^n |a_i b_i| \\ \left(\sum_{i=1}^n |\alpha a_i|^p\right)^{1/p} &= \alpha \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \\ \left(\sum_{i=1}^n |\beta b_i|^q\right)^{1/q} &= \beta \left(\sum_{i=1}^n |b_i|^q\right)^{1/q}\end{aligned}$$

Then the inequality holds for $a, b \in \mathbb{R}^n \iff$ it holds for $\alpha a, \beta b \in \mathbb{R}^n$ for some $\alpha\beta > 0$.
By scaling if needed, we can assume

$$\left(\sum_{i=1}^n |a_i|^p\right)^{1/p} = 1, \left(\sum_{i=1}^n |b_i|^q\right)^{1/q} = 1$$

Lemma 3.

$$|a_i b_i| \leq \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q}, \forall i = 1, \dots, n$$

$$\text{Hence } \sum_{i=1}^n |a_i b_i| \leq \frac{\sum_{i=1}^n |a_i|^p}{p} + \frac{\sum_{i=1}^n |b_i|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 \quad \square$$

Theorem 14. *Minkowski's Inequality:* Let $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$.
Let $1 < p < \infty$, then

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p\right)^{1/p}$$

Proof. Assume $a \neq 0 \neq b$. Let $q/\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned}
\sum_{i=1}^n |a_i + b_i|^p &= \sum_{i=1}^n |a_i + b_i| |a_i + b_i|^{p-1} \\
&\leq \sum_{i=1}^n |a_i| |a_i + b_i|^{p-1} + \sum_{i=1}^n |b_i| |a_i + b_i|^{p-1} \\
\sum_{i=1}^n |a_i| |a_i + b_i|^{p-1} &\leq \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n (|a_i + b_i|^{p-1})^q \right)^{1/q} = \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1/q} \\
\text{Similarly, } \sum_{i=1}^n |b_i| |a_i + b_i|^{p-1} &\leq \left(\sum_{i=1}^n |b_i|^p \right)^{1/p} \left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1/q} \\
\sum_{i=1}^n |a_i + b_i|^p &\leq \left(\left(\sum_{i=1}^n |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |b_i|^p \right)^{1/p} \right) \left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1/q} \\
\left(\sum_{i=1}^n |a_i + b_i|^p \right)^{1-1/p} &\leq \|a\|_p + \|b\|_p
\end{aligned}$$

□

Examples: sequence space

1. Let $l_1 = \{\{x_n\} | \sum_{i=1}^{\infty} |x_n| < \infty\}$ Then $\|\{x_n\}\|_1 = \sum_{i=1}^{\infty} |x_n|$. Let $\{x_n\}, \{y_n\} \in l_1$. Claim that $\{x_n + y_n\} \in l_1$. Let $k \in \mathbb{N}$

$$\sum_{n=1}^k |x_n + y_n| \leq \sum_{n=1}^k |x_n| + \sum_{n=1}^k |y_n| \leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| < \infty$$

By MCT, $\{\sum_{i=1}^k |x_n + y_n|\}$ convergent then $\sum_{n=1}^{\infty} |x_n + y_n|$ convergent. Hence $\{x_n + y_n\} \in l_1$.

Moreover,

$$\|\{x_n + y_n\}\|_1 \leq \|\{x_n\}\|_1 + \|\{y_n\}\|_1$$

This implies $\|\cdot\|_1$ is a norm.

2. Let $1 < p < \infty$,

$$l_p = \{\{x_n\} | \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

$\|\{x_n\}\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ Prove that $\{x_n\}, \{y_n\} \in l_p$ and then $\{x_n + y_n\} \in l_p$ and $\|\cdot\|_p$ is norm.

3. $l_\infty = \{\{x_n\} | \sup\{|x_n|\} < \infty\}$. $\|\{x_n\}\|_\infty = \sup\{|x_n|\}$. This is a norm.

Examples Continuous function space

1. $C([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} | f \text{ is continuous}\}$. $\|f\|_\infty = \max\{|f(x)| | x \in [a, b]\}$. Let $f, g \in C([a, b])$, $x \in [a, b]$.

$$|(f+g)(x)| = |f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \sup_{x \in [a, b]} |f(x)| + \max_{x \in [a, b]} |g(x)| = \|f\|_\infty + \|g\|_\infty$$

$$\|f+g\|_\infty = \max_{x \in [a, b]} |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

2. $C([a, b])$, $\|f\|_1 = \int_a^b |f(t)| dt$.

3. $C([a, b])$, $\|f\|_p = (\int_a^b |f(t)|^p dt)^{1/p}$

Theorem 15. *Holder's inequality II: Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g \in C[a, b]$.*

$$\int_a^b |f(t)g(t)| dt \leq (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |g(t)|^q dt)^{1/q}$$

Theorem 16. *Minkowski's Inequality II: If $f, g \in C([a, b])$ and $1 < p < \infty$*

$$(\int_a^b |(f+g)(t)|^p dt)^{1/p} \leq (\int_a^b |f(t)|^p dt)^{1/p} + (\int_a^b |g(t)|^p dt)^{1/p}$$

Then $f \neq 0 \neq g$.

Proof.

$$\begin{aligned} \int_a^b |f(t) + g(t)|^p dt &= \int_a^b |(f+g)(t)| |(f+g)(t)|^{p-1} dt \\ &\leq \int_a^b |f(t)| |(f+g)(t)|^{p-1} dt + \int_a^b |g(t)| |(f+g)(t)|^{p-1} dt \\ &\leq (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |f(t) + g(t)|^{(p-1)q} dt)^{1/q} \\ &\quad + (\int_a^b |g(t)|^p dt)^{1/p} (\int_a^b |f(t) + g(t)|^{(p-1)q} dt)^{1/q} \\ \int_a^b |f(t) + g(t)|^p dt &\leq [(\int_a^b |f(t)|^p dt)^{1/p} + (\int_a^b |g(t)|^p dt)^{1/p}] (\int_a^b |f(t) + g(t)|^p dt)^{1/q} \\ (\int_a^b |f(t) + g(t)|^p dt)^{1-1/q} &\leq \|f\|_p + \|g\|_p \end{aligned}$$

□

Example: Bounded operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \rightarrow Y$, linear. $\|T\| := \sup\{\|T(x)\|_Y \mid \|x\|_X \leq 1, x \in X\}$. $B(X, Y) = \{T : X \rightarrow Y \text{ linear} \mid \|T\| < \infty\}$.

Claim: $B(X, Y)$ is a vector space and $\|\cdot\|$ is a norm.

- $T, S \in B(X, Y) \implies T + S \in B(X, Y), x \in X, \|x\|_X \leq 1$.

$$\begin{aligned} \|(T + S)(x)\|_Y &= \|T(x) + S(x)\|_Y \\ &\leq \|T(x)\|_Y + \|S(x)\|_Y \\ &\leq \|T\| + \|S\| \end{aligned}$$

$$\begin{aligned} \|T + S\| &= \sup\|(T + S)(x)\| \leq \|T\| + \|S\| < \infty, x \in X, \|x\|_X \leq 1 \\ \implies T + S &\in B(X, Y) \text{ and } \|T + S\| \leq \|T\| + \|S\| \end{aligned}$$

- $\alpha \in \mathbb{R}, T \in B(X, Y)$

$$\begin{aligned} \|\alpha T\| &= \sup_{x \in X, \|x\|_X \leq 1} \|\alpha T(x)\|_Y = |\alpha| \sup_{x \in X, \|x\|_X \leq 1} \|T(x)\|_Y = |\alpha| \|T\| < \infty \\ \implies \alpha T &\in B(X, Y) \text{ and } \|\alpha T\| = |\alpha| \|T\| \end{aligned}$$

Note $B(X, Y) \leq \mathcal{L}(X, Y)$, $0 \in B(X, Y) \implies B(X, Y)$ subspace of $\mathcal{L}(X, Y)$. $\|T\| \geq 0$ and $\|T\| = 0 \iff \|T(x)\|_Y = 0, \forall x \in X, \|x\|_X \leq 1$.

2.1 Topology of Metric Spaces

Definition 17. Let (X, d) be a metric space. Let $x_0 \in X$ and $\epsilon > 0$. The open ball centered at x_0 with radius ϵ is

$$B(x_0, \epsilon) = \{x \in X \mid d(x, x_0) < \epsilon\}$$

The closed ball centered at x_0 with radius ϵ is

$$B[x_0, \epsilon] = \{x \in X \mid d(x, x_0) \leq \epsilon\}$$

A subset $U \subseteq X$ is open if $\forall x \in U, \exists \epsilon > 0 \mid B(x, \epsilon) \subseteq U$. A subset $F \subseteq X$ is closed if F^C is open.

Proposition 6. Let (X, d) be a metric space. Then

1. X, \emptyset are open.
2. If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then the union of all the sets in this collection is open.
3. If $\{U_1, U_2, \dots, U_n\}$ are open, then $\cap_{i=1}^n U_i$ is open.

Example

1. If $x \in X$, any $\epsilon > 0$, $B(x, \epsilon) \subseteq X \implies X$ is open. \emptyset is “trivially” open.
2. If $x \in \cup_{\alpha \in I} U_\alpha$, then $\exists \alpha \in I$ such that $x \in U_\alpha$. Since U_α is an open set and $x \in U_\alpha$, $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq U_\alpha \subseteq \cup_{\alpha \in I} U_\alpha \implies \cup_{\alpha \in I} U_\alpha$ is open.
3. If $x \in \cap_{i=1}^n U_i$, $\forall i \in \{1, \dots, n\}$, $\exists \epsilon_i > 0$ such that $B(x, \epsilon_i) \subseteq U_i$, let $\epsilon = \min\{\epsilon_i \mid i = 1, \dots, n\} > 0$, $B(x, \epsilon) \subseteq B(x, \epsilon_i), \forall i \implies B(x, \epsilon) \subseteq \cap_{i=1}^n B(x, \epsilon_i) \subseteq \cap_{i=1}^n U_i$.

Proposition 7. Let (X, d) be a metric space. Then

1. X, \emptyset are closed
2. If $\{F_\alpha\}_{\alpha \in I}$ is addition of close sets, then $\cap_{\alpha \in I} F_\alpha$ is closed
3. If F_1, \dots, F_n are closed sets, then the union is also closed.

From this proposition, it flows that if (X, d) is a metric space. $\tau_j = \{U \subseteq X \mid U \text{ is open with respect to } d\}$. τ_j is a topology.

Proposition 8. Let (X, d) be a metric space, then

1. If $x_0 \in X, \epsilon > 0 \implies B(x_0, \epsilon)$ is open
2. $U \subseteq X$ is open $\iff U$ is the union of open balls
3. If $x_0 \in X, \epsilon > 0 \implies B[x_0, \epsilon]$ is closed
4. If $x \in X, \{x\}$ is closed. Every finite subset is closed.

Proof. 1. Let $x \in B(x_0, \epsilon)$, then $d(x, x_0) = \delta < \epsilon$. Let $\epsilon' = \epsilon - \delta$. Claim $B(x, \epsilon') \subseteq B(x_0, \epsilon)$.
Let $x \in B(x, \epsilon')$ and $d(x_0, z) \leq d(x_0, x) + d(x, z) < \epsilon + \epsilon - \delta = \epsilon$. This proves that $B(x_0, \epsilon)$ is open.

2. \implies follows (1). \rightarrow If $x \in U$, $\exists \epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$, $\cup_{x \in U} B(x, \epsilon_x) = U$.

3. Let $x \in (B[x_0, \epsilon])^C$. $d(x, x_0) = \delta > \epsilon$. Let $\epsilon' = \delta - \epsilon$. Claim $B(x, \epsilon') \subseteq (B[x_0, \epsilon])^C$. Let $z \in B(x, \epsilon')$ assume $z \in B[x_0, \epsilon]$, $d(x, x_0) \leq d(x, z) + d(z, x_0) < \epsilon' + \epsilon = \delta - \epsilon + \epsilon = \delta$. This implies $z \in (B[x_0, \epsilon])^C$.

4. If $y \in \{x\}^C$, then $y \neq x$ and $d(y, x) > 0$ and $B(y, d(x, y)) \implies \{x\}^C$ is open. □

Open sets in \mathbb{R} .

Recall $I \subseteq \mathbb{R}$ is an interval if $x, y \in I$ and z such that $x < z < y \implies z \in I$.

- Open finite intervals (a, b)

- Closed finite intervals $[a, b]$.
- Half open finite set $(a, b]$.
- Open rays $(a,)$
- Closed rays

Example: Cantor set

P_n is obtained from P_{n-1} by removing the open interval of length $1/3^n$ from the middle third of each of the 2^{n-1} subintervals of P_{n-1} . Each P_n is closed. It's the union of 2^n closed intervals of length $1/3^n$.

$$P = \bigcap_{n=1}^{\infty} P_n \text{ Cantor (ternary) set}$$

- P is closed
- P is uncountable ($x \in P \rightarrow x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0, 2$).
- P contains no interval of positive length

Example: Discrete metric

X set, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ $x \in X, B(x, 2) = X, B(x, 1) = \{x\}$ is an open set.

If $U = X, U = \bigcup_{x \in U} \{x\} = \bigcup_{x \in U} B(x, 1)$ open. U is also closed.

