Finite Element Approximations for the Vlasov Equations

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The Vlasov Systems

The Vlasov equation describes the time evolution of the plasma distribution function under self-consistent electromagnetic fields. Let $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{v} \in \mathbb{R}^3$, and $t \in \mathbb{R}^+$; the distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ satisfies the normalized Vlasov equation:

$$\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f = 0, \tag{1}$$

where $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are determined by Maxwell's equations:

$$\partial_t \boldsymbol{E} = \nabla_{\boldsymbol{x}} \times \boldsymbol{B} - \boldsymbol{J}, \qquad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho,$$
$$\partial_t \boldsymbol{B} = -\nabla_{\boldsymbol{x}} \times \boldsymbol{E}, \qquad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0,$$

with charge density $\rho(\boldsymbol{x}, t)$ and current density $\boldsymbol{J}(\boldsymbol{x}, t)$ given by

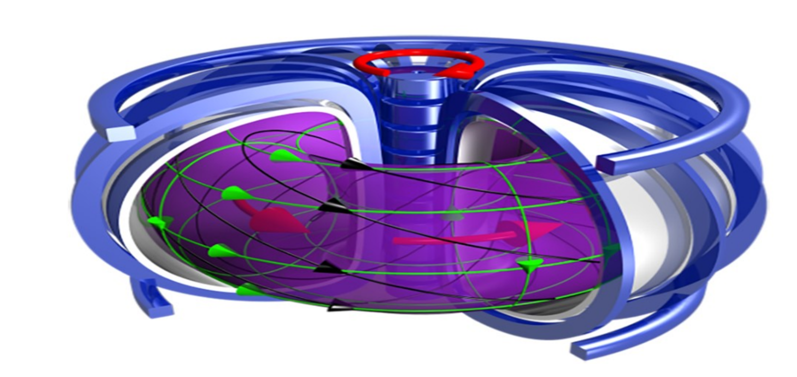
$$\rho = \int_{\mathbb{R}^3} f \, \mathrm{d}\boldsymbol{v} - \rho_0, \qquad \boldsymbol{J} = \int_{\mathbb{R}^3} f \boldsymbol{v} \, \mathrm{d}\boldsymbol{v}.$$

The above system is known as the **Vlasov–Maxwell** system. It is also common to set $\boldsymbol{B} \equiv 0$ and solve for the electric field using $\boldsymbol{E} = -\nabla_{\boldsymbol{x}} \Phi$, where the electric potential is obtained by solving the Poisson equation $-\nabla_{\boldsymbol{x}}^2 \Phi = \rho$. The resulting simplified model is referred to as the **Vlasov–Poisson** system.

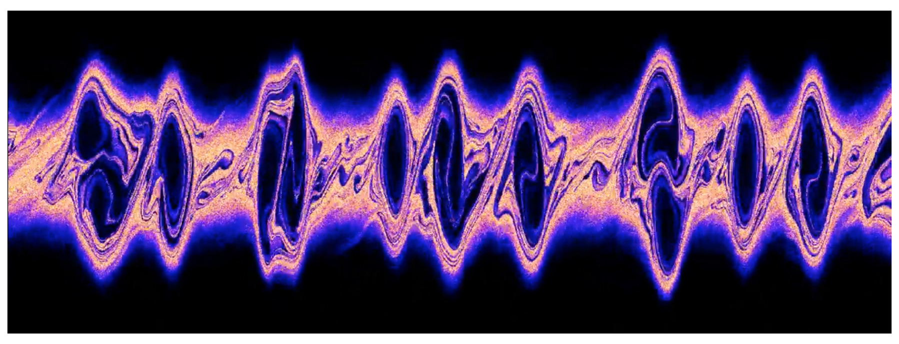
Vlasov systems have broad applications in plasma physics, astrophysics, and fusion energy research.

