Elliptic Curve Cryptography

an introduction which is entirely too short

by Giacomo Fenzi (ETH Zurich)
on 6 January 2022

2022-01-07

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Often very hard, and in general undecidable!
Let us see what we can do...

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» One variable

$$a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a = 0$$

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- Given a rational point, all of them can be found geometrically
- Hasse principle allows us to test if a rational point exists

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» Fields

Definition

A field K is set together with two operations $+, \cdot$ such that

* K is an abelian group under $+$ with identity 0
* $K - \{0\}$ is an abelian group under multiplication with identity 1.
* For every $a, b, c \in K$ we have that $a(b + c) = ab + ac$
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» Finite Fields

We are mostly interested in finite fields.:

Theorem

For every prime p, and every $n \in \mathbb{Z}^+$ there is an unique field of size p^n, which we denote by either $\mathbb{GF}(p^n)$ or $\mathbb{F}_{p^n}$

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If $n = 1$, then $\mathbb{F}_p = \mathbb{Z}_p$, if not we can write them as

$$\mathbb{F}_{p^n} = \frac{\mathbb{F}_p[X]}{(f(x))}$$

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» Characteristic

For any field, $\mathrm{char}(\mathbb{F})$ is the least integer ℓ such that

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or ∞ if no such integer exists. We have that $\mathrm{char}(\mathbb{F}_{p^n}) = p$.

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1. If two fields have different char, no map between them (apart 0)
2. Freshman's dream

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Let k, K be two fields. If there is an homomorphism $k \to K$, we can identify k with a subfield of K. In that case, K is a **field extension** of k which we denote by $k \subseteq K$.

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2. Can approximate $\bar{\mathbb{F}}$ with $\mathbb{F}_{p^{n!}}$
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Some examples:

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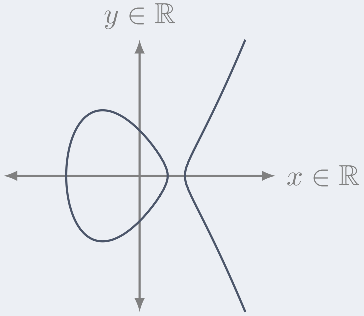

$y^2 = x^3 - 2x + 1$ over $\mathbb{R}$

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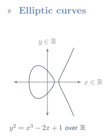
└ Elliptic Curves
  └ Representation and Group Law
    └ Elliptic curves



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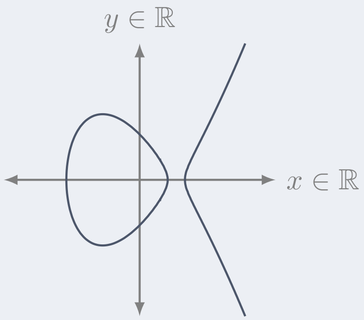

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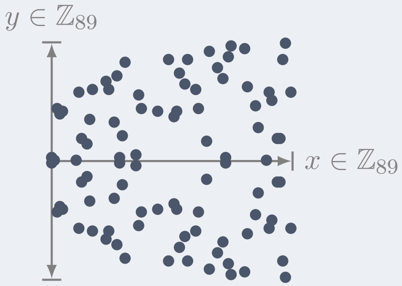
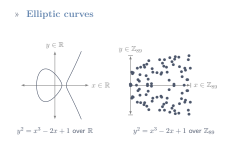
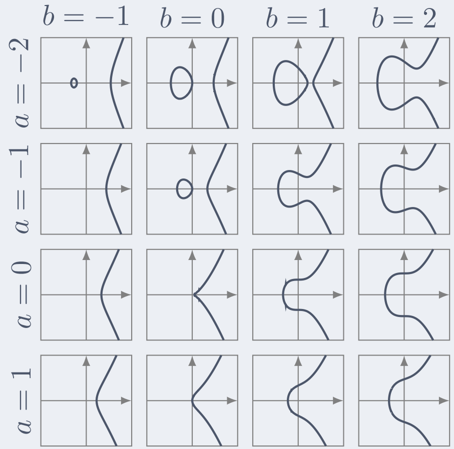