2.2 Boundaries, interiors and closures

Definition 18. Let (X, d) metric space,

1. $A \subseteq \implies$ The closure of A is

$$A = \bigcap \{F \text{ closed in } X \mid A \subseteq F\}$$

It's the smallest closed set that contains A .

2. The interior of A is

$$\text{int}(A) = \bigcup \{U \text{ open in } X \mid U \subseteq A\}$$

It is the largest open set inside A .

3. Let $x \in X, N \subseteq X$, we say that N is a neighborhood of x ($N \subset \mathcal{N}_x$). If $x \in \text{int}(N)$.

4. Given $A \subseteq X, x \in X$ is a boundary point of A . If for every neighbor N of x , we have $N \cap A \neq \emptyset$ and $N \cap A^C \neq \emptyset$. Equivalently, x is a boundary point of A , if $\forall \epsilon > 0, B(x, \epsilon) \cap A \neq \emptyset$ and $B(x, \epsilon) \cap A^C \neq \emptyset$.

$$(\partial A)bdy(A) = \{x \in X | x \text{ is a boundary point of } A\}$$

Proposition 9. (X, d) metric space, $A \subseteq X$

1. A is closed $\iff bdy(A) \subseteq A$
2. $\bar{A} = A \cup bdy(A)$.

Proof. 1. (\implies) A is closed if and only if A^C is open. If $x \in A^C$, $\exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq A^C$ and then $B(x, \epsilon) \cap A = \emptyset \implies x \notin bdy(A)$.

\leftarrow Let $x \in A^C$, then $x \notin bdy(A)$. This implies $\exists \epsilon > 0$ such that $B(x, \epsilon) \cap A = \emptyset$. This implies $B(x, \epsilon) \subseteq A^C$. By definition, A^C is open.

2. Claim that $bdy(A) \subseteq \bar{A}$. Let $x \in (\bar{A})^C$. There exists $\exists \epsilon > 0$ such that $B(x, \epsilon) \cap \bar{A} = \emptyset$. This implies that $B(x, \epsilon) \cap A = \emptyset \implies x \notin bdy(A)$. This implies $F = bdy(A) \cup A \subseteq \bar{A}$. Claim that F is closed.

□

Definition 19. Let (X, d) metric space, $A \subseteq X$ and $x \in X$. We say that x is a limit point of A , if for all neighborhood N of x , we have $N \cap (A \setminus \{x\}) \neq \emptyset$. Equivalently, $\forall \epsilon > 0, B(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$. The set of limit points of A is $Lim(A)$ cluster points.

Note: $A = [0, 1] \subseteq \mathbb{R}, bdy(A) = \{0, 1\}, Lim(A) = A$. For $B = \{x\} \subseteq \mathbb{R}, bdy(B) = B, Lim(B) = \emptyset$.

Proposition 10. Let (X, d) metric space, $A \subseteq X$

1. A is closed $\iff Lim(A) \subseteq A$
2. $\bar{A} = A \cup Lim(A)$.

Proposition 11. 1. $\bar{A} \subseteq \bar{B}$.

2. $int(A) \subseteq int(B)$.
3. $int(A) = A \setminus bdy(A)$.

Proposition 12. Let $A, B \subseteq (X, d)$ metric space.

1. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
2. $int(A \cup B) = int(A) \cup int(B)$

Proof. 1. $A \cup B \subseteq \bar{A} \cup \bar{B}$. Hence, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

Conversely, $A \subseteq \overline{A \cup B} \implies \bar{A} \subseteq \overline{A \cup B}$. Similarly for B.

2. $\text{int}(A) \cap \text{int}(B) \subseteq A \cap B$. and $\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$.

Conversely, $\text{int}(A \cap B) \subseteq A \implies \text{int}(A \cap B) \subseteq \text{int}(A)$. Similar for B.

□

Definition 20. Let (X, d) metric space. $A \subseteq X$ is dense in X if $\bar{A} = X$. We say that (X, d) is separable if X has a countable subset A such that $\bar{A} = X$. Otherwise, X is non-separable.

Examples:

1. \mathbb{R} is separable
2. \mathbb{R}^n is separable.
3. l_1 is separable
4. l_∞ is non-separable.

Question:

Is $(C[a, b], |||_\infty)$ separable?

2.3 Convergence of sequences and topology in a metric space

Definition 21. (X, d) metric space, $\{x_n\} \subseteq X$ sequence. We say that $\{x_n\}$ converges to a point $x_0 \in X$ if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $d(x_n, x_0) < \epsilon$. Then x_0 is the limit of $\{x_n\}$, $\lim_n x_n = x_0, x_n \rightarrow x_0$. Equivalently, $\lim_n x_n = x_0 \iff \lim_n d(x_n, x_0) = 0$.

Proposition 13. (X, d) metric space, $\{x_n\} \subseteq X$. If $\lim x_n = x_0 = y_0$

Proposition 14. 1. $x_0 \in \text{bdy}(A) \iff \exists$ sequence $\{x_n\} \subseteq A, \{y_n\} \subseteq A^c$ such that $x_n \rightarrow x_0, y_n \rightarrow x_0$.

2. A is closed \iff whenever $\{x_n\} \subseteq A$ with $x_n \rightarrow x_0 \implies x_0 \in A$.

Proof. 1. $x_0 \in \text{bdy}(A), x_n \in B(x_0, \frac{1}{n}) \cap A. y_n \in B(x_0, \frac{1}{n}) \cap A^c$. Conversely, suppose $\{x_n\} \subseteq A, \{y_n\} \subseteq A^c, x_n \rightarrow x_0, y_n \rightarrow x_0$. Given $\epsilon > 0, \exists N \in \mathbb{N}$, such that $x_n(x_0, \epsilon), \forall n \geq N \implies B(x_0, \epsilon) \cap A \neq \emptyset. \exists N' \in \mathbb{N}$, such that $x_n \in B(x_0, \epsilon), \forall n \geq N' \implies B(x_0, \epsilon) \cap A^c \neq \emptyset$. This implies $x_0 \in \text{bdy}(A)$.

2. A is closed, $\{x_n\} \subseteq A, x_n \rightarrow x_0$. Suppose $x_0 \in A^c \implies \exists \epsilon > 0$, such that $B(x_0, \epsilon) \cap A = \emptyset$ but since $x_n \rightarrow x_0, \exists N \in \mathbb{N}$, such that $d(x_0, x_n) < \epsilon, \forall n \geq N$. Contradiction. Then $x_0 \in A$.

Conversely, suppose A is not closed, Then $x_0 \in \text{bdy}(A) \setminus A$. By (1), $\exists \{x_n\} \subseteq A$ such that $x_n \rightarrow x_0 \implies x_0 \in A$. This is a contradiction. Then A is closed.

□

Proposition 15. Let (X, d) metric space, $\{x_n\} \subseteq X$. If $x_0 = \lim_{n \rightarrow \infty} x_n = y_0$, then $x_0 = y_0$.

Proof. Suppose $x_0 \neq y_0 \implies d(x_0, y_0) = \epsilon > 0$. $\frac{\epsilon}{2} > 0, \exists N \in \mathbb{N}$ such that $d(x_n, x_0) < \frac{\epsilon}{2}, \forall n \geq N$, $\exists N' \in \mathbb{N}$ such that $d(x_n, y_0) < \frac{\epsilon}{2}, \forall n \geq N'$. If $n = \max\{N, N'\}$, $\epsilon = d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. \square

Definition 22. We say that x_0 is a limit point of $\{x_n\}$ if \exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow x_0$. $\lim^*(\{x_n\}) = \{x_0 \in X | x_0 \text{ is a limit point of } \{x_n\}\}$ $\lim(\{x_n\}) \leftarrow \{x_n\}$ subset of X .

Example, $x_n = (-1)^n$. $\lim^*(\{x_n\}) = \{-1, 1\}$. $\lim(\{x_n\}) = \emptyset$.

Proposition 16. (X, d) metric space, $A \subseteq X$. $x_0 \in \lim(A) \iff \exists \{x_n\} \subseteq A$, with $x_n \neq x_0$ and $x_n \rightarrow x_0$.

Proof. Let $x_0 \in \lim(A), \forall n \in \mathbb{N}, \exists x_n \in A$ such that $\{x_n\} \cap B(x_0, \frac{1}{n}) \neq \emptyset$. Hence $\{x_n\} \subseteq A, x_n \neq x_0, x_n \rightarrow x_0$.

Conversely. $\forall \epsilon > 0, A \setminus \{x_0\} \cap B(x_0, \epsilon) \neq \emptyset$. Since $\exists N \in \mathbb{N}$, such that $x_n \neq x_0 \in B(x_0, \epsilon), \forall n \geq N$. \square

2.4 Induced metric and the relative topology

Definition 23. Let (X, d) metric space, $A \subseteq X$. Define $d_A : A \times A \rightarrow \mathbb{R}$ such that $d_A(x, y) = d(x, y), \forall x, y \in A$. d_A is a metric, and its called the induced metric. Let $\tau_A = \{W \subset A | W = U \cap A \text{ for some } U \text{ open in } X\}$. τ_A is a topology in A called the relative topology in A inherited from τ_d on X .

Theorem 17. (X, d) metric space, $A \subseteq X$, Then $\tau_A = \tau_{d_A}$.

Proof. Let $W \subseteq A, W \in \tau_A$ and $x \in W$. $\exists U$ open in X such that $U \cap A = W$. $x \in U \implies \exists \epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. $x \in B_{d_A}(x, \epsilon) \subseteq B_d(x, \epsilon) \subseteq U$. $x \in B_{d_A}(x, \epsilon) \subseteq U \cap A = W \in \tau_{d_A}$.

Let $W \subseteq A, W \in \tau_{d_A}, \forall x \in W, \exists \epsilon_x > 0$ such that $B_{d_A}(x, \epsilon_x) \subseteq W$.

$$W = \bigcup_{x \in W} B_{d_A}(x, \epsilon_x)$$

$$X \supseteq U = \bigcup_{x \in W} B_d(x, \epsilon_x) \text{ open in } X$$

Now $W = A \cap U \implies W \in \tau_A$. \square

2.5 Continuity

$(X, d_x), (Y, d_y)$ metric spaces, $f : X \rightarrow Y$ function $f(x)$ is continuous at $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \epsilon)$. Otherwise, $f(x)$ is discontinuous at x_0 . $f(x)$ is continuous if it is continuous at x_0 , for all $x_0 \in X$.

Theorem 18. $(X, d_x), (Y, d_y)$ metric space, $f : X \rightarrow Y$ TFAE

1. $f(x)$ is continuous at $x_0 \in X$.
2. If W is a neighborhood of $g = f(x_0)$, then $v = f^{-1}(W)$ is a neighborhood of x_0 .

Proof. From (1) to (2): $\exists \epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq W$. This implies $\exists \delta > 0$ such that $d(z, x_0) < \delta \implies d_X(f(z), f(x_0)) < \epsilon$. Therefore, $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq W$. But $V = f^{-1}(W)$ Hence $x_0 \in B(x_0, \delta) \subseteq V \implies x_0 \in \text{int}(V)$.

From 2 to 1, let $\epsilon > 0$, Therefore, $B(f(x_0), \epsilon) = W$ neighborhood of $f(x_0)$. Then $f^{-1}(W)$ is a neighborhood of x_0 , i.e. $x_0 \in \text{int}(f^{-1}(W))$ Therefore, $\exists \delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(W)$. \square

Theorem 19. Sequential Characterization of continuous $(X, d_x), (Y, d_y)$ metric space, $f : X \rightarrow Y$, TFAE

1. $f(x)$ is continuous at $x_0 \in X$.
2. If $\{x_n\} \subseteq X, x_n \rightarrow x_0 \implies f(x_n) \rightarrow f(x_0)$.

Proof. From 1 to 2, $f(x)$ is continuous at $x_0, \{x_n\} \subseteq X, x_n \rightarrow x_0$. Fix $\epsilon > 0$, then $\exists \delta > 0$ such that $d_x(x, x_0) < \delta \implies d_y(f(x), f(x_0)) < \epsilon$. Since $x_n \rightarrow x_0$. $\exists N \in \mathbb{N}$, such that if $n \geq N, d_x(x_n, x_0) < \delta \implies d_y(f(x_n), f(x_0)) < \epsilon$.

From 2 to 1, assume $f(x)$ is not continuous at x_0 . $\exists \epsilon_0 > 0$, for every ball $B_x(x_0, \delta)$, $\exists x_\delta \in B_x(x_0, \delta)$ such that $d_Y(f(x_\delta), f(x_0)) \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, $x_n \in B_x(x_0, \frac{1}{n})$ Note: $x_n \rightarrow x_0$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$ i.e. $f(x_n)$ does not converge to $f(x_0)$. \square

Theorem 20. $(X, d_x), (Y, d_y)$ metric space, $f : X \rightarrow Y$, TFAE

1. $f(x)$ is continuous
2. If $W \subseteq Y$ is open, then $f^{-1}(W) = V \subseteq X$ is open
3. If $\{x_n\} \subseteq X, x_n \rightarrow x_0$ for some $x_0 \in X$, then $f(x_n) \rightarrow f(x_0) \in Y$.

