Lossy Trapdoor Functions

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Motivation

Lossy Trapdoor Functions and Their Applications

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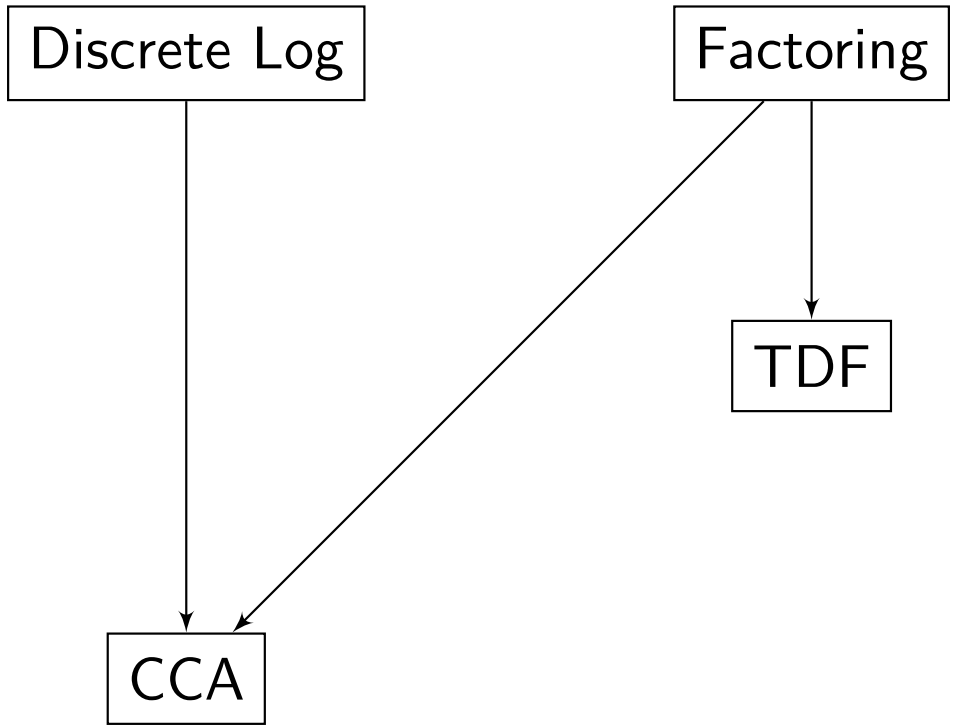
- Trapdoor Functions are a basic primitive, but hard to instantiate
- CCA Security from factoring and discrete log but not lattices

