Elliptic Curve Cryptography

an introduction which is entirely too short

by Giacomo Fenzi (ETH Zurich)
on 6 January 2022

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- Fascinating mathematically

» Outline

* Historical Notes

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* Isogenies

» Diophantine Equations

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Often very hard, and in general undecidable!
Let us see what we can do...

» One variable

$$a_nx^n + a_{n-1}x^{n-1} + \ldots a_1x + a = 0$$

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Quite easy! We can show that:

Theorem

Let $\frac{p}{q} \in \mathbb{Q}$ be a solution of the above equation. Then q divides a_n and p divides a_0.

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Check the finite list of candidates.
Alternatively, solve numerically and find candidate of form $\frac{b}{a_n}$

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These are rational points on a conic.

- Given a rational point, all of them can be found geometrically
- Hasse principle allows us to test if a rational point exists

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What about:

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Elliptic Curves $\neq$ Ellipse

» Fields

Definition

A field K is set together with two operations $+, \cdot$ such that

* K is an abelian group under $+$ with identity 0
* $K - \{0\}$ is an abelian group under multiplication with identity 1.
* For every $a, b, c \in K$ we have that $a(b + c) = ab + ac$
* $0 \neq 1$

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Informally, we can add, subtract, multiply and divide non zero elements.

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We are mostly interested in finite fields.:

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For every prime p, and every $n \in \mathbb{Z}^+$ there is an unique field of size p^n, which we denote by either $\mathbb{GF}(p^n)$ or $\mathbb{F}_{p^n}$

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If $n = 1$, then $\mathbb{F}_p = \mathbb{Z}_p$, if not we can write them as

$$\mathbb{F}_{p^n} = \frac{\mathbb{F}_p[X]}{(f(x))}$$

where $f(x)$ is an irreducible polynomial of degree n.

» Characteristic

For any field, $\mathrm{char}(\mathbb{F})$ is the least integer ℓ such that

$$\underbrace{1 + \ldots 1}_{\ell \text{ times}} = 0,$$

or ∞ if no such integer exists. We have that $\mathrm{char}(\mathbb{F}_{p^n}) = p$.

» Field Extensions

Let k, K be two fields. If there is an homomorphism $k \to K$, we can identify k with a subfield of K. In that case, K is a **field extension** of k which we denote by $k \subseteq K$.

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Given any field K we can construct the algebraic closure $\overline{K}$ which is the smallest algebraically closed extension containing K.
Some examples:

* $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
* $\mathbb{F}_p \subseteq \mathbb{F}_{p^2} \subseteq \mathbb{F}_{p^3} \cdots \subseteq \overline{\mathbb{F}}_p$

» Weierstrass Form

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$$\Big\downarrow \ \mathrm{char}(K) \neq 2, 3$$

$$y^2 = x^3 + ax + b$$

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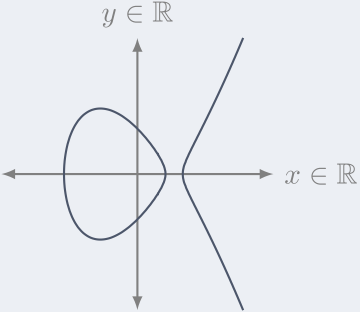
Mathematicians are often interested with $E(\mathbb{Q}) \subseteq E(\mathbb{R}) \subseteq E(\mathbb{C})$ but we mostly consider the finite case.

» Elliptic curves



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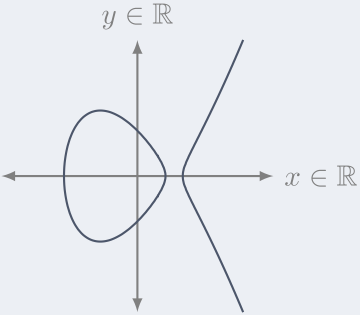

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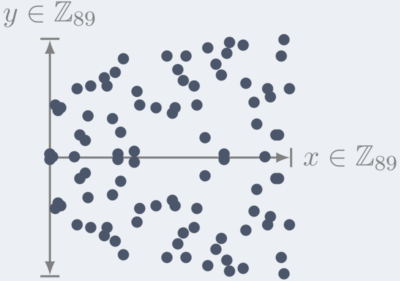
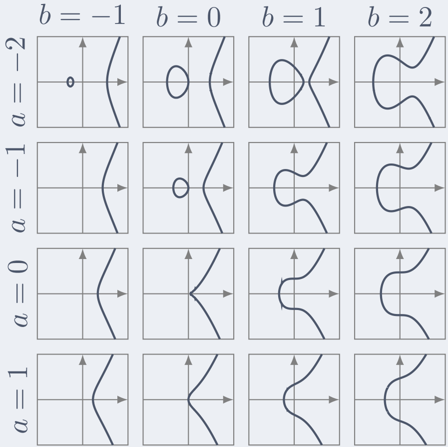