Proof. 3 to 1 is done

1 to 2: Let $W \subseteq Y$ open and $V = f^{-1}(W)$. Let $x_0 \in V, f(x_0) \in W$ open. Therefore, W is a neighborhood of $f(x_0)$. By 1, $f^{-1}(W) = V$ is a neighborhood of x_0 i.e. $x_0 \in \text{int}(V)$ Then $V = \text{int}(V)$ is open.

2 to 3: let $\{x_n\} \subseteq X, x_n \rightarrow x_0$. Let $y_0 = f(x_0)$. Fix $\epsilon > 0$, if $W = B_y(y_0, \epsilon)$ open in Y . Then $f^{-1}(W) \subseteq X$ open. Note: $x_0 \in V \implies \exists \delta > 0$, such that $B_x(x_0, \delta) \subseteq V$. Since $x_n \rightarrow x_0, \exists N$ such that if $n \geq N$, then $d_x(x_n, x_0) < \delta$, i.e. $x_n \in V, \forall n \geq N$. Hence $f(x_n) \subseteq W, \forall n \geq N$. i.e. $d_y(f(x_n), f(x_0)) < \epsilon \iff f(x_n) \rightarrow f(x_0)$. \square

Example: X a set, d discrete metric (Y, d_Y) metric space, $f(X, d) \rightarrow (Y, d_Y)$ is continuous.

Definition 24. $f(X, d_X) \rightarrow (Y, d_Y)$: f is a homeomorphism if f is one-to-one and onto, and both f and f^{-1} are continuous. We say that (X, d_X) and (Y, d_Y) are homeomorphic.

Remark: $f : X \rightarrow Y$ is homeomorphic, $U \subseteq X$ is open $\iff f(U) \subseteq Y$ is open.

Two metric spaces (X, d_X) and (Y, d_Y) are equivalent if \exists a one-to-one and onto map $f : X \rightarrow Y$ and two constants, $c_1, c_2 > 0$, such that $c_1 d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2), \forall x_1, x_2 \in X$. Remark: If X and Y are equivalent, then they are homeomorphic.

2.6 Complete Metric Spaces: Cauchy sequences

Note: If $\{x_n\} \subset (X, d_X), x_n \rightarrow x_0 \in X$ then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n \geq N \implies d(x_0, x_n) < \epsilon/2$, If $n, m \geq N, d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) < \epsilon/2 + \epsilon/2 < \epsilon$.

Definition 25. A sequence $\{x_n\} \subseteq (X, d_x)$ is cauchy in (X, d_x) if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N, d(x_n, x_m) < \epsilon$.

Theorem 21. Let $\{x_n\} \subseteq (X, d_x)$ be a convergent sequence then $\{x_n\}$ is Cauchy.

Does every Cauchy sequence converge? $x_n = \frac{1}{n}, X = (0, 2)$ used metric $\{x_n\}$ is Cauchy but it does not converge.

Definition 26. A metric space (X, d_x) is complete if every Cauchy sequence converges. A set $A \subseteq X$ is bounded if $\exists M > 0$, and $x_0 \in X$ such that $A \subseteq B[x_0, M]$.

Proposition 17. Every Cauchy sequence is bounded $\{x_n\}$ is Cauchy. This implies $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N, d(x_n, x_m) < 1$. In particular, $d(x_N, x_m) < 1, \forall m \geq N$. $M = \max\{d(x_1, x_N), \dots, d(x_{N-1}, x_N), 1\}$. This implies $\{x_n\} \subseteq B[x_N, M]$.

Proposition 18. Assume $\{x_n\}$ is a Cauchy sequence with a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow x_0$. Then $x_n \rightarrow x_0$. Then $x_n \rightarrow x_0$. Let $\epsilon > 0 \implies \exists N \in \mathbb{N}$ such that $n, m \geq N, d(x_n, x_m) < \epsilon/2$ since $x_{n_k} \rightarrow x_0, \exists k \in \mathbb{N}$ such that $\forall n_k \geq k, d(x_{n_k}, x_0) < \epsilon/2$. $M = \max\{N, k\}, \forall n \geq M, d(x_n, x_0) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon/2 + \epsilon/2 < \epsilon$. Pick $n_k > M$.

2.7 Completeness of \mathbb{R}, \mathbb{R}^n and l_p

Theorem 22. *Bolzano-Weierstrass Theorem: every bounded sequence in \mathbb{R} has a convergent subsequence.*

Theorem 23. *Completeness Theorem for \mathbb{R} . Every Cauchy sequence in \mathbb{R} converges. $\{x_n\}$ is Cauchy $\implies \{x_n\}$ is bounded $\implies \{x_n\}$ has a convergent subsequence \implies Then $\{x_n\}$ is convergent.*

Theorem 24. *Let $1 \leq p \leq \infty$, every Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_p)$ converges.*

Lemma 4. *Let $1 \leq p < \infty$, let $\{x_k\}$ be a Cauchy sequence in $(l_p, \|\cdot\|_p)$. Then for each $i \in \mathbb{N}$, the component sequence $\{x_{k,i}\}_k$ is Cauchy in \mathbb{R} .*

Proof. Assume $\{x_k\}_{k \in \mathbb{N}} \subseteq (l_p, \|\cdot\|_p)$ is Cauchy. $x_k = \{x_{k,1}, \dots, x_{k,n}\}$ Since each component sequence $\{x_{k,i}\}_k$ is Cauchy on \mathbb{R} . and \mathbb{R} is complete. Let $x_{0,i} = \lim_{k \rightarrow \infty} x_{k,i} \in \mathbb{R}$ Let $x_0 = \{x_{0,1}, \dots, x_{0,i}, \dots\}$.

Claim: $x_0 \in l_p$ and $x_k \rightarrow x_0$.

Let $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $k, m \geq N_0, \|x_m - x_k\|_p < \frac{\epsilon}{2}$.

Case 1 Let $p = \infty$, $k \geq N_0$, $|x_{m,i} - x_{k,i}| \leq \|x_m - x_k\|_\infty$, $\forall m \geq N_0, \forall i \in \mathbb{N}$. $k \geq N_0$, $|x_{0,i} - x_{k,i}| = \lim_{m \rightarrow \infty} |x_{m,i} - x_{k,i}| \leq \frac{\epsilon}{2} < \epsilon, \forall i \in \mathbb{N}$. This implies $\{x_{0,i} - x_{k,i}\}_i \in l_\infty$. Well $\{x_{k,i}\} \in l_\infty$. This implies $\{x_{0,i}\} \in l_\infty$. Therefore, $\|x_0 - x_k\|_\infty < \epsilon, \forall k \geq N_0$. This implies $x_k \rightarrow x_0$.

Case 2 Let $k \geq N_0$. For each $j \in \mathbb{N}$ such that $(\sum_{i=1}^j |x_{m,i} - x_{k,i}|^p)^{1/p} \leq \|x_m - x_k\|_p < \frac{\epsilon}{2}$. $(\sum_{i=1}^j |x_{0,i} - x_{k,i}|^p)^{1/p} = \lim_{m \rightarrow \infty} (\sum_{i=1}^j |x_{m,i} - x_{k,i}|^p)^{1/p} \leq \frac{\epsilon}{2}$.

$$\left(\sum_{i=1}^{\infty} |x_{0,i} - x_{k,i}|^p\right)^{1/p} \leq \frac{\epsilon}{2} < \epsilon, \forall k \geq N_0$$

Then this implies $\{x_{0,i} - x_{k,i}\} \in l^p$ and $\{x_{k,i}\}_i \in l^p$. Then $\{x_{0,i}\} = x_0 \in l^p$. then $\|x_0 - x_k\|_p < \epsilon, \forall k \geq N_0$, then $x_k \rightarrow x_0$.

□

2.8 Completeness of $(\mathcal{C}_b(X), \|\cdot\|_\infty)$

Definition 27. $(X, d_x), (Y, d_y)$ metric space $\{f_n\}$ sequence of functions $f_n : X \rightarrow Y$. $\{f_n\}$ converges pointwise to $f_0 : X \rightarrow Y$ if $\lim_n f_n(x_0) = f_0(x_0), \forall x_0 \in X$. $\{f_n\}$ converges uniformly to $f_0 : X \rightarrow Y$ if $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that $n \geq N_0, d_Y(f_n(x), f_0(x)) < \epsilon, \forall x \in X$.

Remark: $\{f_n\}$ such that $f_n \xrightarrow{\text{uniform}} f_0 \implies f_n \xrightarrow{\text{pointwise}} f_0(x), \forall x$. Let $f_n(x) = x^n$ on $[0, 1]$. $f_n(x) \rightarrow f_0(x), \forall x$, for $f_0(x) = 1, x = 1$ otherwise 0.

Theorem 25. $(X, d_x), (Y, d_y)$ metric space, $\{f_n\}$ such that $f_n : X \rightarrow Y$ and $f_n \xrightarrow{unit} f_0 : X \rightarrow Y$. If each f_n is continuous at x_0 , so is f_0 .

$f_n \xrightarrow{unit} f_0$. This implies $\exists N_0 \in \mathbb{N}$ such that $n \geq N_0, d_y(f_n(x), f_0(x)) < \frac{\epsilon}{3}, \forall x \in X$.

f_n continuous at $x_0, \forall n \implies$ in particular f_{N_0} is continuous at x_0 . This means $\exists \delta > 0$ such that $x \in B(x_0, \delta) \implies d_y(f_{N_0}(x_0), f_{N_0}(x)) < \frac{\epsilon}{3}$.

Proof. If $x \in B(x_0, \delta)$,

$$d_Y(f_0(x_0), f_0(x)) \leq d_Y(f_0(x_0), f_{N_0}(x_0)) + d_Y(f_{N_0}(x_0), f_{N_0}(x)) + d_Y(f_{N_0}(x), f_0(x)) < \frac{\epsilon}{3} \times 3 = \epsilon$$

□

Definition 28. (X, d_x) metric space, $\mathcal{C}_b(X) := \{f : X \rightarrow \mathbb{R} | f \text{ is continuous on } X \text{ and } f(x) \text{ is bounded}\}$.

$$\|f\|_\infty = \sup\{|f(x)| | x \in X\}$$

$(\mathcal{C}_b(X), \|\cdot\|_\infty)$ is a normed linear space.

Remark: let $\{f_n\} \subseteq \mathcal{C}_b(X), f_n(X, d_x) \rightarrow (\mathbb{R}, \text{usual metric})$. $f_n \xrightarrow{\|\cdot\|_\infty} f_0 \iff f_n \xrightarrow{uniform} f_0$.

Theorem 26. Completeness for $(\mathcal{C}_b(X), \|\cdot\|_\infty)$, $(\mathcal{C}_b(X), \|\cdot\|_\infty)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence. For each $x_0 \in X, |f_n(x_0) - f_m(x_0)| \leq \|f_n - f_m\|_\infty$. It follows, that $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}, \forall x_0 \in X$. $f_0(x) = \lim_{n \rightarrow \infty} f_n(x), \forall x \in X$.

Claim: $f_n \rightarrow f_0$.

Let $\epsilon > 0$, choose N_0 such that $n, m \geq N_0 \implies \|f_n - f_m\|_\infty < \frac{\epsilon}{2}$. If $n \geq N_0$ and $x \in X$, then $|f_n(x) - f_0(x)| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon$. Therefore, $f_n \rightarrow f_0 \implies f_0$ is continuous.

f_0 is bounded. $\{f_n\}$ is Cauchy, then $\{f_n\}$ is bounded. $\exists M > 0$ such that $\|f_n\|_\infty < M, \forall n \in \mathbb{N}$. $\exists n_0$ such that $|f_0(x) - f_{n_0}(x)| < 1, \forall x \in X$. Then $|f_0(x)| \leq |f_0(x) - f_{n_0}(x)| + |f_{n_0}(x)| < 1 + M, \forall x \in X$. Hence $f_0 \in \mathcal{C}_b(X)$ and $f_n \rightarrow f_0$.

Remark: \mathbb{N} , discrete metric space. $(\mathcal{C}_b(\mathbb{N}), \|\cdot\|_\infty) = (l_\infty, \|\cdot\|_\infty)$ and $(\mathcal{C}_b(X), \|\cdot\|_\infty) \implies (l_\infty(X), \|\cdot\|_\infty)$

2.9 Characterizations of Complete Metric Spaces

Note: Theorem fails if we consider open intervals $\{(0, 1/n)\}$.

Note: Theorem fails if we consider unbounded intervals $\{[n, \infty)\}$.

Definition 29. Let $A \subseteq (X, d)$. $\text{diam}(A) := \sup\{d(x, y) | x, y \in A\}$ is the diameter of A .

Proposition 19. Let $A \subseteq B \subseteq (X, d)$, Then:

1. $\text{diam}(A) \leq \text{diam}(B)$
2. $\text{diam}(A) = \text{diam}(\bar{A})$.

Proof. The second: \leq from (1). If $\text{diam}(A) = \infty \implies \text{diam}(\bar{A}) = \infty$. Let $\epsilon > 0$, let $x, y \in \bar{A}$. this implies $\exists x_0, y_0 \in A$ such that $d(x, x_0) < \frac{\epsilon}{2}, d(y, y_0) < \frac{\epsilon}{2}$. $d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}A + \epsilon$. Hence $\text{diam}A \leq \text{diam}\bar{A} \leq \text{diam}A + \epsilon, \forall \epsilon > 0$. \square

Generalization of Nested Interval Theorem to (X, d) is complete.