$y^2 = x^3 - 2x + 1$ over $\mathbb{Z}_{89}$



» Some elliptic curves



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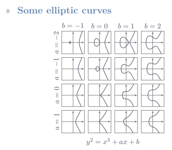
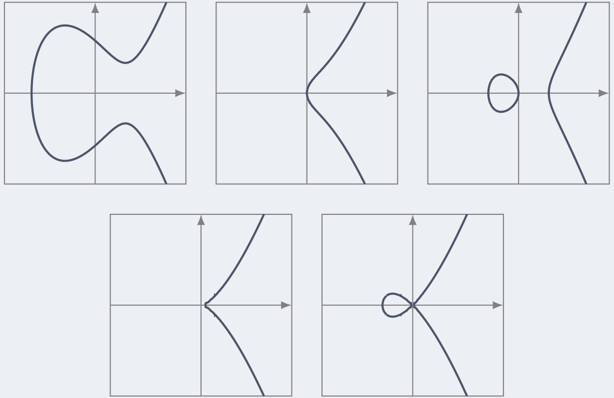


» More elliptic curves

$y^2 = x^3 + -3x + 3$ $\quad$ $y^2 = x^3 + x$ $\quad$ $y^2 = x^3 - x$



$y^2 = x^3 - x$ $\quad$ $y^2 = x^3 + x^2$

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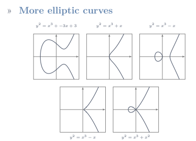

1. Top are non singular
2. First singular has cusp
3. Second has a node

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The **discriminant** of E is

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A curve is **singular** if $\Delta = 0$.

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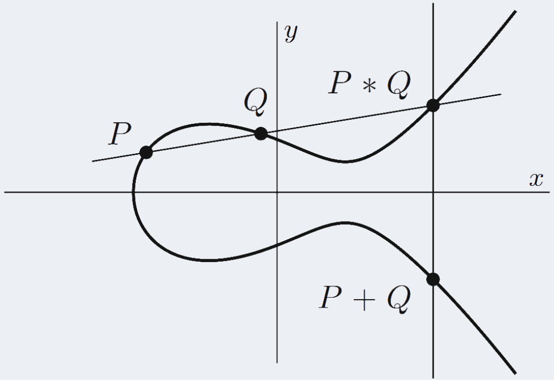

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Elliptic Curve Cryptography
└─Elliptic Curves
  └─Representation and Group Law
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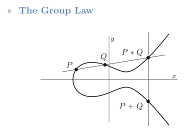

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$K = [x]Q_B$		$K = [y]Q_A$

Correctness follows since:

$$K = [x]Q_B = [x][y]P = [xy]P = [y][x]P = [y]Q_A = K$$

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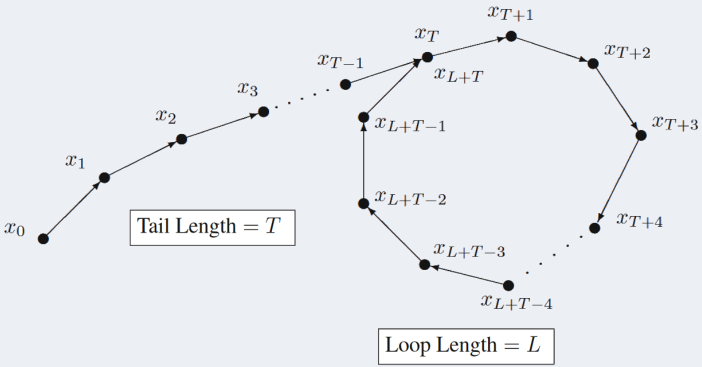

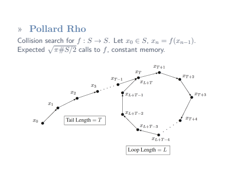

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$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, y \cdot \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \right)$$

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If an isogeny factors trough a Frobenius isogeny it is inseparable. If it is a Frobenius followed by an isomorphisms, it is purely inseparable.

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Let $E/k : y^2 = x^3 + ax + b$, with $\mathrm{char}(k) = p$. Define $E^{(p^r)} : y^2 = x^3 + a^{p^r}x + b^{p^r}$. The map:

$$\pi : E \to E^{(p^r)}, (x, y) \mapsto \left(x^{p^r}, y^{p^r}\right)$$

is the (p^r)-**Frobenius isogeny**. Note if $k = \mathbb{F}_{p^r}$ then $E^{(p^r)} = E$

If an isogeny factors trough a Frobenius isogeny it is inseparable. If it is a Frobenius followed by an isomorphisms, it is purely inseparable.

