Finite Element Approximations of the Vlasov Equations

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Overview

Vlasov–Maxwell equations

- For $\boldsymbol{x} \in \mathbb{R}^3, \boldsymbol{v} \in \mathbb{R}^3, t \in \mathbb{R}^+$, the distribution function of plasma $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is characterised by the Vlasov equation:

$$\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_{\boldsymbol{v}} f = 0,$$

where $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ are the electric and magnetic fields.

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- $\boldsymbol{E}$ and $\boldsymbol{B}$ are governed by Maxwell's equations:

$$\begin{aligned} \partial_t \boldsymbol{E} &= \nabla_{\boldsymbol{x}} \times \boldsymbol{B} - \boldsymbol{J}, \\ \partial_t \boldsymbol{B} &= -\nabla_{\boldsymbol{x}} \times \boldsymbol{E}, \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} &= \rho, \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} &= 0. \end{aligned}$$

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- Charge and current densities are given by:

$$\rho = \int_{\mathbb{R}^3} f \, \mathrm{d}\boldsymbol{v} - \rho_0, \qquad \boldsymbol{J} = \int_{\mathbb{R}^3} f \boldsymbol{v} \, \mathrm{d}\boldsymbol{v}.$$

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- FEM is ideal for coupling Vlasov equations to Maxwell's equations.

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- Choose $w = \psi_i$, $i = 1, \dots, N$, and insert $f_h = \sum_{j=1}^N \psi_j f_j$:

$$\sum_{j=1}^N (\psi_j, \psi_i) \dot{f}_j = -(\boldsymbol{\beta}_h \cdot \nabla f_h, \psi_i), \qquad i = 1, \dots, N.$$

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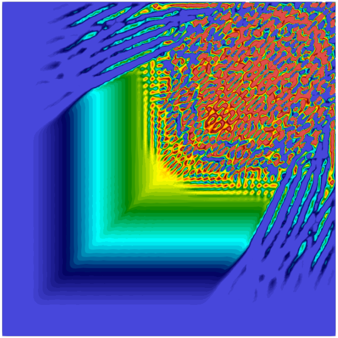
- Robust stabilization.
- Physical invariants:
  - Conservation of mass, momentum, and energy;
  - Positivity of f;
  - Divergence constraints: $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0$.

Overview

1. Introduction
2. Numerical methods
   - Residual-based artificial viscosity
   - Tensor-product finite element method
   - Finite element methods for the Maxwell's equations
3. Conclusions

Artificial viscosity method

- Standard FEM is not stable and produces spurious oscillations when applied to advection-dominated problems.



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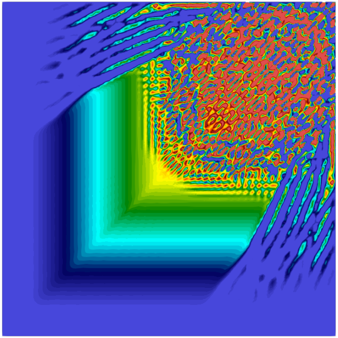

- **Artificial viscosity method:**

$$\partial_t f^\varepsilon + \boldsymbol{\beta} \cdot \nabla f^\varepsilon - \nabla \cdot (\varepsilon \nabla f^\varepsilon) = 0,$$

where $\varepsilon \geq 0$ is a vanishing artificial viscosity.

First-order viscosity

- Lax–Friedrichs method:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + \beta \frac{f_{i+1}^n - f_{i-1}^n}{2h} - \frac{h^2}{2\Delta t} \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{h^2} = 0.$$

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- Considering the CFL condition:$\Delta t = h/|\beta|$, the Lax–Friedrichs method is equivalent to the discretization of

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- **Anisotropic viscosity + FEM:** Let $\mathsf{A}^{\mathsf{L}} = \mathrm{diag}(\varepsilon_1^{\mathsf{L}}, \dots, \varepsilon_d^{\mathsf{L}})$,

$$(\partial_t f_h + \boldsymbol{\beta}_h \cdot \nabla f_h, w) + (\mathsf{A}^{\mathsf{L}} \nabla f_h, \nabla w) = 0,$$

and in each dimension: $\varepsilon_l^{\mathsf{L},i} = \frac{1}{2} h_l \|\beta_l\|_{\mathrm{loc}(i)}$, $l = 1, \dots, d$.

First-order viscosity

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- This scheme is stable but only first order.