Applications:

The tokamak fusion reactor



Plasma instability



Challenges:

- The Vlasov equation is hyperbolic, which can produce spurious oscillations in numerical solutions.
- The system is up to 6D, making it computationally expensive to solve numerically.
- Fundamental physical properties must be preserved, such as $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0$.

Anisotropic artificial viscosity

The Vlasov equation is advection-dominated, so we add artificial viscosity to (1) to stabilize the numerical solution:

$$\partial_t f + \boldsymbol{\beta} \cdot \nabla f - \nabla_{\boldsymbol{x}} \cdot (\mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} f) - \nabla_{\boldsymbol{v}} \cdot (\mathsf{A}_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} f) = 0, \tag{2}$$

where $\mathsf{A}_{\boldsymbol{x}}$ and $\mathsf{A}_{\boldsymbol{v}}$ are the artificial viscosity tensors in physical and velocity space, respectively. The viscosity is anisotropic and independent in each direction, i.e.,

$$\mathsf{A}_{\boldsymbol{x}} = \mathrm{diag}(\varepsilon_{x_1}, \varepsilon_{x_2}, \varepsilon_{x_3}), \qquad \mathsf{A}_{\boldsymbol{v}} = \mathrm{diag}(\varepsilon_{v_1}, \varepsilon_{v_2}, \varepsilon_{v_3}),$$

where the viscosity coefficients ε_{x_i} and ε_{v_i} are non-negative and may vanish in smooth regions.

For Vlasov equations, the components of the convection field can have different units due to their distinct physical meanings. Therefore, it is more appropriate to use a tensor-valued viscosity rather than a scalar one [2].

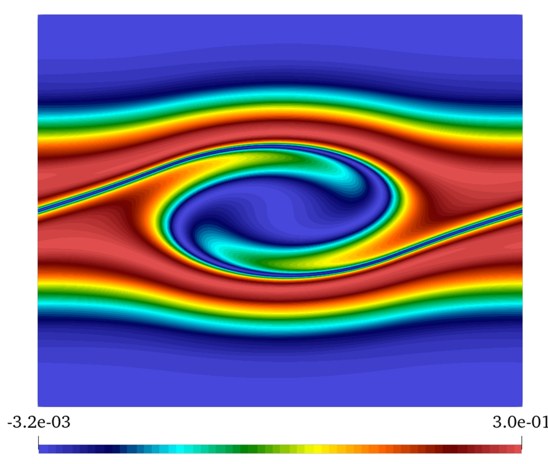
Residual-based artificial viscosity (RV)

The viscosity coefficients are computed based on the residual of the Vlasov equation (1). In each direction, the viscosity is defined as

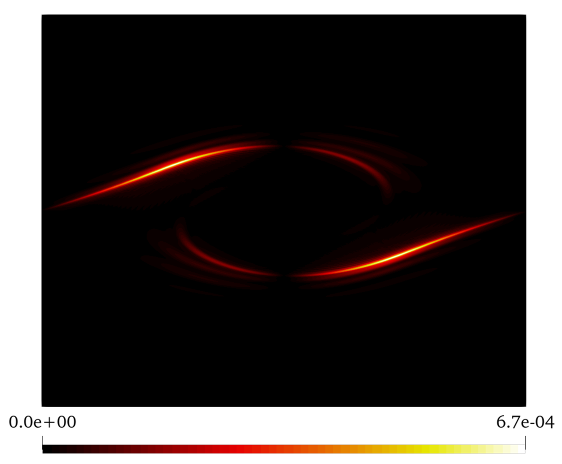
$$\varepsilon_l = \min\left(\frac{1}{2} h_l \|\beta_l\|_{\mathrm{loc}},\ h_l^2 \frac{\|R(f)\|_{\mathrm{loc}}}{\|f - \bar{f}\|}\right), \qquad l = x_1, x_2, x_3, v_1, v_2, v_3,$$

where $R(f)$ is the residual of the Vlasov equation (1), and h_l is the local mesh size.