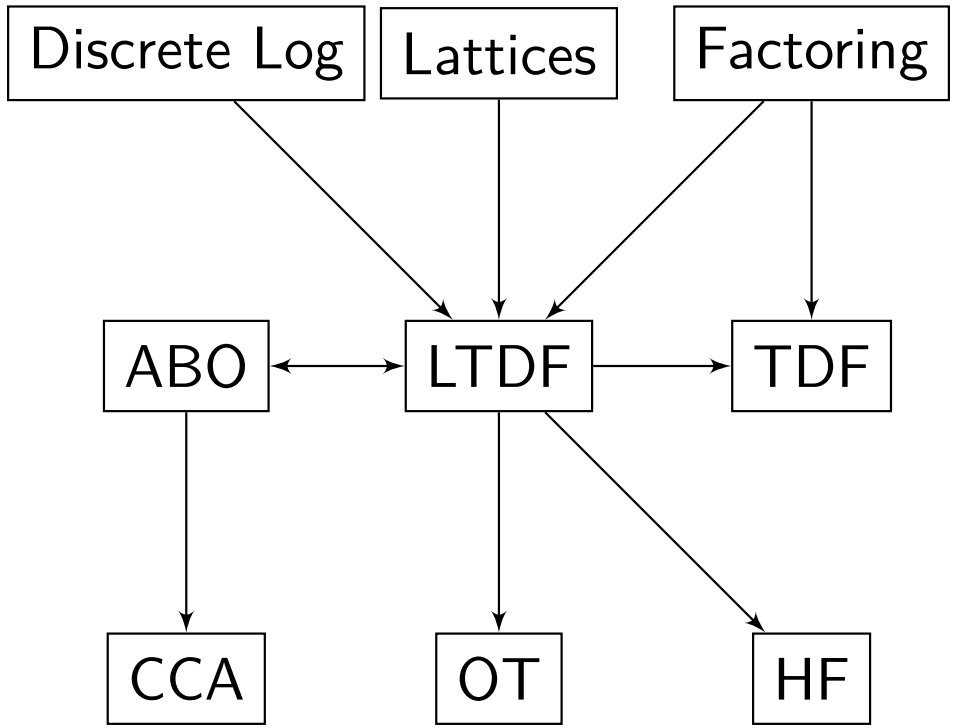
Paper Results

- Introduce Lossy Trapdoor Functions (LTDFs)
- Realize LTDFs from factoring, discrete log *and* lattices
- Show LDTFs imply TDFs
- Black box construction of CCA-secure (witness recovering) cryptosystems, collision-resistant hash functions and oblivious transfer protocols.

Connections



Connections



Trapdoor Functions

Informally, a trapdoor function is family of functions that are hard to invert without access to some additional information called a trapdoor

Definition

A trapdoor function consists of three PPT algorithms (S, F, F^{-1}) such that:

- *Easy to sample and invert with trapdoor.* $S(1^\lambda) \to (s, t)$ such that $F(s, -)$ is an injective function on $\{0,1\}^n$ and $F^{-1}(t, -)$ is its inverse
- *Hard to invert without.* For *any* PPT inverter $\mathcal{A}$ we have that $\mathcal{A}(1^\lambda, s, F(s, x))$ outputs x with negligible probability.

Example of Trapdoor

RSA Encryption! In trapdoor form:

- $S(1^\lambda)$ generates N, e, d as in RSA, set $s := (N, e)$ and $t := (d)$ and returns (s, t)
- $F(s, x)$ computes $x^e \mod N$
- $F^{-1}(t, c)$ computes $c^d \mod N$

Composite Residuosity

- $S(1^\lambda)$ generates $N = pq$ as a product of large primes, select g suitably, $s := (N, g)$, $t := (p, q)$
- $F(s, x)$ splits $x = m_1 + Nm_2$ and returns $g^{m_1} m_2^N \mod N^2$
- $F^{-1}(t, c)$ decrypts using the factorization to compute Carmichael function

Lossy Trapdoors

Informally, you either get an injective trapdoor or a 'lossy' function, and *cannot tell which is which*

Definition

A (n, k)-lossy trapdoor function consists of three PPT algorithms (S, F, F^{-1}). We denote $S_{inj}(-) \triangleq S(-, 0)$ and $S_{lossy}(-) \triangleq S(-, 1)$.

- *Outputs of S_{inj} are easy to compute and easy to invert with trapdoor.* $S_{inj}(1^\lambda) \to (s, t)$ s.t. that $F(s, -)$, $F^{-1}(t, -)$ are in the trapdoor case
- *Outputs of S_{lossy} are easy to compute.* $S_{lossy}(1^\lambda) \to (s, \perp)$ s.t. $F(s, -)$ is a function on $\{0, 1\}^n$ with image size at most 2^{n-k}.
- The first outputs of $S_{inj}(1^\lambda)$ and $S_{lossy}(1^\lambda)$ are computationally indistinguishable.

Subleties

- The definition really relates to a collection of lossy trapdoor functions.
- $k \triangleq k(\lambda) = \text{poly}(\lambda) \leq n$ is a parameter that represents how 'lossy' the collection is.
- We also write $r \triangleq n - k = \text{poly}(\lambda)$ as the *residual leakage*.
- No hardness requirement on inverting outputs of S_{inj}
- Requirements are too strict in lattices, leads to *almost-always* lossy functions.

All-But-One TDFs

Intuition: Most branches[1] are trapdoors, except one which is lossy. You cannot tell which one it is.

Definition

An (n, k)-ABO TDF is a triple of PPT algorithms S, F, F^{-1} such that:

- $S(1^\lambda, b^*) \to (s, t)$ as before
- For any $b \neq b^*$, $F(s, b, -)$, $F^{-1}(t, b, -)$ are as in the previous definition.
- $F(s, b^*, -)$ is a lossy function as before
- For any b, b' the first outputs of $S(1^\lambda, b)$, $S(1^\lambda, b')$ are computationally indistinguishable.