Theorem 27. *Cantor's Intersection Theorem: Let (X, d) be a metric space TFAE*

1. (X, d) is complete.
2. (X, d) satisfies the following proposition.
 - If $\{F_n\}$ is a sequence of non-empty closed sets. such that $F_{n+1} \subseteq F_n, \forall n$, and $\lim_n(\text{diam}F_n) = 0 \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. 1 to 2: $\{F_n\}$ a sequence such that $F_n \neq \emptyset, F_n$ is closed, $F_{n+1} \subseteq F_n, \lim(\text{diam}F_n) = 0$. For each n , choose $x_n \in F_n$. Let $\epsilon > 0, \exists N_0$ such that $\text{diam}F_{N_0} < \epsilon$. If $n, m \geq N_0, \implies x_n, x_m \in F_{N_0}$. $d(x_n, x_m) \leq \text{diam}(F_{N_0}) < \epsilon$. Hence $\{x_n\}$ is a Cauchy sequence and (X, d) is complete. Then $x_n \rightarrow_n x_0 \in X$.

For each $n, \{x_n, x_{n+1}, \dots, x_{n+k}, \dots\} \subseteq F_n$. Then $x_{n+k} \rightarrow_k x_0$ and F_n closed so $x_0 \in F_n, \forall n$. This implies $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

2 to 1: let $\{x_n\} \subseteq X$. Cauchy. For each $n, A_n := \{x_n, x_{n+1}, \dots\}$ Claim: $\text{diam}(A_n) \rightarrow_n 0$. Let $F_n = \bar{A}_n, A_{n+1} \subseteq A_n \implies F_{n+1} \subseteq F_n. \text{diam}(F_n) \rightarrow_n 0$.

This implies $\exists x_0 \in \bigcap_{n=1}^{\infty} F_n$, let $\epsilon > 0$, choose N_0 such that $\text{diam}F_{N_0} < \epsilon$. This implies $F_{N_0} \subseteq B(x_0, \epsilon)$. If $n \geq N_0, d(x_n, x_0) < \epsilon$. This implies $x_n \rightarrow_n x_0$. \square

Definition 30. Define $(X, \|\cdot\|)$ normed space. $\{x_n\} \subseteq X$. A series with terms $\{x_n\}$ is a formal sum $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots$. For each $k \in \mathbb{N}$, define the k th-partial sum of $\sum_{n=1}^{\infty} x_n$ by $s_k = \sum_{n=1}^k x_n \in X$. The series $\sum_{n=1}^{\infty} x_n$ converges if the sequence $\{s_k\}$ converges. Otherwise, diverge.

Definition 31. A normed linear space $(X, \|\cdot\|)$ which is complete under the metric induced is called a Banach space.

Theorem 28. *Generalized Weierstrass M-Test: Let $(X, \|\cdot\|)$ normed linear space TFAE*

1. $(X, \|\cdot\|)$ is a Banach Space.
2. The space $(X, \|\cdot\|)$ satisfies the following property:

Let $\{x_n\} \subseteq X$. If $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R} \implies \sum_{n=1}^{\infty} x_n$ converges in $(X, \|\cdot\|)$.

Proof. 1 to 2: Let $T_k = \sum_{n=1}^k \|x_n\| \implies \{T_k\}$ is Cauchy. Given $\epsilon > 0, \exists N_0$ such that $k > m > N_0$

$$\sum_{n=m+1}^k \|x_n\| = |T_k - T_m| < \epsilon$$

Let $s_k = \sum_{n=1}^k x_n$, let $k > m > N_0$.

$$\|s_k - s_m\| = \left\| \sum_{n=m+1}^k x_n \right\| \leq \sum_{n=m+1}^k \|x_n\| < \epsilon$$

Therefore $\{s_k\}$ is Cauchy. This implies $\{s_k\}$ converges and then $\sum_{n=1}^{\infty} x_n$ converges.

2 to 1: Assume 2 holds and $\{x_n\}$ is Cauchy. Choose n_1 if $i, j > n_1 \implies \|x_i - x_j\| < \frac{1}{2}$ and choose n_2 , such that if $i, j > n_2 \implies \|x_i - x_j\| < \frac{1}{2^2}$.

If we have $n_k > n_{k-1} > \dots > n_2 > n_1$ such that if $i, j > n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. Choose $n_{k+1} > n_k$ such that if $i, j > n_{k+1} \implies \|x_i - x_j\| < \frac{1}{2^{k+1}}$. By induction, $\{n_k\}_k$ is an increasing sequence of \mathbb{N} such that $i, j > n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. In particular $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \implies g_k = x_{n_k} - x_{n_{k+1}} \in X, \forall k$.

$$\sum_{k=1}^{\infty} \|g_k\| = \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

Hence $\sum_{k=1}^{\infty} \|g_k\|$ converges. Hence $\sum_{k=1}^{\infty} g_k$ converges in $(X, \|\cdot\|) \iff \{s_k\}_k$ converges $s_k = \sum_{j=1}^k g_j$. $s_k = g_1 + g_2 + \dots + g_k = x_{n_1} - x_{n_2} + x_{n_2} - x_{n_3} + \dots + x_{n_k} - x_{n_{k+1}} = x_{n_1} - x_{n_{k+1}}$. $x_{n_{k+1}} \rightarrow x_{n_1} - \sum_{j=1}^{\infty} g_j$. Therefore $\{x_{n_k}\}$ converges and $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges. \square

Example:

A continuous, nowhere differentiable function

Let $\phi(x) = \begin{cases} x & x \in [0, 1] \\ 2 - x & x \in [1, 2] \end{cases}$. Extend to \mathbb{R} by $\phi(x) = \phi(x + 2)$. Let $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$.

1. Claim 1: $f(x)$ is continuous on \mathbb{R} . $\sum_{n=1}^{\infty} (\frac{3}{4})^n \phi(4^n x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n = L$. Then $f(x)$ is defined. $\sum_{n=1}^k (\frac{3}{4})^n \phi(4^n x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n \rightarrow f(x)$.

2.10 Completion of Metric Space

Proposition 20. (X, d) complete metric space, let $A \subseteq X$, then (A, d_A) is complete $\iff A$ is closed in X .

Proof. Converse: assume $A \subseteq X$ is closed, $\{x_n\} \subseteq A$ Cauchy in (A, d_A) . Then $\{x_n\}$ Cauchy in $(X, d) \implies \exists x_0$ such that $x_n \rightarrow x_0$ and A is closed so $x_0 \in A$.

\implies Suppose A is not closed. This implies $\exists x_0 \in \text{bdy}(A) \setminus A$. This implies $\{x_n\} \subseteq A$ such that $x_n \rightarrow_n x_0$. This means $\{x_n\}$ is Cauchy (A, d_A) . This means A is not complete. Hence contradiction. \square

Definition 32. $(X, d_x), (Y, d_y)$ metric spaces. A map $\phi : X \rightarrow Y$ is an isometry if $d_Y(\phi(x), \phi(y)) = d_X(x, y), \forall x, y \in X$. Note: If ϕ is an isometry, then ϕ is one-to-one. If ϕ is an isometry and ϕ is onto, we say that (X, d_X) and (Y, d_Y) are isometric. A completion of (X, d_X) is a pair $((Y, d_Y), \phi)$ such that (Y, d_Y) is a complete metric space, $\phi : X \rightarrow Y$ is an isometry and $\overline{\phi(X)} = Y$.

Theorem 29. (X, d) metric space. This implies \exists an isometry such that

$$\phi : X \rightarrow (C_b(X), \|\cdot\|_\infty)$$

Proof. Fix $a \in X$, for $u \in X$, let $f_u : X \rightarrow \mathbb{R}$. Then $f_u(x) = d(u, x) - d(x, a)$. f_u is continuous such that f_u is bounded, $|f_u(x)| = |d(u, x) - d(x, a)| \leq d(u, a)$. This implies $f_u \in C_b(X)$. Let $\phi : X \rightarrow C_b(X)$ such that $u \rightarrow f_u$.

$$\begin{aligned} d(f_u, f_v) &= \|f_u - f_v\|_\infty = \sup_{x \in X} \{|f_u(x) - f_v(x)|\} \\ &= \sup_{x \in X} \{|d(u, x) - d(x, a) - d(v, x) + d(x, a)|\} \leq d(u, v) \\ |f_u(v) - f_v(v)| &= d(u, v) \implies \|f_u - f_v\|_\infty = d(u, v) \end{aligned}$$

\square

Corollary 7. Every metric space has a completion. Let $\phi : X \rightarrow (C_b(X), \|\cdot\|_\infty)$ and $Y = \overline{\phi(X)}$. $((Y, d_Y), \phi)$ is complete.

2.11 Banach Contractive Mapping Theorem

Question: can we find $f \in C[0, 1]$ such that $f(x) = e^x + \int_0^x \sin(t)/2f(t)dt$?

Strategy: define $\Gamma : C[0, 1] \rightarrow C[0, 1]$. $\Gamma(g)(x) = e^x + \int_0^x \sin(t)/2g(t)dt \in C([0, 1])$. $\exists! f \in C[0, 1]$ such that Γ fixes f , i.e., $\Gamma(f) = f$.

Definition 33. (X, d_X) metric space, let $\Gamma : X \rightarrow X$. We call $x_0 \in X$ a fixed point of Γ if $\Gamma(x_0) = x_0$. We say that Γ is Lipchitz if $\exists \alpha \geq 0$ such that $d(\Gamma(x), \Gamma(y)) \leq \alpha d(x, y), \forall x, y \in X$ and Γ is a contraction if $\exists k$ such that $0 \leq k < 1$ such that $d(\Gamma(x), \Gamma(y)) \leq kd(x, y), \forall x, y \in X$.

Theorem 30. *Banach Contractive Mapping Theorem (or Banach fixed point Theorem).*
Let (X, d) be a complete metric space. This implies Γ has a unique fixed point $x_0 \in X$.

1. If such x_0 exists, it's unique: suppose $\Gamma(x_0) = x_0$ and $\Gamma(y_0) = y_0$, $\Gamma \neq 0$. This implies $d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leq kd(x_0, y_0)$ This implies $d(x_0, y_0) = 0$.
2. Let $x_1 \in X$ and $x_2 = \Gamma(x_1)$, $x_3 = \Gamma(x_2)$, \dots , $x_{n+1} = \Gamma(x_n)$.

$$d(x_2, x_3) = d(\Gamma(x_1), \Gamma(x_2)) \leq kd(x_1, x_2)$$

$$d(x_4, x_3) = d(\Gamma(x_3), \Gamma(x_2)) \leq kd(x_3, x_2) \leq k^2d(x_1, x_2)$$

By induction, $d(x_{n+1}, x_n) \leq k^{n-1}d(x_1, x_2)$. If $m > n$, $d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \leq k^{m-2}d(x_2, x_1) + k^{m-3}d(x_2, x_1) + \dots + k^nd(x_1, x_2) + k^{n-1}d(x_2, x_3) = \frac{k^{n-1}}{1-k}d(x_2, x_1)$.

Remark: If $d(\Gamma(x), \Gamma(y)) < d(x, y)$, theorem fails.

Example: Show that there exists a unique $f \in C[0, 1]$ such that

$$f(x) = e^x + \int_0^x \frac{\sin(t)}{2} f(t) dt$$

Let $\Gamma(g)(x) = e^x + \int_0^x \frac{\sin(t)}{2} g(t) dt$. $(C[0, 1], \|\cdot\|_\infty)$ is complete. Let $f(x), g(x) \in C[0, 1]$ and $x \in [0, 1]$.

$$\begin{aligned} |\Gamma(g)(x) - \Gamma(f)(x)| &= |e^x + \int_0^x \frac{\sin(t)}{2} g(t) dt - e^x - \int_0^x \frac{\sin(t)}{2} f(t) dt| \\ &= |\int_0^x \frac{\sin(t)}{2} (g(t) - f(t)) dt| \\ &\leq \int_0^x |\frac{\sin(t)}{2}| |g(t) - f(t)| dt \leq \|g - f\|_\infty \int_0^1 \frac{1}{2} dt = \frac{1}{2} \|g - f\|_\infty \\ &\implies \|\Gamma(g) - \Gamma(f)\|_\infty \leq \frac{1}{2} \|g - f\|_\infty \implies \Gamma \text{ is a contraction} \\ &\implies \exists! f(x) \in C[0, 1] \end{aligned}$$

Example: Show that there exists a unique $f_0(x) \in C[0, 1]$ such that

$$f_0(x) = x + \int_0^x t^2 f_0(t) dt$$

Find a power series representation for $f_0(x)$. Let $\Gamma(g)(x) = x + \int_0^x t^2 g(t) dt$ Note $(C[0, 1], \|\cdot\|_\infty)$ is complete. Let $f, g \in C[0, 1], x \in [0, 1]$.

$$\begin{aligned} |\Gamma(g)(x) - \Gamma(f)(x)| &= \left| \int_0^x t^2 (g(t) - f(t)) dt \right| \\ &\leq \int_0^1 t^2 |g(t) - f(t)| dt \leq \|g - f\|_\infty \int_0^1 t^2 dt = \frac{1}{3} \|g - f\|_\infty, x \in [0, 1] \\ \|\Gamma(g) - \Gamma(f)\|_\infty &\leq \frac{1}{3} \|f - g\|_\infty, \forall f, g \in C[0, 1] \end{aligned}$$

Therefore, Γ is a contraction. By BCM theorem, $\exists! f_0 \in C[0, 1]$ such that $\Gamma(f_0) = f_0$. Let $f_1 = 0, f_{n+1} = \Gamma(f_n)$. Therefore,

$$\begin{aligned} f_2(x) &= x + \int_0^x t^2 \theta dt = x \\ f_3(x) &= x + \int_0^x t^2 t dt = x + \frac{x^4}{4} \\ &\dots \\ f(x) &= \sum_{n=0}^{\infty} \frac{x^{3n+1}}{1, 47(3n+1)} \end{aligned}$$

Theorem 31. *Picard-Lindelof Theorem: Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and Lipchitz in y , i.e., $1 > \alpha \geq 0$, such that*

$$|f(t, y) - f(t, z)| \leq \alpha |y - z|, \forall y, z \in \mathbb{R}$$

Let $y_0 \in \mathbb{R}$, $\implies \exists! y(t) \in C[0, b]$ such that $y'(t) = f(t, y(t)) \forall t$ and $y(0) = y_0$.