Residual-based viscosity (RV)

- The PDE residual $R := \partial_t f + \boldsymbol{\beta} \cdot \nabla f$ tracks discontinuities and shocks.

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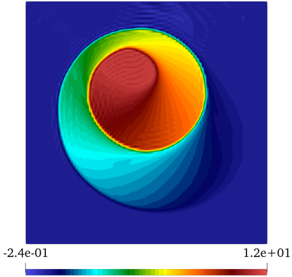
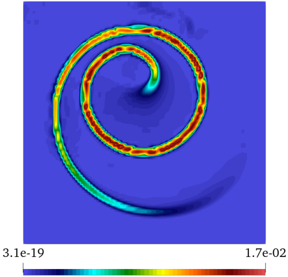
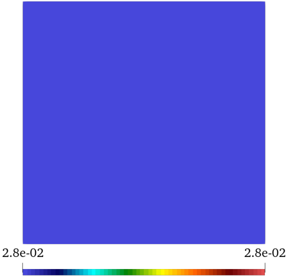
- **Idea**: Compute the viscosity based on the residual, so that it is activated near sharp gradients and vanishes in smooth regions.
- Example: Kurganov-Petrova-Popov (KPP) problem with $\boldsymbol{\beta} = (\cos(f), -\sin(f))^\mathsf{T}$:



(a) RV solution



(b) Residual viscosity



(c) First-order viscosity

High-order viscosity

- Compute the residual $R_h(f_h^n) \in \mathcal{V}$ using L^2-projection:

$$(R_h(f_h^n), w) = (|D_t f_h^n + \boldsymbol{\beta}_h^n \cdot \nabla f_h^n|, w), \qquad \forall w \in \mathcal{V}.$$

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$$\text{BDF2} = \frac{3f_h^n - 4f_h^{n-1} + f_h^{n-2}}{2\Delta t}.$$

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- **High-order viscosity:**

$$\varepsilon_l^{\mathsf{H},i} = \min\left(\varepsilon_l^{\mathsf{L},i}, h_l^2 \frac{|R_i|}{n}\right), \qquad l = 1, \dots, d,$$

where $n = \|f_h^n - \overline{f_h^n}\|$ is the normalization function.

Time stepping

- Using forward Euler:

$$\sum_{j=1}^{N} m_{ij} \frac{f_j^{n+1} - f_j^n}{\Delta t} + \sum_{j=1}^{N} c_{ij} f_j^n + \sum_{j=1}^{N} d_{ij}^{\mathsf{H},n} f_j^n = 0, \qquad i = 1 \ldots, N,$$

where $m_{ij} = (\psi_j, \psi_i)$, $c_{ij} = (\boldsymbol{\beta}_h^n \cdot \nabla \psi_j, \psi_i)$, $d_{ij} = (\mathsf{A}^{\mathsf{H}} \nabla \psi_j, \nabla \psi_i)$.

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- Higher-order SSPRK schemes enable improved accuracy.
- Mass conservation holds: $\sum_{i=1}^{N} m_i f_i^{n+1}=\sum_{i=1}^{N} m_i f_i^n$, where $m_i=\sum_{j=1}^{N} m_{ij}$.

Landau damping

Domain: $\Omega_x \times \Omega_v = [0, 2\pi/\theta] \times [-6, 6]$, and Initial data:

$$f_0(x,v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right)(1 + \alpha\cos(\theta x)), \qquad \theta = 0.5.$$

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When $\alpha = 0.01$:

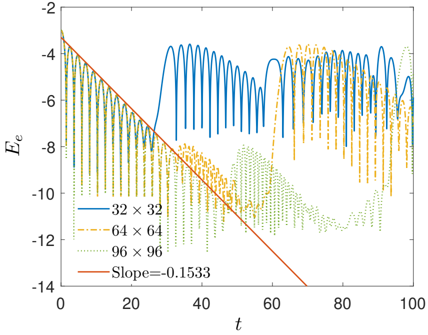

Figure 2: Time evolution of electric energy

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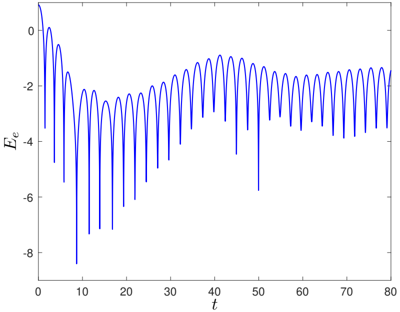
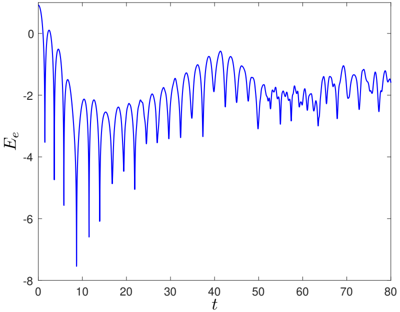
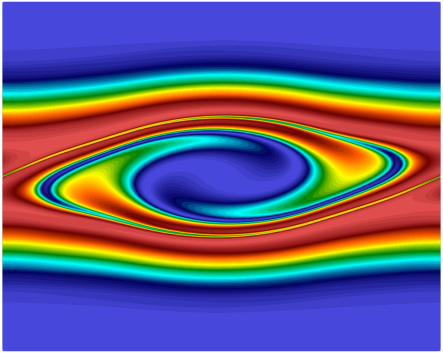


(a) RV solution



(b) FEM solution

Two-stream instability



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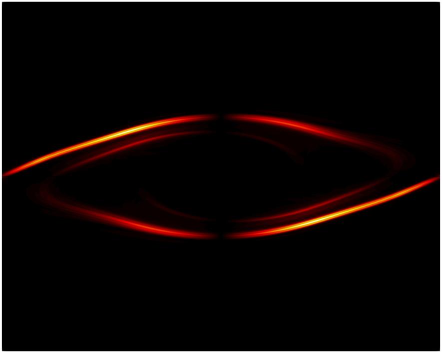


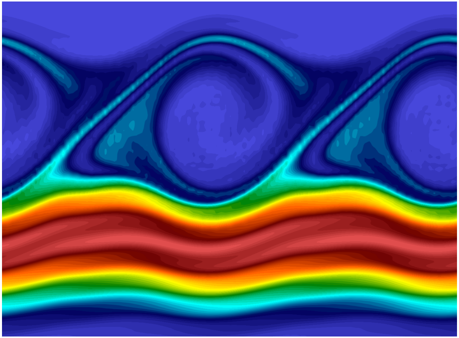
(b) viscosity

Two-stream instability

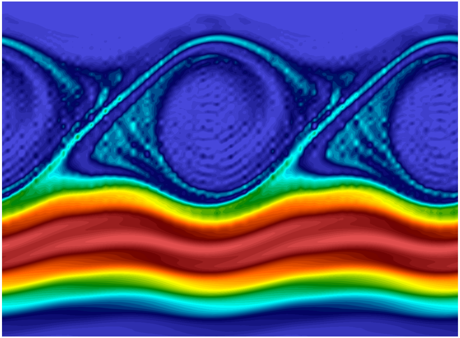
Convergence rates of the RV method, run until $t = 5$ and then back to $t = 10$.

	$N_x \times N_v$	L^1-error	Rate	L^2-error	Rate	L^∞-error	Rate
Q_1	31×31	2.21E−02	–	2.36E−02	–	3.73E−02	—
	61×61	5.46E−03	2.01	5.88E−03	2.01	9.94E−03	1.91
	121×121	1.32E−03	2.05	1.42E−03	2.05	2.43E−03	2.03
	241×241	3.24E−04	2.02	3.50E−04	2.02	5.98E−04	2.02
Q_2	31×31	7.93E−03	–	8.92E−03	–	1.62E−02	—
	61×61	1.06E−03	2.91	1.07E−03	3.06	1.89E−03	3.10
	121×121	1.36E−04	2.95	1.54E−04	2.80	3.79E−04	2.32
	241×241	1.61E−05	3.08	1.69E−05	3.18	3.37E−05	3.49
Q_3	31×31	2.67E−03	–	2.52E−03	–	3.33E−03	—
	61×61	2.23E−04	3.58	2.30E−04	3.46	3.94E−04	3.08
	121×121	1.42E−05	3.97	1.48E−05	3.95	2.91E−05	3.76
	241×241	9.02E−07	3.98	9.38E−07	3.98	1.89E−06	3.94