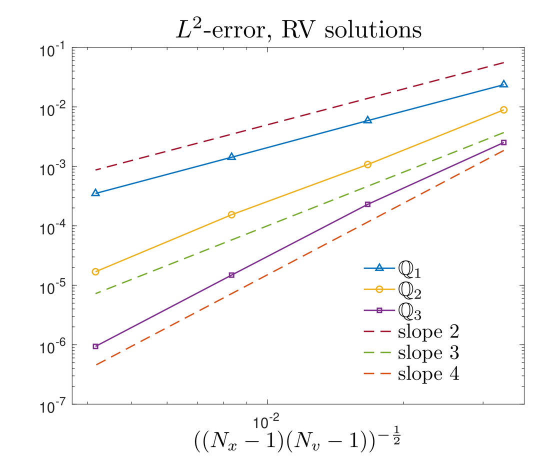
- The first-order viscosity $\frac{1}{2} h_l \|\beta_l\|_{\mathrm{loc}}$ serves as an upper bound for the artificial viscosity.
- A normalization function $\|f - \bar{f}\|$ is used to remove the unit dependence of f in the residual.
- The viscosity is large in regions with high residuals (e.g., near discontinuities) and small in smooth regions, leading to a more accurate and stable solution.



Solution



Residual viscosity



Convergence rates of RV

A novel residual viscosity method

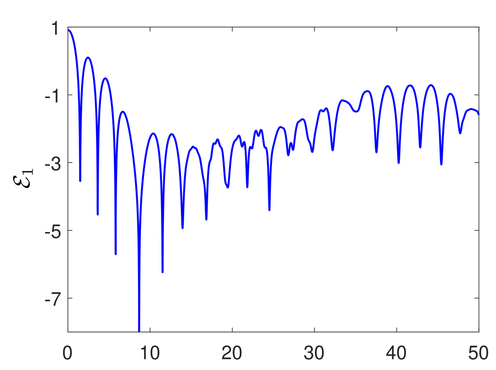
Let $\mathsf{A}_{\boldsymbol{x}} := \mathsf{A}_{\boldsymbol{x}}(\boldsymbol{x}, t)$ and $\mathsf{A}_{\boldsymbol{v}} := \mathsf{A}_{\boldsymbol{v}}(\boldsymbol{v}, t)$. Integrating the Vlasov equation (2) over velocity and spatial space gives:

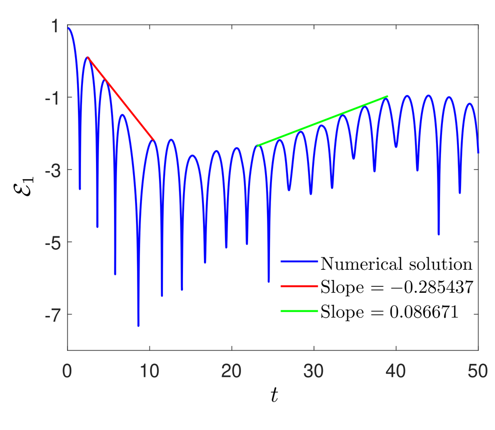
$$\partial_t u_x + \nabla_{\boldsymbol{x}} \cdot \mathbf{F}_{\boldsymbol{x}}(u_x) - \nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{x}} u_x) = 0,$$
$$\partial_t u_v + \nabla_{\boldsymbol{v}} \cdot \mathbf{F}_{\boldsymbol{v}}(u_v) - \nabla_{\boldsymbol{v}} \cdot (\nabla_{\boldsymbol{v}} u_v) = 0,$$

and the residual-based viscosities are computed as:

$$\varepsilon_{x_i}^{\mathsf{H}} = h_{x_i}^2 \frac{\|R(u_x)\|_{\mathrm{loc}}}{\|u_x - \bar{u}_x\|} \frac{d_{\boldsymbol{x}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}},$$
$$\varepsilon_{v_i}^{\mathsf{H}} = h_{v_i}^2 \frac{\|R(u_v)\|_{\mathrm{loc}}}{\|u_v - \bar{u}_v\|} \frac{d_{\boldsymbol{v}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}}.$$