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2. In our case they are a bit of a nuisance for a theorem later on

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» Computing large degree isogenies

* Velu's formula are too slow for large degree

4- Take $H \cong \mathbb{Z}_{\ell^k}$. Set $\ker \psi_i = [\ell^{k-i}](\psi_{i-1} \circ \cdots \circ \psi_1)(H)$. Then $\deg(\psi_i) = \ell$ and

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└Elliptic Curves
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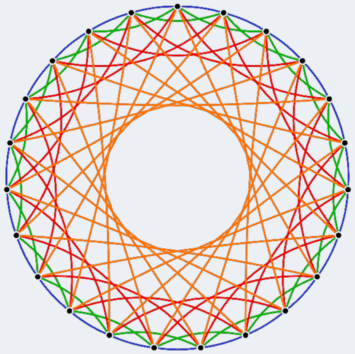
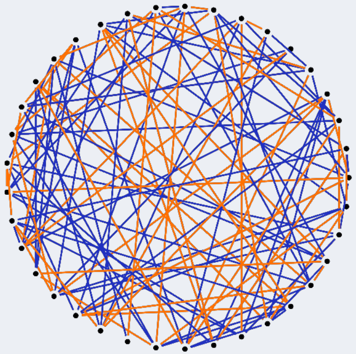
Look something like this! We focus on the second





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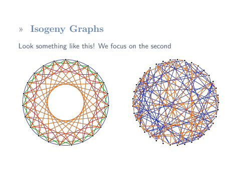

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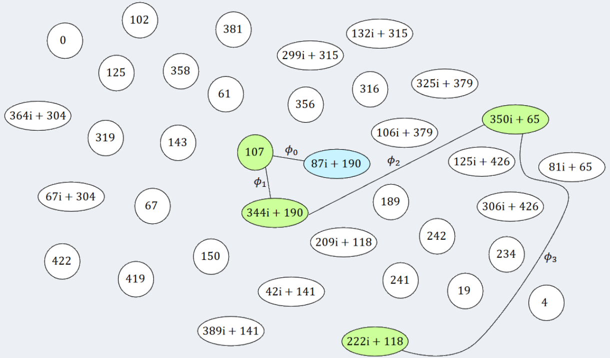

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Elliptic Curve Cryptography
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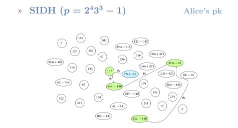

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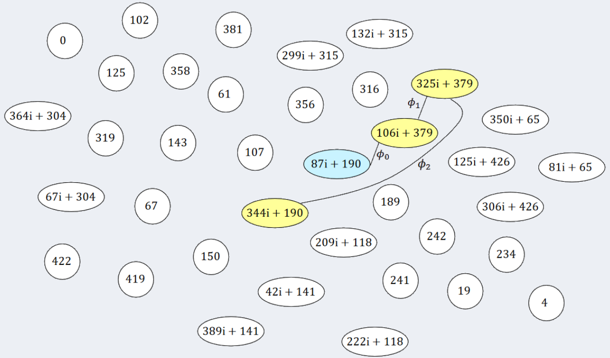

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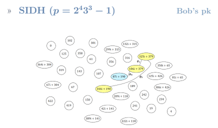

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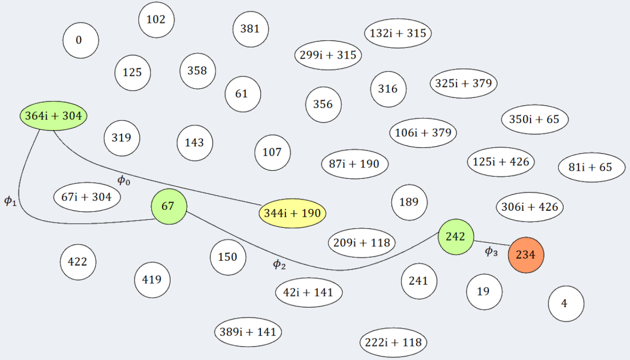

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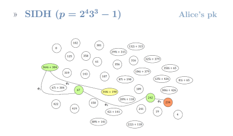