Anisotropic vs Isotropic



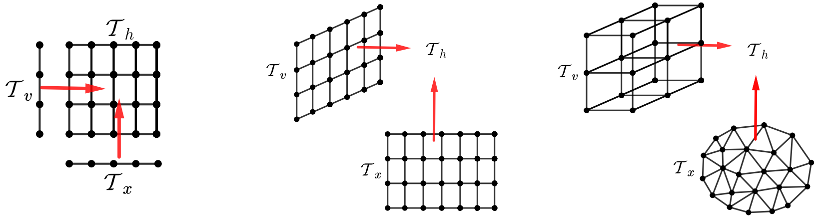
(a) Anisotropic viscosity



(b) Isotropic viscosity

Tensor product finite element

Tensor-product FE constructs high-dimensional spaces from multiple low-dimensional ones:



$\mathcal{T}_h := \mathcal{T}_{\boldsymbol{x}} \times \mathcal{T}_{\boldsymbol{v}}$, define $\mathcal{V}_{\boldsymbol{x}} \in \mathcal{T}_{\boldsymbol{x}}$ and $\mathcal{V}_{\boldsymbol{v}} \in \mathcal{T}_{\boldsymbol{v}}$, there exists $\mathcal{V}$ on $\mathcal{T}_h$ such that $\mathcal{V} := \mathcal{V}_{\boldsymbol{x}} \times \mathcal{V}_{\boldsymbol{v}}$.

Tensor-product FEM for Vlasov

- FEM for the Vlasov–Maxwell equation:

$$\left(\partial_t \sum_{i=1}^{N} f_i\psi_i, \psi_j\right) + \left(\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} \sum_{i=1}^{N} f_i\psi_i, \psi_j\right)$$

$$\left((\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} \sum_{i=1}^{N} f_i\psi_i, \psi_j\right) = 0, \qquad \forall \psi_j \in \mathcal{V}.$$

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- It is equivalent to:

$$\left(\partial_t \sum_{i=1}^{N_x}\sum_{j=1}^{N_v} f_{ij}\phi_i\varphi_j, \phi_k\varphi_l\right) + \left(\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} \sum_{i=1}^{N_x}\sum_{j=1}^{N_v} f_{ij}\phi_i\varphi_j, \phi_k\varphi_l\right)$$
$$\left((\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}} \sum_{i=1}^{N_x}\sum_{j=1}^{N_v} f_{ij}\phi_i\varphi_j, \phi_k\varphi_l\right) = 0,$$
$$\forall\phi_k \in \mathcal{V}_{\boldsymbol{x}}, \varphi_l \in \mathcal{V}_{\boldsymbol{v}}.$$

Tensor-product FEM for Vlasov

- With any functions $\mathcal{L}_x := \mathcal{L}(\boldsymbol{x})$ and $\mathcal{L}_v := \mathcal{L}(\boldsymbol{v})$, we can decompose:

$$(f_{ij}\mathcal{L}_x\mathcal{L}_v\phi_i\varphi_j, \phi_k\varphi_l) = f_{ij}\left(\int_{\Omega_{\boldsymbol{x}}} \mathcal{L}_x\phi_i\phi_k \, \mathrm{d}\boldsymbol{x}\right)\left(\int_{\Omega_{\boldsymbol{v}}} \mathcal{L}_v\varphi_j\varphi_l \, \mathrm{d}\boldsymbol{v}\right).$$

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- We obtain the following system from Vlasov–Maxwell

$$(\mathbb{M}^{\boldsymbol{x}}\otimes\mathbb{M}^{\boldsymbol{v}})\dot{\mathbb{f}}+\sum_{l=1}^{3}\left(\mathbb{A}^{\boldsymbol{x},l}\otimes\mathbb{C}^{\boldsymbol{v},l}+\mathbb{C}^{\boldsymbol{x},l}(\boldsymbol{E})\otimes\mathbb{A}^{\boldsymbol{v},l}+\mathbb{C}^{\boldsymbol{x},l}(\boldsymbol{B})\otimes\mathbb{G}^{\boldsymbol{v},l}\right)\mathbb{f} = 0,$$

where

$$\mathbb{A}^{\boldsymbol{x},l}_{ij} = \int_{\Omega_{\boldsymbol{x}}}(\partial_{x_l}\phi_j)\phi_i \, \mathrm{d}\boldsymbol{x}, \qquad \mathbb{C}^{\boldsymbol{x},l}_{ij}(\boldsymbol{w}) = \int_{\Omega_{\boldsymbol{x}}} w_l\phi_j\phi_i \, \mathrm{d}\boldsymbol{x},$$