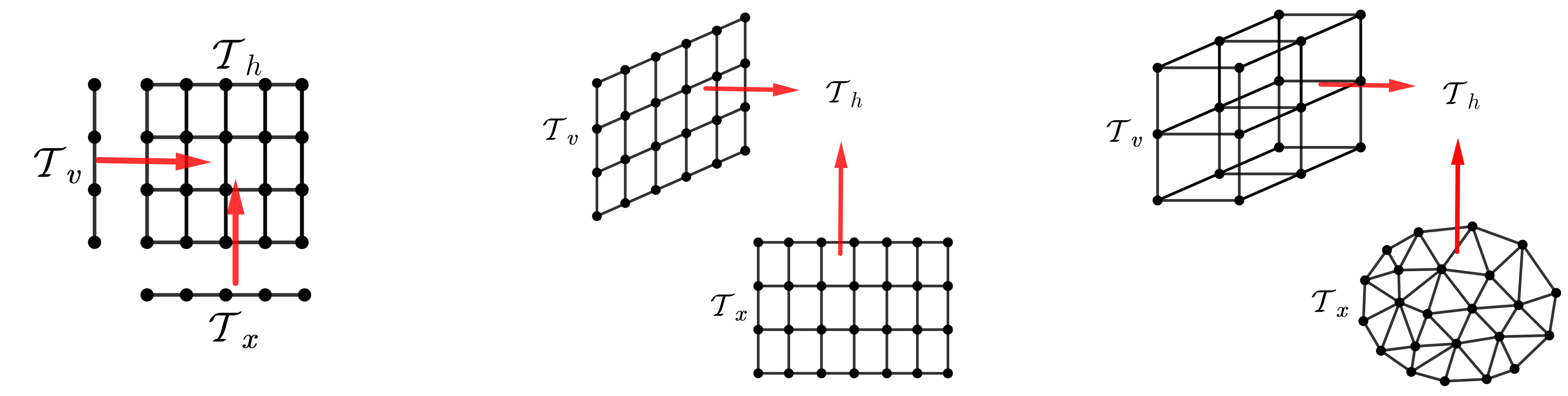
- The parameters $\frac{d_{\boldsymbol{x}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}}$ and $\frac{d_{\boldsymbol{v}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}}$ balance the viscosity between physical and velocity spaces.
- The low-dimensional residuals are cheaper to compute compared to the full residual in phase space.





Tensor-product finite element method

We discretize the Vlasov equation using **tensor-product** finite element spaces: $\mathcal{V}_{\mathrm{phase}} = \mathcal{V}_{\boldsymbol{x}} \times \mathcal{V}_{\boldsymbol{v}}$, where $\mathcal{V}_{\boldsymbol{x}}$ and $\mathcal{V}_{\boldsymbol{v}}$ are continuous polynomial spaces in physical and velocity domains, respectively. Different polynomial degrees can be used in each space, and either $\mathbb{P}_k$ or $\mathbb{Q}_k$ elements may be employed.



