

2. 由题意知: (1)  $A_1 \bar{A}_2 \bar{A}_3 + \bar{A}_1 A_2 \bar{A}_3 + \bar{A}_1 \bar{A}_2 A_3$

(2)  $A_1 \cup A_2 \cup A_3$

(3)  $A_1 \cap (A_2 \cup A_3)$

(4)  $A_1 \bar{A}_2 \bar{A}_3 + \bar{A}_1 A_2 \bar{A}_3 + \bar{A}_1 \bar{A}_2 A_3 + \bar{A}_1 \bar{A}_2 \bar{A}_3$

(表示方法有多种, 答案不唯一)

8. 由  $P(AC) = 0$ , 且  $ABC \subset AC$ , 有  $P(ABC) = 0$ . 再由加法公式, 得  $A, B, C$  至少发生一个的概率为

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \\ &= 1 - \frac{2}{8} = \frac{3}{4} \end{aligned}$$

12. 三局两胜时, 甲获胜的概率为

$$P_1 = p^2 + C_2^1 p^2 (1-p) = p^2 (3-2p)$$

五局三胜时, 甲获胜的概率为

$$P_2 = p^3 + C_3^1 p^3 (1-p) + C_4^2 p^3 (1-p)^2 = p^3 (6p^2 - 15p + 10).$$

当  $\frac{1}{2} < p \leq 1$  时, 易得  $P_1 < P_2$ , 所以, 五局三胜对甲更有利。

13. 记事件  $A_i$  为甲掷硬币的次数为  $i$ , 则  $P(A_i) = (1/2)^i$ . 记事件  $B$  为甲获胜, 则

$$P(B) = \sum_{i=1}^n P(B | A_i) P(A_i) = \sum_{i=1}^n \frac{1}{2^i} \cdot \frac{1}{i+1}$$

当  $n \rightarrow \infty$  时,  $P(B) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{i+1} < \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{1}{2} = \frac{1}{2}$ , 所以该规则对乙更有利。

17. 由题意可知, 甲乙两人可乘 3 : 15, 3 : 30, 3 : 45 和 4 : 00 班次的公交车, 且乘任一班次的车时等可能的。记  $A :=$  '甲乙两人同乘一辆车', 所以

$$P(A) = \frac{C_4^1}{4^2} = \frac{1}{4}.$$

20. 掷一枚均匀的骰子, 记事件  $A$  为投出的点数大于 2, 事件  $B$  为投出的点数

为4或6，事件C为投出的点数为偶数，则

$$P(A) = \frac{2}{3}, \quad P(B) = \frac{1}{3}, \quad P(C) = \frac{1}{2}, \quad P(AB) = P(AC) = P(BC) = \frac{1}{3}.$$

所以

$$P(A|B) = \frac{P(AB)}{P(B)} = 1 > P(A), \quad P(B|C) = \frac{P(BC)}{P(C)} = \frac{2}{3} > P(B).$$

但是

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{2}{3} = P(A).$$

23.(1)记事件A 为谣言传播r次还没回到第一个造谣者所在群，则

$$P(A) = \frac{n(n-1)^{r-1}}{n^r} = \left(\frac{n-1}{n}\right)^{r-1}.$$

(2)记事件 $B_k$ 为传播k次没有一个微信群两次收到谣言， $k = 1, 2, \dots, n$ 。则

$$P(B_k) = \frac{A_n^k}{n^k}, \quad k = 1, 2, \dots, n.$$

(3)每次随机向m个群传播谣言，记事件 $A'$  为谣言传播r次还没回到第一个造谣者所在群，记事件 $B'_k$ 为传播k次没有一个微信群两次收到谣言，此时谣言经历了 $m+m^2+\dots+m^r = (m^{r+1}-m)/(m-1)$ 个群，要令没有一个群两次收到谣言，则经历的群数小于等于n，有 $k = 1, 2, \dots, \lfloor \log_m(nm + m - n) \rfloor - 1$ ，则

$$P(A') = \frac{C_n^m (C_{n-1}^m)^m \dots (C_{n-1}^m)^{m^{r-1}}}{C_n^m (C_n^m)^m \dots (C_n^m)^{m^{r-1}}} = \frac{n}{n-m} \left(\frac{C_{n-1}^m}{C_n^m}\right)^{\frac{m^r-1}{m-1}} = \left(\frac{n-m}{n}\right)^{\frac{m^r-m}{m-1}}.$$

$$P(B'_k) = \frac{C_n^m (C_{n-m}^m \dots C_{n-m^2-m}^m) \dots (C_{n-m^{k-1}-\dots-m}^m \dots C_{n-m^k-\dots-m}^m)}{C_n^m (C_n^m)^m \dots (C_n^m)^{m^{r-1}}} = \frac{A_n^{(m^{k+1}-m)/(m-1)}}{(n!/(n-m)!)^{\frac{m^k-1}{m-1}}}.$$

24.记事件 $A_i$ 为第i次取到的零件为一等品，事件 $B_i$ 为从第i个箱子中取样， $i = 1, 2$ 。则

(1)

$$P(A_1) = P(A_1|B_1)P(B_1) + P(A_1|B_2)P(B_2) = \frac{1}{2} \times \frac{1}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{2}{5}.$$

(2)

$$P(A_2|A_1) = \frac{P(A_1A_2)}{P(A_1)} = \frac{P(A_1A_2B_1) + P(A_1A_2B_2)}{P(A_1)} = \frac{\frac{1}{2} \times \frac{1}{5} \times \frac{9}{49} + \frac{1}{2} \times \frac{3}{5} \times \frac{17}{29}}{\frac{2}{5}} \approx 0.49.$$

30. 记  $A :=$  “该人为带菌者”,  $B_i :=$  “第  $i$  次检测为阳性”, 则

(1)

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.95 \times 0.1}{0.95 \times 0.1 + 0.01 \times 0.9} = \frac{0.095}{0.104} = 0.91.$$

(2)

$$P(A|B_1B_2) = \frac{P(AB_1B_2)}{P(B_1B_2)} = \frac{0.95^2 \times 0.1}{0.95^2 \times 0.1 + 0.01^2 \times 0.9} = 0.99.$$

33. 记  $A :=$  “从乙袋中取出的球为白球”,  $B_i :=$  “第  $i$  次从甲袋中取出的球为白球”,  $i = 1, 2$ 。则

(1)

$$\begin{aligned} P(A) &= P(AB_1B_2) + P(AB_1\bar{B}_2) + P(A\bar{B}_1B_2) + P(A\bar{B}_1\bar{B}_2) \\ &= \frac{5}{7} \times \frac{4}{6} \times \frac{6}{11} + \frac{5}{7} \times \frac{2}{6} \times \frac{5}{11} + \frac{2}{7} \times \frac{5}{6} \times \frac{5}{11} + \frac{2}{7} \times \frac{1}{6} \times \frac{4}{11} \\ &= \frac{38}{77} \end{aligned}$$

(2)

$$1 - P(\bar{B}_1\bar{B}_2|A) = 1 - \frac{P(A\bar{B}_1\bar{B}_2)}{P(A)} = 1 - \frac{\frac{2}{7} \times \frac{1}{6} \times \frac{4}{11}}{\frac{38}{77}} = 1 - \frac{2}{57} = \frac{55}{57}.$$

## 第二周作业答案

March 29, 2023

37.  $P(A) = P(AC) + P(A\bar{C}) = P(A|C)P(C) + P(A|\bar{C})P(\bar{C}) = 0.55$ , 同理,  $P(B) = 0.5$ , 又

$$\begin{aligned} P(AB) &= P(ABC) + P(AB\bar{C}) = P(AB|C)P(C) + P(AB|\bar{C})P(\bar{C}) \\ &= P(A|C)P(B|C)P(C) + P(A|\bar{C})P(B|\bar{C})P(\bar{C}) = 0.415 \end{aligned}$$

所以  $P(AB) \neq P(A)P(B)$

$$\begin{aligned} 38 : (1) \text{ 恰有一次的射中概率 } P_1 &= 0.5 \times (1-0.6) \times (1-0.8) + 0.5 \times 0.6 \times (1-0.8) \\ &\quad + 0.5 \times (1-0.6) \times 0.8 = 0.26 \end{aligned}$$

$$(2) \text{ 至少有一次射中的概率 } P_2 = 1 - 0.5 \times 0.4 \times 0.2 = 0.96$$

$$39 (4) \text{ 考虑 } A \cdot B \cdot C \text{ 至少有一个正常工作 } P = [1 - (1 - P_A)(1 - P_B)(1 - P_C)] P_D^2$$

(5) 考虑C 是否能正常工作, 则

$$\begin{aligned} P &= (2P_A P_B - P_A^2 P_B^2)(1 - P_C) + (4P_A P_B + P_A^2 P_B^2 - 2P_A P_B^2 - 2P_A^2 P_B) P_C = \\ &= 2P_A(1 - P_A)[1 - (1 - P_B)(1 - P_B P_C)] + P_A^2 [1 - (1 - P_B)^2] \end{aligned}$$

$$40. \text{ 电路断开的概率 } P = 0.3 + 0.4 * 0.6 - 0.3 * 0.4 * 0.6 = 0.468$$

补充题: 取8个球, 分别编号为1,2,2,3,3,12,13,123。  $A_i = \{ \text{随机取一个球,}$

球上有数字 $i$  },  $i = 1, 2, 3$ .

$$\begin{aligned} P(A_1) &= P(A_2) = P(A_3) = \frac{1}{2}, P(A_1 A_2 A_3) = \frac{1}{8}, \\ P(A_1 A_2 A_3) &= P(A_1) P(A_2) P(A_3). \\ P(A_1 A_2) &= \frac{1}{4} = P(A_1) P(A_2) \\ P(A_1 A_3) &= \frac{1}{4} = P(A_1) P(A_3) \\ P(A_2 A_3) &= \frac{1}{8} \neq P(A_2) P(A_3). \end{aligned}$$

2. 考虑 $n$ 次投篮投进次数为 $X_n, X_n \in \{1, 2, \dots, n-1\}$ . 数学归纳法:  $n = 2, P(X_2 = i) = 1, n = k, P(X_k = i) = 1/(k-1), \forall i \in \{1, 2, \dots, k-1\}$ , 当 $n = k+1$ 时, 则 $P(X_{k+1} = i) = P(X_k = i-1 | A) P(A) + P(X_k = i-1 | A^C) P(A^C)$   
 $P(X_{k+1} = i) = \frac{1}{k-1} \times \frac{i-1}{k} + \frac{1}{k-1} \times \frac{k-i}{k} = \frac{k-1}{k(k-1)} = \frac{1}{k}, \forall i \in \{1, 2, \dots, k-1\}$ . 所以 $n = 100, P(X_{100} = i) = 1/99, \forall i \in \{1, 2, \dots, 99\}$

$$10. P = P_4 + P_5 + P_6 + P_7 = 0.6^4 + \binom{4}{1} 0.6^4 0.4 + \binom{5}{2} 0.6^4 0.4^2 + \binom{6}{3} 0.6^4 0.4^3 = 0.71$$

$$P' = P'_2 + P'_3 = 0.6^2 + \binom{2}{1} 0.6^2 0.4 = 0.65$$

$P > P'$ , 所以三局两胜制对乙队更有利。

11. 以 $X$  表示赌徒赌完一局后的收益, 则有

$$\begin{aligned} P(X = -1) &= \left(\frac{5}{6}\right)^3 = \frac{125}{216} \\ P(X = 1) &= \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2 = \frac{75}{216} \\ P(X = 2) &= \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) = \frac{15}{216} \\ P(X = 3) &= \left(\frac{1}{6}\right)^3 = \frac{1}{216} \end{aligned}$$

所以,  $X$  的分布律为

$$\begin{pmatrix} X & -1 & 1 & 2 & 3 \\ P & \frac{125}{216} & \frac{75}{216} & \frac{15}{216} & \frac{1}{216} \end{pmatrix}.$$

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$$\lambda = np = 400 \times 0.02 = 8, X \sim P(8)$$

$$P(X \geq 2) = 1 - P(X = 1) - P(X = 0) = 1 - 8e^{-8} - e^{-8} = 1 - 9e^{-8} \approx 0.997$$

16. 没来的乘客人数可近似为poisson分布,  $\lambda = 52 \times 0.05 = 2.6$

$$P(X \leq 1) = P(X = 0) + P(X = 1) = (1 + 2.6)e^{-2.6} \approx 0.27$$

20.

$$P(1 < X < 2) = \int_1^2 ax dx = \frac{a}{2} x^2 \Big|_1^2 = \frac{3}{2}a,$$

$$P(2 < X < 3) = \int_2^3 b dx = b = \frac{3}{2}a,$$

又因为  $P(1 < X < 2) + P(2 < X < 3) = 1$ , 所以  $a = \frac{1}{3}$ ,  $b = \frac{1}{2}$ .

21.(1)

$$\int_{-\infty}^{+\infty} \frac{a}{1+x^2} dx = a \arctan x \Big|_{-\infty}^{+\infty} = a\pi = 1.$$

所以,  $a = \frac{1}{\pi}$

(2)

$$F(x) = \int_{-\infty}^x \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \arctan x + \frac{1}{2}, \quad x \in R.$$

(3)

$$P(|x| < 1) = \int_{-1}^1 \frac{1}{\pi(1+x^2)} dx = \frac{1}{2}.$$

22.

$$S = \int_0^2 (2x - x^2) dx = [x^2 - \frac{1}{3}x^3] \Big|_0^2 = \frac{4}{3},$$

$$P(X \leq x) = \int_0^x (2x - x^2) dx / S = \frac{3}{4} (x^2 - \frac{1}{3}x^3) (x \in (0, 2)),$$

所以

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{3}{4}x^2 - \frac{1}{4}x^3, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2}x - \frac{3}{4}x^2, & 0 < x < 2 \\ 0, & other \end{cases}$$

27.证: 令  $F(x)$  为  $X$  的分布函数, 对于  $\forall n \in (1, \infty)$ , 由题意得,

$$F(\frac{1}{n}) - F(0) = F(\frac{2}{n}) - F(\frac{1}{n}) = \dots = F(1) - F(\frac{n-1}{n}),$$

又因为

$$\sum_{i=1}^n (F(\frac{i}{n}) - F(\frac{i-1}{n})) = F(1) - F(0) = 1,$$

所以

$$F\left(\frac{m}{n}\right) = \sum_{i=1}^m \left(F\left(\frac{i}{n}\right) - F\left(\frac{i-1}{n}\right)\right) = \frac{m}{n}, \quad m \leq n,$$

所以有  $F(x) = x$ .