$$\mathbb{A}^{\boldsymbol{v},l}_{ij} = \int_{\Omega_{\boldsymbol{v}}}(\partial_{v_l}\varphi_j)\varphi_i \, \mathrm{d}\boldsymbol{v}, \qquad \mathbb{C}^{\boldsymbol{v},l}_{ij} = \int_{\Omega_{\boldsymbol{v}}} v_l\varphi_j\varphi_i \, \mathrm{d}\boldsymbol{v},$$

$$\mathbb{G}^{\boldsymbol{v},l}_{ij} = \int_{\Omega_{\boldsymbol{v}}}\left(\nabla_{\boldsymbol{v}}\times\boldsymbol{v}\right)_l\varphi_j\varphi_i \mathrm{d}\boldsymbol{v}, \qquad l = 1, 2, 3,$$

$$\mathbb{M}^{\boldsymbol{x}}_{ij} = \int_{\Omega_{\boldsymbol{x}}}\phi_j\phi_i \, \mathrm{d}\boldsymbol{x}, \qquad \mathbb{M}^{\boldsymbol{v}}_{ij} = \int_{\Omega_{\boldsymbol{v}}}\varphi_j\varphi_i \, \mathrm{d}\boldsymbol{v}.$$

A novel residual viscosity

- Viscosity solution to Vlasov:

$$\partial_t f + \boldsymbol{\beta}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} f + \boldsymbol{\beta}_{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{v}} f - \nabla_{\boldsymbol{x}} \cdot (\mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} f) - \nabla_{\boldsymbol{v}} \cdot (\mathsf{A}_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} f) = 0$$

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- Assuming $\mathsf{A}_{\boldsymbol{x}} := \mathsf{A}_{\boldsymbol{x}}(\boldsymbol{x})$ and $\mathsf{A}_{\boldsymbol{v}} := \mathsf{A}_{\boldsymbol{v}}(\boldsymbol{v})$, integration in $\Omega_{\boldsymbol{x}}$ and $\Omega_{\boldsymbol{v}}$ gives

$$\partial_t u_v + \nabla_{\boldsymbol{v}} \cdot \mathbf{F}_{\boldsymbol{v}}(u_v) - \nabla_{\boldsymbol{v}} \cdot (\mathsf{A}_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} u_v) = 0,$$
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where

$$u_x = \int_{\Omega_{\boldsymbol{v}}} f \, \mathrm{d}\boldsymbol{v}, \qquad \mathbf{F}_{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{v}}} \boldsymbol{\beta}_{\boldsymbol{x}} f \, \mathrm{d}\boldsymbol{v},$$
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- Compute $\mathsf{A}_{\boldsymbol{x}}$ and $\mathsf{A}_{\boldsymbol{v}}$ using the residuals of low-dimensional conservation laws.

A novel residual viscosity

- Compute the viscosity coefficients for residual viscosity:

$$\varepsilon_{x_l}^{\mathsf{H},i} = h_{x_l}^2 \frac{|R_h(u_x)_i|}{\|u_x - \bar{u}_x\|} \frac{d_{\boldsymbol{x}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}}, \qquad \varepsilon_{v_l}^{\mathsf{H},i} = h_{v_l}^2 \frac{|R_h(u_v)_i|}{\|u_v - \bar{u}_v\|} \frac{d_{\boldsymbol{v}}}{d_{\boldsymbol{x}} + d_{\boldsymbol{v}}}.$$

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- The diffusion terms $(\mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} f_h, \nabla_{\boldsymbol{x}} \phi_i \varphi_j)$ and $(\mathsf{A}_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} f_h, \nabla_{\boldsymbol{v}} \phi_i \varphi_j)$ are obtained using tensor products:

$$(\mathbb{D}^{\boldsymbol{x}} \otimes \mathbb{M}^{\boldsymbol{v}})\mathbb{f} + (\mathbb{M}^{\boldsymbol{x}} \otimes \mathbb{D}^{\boldsymbol{v}})\mathbb{f},$$

where

$$\mathbb{D}_{ij}^{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{x}}} \mathsf{A}_{\boldsymbol{x}} (\nabla_{\boldsymbol{x}} \phi_j) \cdot (\nabla_{\boldsymbol{x}} \phi_i) \, \mathrm{d}\boldsymbol{x}, \qquad \mathbb{D}_{ij}^{\boldsymbol{v}} = \int_{\Omega_{\boldsymbol{v}}} \mathsf{A}_{\boldsymbol{v}} (\nabla_{\boldsymbol{v}} \varphi_j) \cdot (\nabla_{\boldsymbol{v}} \varphi_i) \, \mathrm{d}\boldsymbol{v}.$$