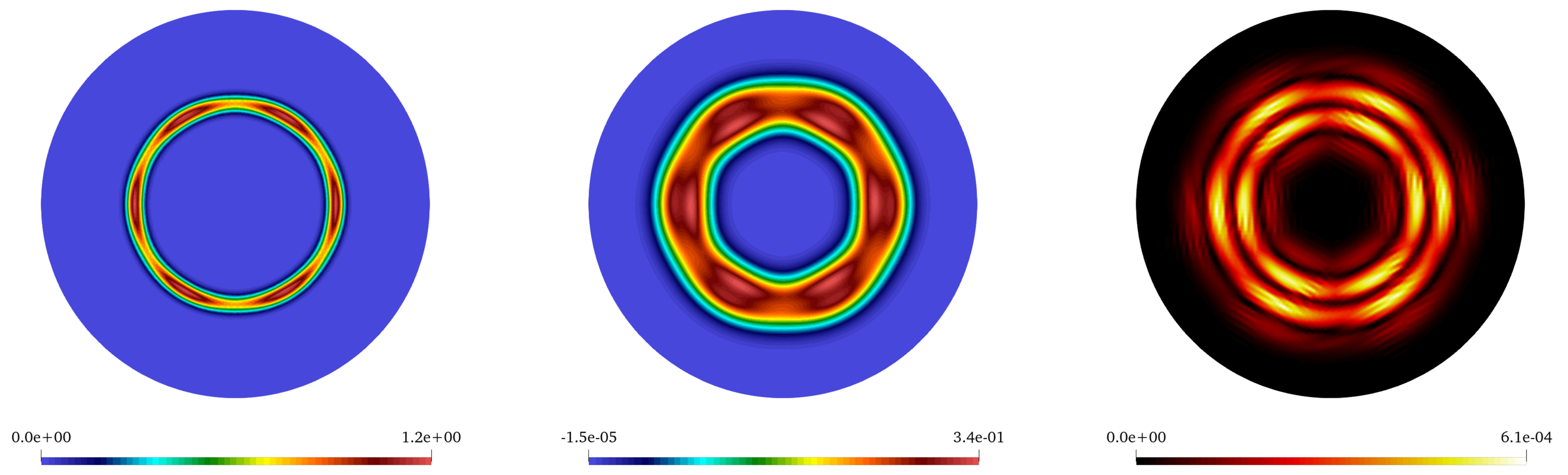
The semi-discrete system reads compactly as

$$(\mathbb{M}^{\boldsymbol{x}} \otimes \mathbb{M}^{\boldsymbol{v}}) \dot{\mathrm{f}} + \sum_{l=1}^{3} \left(\mathbb{A}^{\boldsymbol{x},l} \otimes \mathbb{C}^{\boldsymbol{v},l} + \mathbb{C}^{\boldsymbol{x},l}(\boldsymbol{E}) \otimes \mathbb{A}^{\boldsymbol{v},l} + \mathbb{C}^{\boldsymbol{x},l}(\boldsymbol{B}) \otimes \mathbb{G}^{\boldsymbol{v},l} \right) \mathrm{f} + (\mathbb{D}^{\boldsymbol{x}} \otimes \mathbb{M}^{\boldsymbol{v}}) \mathrm{f} + (\mathbb{M}^{\boldsymbol{x}} \otimes \mathbb{D}^{\boldsymbol{v}}) \mathrm{f} = 0,$$

where the operators are defined as

$$\mathbb{A}_{ij}^{\boldsymbol{x},l} = \int_{\Omega_{\boldsymbol{x}}} (\partial_{x_l} \phi_j) \phi_i \, \mathrm{d}\boldsymbol{x}, \ \mathbb{C}_{ij}^{\boldsymbol{x},l}(\boldsymbol{w}) = \int_{\Omega_{\boldsymbol{x}}} w_l \phi_j \phi_i \, \mathrm{d}\boldsymbol{x}, \qquad l = 1,2,3,$$
$$\mathbb{A}_{ij}^{\boldsymbol{v},l} = \int_{\Omega_{\boldsymbol{v}}} (\partial_{v_l} \varphi_j) \varphi_i \, \mathrm{d}\boldsymbol{v}, \ \mathbb{C}_{ij}^{\boldsymbol{v},l} = \int_{\Omega_{\boldsymbol{v}}} v_l \varphi_j \varphi_i \, \mathrm{d}\boldsymbol{v}, \ \mathbb{G}_{ij}^{\boldsymbol{v},l} = \int_{\Omega_{\boldsymbol{v}}} (\nabla_{\boldsymbol{v}} \times \boldsymbol{v})_l \varphi_j \varphi_i \mathrm{d}\boldsymbol{v}, \qquad l = 1,2,3,$$
$$\mathbb{M}_{ij}^{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{x}}} \phi_j \phi_i \, \mathrm{d}\boldsymbol{x}, \ \mathbb{D}_{ij}^{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{x}}} \mathsf{A}_{\boldsymbol{x}} (\nabla_{\boldsymbol{x}} \phi_j) \cdot (\nabla_{\boldsymbol{x}} \phi_i) \, \mathrm{d}\boldsymbol{x}, \ \mathbb{M}_{ij}^{\boldsymbol{v}} = \int_{\Omega_{\boldsymbol{v}}} \varphi_j \varphi_i \, \mathrm{d}\boldsymbol{v}, \ \mathbb{D}_{ij}^{\boldsymbol{v}} = \int_{\Omega_{\boldsymbol{v}}} \mathsf{A}_{\boldsymbol{v}} (\nabla_{\boldsymbol{v}} \varphi_j) \cdot (\nabla_{\boldsymbol{v}} \varphi_i) \, \mathrm{d}\boldsymbol{v}.$$