28.  $X \sim \exp(\lambda)$ .

(1)

$$P(X > 2) = \int_2^{\infty} e^{-x} dx = 1 - e^{-x} \Big|_2^{\infty} = e^{-2}.$$

(2)

$$P(X > 4) = \int_4^{\infty} e^{-x} dx = e^{-4},$$

$$P(X > 4 \mid X > 2) = \frac{P(X > 4)}{P(X > 2)} = e^{-2}.$$

31.(1)由题意可得:  $\frac{X-1}{2} \sim N(0, 1)$ , 所以

$$P(0 \leq X \leq 4) = P\left(\frac{0-1}{2} \leq \frac{X-1}{2} \leq \frac{4-1}{2}\right) = \Phi(1.5) - \Phi(-0.5) \approx 0.6247.$$

$$P(X > 2.4) = P\left(\frac{X-1}{2} > \frac{2.4}{2}\right) = 1 - \Phi(0.7) \approx 0.2420.$$

$$P(|X| > 2) = 1 - P(-2 \leq X \leq 2) = 1 - \Phi(0.5) + \Phi(-1.5) \approx 0.3753.$$

(2)由题意易知  $1 - \Phi\left(\frac{c-1}{2}\right) = 2\Phi\left(\frac{c-1}{2}\right)$ , 所以  $\Phi\left(\frac{c-1}{2}\right) = \frac{1}{3}$ . 查表可知  $\frac{c-1}{2} \approx -0.4307$ , 所以  $c \approx 0.1386$ .

33.  $X_1 \sim N(30, 100)$ ,  $X_2 \sim N(40, 16)$ ,

(1)

$$P_1 = P(X_1 \leq 50) = P\left(\frac{X_1 - 30}{10} \leq \frac{50 - 30}{10}\right) = \Phi(2),$$

$$P_2 = P(X_2 \leq 50) = P\left(\frac{X_2 - 40}{4} \leq \frac{50 - 40}{4}\right) = \Phi(2.5).$$

所以  $P_2 > P_1$ .

(2)

$$P'_1 = P(X_1 \leq 45) = \Phi(1.5), P'_2 = P(X_2 \leq 45) = \Phi(1.25),$$



所以  $P'_1 > P'_2$ .

34. 记  $X$  为点数之和, 则  $X = 2, 3, \dots, 12$ , 所以

$$P(X = 2) = P(X = 12) = \frac{C_2^2}{6 \times 6} = \frac{1}{36},$$

$$P(X = 3) = P(X = 11) = \frac{C_2^1}{6 \times 6} = \frac{2}{36},$$

$$P(X = 4) = P(X = 10) = \frac{C_2^1 + C_2^2}{6 \times 6} = \frac{3}{36},$$

$$P(X = 5) = P(X = 9) = \frac{C_2^1 + C_2^1}{6 \times 6} = \frac{4}{36},$$

$$P(X = 6) = P(X = 8) = \frac{C_2^1 + C_2^1 + C_2^2}{6 \times 6} = \frac{5}{36},$$

$$P(X = 7) = \frac{C_2^1 + C_2^1 + C_2^1}{6 \times 6} = \frac{6}{36}.$$

所以

$$X \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{6} & \frac{5}{36} & \frac{1}{9} & \frac{1}{12} & \frac{1}{18} & \frac{1}{36} \end{pmatrix}.$$

36. 由  $X$  的分布律, 易得  $Y_1, Y_2, Y_3$  的分布律分别为:

$$Y_1 \sim \begin{pmatrix} -3 & -1 & 1 & 3 \\ 0.4 & 0.1 & 0.3 & 0.2 \end{pmatrix}, \quad Y_2 \sim \begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}, \quad Y_3 \sim \begin{pmatrix} 0 & 1 & 4 \\ 0.1 & 0.7 & 0.2 \end{pmatrix}.$$

37. (1) 由分布函数的有界性:

$$\begin{cases} F(-\infty) = a - \frac{\pi}{2}b = 0 \\ F(\infty) = a + \frac{\pi}{2}b = 1 \end{cases} \Rightarrow \begin{cases} a = 1/2 \\ b = 1/\pi \end{cases}$$

(2) 由分布函数可得  $X$  的密度函数为

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathcal{R}.$$

因为  $y = 3 - \sqrt[3]{x}$  为严格减函数, 其反函数为  $x = (3 - y)^3$ . 所以  $Y = 3 - \sqrt[3]{X}$  的密度函数为

$$f_Y(y) = \frac{3(y-3)^2}{\pi[1+(3-y)^6]}, \quad x \in \mathcal{R}.$$

(3)  $Z = 1/X$  的密度函数为

$$f_Z(z) = \frac{1}{\pi [1 + (1/z)^2]} \cdot \left| -\frac{1}{z^2} \right| = \frac{1}{\pi (1 + z^2)}, \quad z \in \mathcal{R}.$$

所以 $X$  与 $1/X$  具有相同的分布。

## 第四周作业答案

April 12, 2023

40.  $X \sim U(0, 1)$ , 则其密度函数为

$$f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0 & \text{其他.} \end{cases}$$

(1) 因为  $Y_1 = e^X$  的可能取值范围为  $(1, e)$ , 且  $y_1 = e^x$  在  $(0, 1)$  上为严格增函数, 其反函数为  $x = h(y_1) = \ln y_1$ , 对应导数  $h'(y_1) = \frac{1}{y_1}$ . 所以  $Y_1$  的密度函数为

$$f_1(y_1) = \begin{cases} f_X(\ln y_1) \left| \frac{1}{y_1} \right|, & 1 < y_1 < e \\ 0, & \text{其他} \end{cases} = \begin{cases} \frac{1}{y_1}, & 1 < y_1 < e, \\ 0, & \text{其他.} \end{cases}$$

(2)  $Y_2 = X^{-1}$  的可能取值范围为  $(1, \infty)$ , 且  $y_2 = x^{-1}$  在  $(0, 1)$  上为严格减函数, 反函数及对应的导数为  $x = h(y_2) = 1/y_2$ ,  $h'(y_2) = -1/y_2^2$ . 所以  $Y_2$  的密度函数为

$$f_2(y_2) = \begin{cases} f_X(y_2^{-1}) |-1/y_2^2|, & y_2 > 1 \\ 0, & \text{其他} \end{cases} = \begin{cases} \frac{1}{y_2^2}, & y_2 > 1, \\ 0, & \text{其他.} \end{cases}$$

(3)  $Y_3 = -\frac{1}{\lambda} \ln X$  的可能取值范围为  $(0, \infty)$ , 且  $y_3 = -\frac{1}{\lambda} \ln x$  ( $\lambda > 0$ ) 在  $(0, 1)$  上为严格减函数, 反函数及对应的导数为  $x = h(y_3) = e^{-\lambda y_3}$ ,  $h'(y_3) = -\lambda e^{-\lambda y_3}$ . 所以  $Y_3$  的密度函数为

$$f_3(y_3) = \begin{cases} f_X(e^{-\lambda y_3}) |-\lambda e^{-\lambda y_3}|, & y_3 > 0 \\ 0, & \text{其他} \end{cases} = \begin{cases} \lambda e^{-\lambda y_3}, & y_3 > 0, \\ 0, & \text{其他.} \end{cases}$$

42  $Y = F(x)$  的取值范围为  $(0, 1)$ , 且  $F(x)$  严格单调增, 则  $F^{-1}(x)$  也是严格单调增.  $Y$  的分布函数为

$$F_Y(y) = P(Y \leq y) = P(F(x) \leq y) = P(x \leq F^{-1}(y)) = \begin{cases} 0, & y < 0 \\ F(F^{-1}(y)) = y, & 0 \leq y < 1 \\ 1, & y \geq 1 \end{cases}$$

即证  $Y = F(x)$  为均匀分布。

44.  $Y = g(X)$ . 则  $Y$  的密度函数满足

$$f_Y(y) = f(g^{-1}(y)) (g^{-1})'(y) = 2(1 - g^{-1}(y)) (g^{-1})'(y) = e^{-y}.$$

由  $x = g^{-1}(y)$  则

$$2(1 - x) \frac{dx}{dy} = e^{-y}$$

解得  $2x - x^2 = -e^{-y} + C$ , 即  $y = \ln(x^2 - 2x + C)$  ( $0 < x < 1$ ), 由  $Y \in (0, +\infty)$  取  $c = 1$  取  $y = \ln(x^2 - 2x + 1) = -2\ln(1 - x)$ , 即  $g(x) = -2\ln(1 - x)$

48. 由密度函数的可知  $\int_0^3 \frac{1}{a} x^2 dx = \frac{9}{a} = 1$ , 所以  $a = 9$ . (1)  $Y$  的可能取值范围为  $[1, 2]$ , 且

$$P(Y = 1) = P(X > 2) = \int_2^3 \frac{1}{9} x^2 dx = \frac{19}{27},$$

$$P(Y = 2) = P(X \leq 1) = \int_0^1 \frac{1}{9} x^2 dx = \frac{1}{27},$$

$$\forall 1 < y < 2, P(Y \leq y) = P(Y = 1) + P(1 < Y \leq y) = \frac{19}{27} + \int_1^y \frac{1}{9} x^2 dx = \frac{y^3}{27} + \frac{2}{3}.$$

所以  $Y$  的分布函数为

$$F_Y(y) = \begin{cases} 0, & y < 1, \\ \frac{y^3}{27} + \frac{2}{3} & 1 \leq y < 2, \\ 1, & y \geq 2. \end{cases}$$

$$(2) P(X \leq Y) = P(X \leq 1) + P(1 < X < 2) = \int_0^2 \frac{1}{9} x^2 dx = \frac{8}{27}.$$

49. (1)  $Y = \frac{X}{1-X}$ , 在  $(0, 1)$  上严格单调增且  $Y \in (0, +\infty)$ . 由  $X = \frac{Y}{1+Y}$  的导函数为  $X' = \frac{1}{(y+1)^2}$  且  $X \sim U(0, 1)$ , 则  $Y$  的密度函数

$$f_Y(y) = \begin{cases} \frac{1}{(y+1)^2}, & y \in (0, +\infty) \\ 0, & else \end{cases}$$

$$(2) z = xI(a, 1](x) = \begin{cases} 0, & 0 < x \leq a \\ x, & a < x < 1 \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \int_0^a f(x) dx = a, & 0 \leq z < a \\ \int_0^a f(x) dx + \int_z^a f(x) dx = z, & a \leq z < 1 \\ 1, & z \geq 1 \end{cases}$$

$$(3) W = X^2 + X I_{[0,b]}(X), (0 < b < 1)$$

$$X \in (0, b], \quad W = X^2 + X, X = \frac{\sqrt{1+4W} - 1}{2}, \quad X'(W) = \frac{1}{\sqrt{1+4W}}$$

则密度函数为  $f_{W1}(w) = \frac{1}{\sqrt{1+4w}} I_{(0, b^2+b]}(w)$ 。

$$X \in (b, 1), \quad W = X^2, \quad X = \sqrt{W}, \quad X'(W) = \frac{1}{2\sqrt{W}}$$

则密度函数为  $f_{W2}(w) = \frac{1}{2\sqrt{W}} I_{(b^2, 1)}(w)$ 。

当  $b^2 + b < 1$  时,  $W$  的密度函数为

$$f_W(w) = \begin{cases} f_{W1}(w) = \frac{1}{\sqrt{1+4w}}, & 0 < w \leq b^2 \\ f_{W1}(w) + f_{W2}(w) = \frac{1}{\sqrt{1+4w}} + \frac{1}{2\sqrt{W}}, & b^2 < w \leq b^2 + b \\ f_{W2}(w) = \frac{1}{2\sqrt{W}}, & b^2 + b < w < 1 \\ 0, & else \end{cases}$$

当  $b^2 + b \geq 1$  时,  $W$  的密度函数为

$$f_W(w) = \begin{cases} f_{W1}(w) = \frac{1}{\sqrt{1+4w}}, & 0 < w \leq b^2 \\ f_{W1}(w) + f_{W2}(w) = \frac{1}{\sqrt{1+4w}} + \frac{1}{2\sqrt{W}}, & b^2 < w < 1 \\ f_{W1}(w) = \frac{1}{\sqrt{1+4w}}, & 1 \leq w \leq b^2 + b \\ 0, & else \end{cases}$$

1.  $(X, Y)$  的分布律为

X \ Y	Y		
	0	1	2
0	1/5	2/5	1/15
1	1/5	2/15	

$$F(x, y) = \begin{cases} 0, & x < 0 \text{ 或 } y < 0 \\ \frac{1}{5}, & 0 \leq x < 1 \text{ 且 } 0 \leq y < 1 \\ \frac{3}{5}, & 0 \leq x < 1 \text{ 且 } 1 \leq y < 2 \\ \frac{4}{5}, & 0 \leq x < 1 \text{ 且 } y \geq 2 \\ \frac{7}{15}, & x \geq 1 \text{ 且 } 0 \leq y < 1 \\ \frac{14}{15}, & x \geq 1 \text{ 且 } 1 \leq y < 2 \\ 1, & x \geq 1 \text{ 且 } y \geq 2 \end{cases}$$

3.  $(X, Y)$  的分布律为

Y \ X	0	1	2	3
1		3/8	3/8	
3	1/8			1/8

6. (1).  $(X, Y)$  联合分布律

$$P(X = x, Y = y) = \begin{cases} p^2(1-p)^{y-2}, & 0 < x < y, x, y \in N \\ 0, & \text{else} \end{cases}$$

(2)

$$P(X = x) = \sum_{n=x+1}^{\infty} P(X = x, Y = n) = (1-p)^{x-1} \cdot P \cdot \sum_{n=1}^{\infty} (1-p)^{n-1} p = (1-p)^{x-1} \cdot p, \quad x \in N$$

$$P(Y = y) = \sum_{n=1}^{\infty} P(X = n, Y = y) = \sum_{n=1}^{y-1} P(X = n, Y = y) = (y-1)(1-p)^{y-2} p^2, \quad y > 1 \quad \text{and} \quad y \in N$$

9.(1)

$$F(\infty, \infty) = 1.$$

$$a \left( b + \frac{\pi}{2} \right) \left( c + \frac{\pi}{2} \right) = 1$$

$$\begin{cases} F(\infty, -\infty) = 0 \\ F(-\infty, \infty) = 0 \end{cases} \quad \begin{cases} a \left( b + \frac{\pi}{2} \right) \left( c - \frac{\pi}{2} \right) = 0 \\ a \left( b - \frac{\pi}{2} \right) \left( c + \frac{\pi}{2} \right) = 0. \end{cases}$$

$$\begin{cases} a = \frac{1}{\pi^2} \\ b = \frac{\pi}{2} \\ c = \frac{\pi}{2} \end{cases}$$