2.12 Baire's Category Theorem

Example:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}, m \neq 0, \gcd(m, n) = 1 \\ 1 & x = 0 \end{cases}$$

$f(x)$ is discontinuous at $x = r$, for all $r \in \mathbb{Q}$. $f(x)$ is continuous at $x = \alpha$, for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Definition 34. (X, d) metric space, $A \subseteq X$ is said to be on F_σ set if $A = \bigcup_{n=1}^{\infty} F_n$ where $\{F_n\}$ is a sequence of closed sets. This implies $A \subseteq X$ is said to be a G_δ set if $A = \bigcap_{n=1}^{\infty} U_n$ where $\{U_n\} \subseteq X$ is a sequence of open sets.

Remarks:

1. From DeMorgan's Law, A is $F_\sigma \iff A^c$ is G_δ .
2. $[0, 1)$ is both F_σ and G_δ . $[0, 1) = \bigcup_{n=1}^\infty [0, 1 - \frac{1}{n}]$ and $[0, 1) = \bigcap_{r=1}^\infty (-\frac{1}{r}, 1)$.
3. $F \subseteq X$ closed. This implies F is G_δ . $U \subseteq X$ open. This implies U is F_σ .

Definition 35. $(X, d_X), (Y, d_Y)$ metric spaces and $f : X \rightarrow Y$. $D(f) = \{x \in X \mid f \text{ is not continuous at } x\}$.
 $D_n(f) = \{x \in X \mid \forall \epsilon > 0, \exists y, z \in B(x, \delta) \text{ with } d_Y(f(y), f(z)) \geq \frac{1}{n}\}$.

Theorem 32. Let $f : (X, d_X) \rightarrow (Y, d_Y), \forall n \in \mathbb{N}, D_n(f)$ is closed in X . Moreover, $D(f) = \bigcup_{r=1}^\infty D_n(f)$. In particular, $D(f)$ is F_σ .

Proof. $(D_n(f))^c$ open and $x \in (D_n(f))^c \implies \exists \delta > 0, \forall y, z \in B(x, \delta), d_Y(f(y), f(z)) < \frac{1}{n}$.
Let $v \in B(x, \delta), \eta = \delta \cdot d_X(x, v)$. Let $y, z \in B(v, \eta)$. If $y \in B(v, \eta) \implies d(y, x) \leq d(y, v) + d_X(v, x) < \delta - d_X(x, v) + d_X(v, x) < \delta$. This implies $y, z \in B(x, \delta) \implies d_Y(f(x), f(y)) < \frac{1}{n}$.
Hence $B(x, \delta) \subseteq (D_n(f))^c \implies (D_n(f))^c$ is open. \square

Definition 36. (X, d) metric space. A set $A \subseteq X$ is nowhere dense if $\text{int}(\bar{A}) = \emptyset$. A is of first category in X if $A = \bigcup_{n=1}^\infty A_n$ where each A_n is nowhere dense. Otherwise, A is of second category in X . A set C is residual in X if C^c is of first category in X .

Recall: A set $A \subseteq X$ is dense if $\bar{A} = X$. Equivalently, A is dense if $\forall W \subseteq X$ open, $W \cap A \neq \emptyset$. Suppose there exists $W \subseteq X$ open such that $W \cap A = \emptyset$. Let $x \in W \implies x \in X \setminus A$. But $\exists \delta$ such that $B(x, \delta) \subseteq W \implies x \notin \bar{A}$.

Let $x_0 \in X \setminus A$ (want $\exists \{x_n\} \subseteq A \setminus x_n \rightarrow x_0$) since $B(x, \frac{1}{n}) \cap A \neq \emptyset$. This implies $\exists x_n \in B(x, \frac{1}{n}) \cap A \implies \{x_n\} \subseteq A, x_n \rightarrow x_0$.

Theorem 33. Baire Category Theorem 1, (X, d) complete metric space. Let $\{U_n\}$ be a sequence of open, dense sets. Then $\bigcap_{n=1}^\infty U_n$ is dense in X .

Proof. Let $W \subseteq X$ be open and non-empty. Then $\exists x_1 \in X$ and $r_1 < 1, B(x_1, r_1) \subseteq B[x_1, r_1] \subseteq W \cap U_1$. And $\exists x_2 \in X, r_2 < \frac{1}{2}$ such that $B(x_2, r_2) \subseteq B[x_2, r_2] \subseteq B(x_1, r_1) \cap U_2$.

Recursively, we find sequences $\{x_n\} \subseteq X$ and $\{r_n\} \subseteq \mathbb{R}$ such that $0 < r_n < \frac{1}{n}$ and $B(x_{n+1}, r_{n+1}) \subseteq B[x_{n+1}, r_{n+1}] \subseteq B(x_n, r_n) \cap U_{n+1}, \forall n \geq 1$ but $r_n \rightarrow 0, B[x_{n+1}, r_{n+1}] \subseteq B[x_n, r_n]$, X is complete. By Cantor intersection theorem, there exists $x_0 \in \bigcap_{n=1}^\infty B[x_n, r_n] \subseteq W$ and $B[x_n, r_n] \subseteq U_n, \forall n$. This means $x_0 \in W \cap (\bigcap_{n=1}^\infty U_n)$. This implies $\bigcap_{n=1}^\infty U_n$ is dense. \square

Remarks:

1. The Cantor set is nowhere dense in \mathbb{R} , and has cardinality c .
2. A close set F is nowhere dense if and only if $U = F^c$ is dense.

Corollary 8. Baire Category Theorem II: every complete metric space (X, d) is of second category in itself. Assume X is of the first category, i.e. $\exists \{A_n\}$ sequence of nowhere dense sets such that $X = \bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \bar{A}_n$. Let $U_n = (\bar{A}_n)^c \implies U_n$ is open and dense.

But $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (\bar{A}_n)^c = (\bigcup_{n=1}^{\infty} \bar{A}_n)^c = X^c = \emptyset$. Hence contradiction.

Corollary 9. \mathbb{Q} is not a G_δ subset of \mathbb{R} . Suppose $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open. Let $F_n = (U_n)^c, \forall n$. $\mathbb{Q} \subseteq U_n, \forall n$ and $\bar{Q} = \mathbb{R}$ then $\bar{U}_n = \mathbb{R}$. Therefore, F_n is nowhere dense, for all n . Consider $\mathbb{Q} = \{r_1, r_2, \dots\}$ Let $S_n = F_n \cup \{r_n\}$ closed and nowhere dense. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$.

Then $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$, if $x \in \mathbb{Q} \implies x = r_n$ for some n . This implies $x \in S_n$. If $x \in \mathbb{R} \setminus \mathbb{Q} \implies x \in \bigcup_{n=1}^{\infty} U_n^c$. Hence $x \in F_n$ for some n , $x \in S_n$.

Corollary 10. There is no function $f : \mathbb{R} \rightarrow \mathbb{R}$ for which $D(f) = \mathbb{R} \setminus \mathbb{Q}$.

Definition 37. $(X, d_x), (Y, d_y)$ metric space, $\{f_n : X \rightarrow Y\}$ sequence of function $f_n \rightarrow f_0$ pointwise on X . We say that f_n converges uniformly at $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ and $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$ and $d(x, x_0) < \delta \implies d_Y(f_n(x), f_m(x)) < \epsilon$.

Theorem 34. $(X, d_x), (Y, d_y)$ metric space, $\{f_n : X \rightarrow Y\}$ such that $f_n \rightarrow f_0$ point wise on X . Assume that f_n convergence uniformly at x_0 and $\{f_n\}$ is a sequence of continuous function at x_0 . This implies f_0 is continuous at x_0 .

Theorem 35. Let $f_n : (a, b) \rightarrow \mathbb{R}$ be a sequence of continuous functions that converges point wise to f_0 . This implies $\exists x_0 \in (a, b)$ such that f_n converges uniformly at x_0 .

Claim: There exists a closed interval $[\alpha_1, \beta_1] \subset (a, b)$ with $\alpha_1 < \beta_1$ and $N_1 \in \mathbb{N}$ such that if $n, m \geq N$, and $x \in [\alpha_1, \beta_1]$. Then $|f_n(x) - f_m(x)| \leq 1$.

Inductively, we can construct a sequence $\{[\alpha_k, \beta_k]\}$ with $(a, b) \supset [\alpha_1, \beta_1] \supset (\alpha_1, \beta_1) \supset [\alpha_2, \beta_2] \supset (\alpha_2, \beta_2) \supset \dots$ and a sequence $N_1 < N_2 < N_3 < \dots$ such that $n, m \geq N_k$ and $x \in [\alpha_k, \beta_k]$. This implies $|f_n(x) - f_m(x)| \leq \frac{1}{k}$. Let $x_0 \in \bigcap_{k=1}^{\infty} [\alpha_k, \beta_k]$. Given $\epsilon > 0$, if $\frac{1}{k} < \epsilon$, and $n, m \geq N_k$ and $x \in (\alpha_k, \beta_k)$, then

$$|f_n(x) - f_m(x)| \leq \frac{1}{k} < \epsilon$$

. Pick $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (\alpha_k, \beta_k)$. For δ as above, and N_k , the definition of uniform convergence at x_0 is verified.

Corollary 11. $\{f_n\} \subset C[a, b]$ such that $f_n \rightarrow f_0$ point wise on $[a, b]$. This implies \exists a residual set $A \subset [a, b]$ such that f_0 is continuous at each $x \in A$. A^c is first category, i.e. $A^c = \bigcup_{n=1}^{\infty} A_n$, A_n nowhere dense.

$A = \{x \in [a, b] | f_0 \text{ is continuous at } x\}$.

Claim: A is dense in $[a, b]$, i.e. given any $(c, d) \subset [a, b]$, $(c, d) \cap A \neq \emptyset$. Let $(c, d) \subset [a, b]$, then $\exists x_0 \in (c, d)$ such that f_n converges uniformly at x_0 . But each f_n is continuous. Then f_0 is continuous at x_0 . This implies $x_0 \in A \cap (c, d)$. and $A^c = D(f_0)$ is $F_\sigma \implies A$ is G_δ . This implies $A = \bigcap_{n=1}^{\infty} U_n$, U_n open dense $\iff U_n^c$ closed, nowhere dense. i.e. $A^c = \bigcup_{n=1}^{\infty} U_n^c$, i.e., A is residual.

Corollary 12. Suppose $f(x)$ is differentiable on \mathbb{R} . Then $f'(x)$ is continuous for every point in a dense G_δ -subset of \mathbb{R} .

$f_n(x) = \frac{f(x+1/n)-f(x)}{1/n}$ Then $f(x)$ pointwise. Apply Corollary.

2.13 Compactness

Definition 38. An open cover for $A \subseteq X$ is a collection $\{U_\alpha\}_{\alpha \in I}$ of open sets for which $A \subseteq \bigcup_{\alpha \in I} U_\alpha$. Given a cover $\{U_\alpha\}_{\alpha \in I}$ for $A \subseteq X$, a sub cover is a sub collection $\{U_\alpha\}_{\alpha \in J}$, for $J \subseteq I$ such that $A \subseteq \bigcup_{\alpha \in J} U_\alpha$. A sub cover $\{U_\alpha\}_{\alpha \in J}$ is finite if J is finite. We say that $A \subseteq X$ is compact if every open cover of A has a finite sub cover. (X, d) is compact if X is compact. We say that $A \subseteq X$ is sequentially compact if every sequence $\{x_n\} \subseteq A$ has a converging subsequence converging to a point in A . (X, d) is sequentially compact if so is X . We say that X has the Bolzano-Weierstrass property (BWP) if every infinite subset in X has a limit point.

Theorem 36. (X, d) metric space, TFAE

1. X is sequentially compact
2. X has the BWP

Proof. 1 to 2: X sequentially compact and $S \subseteq X$ infinite. S has a countable infinite subset $\{x_1, x_2, \dots\}$. This implies $\exists \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \rightarrow x_0$. $\forall \epsilon > 0$, $(B(x_0, \epsilon) \cap S) \setminus \{x_0\}$ has infinitely many points. Hence $x_0 \in \text{Lim}(S)$.