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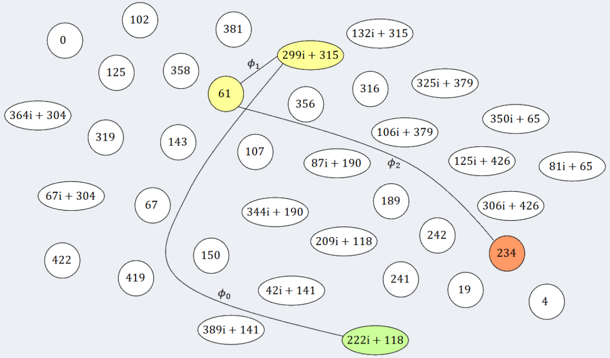

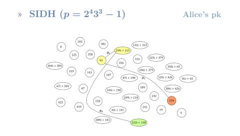

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Picture to keep in mind:

$$
\begin{tikzcd}
E \arrow[r, "\phi_A"] \arrow[d, "\phi_B"'] & E/\langle A\rangle \arrow[d, "\phi'_B"] \\
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Details will follow

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1. Alice computes the left to right ones
2. Bob computes the down arrows

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- Historical Notes follow mostly [0, Introduction]
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- Definition of elliptic curve [1, III.2.2, III.3] or [0, 2.2]
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- Representation example expanded in [6, 5.3.1]
- Complexity estimates from [0, 4.5] and [1, XI.4]
- Diffie Hellman from [everywhere?]
- Singular curves are bad [0, 3.15] and [1, III.2.5] and [6, 5.3.3]
- Small Embedding degree ECDLP [1, XI.6] and [6, 5.2.2]
- Supersingular curves breaking ECDLP [1, XI.6.4] and [6, 5.2.2]
- Anomalous curves breaking ECDLP [1, XI.6.5] and [6, 5.2.2] and [6, 5.3.3]
- Descent methods in [6, 5.2.2]
- Pollard Rho description [1, XI.5.3-5.4]
- Pairings adapted from [1, III.8.1]
- Weil Pairing computation [1, XI.8]
- Modified Weil Pairing and Distorsion map [1, XI.7]

» Detailed References & Credits
- Representation example expanded in [6, 5.3.1]
- Complexity estimates from [0, 4.5] and [1, XI.4]
- Diffie Hellman from [everywhere?]
- Singular curves are bad [0, 3.15] and [1, III.2.5] and [6, 5.3.3]
- Small Embedding degree ECDLP [1, XI.6] and [6, 5.2.2]
- Supersingular curves breaking ECDLP [1, XI.6.4] and [6, 5.2.2]
- Anomalous curves breaking ECDLP [1, XI.6.5] and [6, 5.2.2] and [6, 5.3.3]
- Descent methods in [6, 5.2.2]
- Pollard Rho description [1, XI.5.3-5.4]
- Pairings adapted from [1, III.8.1]
- Weil Pairing computation [1, XI.8]
- Modified Weil Pairing and Distorsion map [1, XI.7]

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- BLS Signatures [1, XI.7.4]
- Isogeny definition [1, III.4]
- Isogeny Example from [3, 2.1]
- Isogeny properties (summary) [3, 2.1]
- Isogeny and Group Hom. [1, III.4.8]
- Isogeny composition, degree and multiplicativity [1, III.4]
- Dual Isogeny [1, III.6]
- Frobenius isogeny and separability [3, 2.1.2]
- Kernels and Velu [3, 2.2] and [1, III.4.12]
- Supersingular curves [1, V.3.1]
- Number of curves [1, V.4.1c]
- Points of supersingular curve [3, 1.8]

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- Isogenous with same number of points [1, Ex. 5.4]
- Graphs from L. Panny's [lekenpraatje]
- Vertices as elements of $\mathbb{F}_{p^2}$ from [1, V.3.1]
- Good mixing properties from [CGL06]
- SIDH diagrams and description from [5]
- SIKE [sike]
- vOW function from [4, 3.1] and [ACV+18]
- vOW description [4, 3.2] and [vOW98]

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» Further Reading

* Attacks on SIDH [torsion] [GPST]
* Mathematics of Isogeny Based Cryptography [deFeo17]
* vOW attack estimation [vOW98] [ACV+18] [CLN+19] [LWS20]
* Verifiable Delay Functions from Isogenies and Pairings [dFMPS19]
* Delfs-Galbraith attack [DG16] [SCS21]

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