Numerical results in 2D2V



Diocotron instability: ρ at $t = 0$ Diocotron instability: ρ at $t = 1$ Diocotron instability: ε_x at $t = 1$

Structure of Maxwell's Equations

Integrating equation (2) over velocity space, we obtain the **continuity equation** for the density ρ:

$$\partial_t \rho + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J} - \nabla_{\boldsymbol{x}} \cdot (\mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \rho) = 0.$$

We then define a modified current $\tilde{\boldsymbol{J}} = \boldsymbol{J} - \mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \rho$ so that the continuity equation is still satisfied.

It follows that

$$\partial_t (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} - \rho) = \nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{x}} \times \boldsymbol{B}) - \nabla_{\boldsymbol{x}} \cdot \tilde{\boldsymbol{J}} - \partial_t \rho = 0,$$
$$\partial_t (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}) = -\nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}) = 0,$$

which implies that if the **initial conditions** satisfy $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0$, these conditions remain **valid for all time**.

Function Spaces for Maxwell's Equations

Define an anisotropic polynomial space $\mathbb{Q}_{\alpha_1, \alpha_2, \ldots, \alpha_{d_{\boldsymbol{x}}}}$ composed of $d_{\boldsymbol{x}}$-variate polynomials whose degree in x_i is at most α_i. Then, we define the following function spaces:

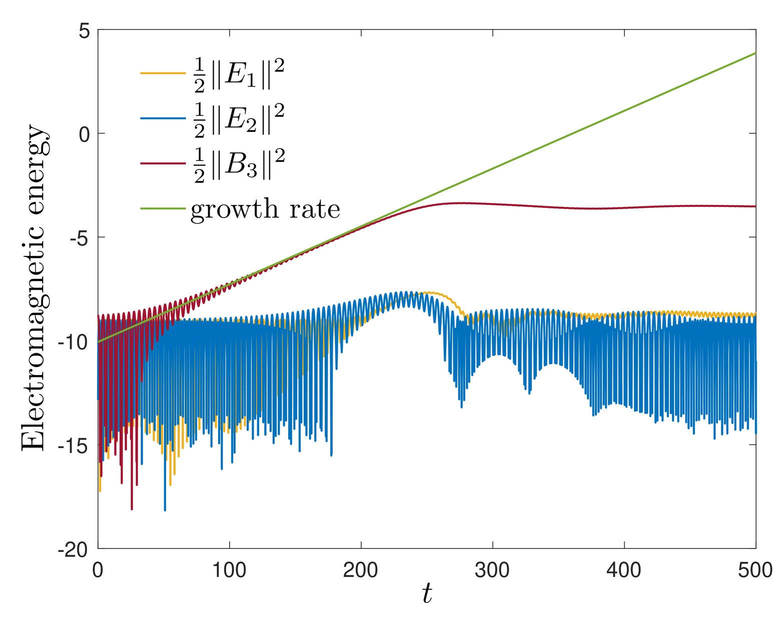
$$\mathbb{E} := \left\{ \boldsymbol{w} \in \boldsymbol{H}(\mathrm{curl}; \Omega_{\boldsymbol{x}}) : [\nabla_{\widehat{\boldsymbol{x}}} \boldsymbol{T}_K(\widehat{\boldsymbol{x}})]^\top \boldsymbol{w}(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) \in \mathcal{N}_{k-1}(\widehat{K}),\ \forall K \in \mathcal{T}_{\boldsymbol{x}} \right\},$$