(2).

$$P(X > 0, Y > 0) = 1 - F(0, +\infty) - F(+\infty, 0) + F(0, 0) = \frac{1}{4}$$

(3)  $F(x, +\infty) = \frac{1}{2} + \frac{1}{\pi} \arctan x$  可见边缘密度函数

$$f_X(x) = \frac{d}{dx} F(x, +\infty) = \frac{1}{\pi(1+x^2)}, \quad x \in R$$

, 同理

$$f_Y(x) = \frac{1}{\pi(1+y^2)}, \quad y \in R$$

$$10. \quad (1) \quad F(x, y) = \int_{-\infty}^x \int_{-\infty}^y \cos t \sin u dt du$$

$$= \begin{cases} \sin x \sin y & (0 \leq x, y < \frac{\pi}{2}) \\ \sin x & (0 \leq x < \frac{\pi}{2}, y \geq \frac{\pi}{2}) \\ \sin y & (0 \leq y < \frac{\pi}{2}, x \geq \frac{\pi}{2}) \\ 1 & (x, y \geq \frac{\pi}{2}) \\ 0 & (x < 0 \text{ 或 } y < 0) \end{cases}$$

$$(2)$$

$$P\left(0 < x < \frac{\pi}{4}, \frac{\pi}{4} < Y < \frac{\pi}{2}\right)$$

$$= \int_0^{\frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos u \cos v du dv$$

$$= \int_0^{\frac{\pi}{4}} \cos v dv \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos u du.$$

$$= \sin v \Big|_0^{\frac{\pi}{4}} \cdot \sin u \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}}{2} \left(1 - \frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}-1}{2}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\int_{A \cap B} f(x) dx}{\int_B f(x) dx}$$

11.(1)  $f_X(x) = \int_0^x f(x, y) dy = xe^{-x}, \quad x > 0$

$$f_{Y|X}(y | x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{1}{x}, & 0 < y < x \\ 0, & \text{其他.} \end{cases}$$

(2) 因为  $f_Y(y) = \int_y^\infty f(x, y) dx = e^{-y}, \quad y > 0$ , 所以有

$$P(X \leq 1, Y \leq 1) = \int_0^1 \int_0^x e^{-x} dy dx = 1 - 2e^{-1},$$

$$P(Y \leq 1) = \int_0^1 e^{-y} dy = 1 - e^{-1},$$

$$P(X \leq 1 | Y \leq 1) = \frac{P(X \leq 1, Y \leq 1)}{P(Y \leq 1)} = \frac{e - 2}{e - 1}.$$

17.(1) 因为  $f(x, y) = f_{Y|X}(y|x)f_X(x)$ , 所以

$$f(x, y) = \begin{cases} \frac{9y^2}{x}, & 0 < y < x < 1 \\ 0, & \text{其他.} \end{cases}$$

(2)

$$f_Y(y) = \int_y^1 f(x, y) dx = \begin{cases} -9y^2 \ln y, & 0 < y < 1, \\ 0, & \text{其他.} \end{cases}$$

18.(1) 由题意可得  $Y | X \sim U(0, X)$ , 所以  $f_{Y|X}(y | x) = \frac{1}{x}, \quad 0 < y < x$ . 则

$$f(x, y) = f_{Y|X}(y | x) \cdot f_X(x) = \frac{1}{x} \cdot xe^{-x} = e^{-x}, \quad 0 < y < x$$

$$f(x, y) = \begin{cases} e^{-x}, & 0 < y < x, \\ 0, & \text{其他.} \end{cases}$$

(2) 因为

$$f_Y(y) = \int_y^\infty f(x, y) dx = \int_y^\infty e^{-x} dx = e^{-y}, \quad y > 0,$$

所以

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0, \\ 0, & \text{其他.} \end{cases}$$



22.(1)由题意可知 $S = 1/2$ , 所以联合密度为

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{其他.} \end{cases}$$

(2)当 $0 \leq x \leq 1$ 时,  $f_X(x) = \int_x^1 2dy = 2(1-x)$ , 所以

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{其他.} \end{cases}$$

当 $0 \leq y \leq 1$ 时,  $f_Y(y) = \int_0^y 2dx = 2y$ , 所以

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1, \\ 0, & \text{其他.} \end{cases}$$

(3)当 $0 \leq x \leq y$ 时, 有 $f_{X|Y}(x|y) = f(x, y)/f_Y(y) = 1/y$ , 所以

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 < x < y, \\ 0, & \text{其他.} \end{cases}$$

(4)

$$P(X \leq 0.5|Y = y) = \int_0^{0.5} \frac{1}{y} dx = \frac{1}{2y}, \quad 0 < y < 1.$$

25.(1)由题意可知,  $(X_1, X_2, \dots, X_n)$ 是多项分布, 则当 $m_1, m_2, \dots, m_n = 1, 2, \dots, m$ 且 $m_1 + m_2 + \dots + m_n = m$ 时,

$$P(X_1 = m_1, X_2 = m_2, \dots, X_n = m_n) = \frac{m!}{m_1!m_2!\dots m_n!} p_1^{m_1} p_2^{m_2} \dots p_n^{m_n}.$$

(2)显然 $X_k \sim b(m, p_k)$ ,  $k = 1, 2, \dots, n$ .

(3)由题意可知,  $(X_1, X_2, X_3 + \dots + X_n) \sim b(m; p_1, p_2, 1 - p_1 - p_2)$ , 所以

$$\begin{aligned} P(X_1 = m_1, X_2 = m_2) &= P(X_1 = m_1, X_2 = m_2, X_3 + \dots + X_n = m - m_1 - m_2) \\ &= \frac{m!}{m_1!m_2!(m - m_1 - m_2)!} p_1^{m_1} p_2^{m_2} (1 - p_1 - p_2)^{m - m_1 - m_2}, \end{aligned}$$

其中  $m_1 + m_2 \leq m, m_1, m_2 \in \mathbf{N}$ .

(4)

$$\begin{aligned} P(X_2 = m_2, \dots, X_n = m_n | X_1 = m_1) &= \frac{P(X_1 = m_1, X_2 = m_2, \dots, X_n = m_n)}{P(X_1 = m_1)} \\ &= \frac{(m - m_1)!}{m_2! \dots m_n!} \left( \frac{p_2}{1 - p_1} \right)^{m_2} \dots \left( \frac{p_n}{1 - p_1} \right)^{m_n}, \sum_i m_i = m, m_i \in \mathbf{N}. \end{aligned}$$

30.(1) 因为  $X, Y$  相互独立, 所以

$$f(x, y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}}, & 0 < x < 1, y > 0 \\ 0, & \text{其他.} \end{cases}$$

(2) 令  $\Delta = 4X^2 - 4Y \geq 0$ , 则有  $Y \leq X^2$ , 所以

$$\begin{aligned} P(Y \leq X^2) &= \int_0^1 \int_0^{x^2} \frac{1}{2}e^{-\frac{y}{2}} dx dy = \int_0^1 (1 - e^{-\frac{x^2}{2}}) dx \\ &= 1 - \sqrt{2\pi} \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 + \sqrt{2\pi}/2 - \sqrt{2\pi}\Phi(1) \approx 0.1446. \end{aligned}$$

32.(1) 由题意可知,  $P(X = 1, Y = -1) = P(X = 1, Y = 1) = 0$ , 所以可得

X \ Y	-1	0	1	
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

(2) 不独立。例如

$$\frac{1}{4} = P(X = 0, Y = -1) \neq P(X = 0)P(Y = -1) = \frac{1}{8}.$$

34. 由题意可知

$$P(X = x) = P(Y = y) = \frac{1}{K}, \quad x, y = 0, 1, \dots, K-1,$$

则

$$P(Z_i = z_i) = P(X + iY = z_i) = \sum_{y=0}^{\lfloor z_i/i \rfloor} P(X = Z_i - iy | Y = y) P(Y = y) = (\lfloor z_i/i \rfloor + 1) \frac{1}{K^2}, \quad z_i = 1, 2, \dots, K.$$

对  $\forall i < k, 0 \leq z_i \leq z_k \leq (k+1)(K-1)$ , 有

$$P(Z_i = z_i, Z_k = z_k) = P(X+iY = z_i, X+kY = z_k) = P\left(X = \frac{kz_i - iz_k}{k-i}, Y = \frac{z_k - z_i}{k-i}\right) = \frac{1}{K^2}$$

显然  $P(Z_i = z_i, Z_k = z_k) \neq P(Z_i = z_i)P(Z_k = z_k)$ 。所以  $Z_n$  之间不独立也不两两独立。

例:

$$P(Z_1 = 2, Z_2 = 3) = P(X+Y = 2, X+2Y = 3) = P(X = 1, Y = 1) = \frac{1}{K^2},$$

$$P(Z_1 = 2) = P(X+Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0) = \frac{3}{K^2},$$

$$P(Z_2 = 3) = P(X+2Y = 3) = P(X = 1, Y = 1) + P(X = 3, Y = 0) = \frac{2}{K^2}.$$

所以  $P(Z_1 = 2, Z_2 = 3) \neq P(Z_1 = 2)P(Z_2 = 3)$ 。所以两两不独立。

37. 由对称性可知,  $X, Y$  的分布相同,  $X^2, Y^2$  的分布相同。(1)

$$f_X(x) = \int_{-1}^1 f(x, y) dy = \begin{cases} \frac{1}{2}, & |x| < 1, \\ 0, & \text{其他.} \end{cases} \quad f_Y(y) = \begin{cases} \frac{1}{2}, & |y| < 1, \\ 0, & \text{其他.} \end{cases}$$

当  $|x| < 1, |y| < 1$  时,

$$f(y|x) = \frac{f(x, y)}{f(x)} = \frac{1+xy}{2},$$

所以

$$f\left(y|X = \frac{1}{2}\right) = \begin{cases} \frac{2+y}{4}, & |y| < 1, \\ 0, & \text{其他} \end{cases}$$

(2) 令  $U = X^2, V = Y^2$ , 取值范围为  $0 \leq U, V \leq 1$ . 则  $(U, V)$  的联合分布为

$$\begin{aligned} P(U \leq u, V \leq v) &= \iint_{x^2 \leq u, y^2 \leq v} f(x, y) dx dy \\ &= \begin{cases} 0, & u, v < 0 \\ \int_{-\sqrt{u}}^{\sqrt{u}} \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1+xy}{4} dx dy, & 0 \leq u, v < 1, \\ \int_{-\sqrt{u}}^{\sqrt{u}} \int_{-1}^1 \frac{1+xy}{4} dx dy, & 0 \leq u < 1, v \geq 1; \\ \int_{-1}^1 \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1+xy}{4} dx dy, & u \geq 1, 0 \leq v < 1; \\ 1, & u, v \geq 1. \end{cases} \quad \begin{cases} 0, & u, v < 0, \\ \sqrt{uv}, & 0 \leq u, v < 1, \\ \sqrt{u}, & 0 \leq u < 1, v \geq 1, \\ \sqrt{v}, & u \geq 1, 0 \leq v < 1, \\ 1, & u, v \geq 1. \end{cases} \end{aligned}$$

再由(1) 中  $X, Y$  的边际密度函数可求

$$F_U(u) = \int_{x^2 \leq u} f_X(x) dx = \begin{cases} 0, & u < 0, \\ \sqrt{u}, & 0 \leq u < 1, \\ 1, & u \geq 1. \end{cases} \quad F_V(v) = \begin{cases} 0, & v < 0, \\ \sqrt{v}, & 0 \leq v < 1, \\ 1, & v \geq 1. \end{cases}$$

$F_{UV}(u, v) = F_U(u)F_V(v)$ , 所以  $U := X^2, V := Y^2$  相互独立.

42.

$$f_X(x) = \int_0^{2\pi} \int_0^{2\pi} f(x, y, z) dy dz = \begin{cases} \frac{1}{2\pi}, & 0 < x < 2\pi, \\ 0, & \text{其他.} \end{cases}$$

$$f(x, y) = \int_0^{2\pi} f(x, y, z) dz = \begin{cases} \frac{1}{4\pi^2}, & 0 < x, y < 2\pi, \\ 0, & \text{其他.} \end{cases}$$

由对称性可知,  $f_Y(y), f_Z(z)$  与  $f_X(x)$ ,  $f(x, z), f(y, z)$  与  $f(x, y)$  有相同的形式, 因为  $f(x, y) = f_X(x)f_Y(y)$ ,

所以  $x, y$  相互独立, 同理可得  $X, Y, Z$  两两独立。

因为  $f(x, y, z) \neq f_X(x)f_Y(y)f_Z(z)$ , 所以  $X, Y, Z$  不相互独立。

## 第六周作业答案

April 21, 2023

44. (1) 设 $Y$  取 $n$ 个值 $a_1, a_2, \dots, a_n$ , 对应概率分别为 $p_1, p_2, \dots, p_n$ , 先求 $U := X + Y$  的分布函数 $F_U(\cdot)$ . 由题意知 $X, Y$ 相互独立, 即

$$\begin{aligned} F_U(u) &= P(X + Y \leq u) \\ &= \sum_{i=1}^n P(Y = a_i) P(X + Y \leq u \mid Y = a_i) \\ &= \sum_{i=1}^n P(Y = a_i) P(X \leq u - a_i) \\ &= \sum_{i=1}^n p_i F_X(u - a_i) \end{aligned}$$

上式对 $u$  求导得 $U$  的密度函数 $h(u) = \sum_{i=1}^n p_i f(u - a_i)$ . 由 $X$ 的密度函数 $f$ 是连续的知 $h$ 也是连续的, 所以 $Z = X + Y$ 是连续型随机变量.