2 to 1: Assume X has the BWP, and $\{x_n\} \subseteq X$. If $\exists x_0 \in X$ appearing infinitely many times in $\{x_n\}$, then $\{x_n\}$ has a constant, converging subsequence. If such an x_0 doesn't exist, viewed as a subset of X , $\{x_n\}$ is infinite. We can assume the terms of $\{x_n\}$ are distinct. Thus $\exists x_0 \in \text{Lim}(\{x_n\})$. This implies $\exists n_1 \in \mathbb{N}$ such that $d(x_0, x_{n_1}) < 1$. Find $n_2 > n_1$ such that $d(x_0, x_{n_2}) < \frac{1}{2}$. If we have $n_1 < n_2 < \dots < n_k$ such that $d(x_0, x_{n_k}) < \frac{1}{k}$. Choose $n_{k+1} > n_k$ such that $d(x_0, x_{n_{k+1}}) < \frac{1}{k+1}$. This implies $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \rightarrow x_0$. \square

Proposition 21. (X, d) metric space, $A \subseteq X$.

1. A compact $\implies A$ is closed and bounded.
2. If A is closed and X is compact, then so is A .
3. If A is sequentially compact. Then A is closed and bounded.
4. A is closed, X is sequentially compact. This implies A is sequentially compact.
5. If X is sequentially compact, then X is complete.

Proof. 1. Bounded pick $x_0 \in A$. This implies $\{B(x_0, n)\}$ is an open cover of A . A compact \implies There exists a finite sub cover $\{B(x_0, n_k)\}$ let $M = \max\{n_j : j = 1, \dots, k\} \implies A(x_0, M)$

Closed: Suppose A is not closed $\implies \exists x_0 \in \text{Lim}(A) \setminus A$, $U_n = (B[x_0, \frac{1}{n}])^c$. $\{U_n\}$ open cover of A , with no finite sub cover but A compact. Then contradiction.

2. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of A . Then $\{U_\alpha\}_{\alpha \in I} \cup \{A^c\}$ is an open cover of X . This implies $\exists \alpha_1, \dots, \alpha_n$ such that $\{U_{\alpha_1}, \dots, U_{\alpha_n}\} \cup \{A^c\}$ covers X . Thus $\{U_{\alpha_n}\}$ covers A . A is compact.

3. Bounded: Assume A is not bounded. Choose $x_1 \in A \implies \exists x_2 \in A, d(x_1, x_2) > 1$. Therefore, $\exists x_3 \in A$ such that $d(x_i, x_3) > 1, i = 1, 2$. Recursively, we define $\{x_n\}$ such that $d(x_n, x_m) > 1$, if $n \neq m$. Therefore, $\{x_n\}$ cannot have a convergent subsequence. Contradiction.

Closed: Assume A is not closed. This means $\exists \{x_n\} \subseteq A$ such that $x_n \rightarrow x_0$ but $x_0 \notin A$. $\implies \{x_n\}$ has no convergent subsequence in A . Contradiction. □

Examples:

- $A \subseteq \mathbb{R}$, A is sequentially compact $\iff A$ is closed and bounded.
- $A \subseteq \mathbb{R}^n$, works too.
- $A \subseteq \mathbb{R}^n$, A compact $\iff A$ is closed and bounded.

Theorem 37. *Heine-Borel Theorem: $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

Notation:

A closed cell in \mathbb{R}^n is a set $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

Proof. 1. A is closed and bounded. Assume A is not compact. Let $F_1 = A$, J_1 be a closed cell such that $A \subseteq J_1$. Bisect each of the intervals $[a_i, b_i]$ of J_1 . This implies we obtain 2^n closed cells $\{J_{11}, J_{12}, \dots, J_{12^n}\}$. Exists some open cover $\{U_\alpha\}_{\alpha \in I}$ such that it does not have a finite sub cover. One of the subcells, call it J_2 , must be such that $F_2 = J_2 \cap A$ does not have a finite sub cover of $\{U_\alpha\}_\alpha$. Recursively, we construct a sequence of closed cells $\{J_n\}$ and closed sets $F_n = J_n \cap A$ such that

- (a) $J_{n+1} \subseteq J_n, \forall n \implies F_{n+1} \subseteq F_n, \forall n$.
- (b) Claim $(J_{n+1}) = \frac{1}{2} \text{diam}(J_n) \implies \text{diam}(F_{n+1}) \leq \frac{\text{diam}(F_n)}{2}$.
- (c) $F_n = J_n \cap A$ cannot be covered by finitely many U_α 's.

2. By Cantor intersection theorem,

$$\bigcap_{n=1}^{\infty} F_n = \{x_0\} \implies x_0 \in A \implies \exists \alpha_0 | x_0 \in U_{\alpha_0} \implies \exists > 0 | B(x_0, \epsilon) \subseteq U_{\alpha_0}$$

Pick n_0 such that $\text{diam} F_{n_0} < \epsilon$. Then $F_{n_0} \subseteq B(x_0, \epsilon) \subseteq U_{\alpha_0}$. $\{U_{\alpha}\}$ covers F_{n_0} .
Contradiction. □

Questions:

$A \subseteq X$ is compact $\iff A$ is closed and bounded?

No, X is infinite set, d is discrete metric space. X is bounded but not compact. But if it is compact, then it is also sequential compact.

Definition 39. X set, a collection $\{A_{\alpha}\}_{\alpha \in I}$, $A_{\alpha} \subseteq X, \forall \alpha$ has finite intersection.

Property: (FIP) if whenever $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ is any finite sub collection, we have

$$\bigcap_{i=1}^n A_{\alpha_i} \neq \emptyset$$

Theorem 38. (X, d) metric space, TFAE

1. X is compact

2. If $\{F_{\alpha}\}_{\alpha \in I}$ is a collection of closed sets of X with the FIP then $\bigcap_{\alpha \in I} F_{\alpha} \neq \emptyset$.

Corollary 13. (X, d) compact metric space, $\{F_n\}$ of non-empty, closed sets such that $F_{n+1} \subseteq F_n, \forall n \in \mathbb{N} \implies \bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$.

Corollary 14. (X, d) compact metric space. Then X has BWP (X is sequentially compact).

Proof. Assume X is compact. Let S be an infinite set. Then exists a sequence $\{x_n\} \subseteq S$ consisting of distinct points. Let $F_n = \{x_n, x_{n+1}, \dots\} \implies \{F_n\}$ has the FIP. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset \implies \exists x_0 \in \bigcap_{n=1}^{\infty} F_n$. For all $\epsilon > 0, B(x_0, \epsilon) \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset, \forall n \in \mathbb{N}$. This implies $B(x_0, \epsilon) \cap S \setminus \{x_0\} \neq \emptyset \implies x_0 \in \text{Lim}(S)$. □

Theorem 39. $(X, d_x), (Y, d_y)$ metric space. Let $f : (X, d_x) \rightarrow (Y, d_y)$ contains. If (X, d_x) sequentially compact. this implies $f(X)$ is sequentially compact. Let $\{y_n\} \subseteq f(X), \implies \forall n, \exists x_n$ such that $y_n = f(x_n)$. This implies $\{x_n\} \subseteq X \implies \exists \{x_{n_k}\}$ such that $x_{n_k} \rightarrow x_0 \in X$. Hence $f(x_{n_k}) \rightarrow f(x_0) \in f(X)$.

Corollary 15. Extreme Value Theorem:

Let $f : (X, d_x) \rightarrow \mathbb{R}$ be continuous. If (X, d_x) is sequentially compact, then there exists $c, d \in X$ such that $f(c) \leq f(x) \leq f(d), \forall x \in X$.

Definition 40. Let $\epsilon > 0$. A collection $\{x_\alpha\}_{\alpha \in I} \subseteq X$ is an ϵ -net for X if $X = \bigcup_{\alpha \in I} B(x_\alpha, \epsilon)$. We say that (X, d) is totally bounded if for each $\epsilon > 0$, X has a finite ϵ -net. Given $A \subseteq X$, A is totally bounded if it is totally bounded in the induced metric. $\forall \epsilon > 0, \exists \{x_1, \dots, x_n\} \subseteq A$ such that $\bigcup_{i=1}^n B(x_i, \epsilon) \supseteq A$.

Proposition 22. If X is sequentially compact, then X is totally bounded. Suppose X is not totally bounded: Then $\exists \epsilon_0 > 0$, with no finite ϵ_0 -net. Then \exists sequence $\{x_n\} \subseteq X$ such that $x_i \notin B(x_j, \epsilon_0)$ if $i \neq j$. Then $\{x_n\}$ has no convergent subsequence. Contradiction.

Remarks:

1. (\mathbb{N}, d) discrete metric (\mathbb{N}, d) is bounded but it is not totally bounded. Then there does not exist finite $1/2$ -net.
2. If $A \subseteq (X, d)$ is totally bounded. Then so is \bar{A} . If $\{x_1, \dots, x_n\}$ is an ϵ -net for A . Then $\{x_1, \dots, x_n\}$ is an ϵ -net for \bar{A} .

Theorem 40. Lebesgue (X, d) compact metric space, $\{U_\alpha\}_{\alpha \in I}$ open cover of X . Then $\exists \epsilon > 0, \forall x \in X$ and $0 < \delta < \epsilon$. there exists $\alpha_0 \in I$ with $B(x, \delta) \subseteq U_{\alpha_0}$.

Proof. If $X = U_\alpha$ for some α , then any $\epsilon > 0$ would work. Assume $X \neq U_\alpha, \forall \alpha$. For each $x \in X$, let $\phi(x) = \sup\{r \in \mathbb{R} | B(x, r) \subseteq U_{\alpha_0}, \text{ for some } \alpha_0 \in I\}$. Then $\phi(x) > 0$. Also, $\phi(x) < \infty$: if $\phi(x) = \infty, \exists \{r_n\} \subseteq \mathbb{R}, \{\alpha_n\} \subseteq I | B(x, r_n) \subseteq U_{\alpha_n}, r_n \rightarrow \infty$. But X sequentially compact. This implies X is bounded and $\exists M > 0, B(x, M) = X$. Pick $r_n > M \implies B(x, r_n) = X \subseteq U_{\alpha_n}$ but $X \neq U_{\alpha_n}$. Contradiction.

If ϕ is continuous: if $x, y \in X, \phi(x) \leq \phi(y) + d(x, y)$:

case 1 $\exists \alpha_0$ and $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$ and $y \in B(x, r)$. $B(y, r - d(x, y)) \subseteq U_{\alpha_0} \implies \phi(y) \geq r - d(x, y) \implies \phi(x) \leq d(x, y) + \phi(y)$.

case 2 $\forall r$ and α such that $B(x, r) \not\subseteq U_\alpha, y \notin B(x, r)$. $r \leq d(x, y), \phi(x) \leq d(x, y)$ and $\phi(x) \leq d(x, y) + \phi(y)$ and $|\phi(x) - \phi(y)| \leq d(x, y) \implies \phi$ is continuous. Therefore, by extreme value theorem, $\epsilon > 0$, such that $\phi(x) \geq \epsilon, \forall x \in X$.

□

Theorem 41. Borel-Lebesgue (X, d) metric space, TFAE

1. X is compact
2. X has the BWP
3. X is sequentially compact.

Proof. 3 to 1: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for X . This implies $\{U_\alpha\}$ has a Lebesgue number $\epsilon > 0$. Since X is totally bounded, there exists finite subset $\{x_1, x_2, \dots, x_n\} \subseteq X$ such that $\bigcup_{i=1}^n B(x_i, \delta) = X$ where $0 < \delta < \epsilon$. But for each $i = 1, 2, \dots, n$, we can find $\alpha_i \in I$ such that $B(x_i, \delta) \subseteq U_{\alpha_i}$. This implies $\{U_{\alpha_i}\}_{i=1, \dots, n}$ is a finite sub cover. This implies X is compact. \square

Theorem 42. *Heine Borel for metric space: (X, d) metric space TFAE*

1. X is compact
2. X is complete and totally bounded.

Proof. 2 to 1 (X is sequentially compact). Let $\{x_n\}$ be a sequence in X . Since X is totally bounded, $\exists y_1, \dots, y_n \in X$ such that $\bigcup_{i=1}^n B(y_i, 1) = X$. Then there exists y_i such that $B(y_1, 1) = S_1$ contains infinitely many terms of $\{x_n\}$. Since X is totally bounded, $\exists y_1^2, \dots, y_{n_2}^2$ such that $\bigcup_{i=1}^{n_2} B(y_i^2, \frac{1}{2}) = X$. Therefore $\exists y_i^2 | B(y_i^2, 1/2) = S_2$ contains infinitely many terms of $\{x_n\}$ in S_1 . Then, we construct sequence of open balls $\{S_k = B(y^k, 1/k)\}$ and each S_{k+1} contains infinitely many terms of $\{x_n\}$ also in $S_1 \cap \dots \cap S_k$. In particular, we can choose $n_1 < n_2 < \dots$ such that $x_{n_k} \in S_1 \cap \dots \cap S_k$. But $\text{diam}(S_k) \rightarrow 0$, this implies $\{x_{n_k}\}$ is cauchy and X is complete. thus $\{x_{n_k}\}$ is convergent. \square

2.14 Compactness and Continuity

Theorem 43. *Let $f : (X, d_x) \rightarrow (Y, d_y)$ be continuous. If (X, d_x) is compact. $f(x)$ is compact.*

Corollary 16. *Extreme Value Theorem: Let $f : (X, d_x) \rightarrow \mathbb{R}$ be continuous. If (X, d_x) is compact. There exists $c, d \in X$ such that $f(x) \leq f(d), \forall x \in X$.*

Theorem 44. *Sequential characterization of uniform continuity: suppose $f : (X, d_x) \rightarrow (Y, d_y)$ function TFAE*

1. f is uniformly continuous on X
2. If $\{x_n\}, \{z_n\}$ in X with $\lim_n d(x_n, z_n) = 0 \implies \lim_n d_Y(f(x_n), f(z_n)) = 0$.