$y^2 = x^3 - 2x + 1$ over $\mathbb{Z}_{89}$

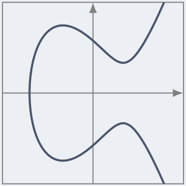
» Some elliptic curves



$$y^2 = x^3 + ax + b$$

» More elliptic curves

$y^2 = x^3 + -3x + 3$



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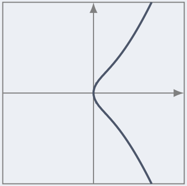

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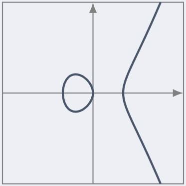

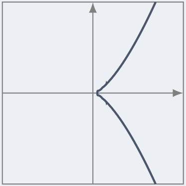

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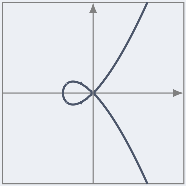

$y^2 = x^3 + x^2$

» Discriminant

Definition

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve.
The **discriminant** of E is

$$\Delta = -16(4a^3 + 27b^2)$$

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From now on, all curves are assumed non singular.

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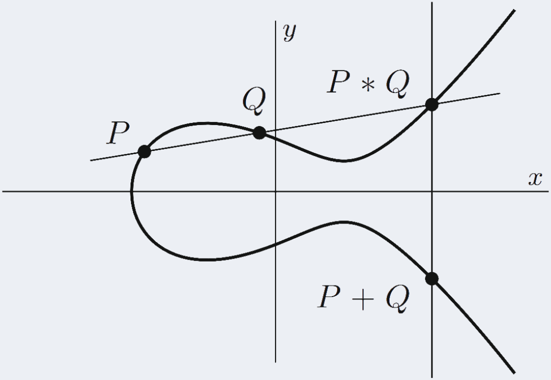
for $u \in \overline{K}^*$ and this yields:

Theorem

Let E, E' be two elliptic curves over K. Then $E \cong E'$ over $\overline{K}$ if and only if $j(E) = j(E')$.

» The Group Law



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* Let $x_3 = \lambda^2 - x_1 - x_2$, $y_3 = \lambda(x_1 - x_3) - y_1$ where λ is:

$$\lambda = \begin{cases} \frac{y_2 - y_1}{x_2 - x_1}, & x_1 \neq x_2 \\ \frac{3x_1^2 + a}{2y_1}, & \text{otherwise} \end{cases}$$

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This makes E into an abelian group with identity ∞

» Scalar multiplication

For $n > 0, P \in E$ we write $[n]P = \underbrace{P + \cdots + P}_{n \text{ times}}$. We then extend the notation by letting $[0]P = \infty$ and $[-n]P = [n](-P)$.

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For $m \in \mathbb{Z}$ we define a map $[m] : E \to E$ accordingly, and write:

$$E[m] := \ker[m]$$

to be the m-**torsion subgroup** of E.

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Exact value can be efficiently found using Schoof's algorithm in $O((\log q)^8)$.

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Definition

Let $\mathrm{Gen}(1^\lambda)$ be a p.p.t. algorithm that returns a group description $\mathbb{G} = (+, P, q)$, where $\mathbb{G} = \langle P \rangle$ and $q = \#\mathbb{G}$. For an attacker $\mathcal{A}$, define

$$\mathsf{Adv}_{\mathcal{A}}^{\mathrm{dlp}}(\lambda) = \Pr\left[\mathcal{A}\left(1^\lambda, \mathbb{G}, [k]P\right) = k \;\middle|\; \begin{array}{c} \mathbb{G} \leftarrow\$ \, \mathrm{Gen}(1^\lambda) \\ k \leftarrow\$ \, \mathbb{Z}_q \end{array}\right]$$

We say that the **discrete logarithm assumption** hold with respect to Gen if, for every p.p.t. attacker $\mathcal{A}$, $\mathsf{Adv}_{\mathcal{A}}^{\mathrm{dlp}}(\cdot)$ is negligible.

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Representation matters! $\mathbb{Z}_{p-1} \cong \mathbb{Z}_p^*$ as groups but the discrete logarithm is trivial in the former, assumed hard in the latter.

» Why elliptic curves?