(2) 先求 $W := XY$  的分布函数 $F_W(\cdot)$ , 其中 $X \sim N(\mu, \sigma^2), Y \sim B(1, p)$ . 由题意知 $XY$ 相互独立, 当 $w < 0$ 时

$$\begin{aligned} F_W(w) &= P(XY \leq w) \\ &= P(Y = 1) P(X \leq w \mid Y = 1) \\ &= p P(X \leq w) \\ &= p F_X(w) \end{aligned}$$

当 $w \geq 0$ 时

$$\begin{aligned} F_W(w) &= P(XY \leq w) \\ &= P(Y = 1) P(X \leq w \mid Y = 1) + P(Y = 0) \\ &= p P(X \leq w) + 1 - p \\ &= p F_X(w) + 1 - p \end{aligned}$$

可知 $F_W(\cdot)$ 在 $w = 0$ 处不连续, 所以不是连续型随机变量。

47.  $(X, Y) \sim N(a, b, \sigma_1^2, \sigma_2^2, \rho)$ , 其联合密度为

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(x-a)^2}{\sigma_1^2} - 2\rho \frac{(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2} \right] \right\}, -\infty < x, y < \infty,$$

$$\begin{cases} U = X + cY, \\ V = X - cY, \end{cases} \text{ 的反函数为 } \begin{cases} X = \frac{1}{2}(U + V), \\ Y = \frac{1}{2c}(U - V). \end{cases} \quad \text{对应的雅可比行列式为}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2c} \\ \frac{1}{2} & -\frac{1}{2c} \end{vmatrix} = -\frac{1}{2c}$$

则 $(U, V)$  的联合密度为

$$\begin{aligned} f_{UV}(u, v) &= f \left( \left( \frac{1}{2}(u+v), \frac{1}{2c}(u-v) \right) \cdot |J| \right) \\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{\left( \frac{1}{2}(u+v) - a \right)^2}{\sigma_1^2} - 2\rho \frac{\left( \frac{1}{2}(u+v) - a \right) \left( \frac{1}{2c}(u-v) - b \right)}{\sigma_1\sigma_2} + \frac{\left( \frac{1}{2c}(u-v) - b \right)^2}{\sigma_2^2} \right] \right\} \cdot \left| \frac{1}{2c} \right| \\ &= \frac{1}{4|c|\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} [A_2u^2 + A_1v + B_2v^2 + B_1v + C_2uv + D] \right\}, \\ &\quad -\infty < u, v < \infty. \text{ 其中 } C_2 = \left( \frac{1}{2\sigma_1^2} - \frac{1}{2c^2\sigma_2^2} \right). \end{aligned}$$

由定理3.2, 要使 $U, V$  独立, 即要求 $uv$  项系数 $C_2$  为0 即可:

$$C_2 = \left( \frac{1}{2\sigma_1^2} - \frac{1}{2c^2\sigma_2^2} \right) = 0 \Rightarrow c = \frac{\sigma_1}{\sigma_2} \left( \text{或} -\frac{\sigma_1}{\sigma_2} \right)$$

48.(1)  $(X, Y) \sim N(0, 0; 1, 1; \rho)$ ,  $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ , 则 $Y = \sqrt{1-\rho^2}Z + \rho X$  且

$$J^{-1} = \frac{\partial(x, z)}{\partial(x, y)} = \begin{vmatrix} 1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{vmatrix} = \frac{1}{\sqrt{1-\rho^2}}.$$

所以 $(X, Z)$  的密度函数为

$$\begin{aligned}
f(x, z) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\} \cdot \sqrt{1-\rho^2} \left( \text{其中 } y = \sqrt{1-\rho^2}z + \rho x \right) \\
&= \frac{1}{2\pi} \exp \left\{ -\frac{(y-\rho x)^2 + x^2 - \rho^2 x^2}{2(1-\rho^2)} \right\} \\
&= \frac{1}{2\pi} \exp \left\{ -\frac{\left(\sqrt{1-\rho^2}z\right)^2 + (1-\rho^2)x^2}{2(1-\rho^2)} \right\} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+z^2)}, \quad x, z \in R.
\end{aligned}$$

由 $(X, Z)$  的密度函数可知:  $(X, Z) \sim N(0, 0; 1, 1; 0)$ , 两者相关系数为0, 所以 $X, Z$  相互独立. 或求得密度函数:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, z) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2+z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in R.$$

同理

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, z \in R$$

所以 $f(x, z) = f_X(x)f_Z(z)$ ,  $X, Z$  相互独立. 对于 $Y, Z$ , 与(1) 过程类似可求得 $(Y, Z)$  密度函数为

$$g(y, z) = \frac{1}{2\pi|\rho|} \exp \left\{ -\frac{y^2 - 2\sqrt{1-\rho^2}yz + z^2}{2\rho^2} \right\}, y, z \in R.$$

可见 $(Y, Z) \sim N(0, 0; 1, 1; \sqrt{1-\rho^2})$  ( $\rho \neq 0, \pm 1$ ), 相关系数不为0, 所以 $Y, Z$  不独立. (2)

$$\begin{aligned}
P(X > 0, Y > 0) &= P\left(X > 0, \sqrt{1-\rho^2}Z + \rho X > 0\right) \\
&= P(X > 0, Z > 0) + P\left(Z > 0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0\right) \\
&= \frac{1}{4} + P\left(X > 0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0\right).
\end{aligned}$$

下面求 $P\left(X > 0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0\right)$ , 令 $X = r \cos \theta, Z = r \sin \theta$ . 代入 $X >$

$0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0$  得  $\theta$  的取值范围为  $\left(-\arctan \frac{\rho}{\sqrt{1-\rho^2}}, 0\right)$ . 所以

$$\begin{aligned} P\left(X > 0, -\frac{\rho X}{\sqrt{1-\rho^2}} < Z < 0\right) &= \int_0^\infty \int_{-\arctan \frac{\rho}{\sqrt{1-\rho^2}}}^0 \frac{1}{2\pi} e^{-\frac{r^2}{2}} \cdot r d\theta dr \\ &= \arctan \frac{\rho}{\sqrt{1-\rho^2}} \int_0^\infty \frac{r}{2\pi} e^{-\frac{r^2}{2}} dr \\ &= \frac{1}{2\pi} \arcsin \rho \end{aligned}$$

即得

$$P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho$$

所以由XY对称性可知

$$\begin{aligned} P(XY < 0) &= 1 - P(X > 0, Y > 0) - P(X < 0, Y < 0) = 1 - 2P(X > 0, Y > 0) \\ &= 1 - \frac{1}{2} - \frac{1}{\pi} \arcsin \rho = \frac{1}{\pi} \arccos \rho \end{aligned}$$

**补充题:**  $X, Y \sim iid \text{Exp}(\lambda)$ , 求证  $\min\{X, Y\}$  与  $\max\{X, Y\} - \min\{X, Y\}$  相互独立, 并求它们的分布。

proof: 由  $X, Y \sim iid \text{Exp}(\lambda)$  知  $X, Y$  的联合密度函数为

$$f(x, y) = \lambda^2 e^{-\lambda(x+y)}$$

联合分布函数为  $F(x, y)$ . 记  $U = \min\{X, Y\}, V = \max\{X, Y\}$ , 所以  $U, V$  的联合分布函数为

$$G(u, v) = P(X \leq u, Y \leq v) + P(Y \leq u, X \leq v) = 2F(u, v)$$

联合密度函数为

$$g(u, v) = 2f(u, v) = 2\lambda^2 e^{-\lambda(u+v)}$$

令  $Z = U, W = V - U$ , 有  $U = Z, V = Z + W$ , 所以雅克比行列式为

$$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

所以  $Z, W$  的联合密度函数为

$$f_{Z,W}(z, w) = g(z, z+w) = 2\lambda^2 e^{-\lambda(2z+w)}$$



$$= 2\lambda e^{-2\lambda z} \times \lambda e^{-\lambda w} = f_Z(z)f_W(w)$$

所以  $Z = \min\{X, Y\}$ ,  $W = \max\{X, Y\} - \min\{X, Y\}$  相互独立, 且

$$f_Z(z) = 2\lambda e^{-2\lambda z}, f_W(w) = \lambda e^{-\lambda w}$$

2. 证明: (1)  $X$  为非负整值随机变量,

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X=k) = \sum_{k=1}^{\infty} \sum_{n=1}^k P(X=k) \\ &= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(X=k) = \sum_{n=1}^{\infty} P(X \geq n) \quad (\text{第一个“=”成立}) \\ &= \sum_{n=1}^{\infty} P(X > n-1) = \sum_{n=0}^{\infty} P(X > n) \quad (\text{第二个“=”成立}) \end{aligned}$$

(2)  $X$  为非负连续型随机变量且  $X \sim F$ , 设对应概率密度为  $f(\cdot)$ .

$$\begin{aligned} E(X) &= \int_0^{\infty} x f(x) dx = \int_0^{\infty} \left( \int_0^x f(x) dt \right) dx \\ &= \int_0^{\infty} \left( \int_t^{\infty} f(x) dx \right) dt \\ &= \int_0^{\infty} F(x) \Big|_t^{\infty} dt = \int_0^{\infty} 1 - F(t) dt \\ &= \int_0^{\infty} 1 - F(x) dx \end{aligned}$$

(3)  $X$  为非负随机变量,

$$\begin{aligned} E(X) &= E \left[ \int_0^X 1 dx \right] = E \left[ \int_0^{\infty} I_{(X>x)} dx \right] \\ &= \int_0^{\infty} E(I_{(X>x)}) dx = \int_0^{\infty} P(X > x) dx \\ &= \int_0^{\infty} 1 - F(x) dx \end{aligned}$$

3. 记  $\phi(x)$  为标准正态分布的密度函数, 则  $X$  的密度函数为  $f(x) = 0.5\phi(x) +$

$0.25 \cdot \phi\left(\frac{x-4}{2}\right)$ , 则

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} x f(x) dx = 0.5 \int_{-\infty}^{\infty} x \phi(x) dx + 0.25 \int_{-\infty}^{\infty} x \phi\left(\frac{x-4}{2}\right) dx \\
 &= 0.25 \int_{-\infty}^{\infty} (2y+4) \phi(y) d(2y+4) \\
 &= 0.5 \int_{-\infty}^{\infty} 2y \phi(y) dy + 0.5 \int_{-\infty}^{\infty} 4 \phi(y) dy \\
 &= 2 \int_{-\infty}^{\infty} \phi(y) dy = 2
 \end{aligned}$$

7. (1) 记  $X_i = \begin{cases} 1, & \text{第 } i \text{ 个盒子为空} \\ 0, & \text{第 } i \text{ 个盒子非空} \end{cases}, i = 1, \dots, n$  则空盒子总数为  $Y = \sum_{i=1}^n X_i$ , 因为  $P(X_i = 1) = \left(1 - \frac{1}{n}\right)^n$  所以  $Y \sim b\left(n, \left(1 - \frac{1}{n}\right)^n\right)$ ,  $EY = n\left(1 - \frac{1}{n}\right)^n$   
(2)  $n \rightarrow \infty$  时,

$$\lim_{n \rightarrow \infty} \frac{n\left(1 - \frac{1}{n}\right)^n}{n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{-n \cdot (-1)} = e^{-1}$$

8.  $X = n, n+1, \dots$ , 记  $Y_j$  为抽到  $i-1$  种卡后, 抽到新卡所需的次数, 则  $X_n = \sum_{j=1}^n Y_j$

$$\begin{aligned}
 P(Y_j = k) &= \frac{n-j+1}{n} \cdot \left(\frac{j-1}{n}\right)^{k-1}, \\
 EY_j &= \frac{n-j+1}{n} \sum_{k=1}^{\infty} k \left(\frac{j-1}{n}\right)^{k-1} = \frac{n-j+1}{n} \cdot \frac{n^2}{(n-j+1)^2} = \frac{n}{n-j+1},
 \end{aligned}$$

所以

$$EX_n = E \sum_{j=1}^n Y_j = \sum_{j=1}^n EY_j = \sum_{j=1}^n \frac{n}{n-j+1} = \sum_{k=1}^n \frac{n}{k}$$

(1)  $n = 12$  时,

$$EX_n = \sum_{k=1}^n \frac{n}{k} = 12 \sum_{k=1}^{12} \frac{1}{k} \approx 37.24$$

(2)

$$\lim_{n \rightarrow \infty} \frac{EX_n}{n \ln n} = \lim_{n \rightarrow \infty} \frac{n \sum_{k=1}^n \frac{1}{k}}{n \ln n} = \lim_{n \rightarrow \infty} \frac{\ln n + \gamma + \frac{1}{2n}}{\ln n} = 1.$$

17. 已知  $X$  的密度函数, 有

$$\begin{aligned}
 E[\min\{|X|, 1\}] &= \int_{|x|>1} f(x)dx + \int_{|x|\leq 1} |x| \cdot f(x)dx \\
 &= 2 \int_1^\infty \frac{1}{\pi(1+x^2)} dx + 2 \int_0^1 \frac{x}{\pi(1+x^2)} dx \quad (\text{对称性}) \\
 &= \frac{2 \arctan x}{\pi} \Big|_1^\infty + \int_0^1 \frac{d(1+x^2)}{\pi(1+x^2)} = \frac{1}{2} + \frac{1}{\pi} \ln(1+x^2) \Big|_0^1 \\
 &= \frac{1}{2} + \frac{\ln 2}{\pi}
 \end{aligned}$$

18. (1) 当  $X = 1$  时,  $Y \in (0, 1)$ , 当  $X = 2$  时,  $Y \in (0, 2)$ , 所以

$$\begin{aligned}
 P(Y < y) &= P(x=1) \cdot P(Y < y | x=1) + P(x=2)P(Y < y | x=2) \\
 &= \frac{1}{2} \int_0^y du + \frac{1}{2} \int_0^y \frac{1}{2} du = \frac{3}{4}y \quad (0 < y < 1)
 \end{aligned}$$

$$\begin{aligned}
 P(Y < y) &= P(Y < 1) + P(X=2)P(1 \leq Y < 2) \\
 &= \frac{3}{4} + \frac{1}{2} \int_1^y \frac{1}{2} du = \frac{1}{2} + \frac{1}{4}y \quad (1 \leq y < 2)
 \end{aligned}$$

$$\text{所以 } F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{3}{4}y, & 0 \leq y < 1 \\ \frac{1}{2} + \frac{1}{4}y, & 1 \leq y < 2 \\ 1, & y \geq 2 \end{cases}$$

(2)

$$EY = \int y dF(y) = \int_0^1 \frac{3}{4}y dy + \int_1^2 \frac{y}{4} dy = \frac{3}{4}.$$

19.  $(X, Y) \sim N(1, 1, 0.5, 0.5, 0.5)$ , 则  $(X, Y)$  的联合密度为

$$f(x, y) = \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{2}{3} [2(x-1)^2 - 2(x-1)(y-1) + 2(y-1)^2] \right\}.$$

(1)  $Z = |X-Y|$ , 可以先求  $X-Y$  的密度。令  $\begin{cases} S = X-Y \\ T = Y \end{cases}$  则  $\begin{cases} X = S+T \\ Y = T \end{cases}$ , 对应的雅可比行列式为