Theorem 45. *$f : (X, d_X) \rightarrow (Y, d_Y)$ continuous if (X, d_x) is compact. This implies $f(x)$ is uniformly continuous. Suppose $f(x)$ is not uniformly continuous. This implies $\exists \epsilon_0 > 0$ and $\{x_n\}, \{y_n\} \subseteq X$ such that $\lim_n d(x_n, z_n) = 0$ but $d_Y(f(x_n), f(z_n)) \geq \epsilon_0, \forall n \geq n_0$. X compact $\implies \exists \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that it converges to x_0 . $\exists \{z_{n_k}\}$ subsequence of $\{z_n\}$ such that it converges to x_0 .*

f is continuous, then $f(x_{n_k}) \rightarrow f(x_0)$ and $f(z_{n_k}) \rightarrow f(x_0)$. contradiction

Theorem 46. $(X, d_x), (Y, d_y)$ metric space, X is compact. Then let $\Phi : X \rightarrow Y$ be one-to-one, onto and continuous. then Φ^{-1} is also continuous.

If Φ is continuous $\iff (U \subseteq X \text{ open} \implies \Phi(U) \subseteq Y \text{ is open})$. $U \subseteq X$ is open, then $U^c = F \subseteq X$ closed and X is compact. Then F is compact. Therefore, $\Phi(F) \subseteq Y$ compact $\implies \Phi(F) \subseteq Y$ is closed there fore $\Phi(U^c) = (\Phi(U))^c$

3 The Space $(C(X), \|\cdot\|_\infty)$

We assume (X, d) is a compact metric space. Then every continuous function is bounded $(C(X), \|\cdot\|_\infty) = (C_b(X), \|\cdot\|_\infty)$. In $C(X)$, unless otherwise stated, the norm is $\|\cdot\|_\infty$

3.1 Weierstrass Approximation Theorem

Problem: Given $h \in C([a, b])$ and $\epsilon > 0$. Exists $p(x)$ polynomial on $[a, b]$ such that $\|h - p\|_\infty < \epsilon$?

Remarks

1. We can assume that $[a, b] = [0, 1]$. Assume $f, g \in C([0, 1])$ and $\|f - g\|_\infty < \epsilon$.

Define $\Phi : [a, b] \rightarrow [0, 1]$ and $\Phi(x) = \frac{x-a}{b-a}$, Φ is one-to-one, onto. Then $\Phi' : [0, 1] \rightarrow [a, b]$ then $\Phi^{-1}(x) = (b-a)x + a$. Then $f \circ \Phi, g \circ \Phi \in C([a, b])$. In fact, $\|f \circ \Phi - g \circ \Phi\|_\infty = \|f - g\|_\infty$. Then the map $\Gamma : C[0, 1] \rightarrow C[a, b]$ $\implies (C[a, b], \|\cdot\|_\infty)$. Then $\Gamma(f) = f \circ \Phi$ is an isometric isomorphism with inverse $\Gamma^{-1}(h) = h \circ \Phi^{-1}, \forall h \in C[a, b]$. Also, $\Gamma(p(x))$ is a polynomial if and only if $p(x)$ is a polynomial.

2. We can assume $f(0) = 0, f(1) = 0$. If $f \in C[0, 1]$, let $g(x) = f(x) - [(f(1) - f(0))x + f(0)]$. Then $g(x) \in C[0, 1], g(0) = 0 = g(1)$. if we approximate $g(x)$ uniformly with error at most ϵ by a polynomial, then we can do so for $f(x)$. $\epsilon > |g(x) - p(x)| = |f(x) - \{(f(1) - f(0))x + f(0)\} + p(x)| = |f(x) - p_1(x)|$

Lemma 5. If $n \in \mathbb{N}, (1 - x^2)^n \geq 1 - nx^2, \forall x \in [0, 1]$. Let $f(x) = (1 - x^2)^n - (1 - nx^2)$. $f(0) = 0, f'(x) = \dots > 0$ on $(0, 1)$. Then the inequality follows.

Theorem 47. Weierstrass Approximation Theorem: let $f \in C[a, b]$. Then there exists a sequence $\{p_n(x)\}$ of polynomials such that

$$p_n(x) \rightarrow f(x) \text{ uniformly on } [a, b]$$

Proof. Assume that $[a, b] = [0, 1]$ and $f(0) = 0 = f(1)$. We can extend $f(x)$ to a uniformly continuous function on \mathbb{R} by setting $f(x) = 0$ if x in $(-\infty, 0) \cup (1, \infty)$. Note that $\int_{-1}^1 (1 - x^2)^n dx \neq 0, \forall n$. Pick c_n such that $\int_{-1}^1 c_n (1 - x^2)^n dx = 1$. Let $Q_n(x) = c_n (1 - x^2)^n$. Since $(1 - x^2)^n \geq 1 - nx^2, \forall x \in [0, 1]$.

$$\int_{-1}^1 (1 - x^2)^n dx = 2 \int_0^1 (1 - x^2)^n dx \geq 2 \int_0^{1/\sqrt{n}} (1 - nx^2) dx = \frac{4}{3\sqrt{n}} \geq 1/\sqrt{n}$$

Then $c_n > \sqrt{n}$. If $0 < \delta < 1 \implies \forall x \in [-1, \delta] \cup [\delta, 1]$,

$$c_n(1 - x^2)^n \geq \sqrt{n}(1 - \delta^2)^n$$

$$\text{Let } p_n(x) = \int_{-1}^1 f(x+t)Q_n(t)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt \begin{cases} t < -x \\ t+x < 0 \\ f(t+x) = 0 \end{cases} = \int_0^1 f(u)Q_n(u-x)du$$

$$x)du \begin{cases} u = x+t \\ du = dt \end{cases}$$

$$p_n(x) = \int_0^1 f(u)Q_n(u-x)du$$

$$\frac{d^{2n+1}p(x)}{dx^{2n+1}} \stackrel{\text{leibniz rule}}{=} \int_0^1 f(u) \frac{d^{2n+1}Q_n(u-x)}{dx^{2n+1}} = 0$$

$p_n(x)$ is a polynomial of degree $2n+1$ or less. Let $M = \|f\|_\infty \neq 0$. Let $\epsilon > 0$, choose $0 < \delta < 1$ so that if $|x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$. Since $\int_{-1}^1 Q_n(t)dt = 1$, this implies $f(x) = \int_{-1}^1 f(x)Q_n(t)dt$. If $x \in [0, 1]$,

$$\begin{aligned} |p_n(x) - f(x)| &= \left| \int_{-1}^1 f(x+t)Q_n(t)dt - \int_{-1}^1 f(x)Q_n(t)dt \right| \\ &= \left| \int_{-1}^1 (f(x+t) - f(x))Q_n(t)dt \right| \leq \int_{-1}^1 |f(x+t) - f(x)|Q_n(t)dt \\ &= \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{\delta}^1 |f(x+t) - f(x)|Q_n(t)dt \\ &\leq 2\sqrt{n}(1 - \delta^2)^{n+1}\|f\|_\infty + \frac{\epsilon}{2} + 2\sqrt{n}(1 - \delta^2)^{n+1}\|f\|_\infty \end{aligned}$$

$$|P_n(x) - f(x)| \leq 4M\sqrt{n}(1 - \delta^2)^{n+1} + \frac{\epsilon}{2}$$

Choose n large enough so that

$$4M\sqrt{n}(1 - \delta^2)^{n+1} < \frac{\epsilon}{2} \implies \|p_n - f\|_\infty < \epsilon$$

□

Corollary 17. Let $f(x) \in C[0, 1]$ such that $\int_0^1 f(t)dt = 0, \int_0^1 f(t)t^n dt = 0, \forall n$. This implies $f(x) = 0, \forall x \in [0, 1]$.

Corollary 18. $(C[a, b], \|\cdot\|_\infty)$ is separable. $\forall n \in \mathbb{N}$,

$$P_n = \{a_0 + a_1x + \dots + a_nx^n | a_i \in \mathbb{R}\}$$

$$Q_n = \{r_0 + r_1x + \cdots + r_nx^n | r_1 \in \mathbb{Q}\} \implies \bar{Q}_n = P_n$$

but also

$$\overline{\bigcup_{n=1}^{\infty} P_n} = C[a, b] \implies \overline{\bigcup Q_n} = C[a, b]$$

. Q_n is countable.

3.2 Stone-Weierstrass Theorem

(X, d) compact metric space:

Definition 41. (X, d) compact metric space, $\Phi \subseteq C(X)$ and Φ is a point separating if whenever $x, y \in X$ and $x \neq y$, there exists $f \in \Phi$ such that $f(x) \neq f(y)$.

Remarks

1. $a, b \in X, a \neq b$. $f(x) = d(x, a) \implies f(x) \in C(X)$ and $f(a) \neq f(b)$ Then $C(X)$ is point separating.
2. Suppose X has at least 2 points and $\Phi \subseteq C(X)$. Suppose $f(x) = f(y), \forall f \in \Phi, \forall x, y \in X \implies g(x) = g(y), \forall g \in \Phi, \forall x, y \in X$. Then if Φ is dense in $C(X)$; Φ must be point separating.

Definition 42. A linear subspace $\Phi \subseteq C(X)$ is a lattice if $\forall f, g \in \Phi$ then $(f \vee g)(x) = \max\{f(x), g(x)\} \in \Phi$ and $(f \wedge g)(x) = \min\{f(x), g(x)\} \in \Phi$.

Remarks

Let $f, g \in C(X)$, $(f \vee g)(x) = \frac{(f(x)+g(x))+|f(x)-g(x)|}{2}$ and $(f \wedge g)(x) = -((f \vee g)(x)) \implies f \vee g, f \wedge g \in C(X)$ Then $C(X)$ is a lattice.

If $\Phi \subseteq C(X)$, Φ is a linear subspace. Then Φ is a lattice if $f \vee g \in \Phi, \forall f, g \in \Phi$.

Examples

$f : [a, b] \rightarrow \mathbb{R}$ is piecewise linear if there exists a partition $\mathcal{P} = \{a = t_0 < \cdots < t_n = b\}$ such that $f|_{[t_{i-1}, t_i]} = m_i + d_i, \forall i = 1, \dots, n$.

$f : [a, b] \rightarrow \mathbb{R}$ is piecewise polynomial if $\exists \mathcal{P} = \{a = t_0 < \cdots < t_n = b\}$ such that $f|_{[t_{i-1}, t_i]} = c_{0,i} + c_{1,i}x + \cdots + c_{n,i}x^n$

Theorem 48. Stone-Weierstrass Theorem (Lattice version): (X, d) is compact metric space, $\Phi \subseteq (C(X), \|\cdots\|_{\infty})$ linear subspace such that

1. the constant function $1 \in \Phi$
2. Φ separates points.
3. If $f, g \in \Phi \implies (f \vee g) \in \Phi$

Hence, Φ is dense in $C(X)$.

Note that if $\alpha, \beta \in \mathbb{R}$, and $x \neq y \in X$, then there exists $g \in \Phi$ such that $g(x) = \alpha$ and $g(y) = \beta$. Let $h \in \Phi$ such that $h(x) \neq h(y)$. Let $g(t) = \alpha + (\beta - \alpha) \frac{h(t) - h(x)}{h(y) - h(x)} \implies g \in \Phi$. Let $f \in C(X)$ and $\epsilon > 0$.

Step 1 Fix $x \in X$. For each $y \in X$, $\exists h_{x,y}(t) \in \Phi$ and $h_{x,y}(x) = f(x), h_{x,y}(y) = f(y)$. Since $h_{x,y}(y) - f(y) = 0, \forall y$, we can find $\delta_y > 0$ such that $t \in B(y, \delta_y)$ and $-\epsilon < h_{x,y}(t) - f(t) < \epsilon$. $\{B(y, \delta_y)\}$ open cover of $X \implies \exists$ points y_1, y_2, \dots, y_n such that $\{B(y_i, \delta_{y_i})\}$ cover X .

$$h_x(t) = h_{x,y_1} \vee \dots \vee h_{x,y_n}$$

Now if $z \in X$, $\exists i$ such that $z \in B(y_i, \delta_{y_i})$. $f(z) - \epsilon < h_{x,y_i}(z) \leq h_x(t)$.