[1] Subsequent work often calls these tags

LTDF $\Longrightarrow$ TDF

- Completeness: Use the injective functions generated by S_{inj}.
- Soundness: We cannot (information theoretically) invert the lossy branch, so if we could invert the injective trapdoors we could distinguish outputs of S_{inj}, S_{lossy}, contradicting LDTFness.
- Formally, let $\mathcal{A}$ be an inverter. We build $\mathcal{D}$

$$\underline{\mathcal{D}^{\mathcal{A}}(s)}$$

$$x \leftarrow\!\!\$\ \{0,1\}^n$$
$$y = F(s, x)$$
$$x' = \mathcal{A}(s, y)$$
$$\textbf{return } x = x'$$

If $s \leftarrow S_{inj}(1^\lambda)$ then it succeeds nonneglibly, while otherwise it will fail

LTDF $\implies$ CCA

Requirements

We will have some requirements primitives[2]. We note that our cryptosystem will have message space $\{0,1\}^{\ell}$.

- We have $\Sigma = (\mathrm{Gen}, \mathrm{Sign}, \mathrm{Vfy})$ a strongly unforgeable one-time signature scheme. We require that the public keys are in $\{0,1\}^{v}$.
- $F = (S_{ltdf}, F_{ltdf}, F_{ltdf}^{-1})$ is a (n,k)-lossy trapdoor function.
- $G = (S_{abo}, F_{abo}, F_{abo}^{-1})$ is a (n,k')-ABO trapdoor function with branch space $\{0,1\}^{v}$.
- $\mathcal{H}$ is a collection of pairwise independent hash functions $\{0,1\}^{n} \to \{0,1\}^{\ell}$.
- We require that $k + k' \geq n + \kappa$ for some $\kappa = \omega(\log n)$ and that $\ell \leq \kappa - 2\lg(1/\epsilon)$ from $\epsilon = \mathsf{negl}(\lambda)$

[2] All of these reduce to LTDFs

LTDF $\Longrightarrow$ CCA

Encryption Scheme

$\mathcal{G}(1^\lambda)$

$(s,t) \leftarrow S_{inj}(1^\lambda)$
$(s',t') \leftarrow S_{abo}(1^\lambda, 0^v)$
$h \leftarrow_\$ \mathcal{H}$
$pk := (s, s', h)$
$sk := (t, t', pk)$
return (pk, sk)

$\mathcal{E}(pk, m)$

$(vk, sk_\sigma) = \mathrm{Gen}(1^\lambda)$
$x \leftarrow_\$ \{0,1\}^n$
$c_1 = F_{ltdf}(s, x)$
$c_2 = G_{abo}(s', vk, x)$
$c_3 = m \oplus h(x)$
$\omega \leftarrow \mathrm{Sign}(sk_\sigma, (c_i)_{i=1}^3)$
return $(vk, c_1, c_2, c_3, \sigma)$

$\mathcal{D}(sk, c)$

if $\neg\mathrm{Vfy}(vk, (c_i)_{i=1}^3, \sigma)$
  return $\perp$
fi
$x = F^{-1}(t, c_1)$
if $c_1 \neq F_{ltdf}(s, x) \vee$
  $c_2 \neq G_{abo}(s, vk, x)$
  return $\perp$
fi
return $c_3 \oplus h(x)$

LTDF $\implies$ CCA

CCA Game

Correctness is easy to check. We next show security in the single encryption CCA security game. Below we show the formal game definition. Setup is to be called once at the beginning of the game, and the attacker is allowed a single query to EncO and oracle access to DecO. The attacker wins if it outputs $b' = b$.

$\mathrm{Setup}(\lambda)$	$\mathrm{EncO}(m_0, m_1)$	$\mathrm{DecO}(c^*)$
$b \leftarrow\!\!\$ \{0,1\}$	$c \leftarrow \mathcal{E}(pk, m_b)$	**if** $c^* \in \mathcal{T}_{enc}$
$\mathcal{T}_{enc} = \emptyset$	$\mathcal{T}_{enc} := \mathcal{T}_{enc} \cup \{c\}$	**return** $\perp$
$pk, sk \leftarrow \mathcal{G}(\lambda)$	**return** c	**fi**
return pk		**return** $\mathcal{D}(sk, c^*)$

LTDF $\Longrightarrow$ CCA

Game Hops

We proceed by a sequence of games. We note that, since a single query is made to EncO we move the signature scheme generation in Setup and denote that verification key as vk^*.