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Assumption	Group	Best Algorithm	$\approx$ Complexity
RSA	$\mathbb{Z}_N$	Number Field Sieve	$\exp(c^3\sqrt{\log N})$
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* Shorter keysizes (≈ 256 vs 3072 bits)
* Faster computation

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Diffie Hellman

Alice		**Bob**
$x \xleftarrow{\$} \mathbb{Z}_q$		$y \xleftarrow{\$} \mathbb{Z}_q$
$Q_A = [x]P$		$Q_B = [y]P$
	$\xrightarrow{Q_A}$	
	$\xleftarrow{Q_B}$	
$K = [x]Q_B$		$K = [y]Q_A$

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$K = [x]Q_B$		$K = [y]Q_A$

Correctness follows since:

$$K = [x]Q_B = [x][y]P = [xy]P = [y][x]P = [y]Q_A = K$$

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* Curves defined over $\mathbb{F}_{p^k}$ for k with small factors. GHS Method, Diem's Analysis.

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Collision search for $f : S \to S$. Let $x_0 \in S$, $x_n = f(x_{n-1})$.

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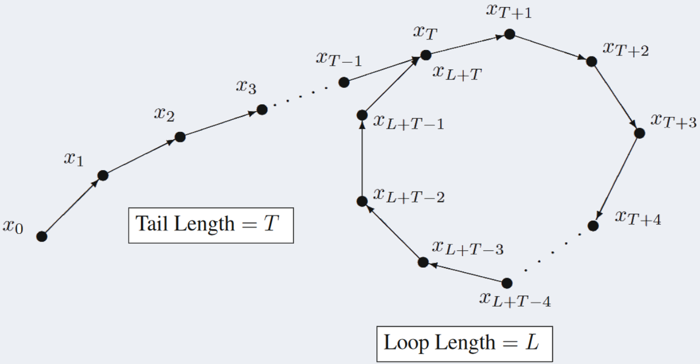

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Let $X_0 = \infty$, then $X_i = [\alpha_i]P + [\beta_i]Q$ and we can track α_i, β_i. A collision $X_j = X_{j+\ell}$ with $\gcd(\beta_{j+\ell} - \beta_j, N) = 1$ allows us to solve the DLP with

$$k \equiv \frac{\alpha_j - \alpha_{j+\ell}}{\beta_{j+\ell} - \beta_j} \pmod{N}$$

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* Alternating:

$$e(T, T) = 1$$

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It is degenerate on cyclic subgroups of $E[m]$, so use modified Weil pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : E[m] \times E[m] &\to \mu_m \\ \langle P, Q \rangle &= e_m(S, \phi(Q)) \end{aligned}$$

For $\phi : E \to E$ a distorsion map

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Let $\mathbb{G}, \mathbb{G}_T$ be cyclic groups of prime order p. Let P be a generator of $\mathbb{G}$, and e a non degenerate pairing. Also, let $H : \{0,1\}^* \to \mathbb{G}$

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Correctness by:

$$e(\sigma, P) = e([x]Q, P) = e(Q, P)^x = e(Q, [x]P) = e(H(m), [x]P)$$

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- **Isogenies!**

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For example, the curves $y^2 = x^3 + x$ and $y^2 = x^3 - 3x + 3$ are isogenous over $\mathbb{F}_{71}$ via the isogeny

$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, y \cdot \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \right)$$

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* An isogeny between two Weierstrass curves has the form
$$(x, y) \mapsto \left(\frac{f}{h^2}(x), y \cdot \frac{g}{h^3}(x)\right)$$

» Separable and Inseparable Isogenies

Definition

Let $E/k : y^2 = x^3 + ax + b$, with $\mathrm{char}(k) = p$. Define $E^{(p^r)} : y^2 = x^3 + a^{p^r}x + b^{p^r}$. The map:

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» Kernel and Velu

Theorem

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Let E/k, with k a finite field. For any subgroup $H \leq E$ we can find an isogeny with kernel H in $\Theta(\#H)$ using Velu's formulas. We denote the target of that isogeny by E/H

» Computing large degree isogenies

* Velu's formula are too slow for large degree

4- Take $H \cong \mathbb{Z}_{\ell^k}$. Set $\ker \psi_i = [\ell^{k-i}](\psi_{i-1} \circ \cdots \circ \psi_1)(H)$. Then $\deg(\psi_i) = \ell$ and