Step 2 For each $x \in X$, $h_x(x) - f(x) = 0$. For each $x \in X$, $\exists \delta_x > 0$ such that $t \in B(x, \delta_x)$, then $-\epsilon < h_x(t) - f(t) < \epsilon$. As we did before, we can find $\{x_1, x_2, \dots, x_k\}$ such that $\{B(x_j, \delta_{x_j})\}$ is a cover for X . Let $h(t) = h_{x_1} \wedge \dots \wedge h_{x_k} \in \Phi$. Then if $z \in X$, then $f(z) - \epsilon < h(z) < f(z) + \epsilon$.

Corollary 19. Let $\Phi_1 = \{f \in C[a, b] | f \text{ is piecewise linear}\}$ and $\Phi_2 = \{f \in C[a, b] | f \text{ is piecewise polynomial}\}$. Then Φ_i is dense in $C(X)$, $i = 1, 2, \dots$.

Definition 43. A subspace $\Phi \subseteq C(X)$ is said to be a sub algebra if $f \cdot g \in \Phi$, for every $f, g \in \Phi$.

Example: If P is the collection of all polynomials on $[a, b]$, P is a sub algebra of $C([a, b])$.

Remark:

If $\Phi \subseteq C(X)$ is a sub algebra, then so is Φ . Let $\{f_n\}, \{g_n\} \subseteq \Phi | f_n \rightarrow f, g_n \rightarrow g$. Note that $fg \in C(X)$ Note also $\{g_n\}$ is bounded.

$$\|f_n g_n - fg\|_\infty = \|(f_n g_n - f g_n) + (f g_n - fg)\|_\infty \leq \|g_n\|_\infty \|f_n - f\|_\infty + \|f\|_\infty \|g_n - g\|_\infty \rightarrow 0$$

.

Theorem 49. Subalgebra version) Stone-Weierstrass: (X, d) compact metric space. Let Φ be a linear subspace of $(C(X), \|\cdot\|_\infty)$ such that

1. $1 \in \Phi$.
2. Φ separates points
3. Φ is a subalgebra

Then Φ is dense in $C(X)$.

Proof. Step 1 If $f \in \Phi$, then $|f| \in \bar{\Phi}$. Fix $\epsilon > 0$, since X is compact, $\exists M > 0$ such that $|f(x)| < M, \forall x \in X$. We consider the function $g(t) = |t|$ on $[-M, M]$. By W.A Theorem, $\exists p(t) = c_0 + c_1 t + \dots + c_n t^n$ such that

$$|g(t) - p(t)| = ||t| - p(t)| < \epsilon, \forall t \in [-M, M]$$

but $pf = c_0 1 + c_1 f + c_2 f^2 + \dots + c_n f^n \in \Phi$. If $x \in X, f(x) \in [-M, M]$ and then $||f(x)| - p(f(x))| < \epsilon, \forall x \in X$. This implies $|f| \in \bar{\Phi}$.

Step 2 $hg \in \Phi \implies h \vee g \in \bar{\Phi}$. Then $g \vee h(x) = \frac{(g(x)+h(x))-|g(x)-h(x)|}{2} \in \bar{\Phi}$. Then

1. $1 \in \bar{\Phi}$
2. $\bar{\Phi}$ separates points
3. $\bar{\Phi}$ is a lattice.

Therefore, $\overline{\bar{\Phi}} = C(X) = \bar{\Phi}$.

□

3.3 Complex Version

$C(X, \mathbb{C}) = \{f : X \rightarrow \mathbb{C} | f(x) \text{ is continuous on } X\}$. $\|f\|_\infty = \sup\{|f(x)| | x \in X\}$ A subspace $\Phi \subseteq C(X, \mathbb{C})$ is self-adjoint if $f \in \Phi$ implies that $\bar{f} \in \Phi$.

Theorem 50. *Stone-Weirstrass \mathbb{C} -version* (X, d) compact metric space. If $\Phi \subset C(X, \mathbb{C})$ is a self-adjoint linear subspace such that

1. $1 \in \Phi$
2. Φ separates points
3. Φ is a subalgebra

This implies $\bar{\Phi} = C(X, \mathbb{C})$.

Example

Let $\pi = \{\lambda \in \mathbb{C} | |\lambda| = 1\}$. Let $\phi : \pi \rightarrow [0, 2\pi), e^{i\Theta} \rightarrow \Theta$. On $[0, 2\pi)$ we consider the metric $d_*(\Theta_1, \Theta_2) =$ the shortest at-length between $e^{i\Theta_1}$ and $e^{i\Theta_2}$. Thus ϕ is a homeomorphism. This implies $([0, 2\pi), d_*)$ is compact. $C(\pi) \approx \{f \in C([0, 2\pi)) | f(0) = f(2\pi)\}$. A trigonometric polynomial is an element of

$$Trig_{\mathbb{C}}([0, 2\pi)) = \text{span}\{f(\theta) = e^{in\theta} | n \in \mathbb{Z}\}$$

. This implies $\overline{Trig_{\mathbb{C}}([0, 2\pi))} = C([0, 2\pi))$.

Example:

$\Psi = \{F(x, y) \in C([0, 1] \times [0, 1]) | F(x, y) = \sum_{i=1}^k f_i(x)g_i(y) \text{ for } f_i, g_i \in C[0, 1]\}$. Then to prove that $\bar{\Psi} = C([0, 1] \times [0, 1])$

3.4 Compactness in $(C(X), \|\cdot\|_\infty)$ and the Ascoli-Arzelà Theorem

Definition 44. (X, d) metric space. $A \subseteq X$ is relatively compact if \bar{A} is compact. Remark: Assume (X, d) is complete. Recall; if A is totally bounded, then \bar{A} is totally bounded. Then $A \subseteq X$ is relatively compact $\iff A$ is totally bounded.

Theorem 51. Arzelà-Ascoli Theorem: Let (X, d) be a compact metric space. Let $\mathcal{F} \subseteq (C(X), \|\cdot\|_\infty)$. Then, TFAE:

1. \mathcal{F} is relative compact
2. \mathcal{F} is equicontinuous and pointwise bounded.

Proof. 1 to 2: \mathcal{F} is relative compact. This implies that \mathcal{F} is bounded. This implies \mathcal{F} is point wise bounded. Fix $\epsilon > 0$. \mathcal{F} is relative compact. This implies \mathcal{F} is totally bounded. This implies there exists an $\frac{\epsilon}{3}$ -net $\{f_1, \dots, f_n\} \subseteq \mathcal{F}$. Since $\{f_1, \dots, f_n\}$ is finite, it's equicontinuous. Given $\frac{\epsilon}{3}$, there exists $\delta > 0$ such that $d(x, y) < \delta$. This implies $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}, \forall i = 1, 2, \dots, n$. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $d(x, y) < \delta$. This implies $\exists i_0 \in \{1, \dots, n\}$ such that $\|f_{i_0} - f\| < \frac{\epsilon}{3}$. Then $|f(x) - f(y)| \leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(y) - f(y)| < \frac{\epsilon}{3} \times 3 = \epsilon$. \square

Definition 45. Compact operators $\Gamma : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$ linear map is compact if $\Gamma(\{x \in X \mid \|x\|_X \leq 1\})$ is relatively compact.

Remark:

Γ is compact $\implies \Gamma$ is continuous.

Example: $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y) = (C([a, b]), \|\cdot\|_\infty)$. Let $K : [a, b] \times [a, b] \implies [a, b]$ continuous. If $f \in C([a, b])$. $\Gamma(f)(x) = \int_a^b k(x, y)f(y)dy$. Clearly, Γ is linear.

Claim: $\Gamma(f) \in C([a, b])$. If $f = \theta$, $\Gamma(f) \in C[a, b]$. If $f \neq \theta$, since K is uniformly continuous given $\epsilon > 0, \exists \delta > 0$ such that $\|(x_1, y_1) - (x_2, y_2)\|_2 < \delta \implies |K(x_1, y_1) - K(x_2, y_2)| < \frac{\epsilon}{(b-a)\|f\|_\infty}$. Now if $|x - z| < \delta$, then $|\Gamma(f)(x) - \Gamma(f)(z)| = |\int_a^b (K(x, y) - K(z, y))f(y)dy| \leq \int_a^b |K(x, y) - K(z, y)||f(y)|dy < \frac{\epsilon}{(b-a)\|f\|_\infty} \|f\|_\infty (b-a) = \epsilon$

Claim: $\Gamma(B_x[0, 1])$ is uniformly equicontinuous. Fix $\epsilon > 0$. there exists $\delta_1 > 0$ such that $|x - z| < \delta_1 \implies |K(x, y) - K(z, y)| < \frac{\epsilon}{b-a}, \forall y \in [a, b]$.

let $|x - z| < \delta_1$ and $f \in C([a, b])$ such that $\|f\|_\infty \leq 1$. $|\Gamma(f)(x) - \Gamma(f)(z)| \leq \int_a^b |K(x, y) - K(z, y)||f(y)|dy < \epsilon$

Claim: $\Gamma(B_x[\theta, 1])$ is uniformly bounded. Let $M > 0$ such that $|K(x, y)| \leq M, \forall (x, y) \in [a, b] \times [a, b]$. Let $f \in C[a, b]$ such that $\|f\|_\infty \leq 1$. $|\Gamma(f)(x)| \leq \int_a^b |K(x, y)||f(y)|dy \leq M \int_a^b dy = M(b-a), \forall x \in [a, b]$. Therefore, for all $f \in [a, b]$ such that $\|f\|_\infty < 1$. This implies $\Gamma(B_x[\theta, 1])$ is relatively compact by Arzelà Ascoli Theorem,. Therefore, Γ is compact.

Theorem 52. *Peano's Theorem: Let f be continuous on an open subset D of \mathbb{R}^2 . Let $(x_0, y_0) \in D$. Then the differential equation*

$$y' = f(x, y)$$

has a local solution through the point (x_0, y_0) . Let R be a closed rectangle, $R \subseteq D$, with $(x_0, y_0) \in \text{int}(R)$. f is continuous on R , R compact; then there exists $M \geq 1$ such that $|f(x, y)| \leq M, \forall (x, y) \in R$. Let $W = \{(x, y) \in R \mid |y - y_0| \leq M|x - x_0|\}$ and $I = [a, b] = \{x \mid (x, y) \in W \text{ for some } y\}$. By uniform continuity, given $\epsilon > 0, \exists 0 < \delta < 1$, such that if $(x_1, y_1), (x_2, y_2) \in W, |x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta \implies |f(x_1, y_1) - f(x_2, y_2)| < \epsilon$. Choose $a = x_0 < x_1 < \dots < x_n = b$, with $|x_j - x_{j-1}| < \frac{\delta}{M}, \forall j$. On $[x_0, b]$, we define a function $k_\epsilon(x)$:

$$k_\epsilon(x_0) = y_0$$

, and on $[x_0, x_1]$, $k_\epsilon(x)$ is linear and has slope $f(x_0, y_0)$. On $[x_1, x_2]$, $k_\epsilon(x)$ is linear and has slope $f(x, k_\epsilon(x_1))$ and proceed like this to define a piecewise linear function $k_\epsilon(x)$ on $[x_0, b]$.

Note: the graph of $k_\epsilon(x)$ is contained in W and $|k_\epsilon(x) - k_\epsilon(\bar{x})| \leq M|x - \bar{x}|, \forall x, \bar{x} \in [x_0, b]$. Let $x \in [x_0, b], x \neq x_j, j = 0, 1, \dots, n$. This implies there exists j such that $x_{j-1} < x < x_j$.

$$|k_\epsilon(x) - k_\epsilon(x_{j-1})| \leq M|x - x_{j-1}| < M\frac{\delta}{M} = \delta$$

This implies by uniform continuity of f ,

$$|f(x_{j-1}, k_\epsilon(x_{j-1})) - f(x, k_\epsilon(x))| < \epsilon$$

but $k_\epsilon^{+'}(x_{j-1}) = f(x_{j-1}, k_\epsilon(x_{j-1}))$ (slope approaching by the right). This implies $|k_\epsilon^{+'}(x_{j-1}) - f(x, k_\epsilon(x))| < \epsilon, \forall x \in [x_0, b]$ such that $x \neq x_i, i = 0, 1, \dots, n$. Let $K = \{k_\epsilon \mid \epsilon > 0\}$. K is pointwise bounded: $(k_\epsilon(x) \in W \subseteq R \text{ compact})$ K is equicontinuous. () By Arzela-Asidli, K is compact. Let $x \in [x_0, b], k_\epsilon(x) = y_0 + \int_{x_0}^x k'_\epsilon(t) dt = y_0 + \int_{x_0}^x f(t, k_\epsilon(t)) dt + [(k'_\epsilon(t) - f(t, k_\epsilon(t)))] dt$. Consider the sequence $\{k_{\frac{1}{n}}(x)\}_n \subseteq \bar{K}$. This implies \exists subsequence $\{k_{\frac{1}{n_k}}(x)\}_k$ converging uniformly on $[x_0, b]$ to some $k(x)$. f uniformly continuous on W . This implies $\{f(t, k_{\frac{1}{n_k}}(t))\}$ converges uniformly to $f(t, k(t))$ on $[x_0, b]$. $k_\epsilon(t) = y_0 + \int_{x_0}^x f(t, k_\epsilon(t)) dt$. This implies $k(x)$ is a solution to the DE on $[x_0, b]$. Similarly we can find a solution $k^*(x)$ on $[a, x_0]$*