Continuity equation

- Integral the Vlasov equation in $\Omega_{\boldsymbol{v}}$:

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$$\begin{aligned} \partial_t(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E}) &= \nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{x}} \times \boldsymbol{B}) - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J} = \partial_t \rho, \\ \partial_t(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}) &= -\nabla_{\boldsymbol{x}} \cdot (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}) = 0, \end{aligned}$$

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  which imply $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{E} = \rho$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B} = 0$ hold all the time.
- Due to the artificial viscosity:

$$\partial_t \rho + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J} - \nabla_{\boldsymbol{x}} \cdot (\mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \rho) = 0,$$

  we hence replace: $\widetilde{\boldsymbol{J}} = \boldsymbol{J} - \mathsf{A}_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \rho.$

Finite Element Exterior Calculus

- Given that $f_h \in \mathcal{V} := \mathcal{V}_{\boldsymbol{x}} \times \mathcal{V}_{\boldsymbol{v}}$, compute the densities as follows:

$$\rho_h = \int_{\Omega_{\boldsymbol{v}}} f_h \,\mathrm{d}\boldsymbol{v}, \qquad \boldsymbol{J}_h = \int_{\Omega_{\boldsymbol{v}}} \boldsymbol{v} f_h \,\mathrm{d}\boldsymbol{v},$$

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- Define:

$$\mathbb{E} : = \{\boldsymbol{w} \in \boldsymbol{H}(\text{curl}; \Omega_{\boldsymbol{x}}) : \boldsymbol{w}(K) \in \mathcal{N}_{k-1}(K), \forall K \in \mathcal{T}_{\boldsymbol{x}}\},$$
$$\mathbb{B} : = \{\boldsymbol{w} \in \boldsymbol{H}(\text{div}; \Omega_{\boldsymbol{x}}) : \boldsymbol{w}(K) \in \mathcal{RT}_{k-1}(K), \forall K \in \mathcal{T}_{\boldsymbol{x}}\},$$

  where $\mathcal{N}_k$ and $\mathcal{RT}_k$ are the Nédélec and Raviart-Thomas elements.

Solution to the Maxwell's equations

- Using forward Euler:

$$
\begin{aligned}
\Big(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\Delta t}, \boldsymbol{\eta}\Big) &= \Big(\boldsymbol{B}_h^n, \nabla_{\boldsymbol{x}} \times \boldsymbol{\eta}\Big) - \Big(\big(\boldsymbol{J}_h^n - \mathsf{A}_{\boldsymbol{x}}^n \nabla_{\boldsymbol{x}} \rho_h^n\big),\ \boldsymbol{\eta}\Big), \quad \forall \boldsymbol{\eta} \in \mathbb{E}, \\
\frac{\boldsymbol{B}_h^{n+1}-\boldsymbol{B}_h^n}{\Delta t} &= -\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_h^n,
\end{aligned}
$$

where $\boldsymbol{E}_h \in \mathbb{E}$ and $\boldsymbol{B}_h \in \mathbb{B}$.

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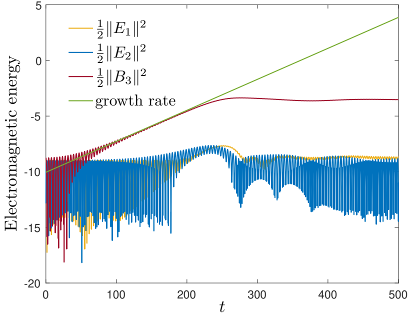
  where $\boldsymbol{E}_h \in \mathbb{E}$ and $\boldsymbol{B}_h \in \mathbb{B}$.