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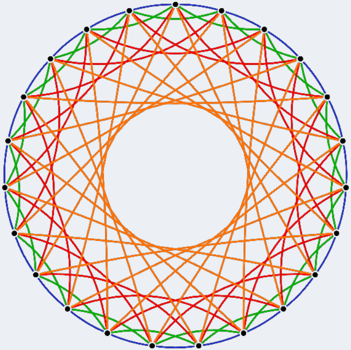
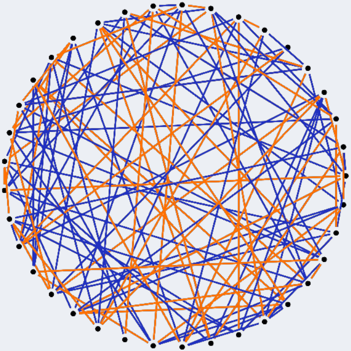
Definition

The **computational supersingular isogeny problem** is as follows: Given two supersingular elliptic curves E, E', find an isogeny between them.

» Isogeny Graphs

Look something like this! We focus on the second





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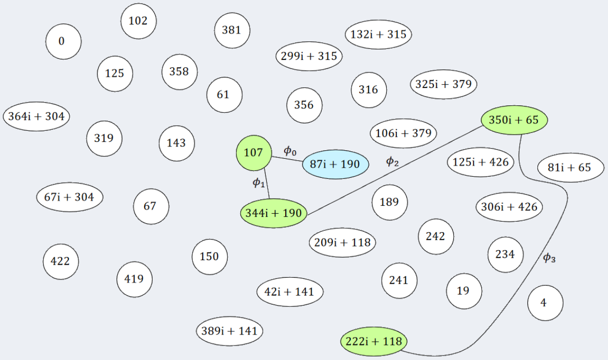
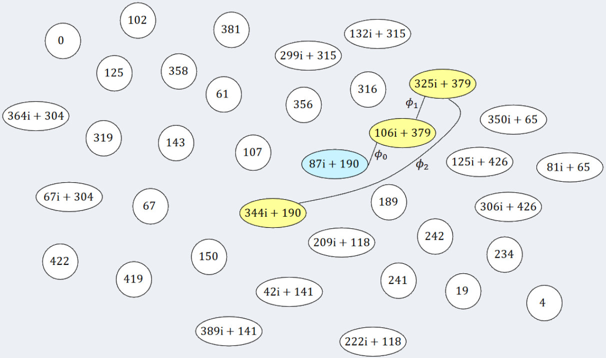
* Most vertices have degree $\ell + 1$

» SIDH ($p = 2^4 3^3 - 1$) Alice's pk



» SIDH ($p = 2^4 3^3 - 1$) Bob's pk



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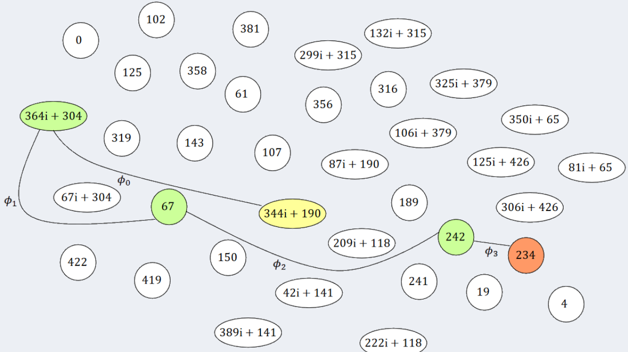

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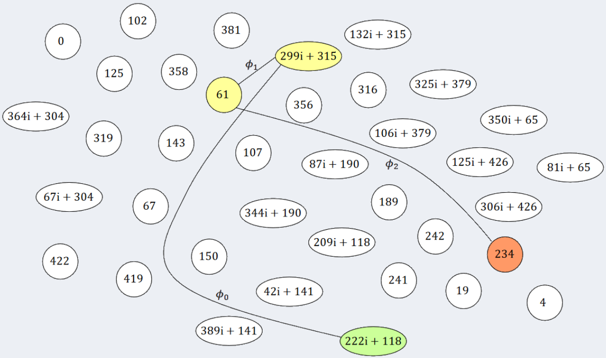

» SIDH

Picture to keep in mind:

$$\begin{tikzcd} E \arrow[r, "\phi_A"] \arrow[d, "\phi_B"'] & E/\langle A\rangle \arrow[d, "\phi'_B"] \\ E/\langle B\rangle \arrow[r, "\phi'_A"'] & E/\langle A, B\rangle \end{tikzcd}$$

Details will follow

» SIDH

Parties select $p = 2^{e_A}3^{e_B} - 1$ prime,

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- The final secret is $j(E_{AB}) = j(E_{BA})$