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

所以  $(S, T)$  的联合密度为

$$\begin{aligned} f_{ST}(s, t) &= f(s+t, t) \cdot |J| \\ &= \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{2}{3} [2(s+t-1)^2 - 2(s+t-1)(t-1) + 2(t-1)^2] \right\} \\ &= \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{4}{3}s^2 - \frac{4}{3}s(t-1) - \frac{4}{3}(t-1)^2 \right\}, -\infty < s, t < \infty. \end{aligned}$$

则  $U$  的边际密度为

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{4}{3}s^2 - \frac{4}{3}s(t-1) - \frac{4}{3}(t-1)^2 \right\} dt \\ &= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{4}{3} \left[ \left( t + \frac{s-2}{2} \right)^2 + \frac{3}{4}s^2 \right] \right\} dt \\ &= \frac{1}{\sqrt{\pi}} e^{-s^2} \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}\pi} \exp \left\{ -\frac{\left( t + \frac{s-2}{2} \right)^2}{3/4} \right\} dt \\ &= \frac{1}{\sqrt{\pi}} e^{-s^2}, \quad -\infty < s < \infty \end{aligned}$$

对于  $Z = |S|$ ,  $P(Z \leq z) = P(-z \leq S \leq z) = 2 \int_0^z \frac{1}{\sqrt{\pi}} e^{-s^2} ds, (z > 0)$ , 可得  $Z$  的密度函数为

$$f_Z(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}, z > 0.$$

期望为

$$E(Z) = \int_0^{\infty} z \cdot \frac{2}{\sqrt{\pi}} e^{-z^2} dz = -\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} d(-z^2) = \frac{1}{\sqrt{\pi}}.$$

(2)  $U = \max(X, Y), V = \min(X, Y)$ , 则有  $\begin{cases} U + V = X + Y \\ U - V = |X - Y| \end{cases}$  由期望的

性质有:

$$\begin{cases} E(U + V) = E(X + Y) \\ E(U - V) = E|X - Y| \end{cases} \Rightarrow \begin{cases} E(U) + E(V) = E(X) + E(Y) = 2 \\ E(U) - E(V) = EZ = \frac{1}{\sqrt{\pi}} \end{cases}$$

$$\text{有} \begin{cases} E(U) = 1 + \frac{1}{2\sqrt{\pi}} \\ E(V) = 1 - \frac{1}{2\sqrt{\pi}} \end{cases}$$

21.(1)由泊松分布的可加性可知,  $X + Y \sim P(\lambda + \mu)$ , 所以

$$\begin{aligned} P(X = k | X + Y = m) &= \frac{P(X = k, X + Y = m)}{P(X + Y = m)} = \frac{P(X = k)P(Y = m - k)}{P(X + Y = m)} = \frac{\frac{\lambda^k}{k!} e^{-\lambda} \frac{\mu^{m-k}}{(m-k)!} e^{-\mu}}{\frac{(\lambda + \mu)^m}{m!} e^{-(\lambda + \mu)}} \\ &= C_m^k \left( \frac{\lambda}{\lambda + \mu} \right)^k \left( 1 - \frac{\lambda}{\lambda + \mu} \right)^{m-k}, \quad k = 0, 1, \dots, m. \end{aligned}$$

所以  $X | X + Y = m \sim b(m, \lambda/(\lambda + \mu))$ ,  $E(X | X + Y = m) = m\lambda/(\lambda + \mu)$ 。

(2)由二项分布的可加性可知,  $X + Y \sim b(2n, p)$ , 所以

$$\begin{aligned} P(X = k | X + Y = m) &= \frac{P(X = k)P(Y = m - k)}{P(X + Y = m)} = \frac{C_n^k p^k (1 - p)^{n-k} C_n^{m-k} p^{m-k} (1 - p)^{n-m+k}}{C_{2n}^m p^m (1 - p)^{2n-m}} \\ &= \frac{C_n^k C_n^{m-k}}{C_{2n}^m}, \quad k = 0, 1, \dots, m. \end{aligned}$$

由题意可知,  $X, Y$  独立同分布。所以  $E(X | X + Y = m) = E(Y | X + Y = m)$ 。

又因为  $E(X + Y | X + Y = m) = m$ , 所以  $E(X | X + Y = m) = m/2$ 。

22.

$$X \sim \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, Y | X = k \sim N(k, 1)$$

(1)  $Y$  的概率密度函数为

$$f_Y(y) = \sum_{k=0}^2 f_Y(y | X = k) P(X = k) = \frac{1}{3} \sum_{k=0}^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-k)^2}{2}}, -\infty < y < \infty.$$

期望为

$$E(Y) = \frac{1}{3} \sum_{k=0}^2 \left[ \int_{-\infty}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-k)^2}{2}} dy \right] = \frac{1}{3} \sum_{k=0}^2 k = 1$$

(2)  $Z := X + Y$ , 则其分布函数为

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X + Y \leq z) \\ &= \sum_{k=0}^2 P(Y \leq z - X | X = k) P(X = k) \\ &= \frac{1}{3} \sum_{k=0}^2 P(Y \leq z - X | X = k) \\ &= \frac{1}{3} \sum_{k=0}^2 \Phi(z - 2k), -\infty < z < \infty. \quad (\text{其中 } \Phi(\cdot) \text{ 为标准正态的分布函数.}) \end{aligned}$$

4.(1)

$$\begin{aligned}
EX &= \int_0^\infty x \cdot \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2\pi}\sigma} \int x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sigma} \sqrt{\frac{\pi}{2}} \sigma^2 = \sigma \sqrt{\frac{\pi}{2}} \\
EX^2 &= \int_0^\infty x^2 \cdot \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = - \int_0^\infty x^2 d\left(e^{-\frac{x^2}{2\sigma^2}}\right) \\
&= - \left. x^2 e^{-\frac{x^2}{2\sigma^2}} \right|_0^\infty + 2 \int_0^\infty x e^{-\frac{x^2}{2\sigma^2}} dx = 0 - 2\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^\infty = 2\sigma^2 \\
\text{Var}(X) &= EX^2 - (EX)^2 = \frac{4-\pi}{2} \sigma^2
\end{aligned}$$

(2)

$$\begin{aligned}
EX &= \int_0^1 x \cdot \frac{T(\alpha+\beta)}{T(\alpha)T(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{T(\alpha+\beta)}{T(\alpha)T(\beta)} \int_0^1 x^{\alpha-1+1} (1-x)^{\beta-1} dx \\
&= \frac{T(\alpha+\beta)}{\Gamma(\alpha)T(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta} \\
EX^2 &= \int_0^1 x^2 \frac{T(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha+\beta)}{T(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+2-1} (1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)T(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\
\text{Var}(X) &= EX^2 - (EX)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}$$

(3)

$$\begin{aligned}
EX &= \int_0^\infty x \cdot \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} dx = \lambda \int_0^\infty \left[\left(\frac{x}{\lambda}\right)^k\right]^{\frac{1}{k}+1-1} e^{-\left(\frac{x}{\lambda}\right)^k} d\left(\frac{x}{\lambda}\right)^k \\
&= \lambda \Gamma\left(1 + \frac{1}{k}\right) \\
EX^2 &= \int_0^\infty x^2 \cdot \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} dx = \lambda^2 \int_0^\infty \left[\left(\frac{x}{\lambda}\right)^k\right]^{\frac{2}{k}+1-1} e^{-\left(\frac{x}{\lambda}\right)^k} d\left(\frac{x}{\lambda}\right)^k \\
&= \lambda^2 \Gamma\left(1 + \frac{2}{k}\right) \\
\text{Var}(X) &= EX^2 - (EX)^2 = \lambda^2 \left[ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right]
\end{aligned}$$

24.(1) 令  $Z = X + Y$ , 由题意易知  $Z \sim N(3\mu, 4\sigma^2)$ 。所以

$$\begin{aligned} E((X + Y - 3\mu)_+) &= E(Z - 3\mu | Z > 3\mu) = \int_{3\mu}^{\infty} \frac{z - 3\mu}{\sqrt{2\pi}2\sigma} e^{-\frac{(z-3\mu)^2}{8\sigma^2}} dz \\ &= \int_0^{\infty} \frac{t}{\sqrt{2\pi}2\sigma} e^{-\frac{t^2}{8\sigma^2}} dt = -\frac{8\sigma^2}{2} \frac{1}{\sqrt{2\pi}2\sigma} e^{-\frac{t^2}{8\sigma^2}} \Big|_0^{\infty} = \frac{2\sigma}{\sqrt{2\pi}}. \end{aligned}$$

(2)

$$E((Z - 3\mu)^2 | Z > 3\mu) = \frac{1}{2} E(Z - 3\mu)^2 = \frac{1}{2} \text{Var}(Z) = 2\sigma^2.$$

所以  $\text{Var}((X + Y - 3\mu)_+) = 2\sigma^2 - (\sigma/\sqrt{2\pi})^2 = 2\sigma^2 - 2\sigma^2/\pi$ 。

28.(1) 设投掷的点数大于等于  $n$  时投掷次数的期望为  $E_n$ , 则题中所要求的为  $E_{10}$ . 则

$$E_1 = 1$$

$$E_2 = 1 \times \frac{5}{6} + 2 \times \frac{1}{6} = \frac{7}{6}$$

对于  $3 \leq n \leq 6$ , 对第一次投出的点数取条件, 则有

$$\begin{aligned} E_n &= \sum_{i=1}^6 E_{n|X_1=i} P(X_1 = i) \\ &= P(X_1 = 1)(1 + E_{n-1}) + \cdots P(X_1 = n-1)(1 + E_1) \\ &\quad + P(X_1 = n)(1 + 0) + \cdots P(X_1 = 6)(1 + 0) \\ &= \sum_{i=1}^6 P(X_1 = i) + P(X_1 = 1)E_{n-1} + P(X_1 = 2)E_{n-2} + \cdots P(X_1 = n-1)E_1 \\ &= 1 + \frac{1}{6} \sum_{i=1}^{n-1} E_i \\ E_n &= \frac{7}{6} E_{n-1} = \left(\frac{7}{6}\right)^{n-1} \quad (1 \leq n \leq 6) \end{aligned}$$

$$\begin{aligned} n > 6 \text{ 时, } E_n &= \sum_{i=1}^6 E_{n|X_1=i} P(X_1 = i) = 1 + \sum_{i=1}^6 P(X_1 = i) E_{n-1} = 1 + \frac{1}{6} \sum_{i=1}^6 E_{n-i} \\ \Rightarrow E_{n+1} - E_n &= \frac{1}{6} (E_n - E_{n-6}) \\ E_{n+1} &= \frac{7}{6} E_n - \frac{1}{6} E_{n-6} \quad (n \geq 7) \end{aligned}$$

两式结合, 有  $E_7 = \left(\frac{7}{6}\right)^6$ ,  $E_9 = \left(\frac{7}{6}\right)^7 - \frac{1}{6}$ ,  $E_9 = \left(\frac{7}{6}\right)^8 - \frac{7}{18}$ ,

$$E_{10} = \left(\frac{7}{6}\right)^9 - \frac{49}{72} \approx 3.3237$$

(2) 记直到点数大于等于10 所需的投推次数为  $Y_{10}$ , 由(1) 知  $E(Y_{10}) = E_{10}$ .  
再由Wald 等式有

$$E\left(\sum_{i=1}^{Y_{10}} X_i\right) = E(X_1) E_{10} = \frac{7}{2} E_{10} \approx 11.6329$$

29.(1)由题意易知

$$P(Y < y) = 1 - P(X_1 > y, X_2 > Y) = 1 - P(X_1 > y)P(X_2 > y) = 1 - e^{-\frac{3}{2}y}, \quad y > 0,$$

$$P(Z < z) = P(X_1 < z, X_2 < z) = P(X_1 < z)P(X_2 < z) = 1 - e^{-z} - e^{-\frac{1}{2}z} + e^{-\frac{3}{2}z}, \quad z > 0.$$

所以有

$$EY = \int_0^\infty y F'_Y(y) dy = \int_0^\infty y \frac{3}{2} e^{-\frac{3}{2}y} dy = \frac{2}{3},$$

$$EZ = \int_0^\infty z F'_Z(z) dz = \int_0^\infty z (e^{-z} + \frac{1}{2} e^{-\frac{z}{2}} - \frac{3}{2} e^{-\frac{3}{2}z}) dz = \frac{7}{3}.$$

(2) 因为  $Y \sim EXP(\frac{3}{2})$ , 所以  $\text{Var}(Y) = 4/9$ 。

$$EZ^2 = \int_0^\infty z^2 (e^{-z} + \frac{1}{2} e^{-\frac{z}{2}} - \frac{3}{2} e^{-\frac{3}{2}z}) dz = \frac{82}{9},$$

所以  $\text{Var}(Z) = 82/9 - (7/3)^2 = 33/9 = 11/3$ 。



# 第八周作业答案

May 17, 2023

31. (1)

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$E(X) = \sum_{x=1}^6 P(X=x) \cdot x = \sum_{x=1}^6 \frac{1}{6} x = \frac{7}{2}$$

$$\text{Var}(X) = \sum_{x=1}^6 P(X=x) \cdot x^2 - (EX)^2 = \frac{35}{12}$$

$$Y \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{36} & \frac{1}{12} & \frac{5}{36} & \frac{7}{36} & \frac{7}{36} & \frac{11}{36} \end{pmatrix}$$

$$E(Y) = \sum_{y=1}^6 P(Y=y) \cdot y = \frac{161}{36}$$

$$\text{Var}(Y) = \sum_{y=1}^6 P(Y=y) y^2 - (EY)^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 \approx 1.97$$

(2)

$Y \backslash X$	1	2	3	4	5	6
1	$\frac{1}{36}$					
2	$\frac{1}{36}$	$\frac{2}{36}$				
3	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{3}{36}$			
4	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{4}{36}$		
5	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{5}{36}$	
6	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{6}{36}$

$$EXY = \sum xyP(X=x, Y=y) = \frac{616}{36} = \frac{154}{9}$$

$$\text{Cov}(X, Y) = EXY - EXEY = \frac{35}{24}$$

33.

$$\begin{aligned}\text{Cov}(\alpha X + \beta Y, \alpha X - \beta Y) &= \text{Cov}(\alpha X, \alpha X - \beta Y) + \text{Cov}(\beta Y, \alpha X - \beta Y) \\ &= \text{Cov}(\alpha X, \alpha X) - \text{Cov}(\alpha X, \beta Y) + \text{Cov}(\beta Y, \alpha X) - \text{Cov}(\beta Y, \beta Y) \\ &= \alpha^2 \text{Cov}(X, X) - \beta^2 \text{Cov}(Y, Y)\end{aligned}$$