$G_1(\lambda)$: This is the original CCA Security Game

$G_2(\lambda)$: In DecO if $vk = vk^*$ return $\perp$

$G_3(\lambda)$: In Setup choose the lossy branch of G to be vk^*

$G_4(\lambda)$: In DecO find x using G's trapdoor rather than F's

$G_5(\lambda)$: In Setup replace S_{inj} with S_{lossy}

The hops are as follows:

$$G_1 \approx_{\Sigma} G_2 \approx_{abo} G_3 \equiv G_4 \approx_{ltdf} G_5$$

Finally, an argument as in TDF case shows that in G_5 even an unbounded attacker has only negligible success probability.

Realizations

We can realize LTDF from *any* encryption scheme that is:

- Additively Homomorphic. This allows to encrypt either $\mathbf{I}_n$ or $\mathbf{0}_n$ indistinguishably and to evaluate at $\mathbf{x}$ by computing $\mathbf{xM}$.
- Secure to Reuse Randomness, so that we can use the same randomness with different keys securely.
- Isolated Randomness, so that it is only dependent on the input randomness and not on keys/messages.

We shows next a realization from the DDH assumption, but a similar technique can also be employed with lattices (based on LWE) with some difficulties.

DDH $\implies$ LTDF

Consider the following variant of the ElGamal cryptosystem, with public key $h = g^z$ and secret key z. The encryption function is $E_h(m; r) = (g^r, h^r g^m)$ for randomness r. To decrypt (c_1, c_2) we output $\log_g(c_2/c_1^z)$ which is easy to compute if $m \in \{0, 1\}$. This scheme is semantically secure and it is additively[3] homomorphic i.e.

$$E_h(m; r) \odot E_h(m'; r') = E_h(m + m', r + r')$$

$$E_h(m; r)^x = E_h(mx; rx)$$

[3] All operations done component wise

DDH $\implies$ LTDF

We show how to use the previous scheme to encrypt a matrix $\mathbf{M} = (m_{i,j}) \in \mathbb{Z}_p^{n \times n}$. Select n pk/sk pairs $h_i = g^{z_i}$ and n pieces of randomness r_i. Then the encryption is the matrix $\mathbf{C} = (c_{i,j}) = (E_{h_j}(m_{i,j}; r_i))$ with the z_js as decryption keys. We can represent $\mathbf{C}$ as the following two matrices:

$$\mathbf{C_1} = \begin{bmatrix} g^{r_1} \\ \vdots \\ g^{r_n} \end{bmatrix}, \ \mathbf{C_2} = \begin{bmatrix} h_1^{r_1} g^{m_{1,1}} & \dots & h_n^{r_1} g^{m_{1,n}} \\ \vdots & \ddots & \vdots \\ h_1^{r_n} g^{m_{n,1}} & \dots & h_n^{r_n} g^{m_{n,n}} \end{bmatrix}$$

Via n^2 hybrid games we can show that this encryption produces indistinguishable ciphertext under DDH. We denote this operation as $\mathrm{ME}(\mathbf{z}, \mathbf{M})$ for $\mathbf{z}$ the vector of private keys.

DDH $\Longrightarrow$ LTDF

We build a LTDF from the previous scheme as it follows. Note that in the injective case we encrypt the identity matrix, while in the lossy the all zero matrix.

$S_{inj}(1^\lambda)$	$S_{lossy}(1^\lambda)$	$F_{ltdf}(\mathbf{C}, \mathbf{x})$
$\mathbb{G} \leftarrow \mathcal{G}(1^\lambda)$	$\mathbb{G} \leftarrow \mathcal{G}(1^\lambda)$	$\mathbf{return}\ \mathbf{y} = \mathbf{x} \cdot \mathbf{C}$
$\mathbf{z} \leftarrow\!\!\$\ \mathbb{Z}_p^n$	$\mathbf{z} \leftarrow\!\!\$\ \mathbb{Z}_p^n$	
$\mathbf{C} \leftarrow \mathrm{ME}(\mathbf{z}, \mathbf{I}_n)$	$\mathbf{C} \leftarrow \mathrm{ME}(\mathbf{z}, \mathbf{0}_n)$	$F_{ltdf}^{-1}(\mathbf{z}, \mathbf{y})$
$\mathbf{return}\ (\mathbf{C}, \mathbf{z})$	$\mathbf{return}\ (\mathbf{C}, \perp)$	$x_i = D_{z_i}(y_i)$
		$\mathbf{return}\ \mathbf{x}$