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$$S_{i,k} := \left\{ H \leq E_i[\ell^k] \mid \; H \text{ cyclic}, |H| = \ell^k \right\}$$
$$S := (\{0\} \times S_{0,k}) \sqcup (\{1\} \times S_{1,k})$$
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A collision $g(0, H) = g(1, H')$ will solve CSSI. To enable Pollard-Rho style methods, let $h : \mathbb{F}_{p^2} \to S$ be a hash function, and let:

$$f : S \to S,\; f := h \circ g$$

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$$\frac{2.5}{m}\sqrt{\#S^3/w} \cdot t = O(p^{3/8})$$

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* SIKE leverages isogenies for post quantum security

» Resources

0 J.H. Silverman, J.T. Tate, Rational Points on Elliptic Curves

1 .H. Silverman, The Arithmetic of Elliptic Curves[1]

2 D.A. Cox, Primes of the form $x^2 + ny^2$

3,4 L. Panny, notes: [intro] [isogenies problems]

5 C. Costello, Supersingular isogeny key exchange for beginners

6 R. Granger, A. Joux, Computing Discrete Logarithms [5.2, 5.3]

7 P. Aluffi, Algebra: Chapter 0

8 S. Galbraith, Mathematics of Public Key Cryptography

[1] The bible

» Detailed References & Credits

- Historical Notes follow mostly [0, Introduction]
- Origin of the name elliptic can be found [here]
- Fields discussed in [7, III.1.14, VII]
- Weierstrass form in [1, III.1]
- Definition of elliptic curve [1, III.2.2, III.3] or [0, 2.2]
- Elliptic curves diagram from [iacr] and curves from [1, Fig 3.1, 3.2]
- Discriminant, j-invariant formula from [1, III.1]
- Discriminant interpretation [0, 2.3]
- Isomorphism form [1, III.3.1b]
- Theorem j-invariance [1, III.1.4b]

» Detailed References & Credits

* Group Law diagram [0, Fig 1.16]
* Formulae [1, III.2.3]
* Scalar multiplication notation [1, III.2]
* Multiplication isogeny [1, III.4.1]
* Double and add [1, XI.1]
* Torsion subgroup [1, III.4]
* Hasse's theorem [1, V.1.1]
* Schoof's algorithm [1, XI.3]
* DLP and related assumption [8. III.13]
* Partial Equivalence of CHD and DLP in [Maurer] [Fifield]

Detailed References & Credits

* Representation example expanded in [6, 5.3.1]
* Complexity estimates from [0, 4.5] and [1, XI.4]
* Diffie Hellman from [everywhere?]
* Singular curves are bad [0, 3.15] and [1, III.2.5] and [6, 5.3.3]
* Small Embedding degree ECDLP [1, XI.6] and [6, 5.2.2]
* Supersingular curves breaking ECDLP [1, XI.6.4] and [6, 5.2.2]
* Anomalous curves breaking ECDLP [1, XI.6.5] and [6, 5.2.2] and [6, 5.3.3]
* Descent methods in [6, 5.2.2]
* Pollard Rho description [1, XI.5.3-5.4]
* Pairings adapted from [1, III.8.1]
* Weil Pairing computation [1, XI.8]
* Modified Weil Pairing and Distorsion map [1, XI.7]

» Detailed References & Credits

- BLS Signatures [1, XI.7.4]
- Isogeny definition [1, III.4]
- Isogeny Example from [3, 2.1]
- Isogeny properties (summary) [3, 2.1]
- Isogeny and Group Hom. [1, III.4.8]
- Isogeny composition, degree and multiplicativity [1, III.4]
- Dual Isogeny [1, III.6]
- Frobenius isogeny and separability [3, 2.1.2]
- Kernels and Velu [3, 2.2] and [1, III.4.12]
- Supersingular curves [1, V.3.1]
- Number of curves [1, V.4.1c]
- Points of supersingular curve [3, 1.8]

» Detailed References & Credits

- Isogenous with same number of points [1, Ex. 5.4]
- Graphs from L. Panny's [lekenpraatje]
- Vertices as elements of $\mathbb{F}_{p^2}$ from [1, V.3.1]
- Good mixing properties from [CGL06]
- SIDH diagrams and description from [5]
- SIKE [sike]
- vOW function from [4, 3.1] and [ACV+18]
- vOW description [4, 3.2] and [vOW98]

Further Reading

- Attacks on SIDH [torsion] [GPST]
- Mathematics of Isogeny Based Cryptography [deFeo17]
- vOW attack estimation [vOW98] [ACV+18] [CLN+19] [LWS20]
- Verifiable Delay Functions from Isogenies and Pairings [dFMPS19]
- Delfs-Galbraith attack [DG16] [SCS21]