$(X, Y) \sim N(\mu, \mu, \sigma^2, \sigma^2, \rho)$ , 则  $X, Y \sim N(\mu, \sigma^2)$  有

$$\begin{aligned}\text{Cov}(\alpha X + \beta Y, \alpha X - \beta Y) &= (\alpha^2 - \beta^2) \sigma^2 = 0 \\ \Rightarrow \alpha &= \pm \beta.\end{aligned}$$

37 (1)

$$\begin{aligned}\text{Cov}(X_1, X_2) &= E(X_1 X_2) - E(X_1)E(X_2) \\ \text{Cov}(X_1, E(X_2 | X_1)) &= E(X_1 E(X_2 | X_1)) - E(X_1)E(E(X_2 | X_1)) \\ &= E(X_1 E(X_2 | X_1)) - E(X_1)E(X_2) = E(X_1 X_2) - E(X_1)E(X_2) \\ \text{所以 } \text{Cov}(X_1, X_2) &= \text{Cov}(X_1, E(X_2 | X_1)).\end{aligned}$$

(2)

$$\begin{aligned}\text{Cov}(X_1, X_2) &= \text{Cov}(X_1, E(X_2 | X_1)) \\ &= \text{Cov}(X_1, 1 + cX_1) = c\text{Var}(X_1) \\ \text{所以 } c &= \text{Cov}(X_1, X_2)/\text{Var}(X_1).\end{aligned}$$

39 (1) 由  $X_i \stackrel{iid}{\sim} Ge(p), i = 1, 2, \dots, N \sim B(n, q)$ , 且  $X_i, N$  相互独立,

$$\begin{aligned}E(S | N = n) &= E\left(\sum_{i=1}^N X_i | N = n\right) = \sum_{i=1}^n E(X_i) = n/p \\ \text{Var}(S | N = n) &= \sum_{i=1}^n \text{Var}(X_i | N = n) = \sum_{i=1}^n \text{Var}(X_i) = n(1-p)/p^2 \\ \text{Var}(S) &= E(\text{Var}(S | N)) + \text{Var}(E(S | N)) \\ &= E(N(1-p)/p^2) + \text{Var}(N/p) = nq(1-p)/p^2 + nq(1-q)/p^2\end{aligned}$$

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(1) 由

$$E(T) = a, Var(T) = b > 0$$

$$N(t) \sim poisson(\lambda t), E(N(t)) = \lambda t, Var(N(t)) = \lambda t$$

有

$$E(N(T)) = E(E(N(T) | T)) = E(\lambda T) = \lambda a$$

$$Cov(T, N(T)) = E(TN(T)) - E(T)E(N(T))$$

$$= E(E(TN(T) | T)) - \lambda a^2 = E(\lambda T^2) - \lambda a^2$$

$$= \lambda(b + a^2) - \lambda a^2 = \lambda b$$

(2)

$$Var(N(T)) = E(N^2(T)) - E^2(N(T))$$

$$= E(E(N^2(T) | T)) - (\lambda a)^2$$

$$= E(\lambda T + (\lambda T)^2) - (\lambda a)^2$$

$$= \lambda a + \lambda^2(b + a^2) - (\lambda a)^2 = \lambda a + \lambda^2 b$$

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$$X_n + Y_n \xrightarrow{P} X + Y$$

$$\iff \forall \epsilon > 0, \lim_{n \rightarrow \infty} P(|X_n + Y_n - X - Y| \geq \epsilon) = 0$$

$$\because |X_n + Y_n - X - Y| \leq |X_n - X| + |Y_n - Y|$$

$$\therefore P(|X_n + Y_n - X - Y| \geq \epsilon) \leq P(|X_n - X| + |Y_n - Y| \geq \epsilon)$$

$$\leq P(|X_n - X| \geq \epsilon/2) + P(|Y_n - Y| \geq \epsilon/2)$$

$$\because X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y \quad \therefore \forall \epsilon/2 > 0,$$

$$\lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon/2) = 0, \quad \lim_{n \rightarrow \infty} P(|Y_n - Y| \geq \epsilon/2) = 0$$

$$\lim_{n \rightarrow \infty} P(|X_n + Y_n - X - Y| \geq \epsilon) \leq \lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon/2) + \lim_{n \rightarrow \infty} P(|Y_n - Y| \geq \epsilon/2) = 0$$

45 记 $X$ 为500次实验中A发生的次数,  $X \sim B(500, 0.2)$ ,所以

$$E(X) = 500 * 0.2 = 100, Var(X) = 500 * 0.2 * 0.8 = 80$$

由切比雪夫不等式知 $P(|X - 100| \geq 20) \leq Var(X)/20^2 = 80/400 = 0.2$ . 所以 $X$ 落在80到120之间的概率为

$$P(|X - 100| < 20) = 1 - P(|X - 100| \geq 20) \geq 0.8$$

由中心极限定理知n比较大时可近似认为

$$\frac{X - E(X)}{\sqrt{Var(X)}} \sim N(0, 1)$$

所以

$$P(|X-100| < 20) = P(|X-100|/\sqrt{80} < 20/\sqrt{80}) = \Phi(20/\sqrt{80}) - \Phi(-20/\sqrt{80}) \approx 0.975$$

46.  $X_1, \dots, X_2$  为独立同分布随机变量,

$$\begin{aligned} E\left(\sum_{i=1}^n X_i^2\right) &= nE(X_1^2) = n\alpha_2, \\ \text{Var}\left(\sum_{i=1}^n X_i^2\right) &= n\text{Var}(X_1^2) = n\left(E(X_1^4) - (E(X_1^2))^2\right) = n(\alpha_4 - \alpha_2^2). \end{aligned}$$

则由中心极限定理有

$$\frac{\sum_{i=1}^n X_i^2 - n\alpha_2}{\sqrt{n(\alpha_4 - \alpha_2^2)}} \xrightarrow{d} N(0, 1)$$

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$X_1, \dots, X_n, i.i.d., \sim b(1, 0.9)$

$$E\left(\sum_{i=1}^n X_i\right) = nE(X_1) = 0.9n, \quad \text{Var}\left(\sum_{i=1}^n X_i\right) = n\text{Var}(X_1) = 0.09n$$

(1) 当  $n = 100$ ,  $E(\sum_{i=1}^n X_i) = 90$ ,  $\text{Var}(\sum_{i=1}^n X_i) = 9$ . 由中心极限定理,

$$\begin{aligned} P\left(\sum_{i=1}^n X_i \geq 85\right) &= 1 - P\left(\sum_{i=1}^n X_i \leq 85\right) \\ &= P\left(\frac{\sum_{i=1}^n X_i - 90}{3} \leq \frac{85 - 90}{3}\right) \\ &\approx 1 - \Phi\left(-\frac{5}{3}\right) = \Phi\left(\frac{5}{3}\right) \\ &\approx 0.9522 \end{aligned}$$

(2) 由中心极限定理, 至少有80% 的部件正常工作的概率为

$$\begin{aligned} P\left(\sum_{i=1}^n X_i > 0.8n\right) &= 1 - P\left(\sum_{i=1}^n X_i \leq 0.8n\right) \\ &= 1 - P\left(\frac{\sum_{i=1}^n X_i - 0.9n}{0.3\sqrt{n}} \leq \frac{0.8n - 0.9n}{0.3\sqrt{n}}\right) \\ &\approx 1 - \Phi\left(-\frac{n}{3\sqrt{n}}\right) = \Phi\left(\frac{\sqrt{n}}{3}\right) \end{aligned}$$

要使所求概率不小于0.95, 即

$$\begin{aligned} \Phi\left(\frac{\sqrt{n}}{3}\right) &\geq 0.95 \\ n &\geq 24.35. \end{aligned}$$

所以 $n$  至少取25 .

51 (1)记 $X$ 为生产线上组装每件产品的时间,  $X \sim \exp(1/10), E(X) = 10, Var(X) = 100$ . 由

$$E\left(\sum_{i=1}^{100} X_i\right) = 100 * 10 = 1000, Var\left(\sum_{i=1}^{100} X_i\right) = 100^2$$

由中心极限定理知

$$\begin{aligned} P(15 * 60 \leq \sum_{i=1}^{100} X_i \leq 20 * 60) &= P(-1 \leq (\sum_{i=1}^{100} X_i - 1000)/100 \leq 2) \\ &= \Phi(2) - \Phi(-1) \approx 0.819 \end{aligned}$$

(2)

$$\begin{aligned} P\left(\sum_{i=1}^n X_i \leq 16 * 60\right) &= P\left(\left(\sum_{i=1}^{100} X_i - 10n\right)/(10\sqrt{n}) \leq (960 - 10n)/(10\sqrt{n})\right) \\ \Phi((960 - 10n)/(10\sqrt{n})) &\geq 0.95 \\ (960 - 10n)/(10\sqrt{n}) &\geq 1.645 \end{aligned}$$

解得 $n \leq 81$ .

## 第五章

11.(1)由 $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$ ,所以由卡方分布的密度函数为

$$g_{n-1}(x) = \frac{x^{(n-1)/2-1} e^{-x/2}}{2^{(n-1)/2} \Gamma((n-1)/2)}.$$

由 $\chi_{n-1}^2$ 分布的方差为 $2(n-1)$ ,即

$$\text{var}((n-1)S^2/\sigma^2) = 2(n-1)$$

$$(n-1)^2/\sigma^4 \text{Var}(S^2) = 2(n-1), \text{Var}(S^2) = 2\sigma^4/(n-1)$$

(2)设 $Y = (n-1)S^2/\sigma^2$ , 有 $S = \sigma\sqrt{Y}/\sqrt{n-1}$ .所以

$$\begin{aligned} E(S) &= E(\sigma\sqrt{Y}/\sqrt{n-1}) = \sigma/\sqrt{n-1} \int_0^{+\infty} \sqrt{y} g_{n-1}(y) dy \\ &= \sigma/\sqrt{n-1} \int_0^{+\infty} \sqrt{y} \frac{y^{(n-1)/2-1} e^{-y/2}}{2^{(n-1)/2} \Gamma((n-1)/2)} dy \\ &= \sigma/\sqrt{n-1} \frac{2^{n/2} \Gamma(n/2)}{2^{(n-1)/2} \Gamma((n-1)/2)} \int_0^{+\infty} \frac{y^{n/2-1} e^{-y/2}}{2^{n/2} \Gamma(n/2)} dy \end{aligned}$$

后一个积分即为 $\chi_n^2$ 分布密度函数的积分为1, 所以有

$$E(S) = \frac{\sigma\sqrt{2}\Gamma(n/2)}{\sqrt{n-1}\Gamma((n-1)/2)}$$

14.  $X_1, X_2 \stackrel{iid}{\sim} N(0, 1)$ , 所以 $X_1 - X_2, X_1 + X_2 \sim N(0, 2)$ .  
下证 $X_1 - X_2, X_1 + X_2$ 相互独立。因为正态分布的独立性与协方差为0等价, 即证

$$\text{Cov}(X_1 - X_2, X_1 + X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 0$$

所以 $X_1 - X_2, X_1 + X_2$ 相互独立。所以

$$\frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} \sim F_{1,1}.$$

15.  $X_1, X_2, X_3, X_4$  i.i.d.  $\sim N(0, 2^2)$ , 则有

$$(X_1 - 2X_2) \sim N(0, 20), \quad (3X_3 - 4X_4) \sim N(0, 10^2)$$

要使  $T = a(X_1 - 2X_2)^2 + b(3X_3 - 4X_4)^2$  服从  $\chi^2$  分布,

$$\begin{aligned} \sqrt{a}(X_1 - 2X_2) &\sim N(0, 1), \quad \sqrt{b}(3X_3 - 4X_4) \sim N(0, 1) \\ \Rightarrow a &= \frac{1}{20}, \quad b = \frac{1}{100} \end{aligned}$$

此时  $T \sim \chi^2(2)$ . 或  $(a = 1/20, b = 0)$  及  $(a = 0, b = 1/100)$  也可, 此时  $T \sim \chi^2(1)$

16.  $X_1, X_2, \dots, X_9 \text{ i.i.d. } \sim N(\mu, \sigma^2)$ , 则有

$$\begin{aligned} Y_1 &\sim N\left(\mu, \frac{1}{6}\sigma^2\right), \quad Y_2 \sim N\left(\mu, \frac{1}{3}\sigma^2\right), \quad \sqrt{2}(Y_1 - Y_2) \sim N(0, \sigma^2), \\ \frac{2S^2}{\sigma^2} &\sim \chi^2(2). \quad (Y_2 \text{ 是样本 } X_7, X_8, X_9 \text{ 的样本均值, } S^2 \text{ 为样本方差.}) \\ Z &= \frac{\frac{\sqrt{2}}{\sigma}(Y_1 - Y_2)}{S} = \frac{\sqrt{2}(Y_1 - Y_2)}{\sqrt{\frac{2S^2}{2\sigma^2}}} \sim t(2). \end{aligned}$$

17.  $X_1, X_2, \dots, X_{15} \text{ i.i.d. } \sim N(0, 2^2)$ . 则有

$$\begin{aligned} \frac{1}{4}(X_1^2 + \dots + X_{10}^2) &\sim \chi^2(10), \\ \frac{1}{4}(X_{11}^2 + \dots + X_{15}^2) &\sim \chi^2(5), \\ Y &= \frac{X_1^2 + \dots + X_{10}^2}{2(X_{11}^2 + \dots + X_{15}^2)} = \frac{(\frac{1}{4}(X_1^2 + \dots + X_{10}^2))/10}{(\frac{1}{4}(X_{11}^2 + \dots + X_{15}^2))/5} \sim F(10, 5). \end{aligned}$$

2. 总体为射击的环数，即  $\{0, 1, 2, \dots, 10\}$ 。样本为五次射击的结果，为 8, 9, 7, 10, 6 环。

4. 由题意可知，总体分布为  $B(1, p)$ 。样本分布为

$$P(X_1 = x_1, \dots, X_{10} = x_{10}) = p^{\sum_{i=1}^{10} x_i} (1-p)^{10 - \sum_{i=1}^{10} x_i}, \quad x_i = 0, 1.$$

8. (1) 样本空间为  $\Omega = \{(X_1, X_2, X_3, X_4, X_5) : X_1, \dots, X_5 \in \{0, 1\}\}$ ，抽样分布为

$$P((X_1, X_2, X_3, X_4, X_5) = (x_1, x_2, x_3, x_4, x_5)) = p^{\sum_{i=1}^5 x_i} (1-p)^{5 - \sum_{i=1}^5 x_i}, \quad x_i \in \{0, 1\}, i = 1, \dots, 5$$

(2)  $X_5 + p$  不是统计量，因为含有未知参数。

9.

$$\bar{X} = \frac{1}{7}(74.001 + 74.005 + \dots + 74.002) = 73.9893,$$

$$S = \sqrt{\frac{1}{n-1}((74.001 - 73.9893)^2 + \dots + (74.002 - 73.9893)^2)} = 0.0359.$$

1.

$$\bar{X} = \frac{53 \times 1 + 16 \times 2 + 21 \times 3}{100} = 1.48,$$

$$EX = 2\theta(1 - \theta) + 2\theta^2 + 3(1 - 2\theta) = 3 - 4\theta,$$

$$\text{所以 } \theta = \frac{3 - EX}{4},$$

$$\text{所以 } \hat{\theta} = \frac{3 - \bar{X}}{4} = 0.38$$

4. (1)

$$EX = \int_0^\theta \frac{x}{\theta^2} 2(\theta - x) dx = \frac{1}{\theta^2} \left[ \theta x^2 - \frac{2}{3} x^3 \right]_0^\theta = \frac{\theta}{3},$$

$$\text{所以 } \theta = 3EX, \text{ 所以 } \hat{\theta} = 3\bar{X}.$$

(2)

$$EX = \int_0^1 x(\theta + 1)x^\theta dx = \int_0^1 (\theta + 1)x^{\theta+1} = \frac{\theta + 1}{\theta + 2} x^{\theta+2} \Big|_0^1 = 1 - \frac{1}{\theta + 1},$$



所以  $\theta = \frac{1}{1-EX} - 2$ , 所以  $\hat{\theta} = \frac{1}{1-\bar{X}} = \frac{2\bar{X}-1}{1-\bar{X}}$

(3)

$$\int_0^1 x\sqrt{\theta}x^{\sqrt{\theta}-1}dx = \frac{\sqrt{\theta}}{\sqrt{\theta}+1}x^{\sqrt{\theta}+1}\big|_0^1 = \frac{\sqrt{\theta}}{\sqrt{\theta}+1},$$

所以  $\theta = \left(\frac{EX}{1-EX}\right)^2$ , 所以  $\hat{\theta} = \left(\frac{\bar{X}}{1-\bar{X}}\right)^2$ .