DDH $\Longrightarrow$ LTDF

Subleties

- $\mathcal{G}$ is the group generation algorithms, it returns (G, p, g) where G is a cyclic group of prime order p with generator g. We assume DDH hardness w.r.t. $\mathcal{G}$.
- $\mathbf{xC}$ is computed by the homomorphic property. In fact, if $\mathbf{C} = \mathrm{ME}(\mathbf{z}, \mathbf{M})$ with randomness $\mathbf{r}$ and $h_j = g^{z_j}$

$$y_j = \bigodot_{i=1}^{n} c_{i,j}^{x_i} = E_{h_j}((\mathbf{xM})_j; R \triangleq \langle \mathbf{r}, \mathbf{x} \rangle)$$

- Note that if $\mathbf{M} = \mathbf{I}_n$ then $y_j = E_{h_j}(x_j; R)$
- If instead $\mathbf{M} = \mathbf{0}_n$ then $y_j = E_{h_j}(0; R)$

DDH $\Longrightarrow$ LTDF

Final Checks

Now, we just have to check that the LTDF conditions are satisfied. In particular, the above construction is $(n, n - \lg p)$-lossy.

- The three algorithms are clearly PPT
- A quick thought shows that the injective conditions are met
- Indistinguishability follows from the indistinguishability of ME.
- Finally, for outputs generated by S_{loss} we have that $y_i = E_{h_i}(0; R)$ for some $R \in \mathbb{Z}_p$ that depends on x. R can take at most p values, the residual leakage is at most $\lg p$ and so the loss is $k = n - r \geq n - \lg p$

Things which I did not have time to show

- ABO $\equiv$ LTDF (see extra)
- More efficient ABO construction from DDH
- LDTFs from LWE
- CPA from LTDFs
- SUF one time signatures from LDTFs
- UOWHFs, CRHFs from LDTFs
- OT from LDFTs

Related Work

- All-But-N LTDFs
- All-But-Many LTDFs
- Identity Based LTDFs

Thank You!

Notation and Entropy

- λ is the security parameter, and we will abbreviate $n(\lambda) = \text{poly}(\lambda)$ as simply n
- $f(-)$ denotes the function taking $x \mapsto f(x)$
- Write $H_\infty(X)$ for the min-entropy of X. This corresponds to the optimal probability of guessing X.
- We let $\widetilde{H}_\infty(X|Y)$ be the average min-entropy of X conditioned on Y. This corresponds to the optimal probability of guessing X knowing Y.
- We use the following lemma, if Y takes at most 2^r values then:

$$\widetilde{H}_\infty(X|Y) \geq H_\infty(X) - r$$

LTDF $\Longrightarrow$ TDF

Note that if s is generated by S_{inj} then with some non negligible probability we have that $\mathcal{A}$ succeeds and $\mathcal{D}$ succeeds whenever $\mathcal{A}$ does.
Instead, if s is generated by S_{lossy} even an unbounded adversary would have best possible probability given by $2^{-\widetilde{H}_\infty(x|s,F(s,x))}$. But note that $F(s,-)$ takes at most 2^r values and so by the previous lemma $\widetilde{H}_\infty(x|s,F(s,x)) \geq H_\infty(x|s) - r = n - (n-k) = k$. So the probability is bounded by 2^{-k} and as such is negligible.
From the above it follows that $\mathcal{D}$ will win the distinguishing game with non negligible probability.

ABO $\equiv$ LTDF

- ABOs and LTDFs are equivalent.
- ABO $\implies$ LTDF. Take ABO on $\{0,1\}$ and evaluate always on one of the branches, but switch lossy branch on generation.
- LTDF $\implies$ ABO. Generate an ABO on $\{0,1\}$ by having $s = (s_0, s_1)$ where one of the two is lossy, and evaluation by using s_b
- Finally, we can extend ABOs on $\{0,1\}$ to ABOs on $\{0,1\}^\ell$ at the cost of having residual leakage ℓr. The idea is, for lossy branch $b^* \in \{0,1\}^\ell$, generate ℓ ABOs each with the i-th having lossy branch b^*_i.