- The divergence constraints hold:

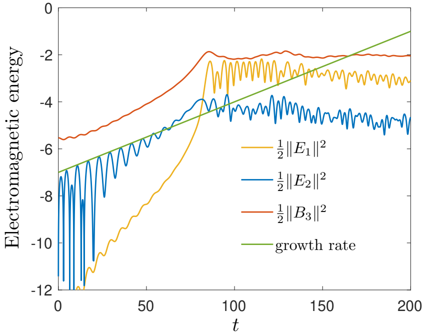
$$\begin{aligned}(\boldsymbol{E}_h^{n+1},\ \nabla_{\boldsymbol{x}}\phi_i) &= -(\rho_h^{n+1},\ \phi_i), \qquad \forall \phi_i \in \mathcal{V}_{\boldsymbol{x}},\\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}_h^{n+1} &= 0,\end{aligned}$$

  if $(\boldsymbol{E}_h^n,\ \nabla_{\boldsymbol{x}}\phi_i) = -(\rho_h^n,\ \phi_i)$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{B}_h^n = 0$.

1D2V Weibel instabilities

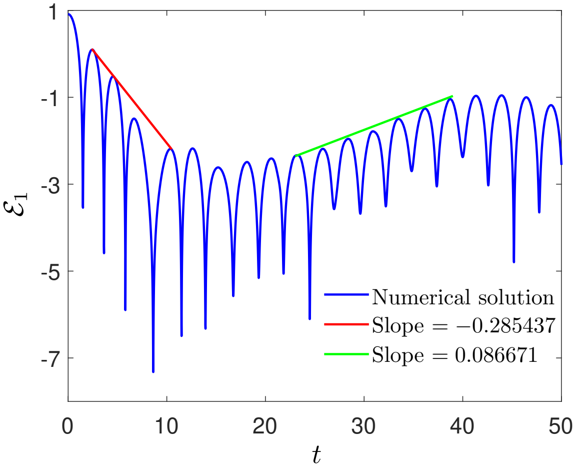


(a) Weibel instability



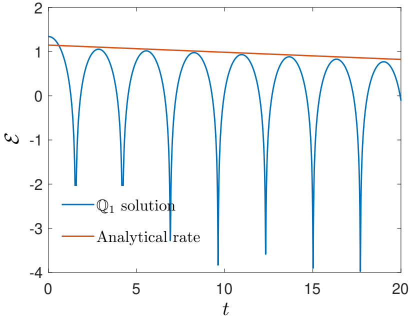
(b) Streaming Weibel instability

1D2V Landau damping

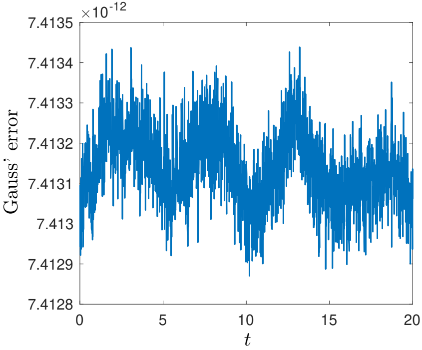


2D2V Landau damping

The solution with $32^2 \times 64^2$ elements:



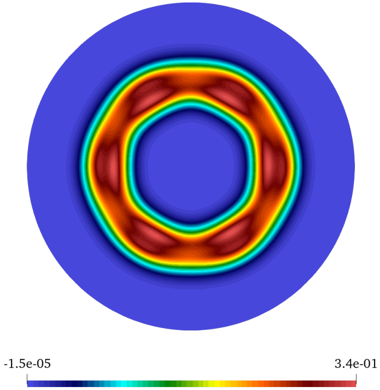
(a) Electric energy



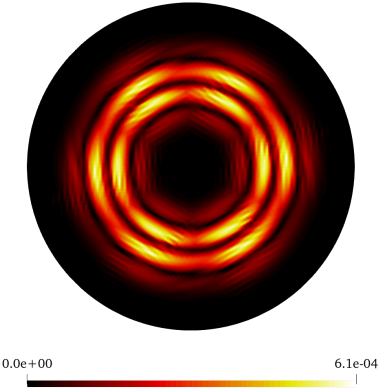
(b) Error of Gauss' law

2D2V Diocotron instability

The solution using $\mathbb{P}_1 \times \mathbb{Q}_1$ elements:



(a) ρ_h at $t = 1$



(b) $\mathsf{A}_{\boldsymbol{x}}$ at $t = 1$

Overview

Conclusion

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- **Future work**: Positivity-preserving, low-rank tensors.

Reference

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- *A structure-preserving finite element framework for the Vlasov-Maxwell system, CMAME 2025.* Katharina Kormann, Murtazo Nazarov, Junjie Wen*

Thanks for listening!
Questions or feedback?