(4)

$$EX = \int_c^\infty x\theta c^\theta x^{-(\theta+1)}dx = \int_c^\infty \theta c^\theta x^{-\theta}dx = \frac{\theta c^\theta}{-\theta+1}x^{-\theta+1}\big|_c^\infty = \frac{\theta c}{\theta-1},$$

所以  $\theta = \frac{EX}{EX-c}$ , 所以  $\hat{\theta} = \frac{\bar{X}}{\bar{X}-c}$

(5)

$$\int_0^\theta \frac{x}{\theta^3} 6x(\theta-x)dx = \frac{1}{\theta^3} [2\theta x^3 - \frac{3}{2}x^4]_0^\theta = \frac{\theta}{2},$$

所以  $\theta = 2EX$ , 所以  $\hat{\theta} = 2\bar{X}$ .

(6)

$$EX = \int_0^\infty x \frac{\theta^2}{x^3} e^{-\frac{\theta}{x}} dx = \int_0^\infty \frac{\theta^2}{x^2} e^{-\frac{\theta}{x}} = \theta e^{-\frac{\theta}{x}}\big|_0^\infty = \theta,$$

所以  $\hat{\theta} = \bar{X}$

5. (1)  $X$  的期望为

$$\begin{aligned} EX &= \int_0^\infty x \cdot \frac{4x^2}{\theta^3\sqrt{\pi}} e^{-\frac{x^2}{\theta^2}} dx = \int_0^\infty \frac{4}{\sqrt{\pi}} y^{\frac{3}{2}} e^{-y} \cdot \frac{\theta}{2\sqrt{y}} dy \\ &= \frac{2\theta}{\sqrt{\pi}} \int_0^\infty y e^{-y} dy = \frac{2\theta}{\sqrt{\pi}} \Gamma(2) = \frac{2\theta}{\sqrt{\pi}} \\ \Rightarrow \theta &= \frac{\sqrt{\pi}}{2} EX \end{aligned}$$

所以,  $\theta$  的矩估计为  $\hat{\theta} = \frac{\sqrt{\pi}}{2} \bar{X}$ . (2) 先求  $X$  的方差:

$$\begin{aligned} EX^2 &= \int_0^\infty x^2 \cdot \frac{4x^2}{\theta^3\sqrt{\pi}} e^{-\frac{x^2}{\theta^2}} dx = \int_0^\infty \frac{4\theta}{\sqrt{\pi}} y^2 e^{-y} \cdot \frac{\theta}{2\sqrt{y}} dy \\ &= \frac{2\theta^2}{\sqrt{\pi}} \int_0^\infty y^{\frac{3}{2}} e^{-y} dy = \frac{2\theta^2}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) = \frac{2\theta^2}{\sqrt{\pi}} \cdot \frac{3\sqrt{\pi}}{4} = \frac{3\theta^2}{2} \\ \text{Var}(X) &= E(X^2) - (EX)^2 = \frac{3\pi-8}{2\pi} \theta^2 \end{aligned}$$

样本均值  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ , 所以  $\text{Var}(\bar{X}) = \frac{1}{n} \text{Var}(X)$ , 由此

$$\text{Var}(\hat{\theta}) = \text{Var}\left(\frac{\sqrt{\pi}}{2} \bar{X}\right) = \frac{\pi}{4} \text{Var}(\bar{X}) = \frac{3\pi-8}{8n} \theta^2.$$

7.(1)

$$EX_1 = \int_0^\infty \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} dx = -\frac{2}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2}} \Big|_0^\infty = \sqrt{\frac{2}{\pi}} E|X_1|,$$

所以  $\sigma = \sqrt{\frac{\pi}{2}} E|X_1|$ , 所以  $\hat{\sigma} = \sqrt{\frac{\pi}{2}} |X|$ 。

(2) 因为  $\sigma^2 = \text{Var}(X_1) = EX_1^2$ , 所以  $\hat{\sigma} = \sqrt{X^2}$ 。

17.(1)

$$E(a_n \bar{X}) = a_n E\bar{X} = a_n \frac{\theta}{2},$$

所以  $a_n = 2$  时,  $\hat{\theta}_1$  无偏。由 46 题可知

$$E(b_n \min\{X_i\}) = b_n E(\min\{X_i\}) = b_n \frac{\theta}{n+1},$$

$$E(c_n \max\{X_i\}) = c_n E(\max\{X_i\}) = c_n \frac{n}{n+1} \theta,$$

所以  $b_n = n+1$ ,  $c_n = \frac{n+1}{n}$ 。

(2)

$$\text{Var}(2\bar{X}) = \frac{4}{n} \text{Var}(\bar{X}) = \frac{\theta^2}{3n},$$

$$\text{Var}((n+1) \max\{X_i\}) = (n+1)^2 \text{Var}(\max\{X_i\}) = \frac{n}{n+2} \theta^2,$$

$$\text{Var}\left(\frac{n+1}{n} \min\{X_i\}\right) = \left(\frac{n+1}{n}\right)^2 \text{Var}(\min\{X_i\}) = \frac{1}{n^2+2n} \theta^2.$$

因为  $\text{Var}(\hat{\theta}_3) \leq \text{Var}(\hat{\theta}_1) \leq \text{Var}(\hat{\theta}_2)$ , 当且仅当  $n=1$  时等号成立, 所以  $\hat{\theta}_3$  更有效。

25. (1)  $p(x; \theta) = 1/\theta$ ,  $x = 0, 1, 2, \dots, \theta-1$ 。则似然函数为

$$L(\theta) = \frac{1}{\theta^n},$$

显然  $L(\theta)$  随着  $\theta$  的增大而减小, 所以  $\theta$  的 MLE 为  $\hat{\theta} = X_{(n)} + 1$ 。

(2)  $p(x; \theta) = \binom{m}{x} \theta^x (1-\theta)^{m-x}$ ,  $x = 0, 1, \dots, m$ 。则似然函数为

$$L(\theta) = \prod_{i=1}^n \binom{m}{x_i} \theta^{\sum_{i=1}^n x_i} (1-\theta)^{\sum_{i=1}^n (m-x_i)},$$

忽略常数项，对数似然函数为

$$\ln L(\theta) = \sum_{i=1}^n x_i \ln \theta + \sum_{i=1}^n (m - x_i) \ln(1 - \theta).$$

对其关于 $\theta$ 求导并令为0,

$$\frac{\sum_{i=1}^n x_i}{\theta} - \frac{\sum_{i=1}^n (m - x_i)}{1 - \theta} = 0.$$

所以， $\theta$ 的MLE为 $\hat{\theta} = \bar{X}/m$ 。

(3)  $p(x; \theta) = (x-1)\theta^2(1-\theta)^{x-2}, x = 2, 3, \dots; 0 < \theta < 1$ . 则似然函数

$$L(\theta) = \theta^{2n}(1-\theta)^{\sum_{i=1}^n x_i - 2n} \prod_{i=1}^n (x_i - 1)$$

对数似然函数为

$$\ln L(\theta) = 2n \ln \theta + \left( \sum_{i=1}^n x_i - 2n \right) \ln(1 - \theta) + C$$

将其关于 $\theta$ 求导并令为0,

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{2n}{\theta} - \frac{\sum_{i=1}^n x_i - 2n}{1 - \theta} = 0$$

所以， $\theta$ 的MLE为 $\hat{\theta} = \frac{2n}{\sum_{i=1}^n x_i} = \frac{2}{\bar{X}}$ .

(4)  $p(x; \theta) = -\frac{1}{\ln(1-\theta)} \frac{\theta^x}{2}, x = 1, 2, \dots$ . 则似然函数为

$$L(\theta) = [-\ln(1-\theta)]^{-n} \theta^{\sum_{i=1}^n x_i} \prod_{i=1}^n x_i^{-1}$$

对数似然函数为

$$\ln L(\theta) = -n \ln(-\ln(1-\theta)) + \ln \theta \sum_{i=1}^n x_i + C$$

将其关于 $\theta$ 求导并令为0,

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{\sum_{i=1}^n x_i}{\theta} + \frac{n}{(1-\theta) \ln(1-\theta)} = 0$$

则

$$\frac{(1-\theta)}{\theta} \ln(1-\theta) = -\frac{n}{\sum_{i=1}^n x_i}.$$

此时MLE无法得出显示解，有

$$\hat{\theta} = \arg \max_{0 < \theta < 1} L(\theta) = \arg \max_{0 < \theta < 1} \ln L(\theta).$$

(5)  $p(x; \theta) = \theta^x e^{-\theta} / x!$ ,  $x = 0, 1, 2, \dots$ , 则似然函数为

$$L(\theta) = \frac{\theta^{\sum_{i=1}^n x_i} e^{-n\theta}}{\prod_{i=1}^n x!},$$

忽略常数项，对数似然函数为

$$\ln L(\theta) = \sum_{i=1}^n x_i \ln \theta - n\theta.$$

对其关于 $\theta$ 求导并令其为0，则 $\sum_{i=1}^n x_i / \theta - n = 0$ 。所以 $\theta$ 的MLE为 $\hat{\theta} = \bar{X}$ 。

30. 因为

$$\int f(x; a, b) dx = \frac{\sqrt{2\pi}}{\sqrt{2b}} c \int \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+a/b)^2}{1/b^2}} dx = \frac{\sqrt{\pi}}{b} = 1$$

所以 $c = \frac{b}{\sqrt{\pi}}$ 。所以 $X \sim N(-a/b, 1/2b^2)$ 。又因为

$$L(\mu, \sigma^2) = \prod_{i=1}^n f(x_i) = \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2}},$$

令 $\frac{\partial \ln L(\mu, \sigma^2)}{\partial \mu} = \frac{\partial \ln L(\mu, \sigma^2)}{\partial \sigma^2} = 0$ ，有

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

所以 $\hat{b} = \sqrt{\frac{n}{2 \sum_{i=1}^n (x_i - \bar{x})^2}}$ 。令 $\frac{\partial \ln(a, b)}{\partial a} = -2 \sum_{i=1}^n a + bx_i = 0$ ，有 $\hat{a} = -\hat{b}\bar{x} = -\bar{x} \sqrt{\frac{n}{2 \sum_{i=1}^n (x_i - \bar{x})^2}}$ 。

42. 由题意可知 $X \sim Ge(p)$ ，所以 $EX = p^{-1}$ ， $\text{Var}(X) = p^{-2}(1 - p)$ 。所以 $p^{-1} = EX$ ， $p^{-2} = \text{Var}(X) + p^{-1} = \text{Var}(X) + EX$ 。由矩估计可得 $\hat{p}^{-1} = \bar{X}$ ， $\hat{p}^{-2} = S^2 + \bar{X}$ 。又 $EX = E\bar{X}$ ， $ES^2 = \text{Var}(X)$ ，所以 $\hat{p}^{-1}$ 和 $\hat{p}^{-2}$ 是无偏估计。

44.(1)

$$EX = \int_{\theta}^{\infty} \frac{x}{\sigma} e^{-\frac{x-\theta}{\sigma}} dx = \int_0^{\infty} \frac{t+\theta}{\sigma} e^{-\frac{t}{\sigma}} dt = \sigma + \theta,$$

所以  $\hat{\theta}_1 = \bar{x} - \sigma$ 。因为

$$L(\theta) = \prod_{i=1}^n f(x_i) = \frac{1}{\sigma^n} e^{-\frac{\sum_{i=1}^n (x_i - \theta)}{\sigma}}, \quad \theta < x_{(1)} \leq \dots \leq x_{(n)}.$$

显然, 当  $\theta$  增加时,  $L(\theta)$  也随之增加。所以  $\hat{\theta}_2 = X_{(1)}$ 。

(2)  $E\hat{\theta}_1 = E\bar{X} - \sigma = EX - \sigma = \theta$ , 所以  $\hat{\theta}_1$  是一个无偏估计。令  $Y = X_{(1)}$ , 所以

$$P(Y \leq y) = 1 - P(X_1 > y, \dots, X_n > y) = 1 - \left( \int_y^{\infty} f(x) dx \right)^n = 1 - e^{-\frac{n(y-\theta)}{\sigma}}, \quad y > \theta,$$

所以

$$EY = \int_{\theta}^{\infty} y dF(y) = \int_{\theta}^{\infty} \frac{ny}{\sigma} e^{-\frac{n(y-\theta)}{\sigma}} dy = \int_0^{\infty} \left( \theta + \frac{t}{n} \right) \frac{1}{\sigma} e^{-\frac{t}{\sigma}} dt = \theta + \frac{\sigma}{n},$$

所以  $\hat{\theta}_2$  不是无偏估计, 可修正为  $\tilde{\theta}_2 = X_{(1)} - \frac{\sigma}{n}$ 。

(3)

$$EX^2 = \int_{\theta}^{\infty} \frac{x^2}{\sigma} e^{-\frac{x-\theta}{\sigma}} dx = \int_0^{\infty} \frac{(t+\theta)^2}{\sigma} e^{-\frac{t}{\sigma}} dt = 2\sigma^2 + 2\theta\sigma + \theta^2,$$

所以

$$\text{Var}(\hat{\theta}_1) = \frac{1}{n} \text{Var}(X) = \frac{1}{n} (EX^2 - (EX)^2) = \frac{\sigma^2}{n}.$$

因为

$$EY^2 = \int_{\theta}^{\infty} \frac{ny^2}{\sigma} e^{-\frac{n(y-\theta)}{\sigma}} dy = \int_0^{\infty} \left( \theta + \frac{t}{n} \right)^2 \frac{1}{\sigma} e^{-\frac{t}{\sigma}} dt = \theta^2 + \frac{2\theta\sigma}{n} + \frac{2\sigma^2}{n^2},$$

所以

$$\text{Var}(\tilde{\theta}_2) = \text{Var}(Y) = EY^2 - (EY)^2 = \frac{\sigma^2}{n^2} < \frac{\sigma^2}{n},$$

所以  $\tilde{\theta}_2$  更优。

## Week11

May 28, 2023

49.  $X_1, X_2, \dots, X_n$  的联合密度函数为

$$f(X_1, X_2, \dots, X_n) = (1/\theta)^n I(\theta \leq X_{(1)} \leq X_{(n)} \leq 2\theta)$$

, 所以  $\theta \geq \min\{X_{(1)}, \frac{X_{(n)}}{2}\}$ .