$$\mathbb{B} := \left\{ \boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}; \Omega_{\boldsymbol{x}}) : \det(\nabla_{\widehat{\boldsymbol{x}}} \boldsymbol{T}_K(\widehat{\boldsymbol{x}})) [\nabla_{\widehat{\boldsymbol{x}}} \boldsymbol{T}_K(\widehat{\boldsymbol{x}})]^{-1} \boldsymbol{w}(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})) \in \mathcal{RT}_{k-1}(\widehat{K}),\ \forall K \in \mathcal{T}_{\boldsymbol{x}} \right\},$$

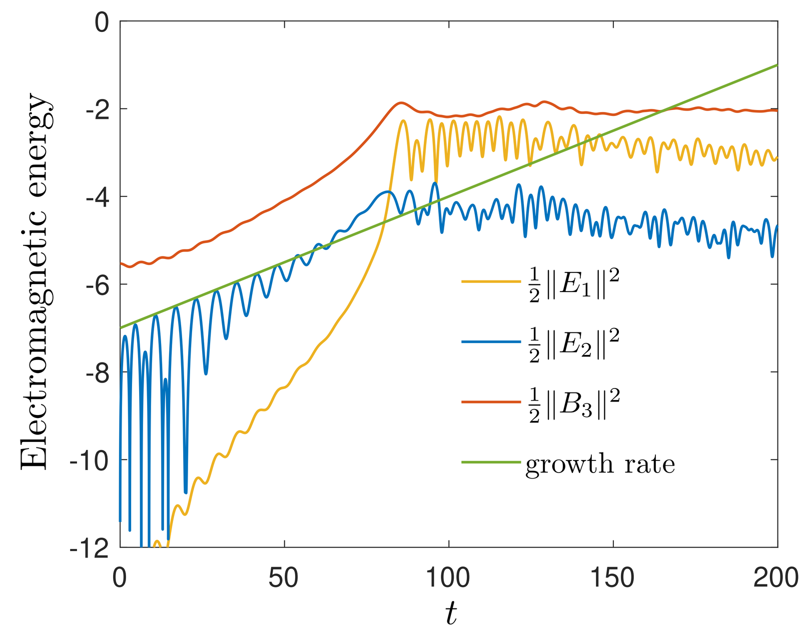
where

$$\mathcal{N}_k(\widehat{K}) := [\mathbb{Q}_{k,k+1,k+1}(\widehat{K}),\ \mathbb{Q}_{k+1,k,k+1}(\widehat{K}),\ \mathbb{Q}_{k+1,k+1,k}(\widehat{K})],$$
$$\mathcal{RT}_k(\widehat{K}) := [\mathbb{Q}_{k+1,k,k}(\widehat{K}),\ \mathbb{Q}_{k,k+1,k}(\widehat{K}),\ \mathbb{Q}_{k,k,k+1}(\widehat{K})].$$

These are the well-known **Nédélec** and **Raviart-Thomas** elements of order k on Cartesian grids.



Weibel instability



Streaming Weibel instability

Reference

[1] Katharina Kormann, Murtazo Nazarov, and Junjie Wen. A structure-preserving finite element framework for the Vlasov-Maxwell system. *Comput. Methods Appl. Mech. Engrg.*, 446:Paper No. 118290, 22, 2025.

[2] Junjie Wen and Murtazo Nazarov. An anisotropic nonlinear stabilization for finite element approximation of Vlasov-Poisson equations. *J. Comput. Phys.*, 536:Paper No. 114079, 21, 2025.