当  $\hat{\theta}_M = \frac{X_{(n)}}{2}$  时上式取到极大值, 所以  $\hat{\theta}_M$  即为极大似然估计。因为  $X_{(n)}$  的密度函数为  $f_n(x) = \frac{n(x-\theta)^{n-1}}{\theta^n}$  所以

$$E(X_{(n)}) = \int_{\theta}^{2\theta} x \frac{n(x-\theta)^{n-1}}{\theta^n} dx = \frac{2n+1}{n+1} \theta$$

所以  $E(\hat{\theta}_M) = \frac{2n+1}{2n+2} \theta$ , 不是无偏估计。可修正为  $\tilde{\theta}_M = \frac{2n+2}{2n+1} \hat{\theta}_M = \frac{n+1}{2n+1} X_{(n)}$ 。

50. 由题意可知, 对数似然函数为

$$\begin{aligned} & \ln L(\mu_1, \mu_2, \sigma^2; X_1, \dots, X_m, Y_1, \dots, Y_n) \\ &= -\frac{1}{2\sigma^2} \sum_{i=1}^m (x_i - \mu_1)^2 - \frac{m}{2} \ln \sigma^2 - \frac{m}{2} \ln 2\pi - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu_2)^2 - \frac{n}{2} \ln \sigma^2 - \frac{n}{2} \ln 2\pi \\ &= -\frac{1}{2\sigma^2} \sum_{i=1}^m (x_i - \mu_1)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu_2)^2 - \frac{m+n}{2} \ln \sigma^2 + C. \\ & \text{令 } \frac{\partial L}{\partial \mu_1} = -\frac{1}{2\sigma^2} \left( \sum_{i=1}^m 2(\mu_1 - x_i) \right) = 0 \\ & \frac{\partial L}{\partial \mu_2} = -\frac{1}{2\sigma^2} \left( 2 \sum_{j=1}^n (\mu_2 - y_j) \right) = 0 \end{aligned}$$

所以

$$\hat{\mu}_{1M} = \bar{x}, \quad \hat{\mu}_{2M} = \bar{y}$$

将 $\sigma^2$ 看成参数, 令

$$\frac{\partial L}{\partial \sigma^2} = -\frac{m+n}{2\sigma^2} + \frac{1}{2\sigma^4} \left( \sum_{i=1}^m (x_i - \mu_1)^2 + \sum_{j=1}^n (y_j - \mu_2)^2 \right) = 0$$

则 $\hat{\sigma}_{MLE}^2 = \frac{1}{m+n} \left( \sum_{i=1}^m (x_i - \mu_1)^2 + \sum_{j=1}^n (y_j - \mu_2)^2 \right)$ . 又因为 $\mu_1, \mu_2$ 未知, 所以将 $\hat{\mu}_{1M}, \hat{\mu}_{2M}$ 代入上式, 得到 $\sigma^2$ 的极大似然估计

$$\hat{\sigma}_{MLE}^2 = \frac{1}{m+n} \left( \sum_{i=1}^m (x_i - \bar{x})^2 + \sum_{j=1}^n (y_j - \bar{y})^2 \right).$$

52. (1) 由题意知, 似然函数为

$$L(\mu) = \prod_{i=1}^n e^{-(x_i - \mu)} I(x_i \geq \mu) = e^{-\sum_{i=1}^n (x_i - \mu)} I(x_{(1)} \geq \mu),$$

要使 $L(\mu)$ 达到最大, 首先示性函数取值应为1, 其次 $e^{-\sum_{i=1}^n (x_i - \mu)}$ 尽可能大, 所以 $\mu$ 取值应尽可能大, 但示性函数为1确定了 $\mu \leq x_{(1)}$ , 由此 $\mu$ 的极大似然估计 $\hat{\mu}^* = X_{(1)}$ . 由最小值的分布结论可知,  $X_{(1)}$ 的密度函数为

$$f_1(x) = \begin{cases} n(1 - F(x))^{n-1} f(x) = ne^{-n(x-\mu)}, & x \geq \mu, \\ 0, & \text{其他.} \end{cases}$$

$$E(X_{(1)}) = \int_{\mu}^{\infty} x \cdot ne^{-n(x-\mu)} dx = \int_0^{\infty} (y + \mu) \cdot ne^{-ny} dy = \mu + \frac{1}{n}$$

所以 $\hat{\mu}^* = X_{(1)}$ 不是 $\mu$ 的无偏估计. 修正之后的无偏估计 $\hat{\mu}^{**} = X_{(1)} - \frac{1}{n}$ .  
(2)

$$E(X) = \int_{\mu}^{\infty} x \cdot e^{-(x-\mu)} dx = \int_0^{\infty} (y + \mu) \cdot e^{-y} dy = \mu + 1.$$

记 $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ , 所以 $\mu$ 的矩估计 $\hat{\mu} = \bar{X} - 1$ , 且

$$E(\hat{\mu}) = E(\bar{X}) - 1 = E(X) - 1 = \mu,$$

$\hat{\mu}$ 是 $\mu$ 的无偏估计.

(3)  $\hat{\mu}^{**}$  及  $\hat{\mu}$  都是  $\mu$  的无偏估计, 比较两者方差

$$\begin{aligned}\text{Var}(\hat{\mu}^{**}) &= \text{Var}\left(X_{(1)} - \frac{1}{n}\right) = \text{Var}(X_{(1)}) \\ &= \int_{\mu}^{\infty} x^2 \cdot n e^{-n(x-\mu)} dx - \left(\mu + \frac{1}{n}\right)^2 = \frac{1}{n^2} \\ \text{Var}(\hat{\mu}) &= \text{Var}(\bar{X} - 1) = \text{Var}(\bar{X}) = \frac{1}{n} \text{Var}(X) \\ &= \frac{1}{n} \left[ \int_{\mu}^{\infty} x^2 \cdot e^{-(x-\mu)} dx - (\mu + 1)^2 \right] = \frac{1}{n}\end{aligned}$$

所以  $\hat{\mu}^{**}$  更有效.

54.

法一: 由 Jensen 不等式有,

$$E\left(\frac{1}{\bar{X}}\right) \geq \frac{1}{E(\bar{X})} = \lambda, \text{ 等号当且仅当 } P(\bar{X} = E(\bar{X})) = 1 \text{ 时成立.}$$

所以  $1/\bar{X}$  不是  $\lambda$  的无偏估计. Jensen 不等式: 如果  $\varphi(\cdot)$  是凸函数,  $X$  是随机变量, 则:

$$E[\varphi(X)] \geq \varphi[E(X)]$$

若  $\varphi(\cdot)$  严格凸, 则等号当且仅当  $P(X = E(X)) = 1$  时成立.

法二: 由指数分布与伽玛分布的关系知:  $X_1, X_2, \dots, X_n \text{ i.i.d. } \sim \text{Exp}(\lambda) = \text{Gamma}(1, \lambda)$ , 再由伽玛分布的可加性, 知

$$Y := X_1 + X_2 + \dots + X_n \sim \text{Gamma}(n, \lambda), \text{ i.e.,}$$

$$f_Y(y) = \begin{cases} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

$$E\left(\frac{1}{\bar{X}}\right) = n E\left(\frac{1}{Y}\right) = n \int_0^{\infty} \frac{1}{y} \cdot \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} dy = \frac{n}{n-1} \lambda$$

所以  $1/\bar{X}$  不是  $\lambda$  的无偏估计. 伽玛分布:  $X \sim \text{Gamma}(\alpha, \lambda)$ , 其中  $\alpha > 0$  为形状参数,  $\lambda > 0$  为尺度参数. 密度函数为

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

利用卷积公式易证其可加性:  $U \sim \text{Gamma}(\alpha_1, \lambda), V \sim \text{Gamma}(\alpha_2, \lambda)$ , 且  $U, V$  相互独立, 则  $U + V \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda)$ .

56. 由题意知, 对于总体  $X$ ,

$$E(X) = \theta + 3\theta + 3(1 - 3\theta) = 3 - 5\theta,$$



$\theta$  的矩估计  $\hat{\theta}_M = \frac{3-\bar{X}}{5}$ . 记  $n_i$  为样本取到  $i$  的次数, 且  $n = \sum_{i=0}^3 n_i$ . 则似然函数, 对数似然为

$$\begin{aligned} L(\theta) &= \left(\frac{\theta}{2}\right)^{n_0} \theta^{n_1} \left(\frac{3\theta}{2}\right)^{n_2} (1-3\theta)^{n_3} \\ \ln L(\theta) &= n_0(\ln \theta - \ln 2) + n_1 \ln \theta + n_2(\ln \theta + \ln(3/2)) + n_3 \ln(1-3\theta) \\ \frac{\partial \ln L(\theta)}{\partial \theta} &= \frac{n_0 + n_1 + n_2}{\theta} - \frac{3n_3}{1-3\theta} = 0 \Rightarrow \theta_0 = \frac{n_0 + n_1 + n_2}{3n} \\ &\text{且可验证 } \frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \Big|_{\theta = \theta_0} < 0. \end{aligned}$$

$\theta$  的极大似然估计  $\hat{\theta}_L = \frac{n_0 + n_1 + n_2}{3n} = \frac{n - n_3}{3n}$ . 由观测值可计算具体的估计值:

$$\hat{\theta}_M = \frac{2}{5}, \hat{\theta}_L = \frac{4}{15}.$$

(2)  $E(\hat{\theta}_M) = \frac{3-E(\bar{X})}{5} = \theta, \hat{\theta}_M$  显然无偏. 对于  $\hat{\theta}_L = \frac{n-n_3}{3n}$ , 这里  $n = 10$ , 且  $n_3 \sim b(n, 1-3\theta), E(n_3) = n(1-3\theta)$ .

$$E(\hat{\theta}_L) = \frac{n - E(n_3)}{3n} = \frac{n - n(1-3\theta)}{3n} = \theta.$$

所以  $\hat{\theta}_L$  也是无偏估计.

(3) 对于  $\hat{\theta}_M$ ,

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (EX)^2 = 9 - 20\theta - (3-5\theta)^2 = 10\theta - 25\theta^2 \\ \text{Var}(\hat{\theta}_M) &= \frac{1}{25} \text{Var}(\bar{X}) = \frac{1}{25n} \text{Var}(X) = \frac{\theta(2-5\theta)}{5n} \end{aligned}$$

对于  $\hat{\theta}_L$ , 由  $n_3 \sim b(n, 1-3\theta)$

$$\begin{aligned} \text{Var}(n_3) &= 3n\theta(1-3\theta) \\ \text{Var}(\hat{\theta}_L) &= \frac{1}{(3n)^2} \text{Var}(n_3) = \frac{\theta(1-3\theta)}{3n} \end{aligned}$$

比较两者方差, 可以发现  $\hat{\theta}_L$  更有效.

## Week13

June 21, 2023

3. (1) 犯第一类错误的概率为

$$\alpha_1 = P(4|\bar{X}| \geq u_{0.05} | H_0) = 2(1 - \Phi(u_{0.05})) = 0.1$$

$$\alpha_2 = P(4|\bar{X}| \geq u_{0.45} | H_0) = 2\Phi(u_{0.45}) - 1 = 0.1$$

$$\alpha_3 = P(4\bar{X} \geq u_{0.10} | H_0) = 1 - \Phi(u_{0.10}) = 0.1$$

$$\alpha_4 = P(4\bar{X} \leq -u_{0.10} | H_0) = 1 - \Phi(u_{0.10}) = 0.1$$

(2) 犯第二类错误的概率为

$$\beta_1 = P(4|\bar{X}| < u_{0.05} | H_1) = P(-u_{0.05} - 4 < 4\bar{X} - 4 < u_{0.05} - 4) = 0.009258$$

$$\beta_2 = P(4|\bar{X}| > u_{0.45} | H_1) = P(\bar{X} - 4 > u_{0.45} - 4) + P(\bar{X} - 4 < -u_{0.45} - 4) = 0.9999$$

$$\beta_3 = P(4\bar{X} < u_{0.10} | H_1) = \Phi(u_{0.10} - 4) = 0.003279$$

$$\beta_4 = P(4\bar{X} > -u_{0.10} | H_1) = 1 - \Phi(-u_{0.10} - 4) = 0.99999$$

所以 $V_3$ 的二类错误最小。

4. 由题意知, 要求的值为

$$\alpha = P(X > 1/2 | H_0) = \int_{1/2}^1 6x^5 dx = \frac{63}{64}$$

$$\beta = P(X \leq 1/2 | H_1) = \int_0^{1/2} 4x^3 dx = \frac{1}{16}$$

$$g(\theta) = P(X > 1/2 | \theta = 2) = \int_{1/2}^1 3x^2 dx = \frac{7}{8}$$

7. (1) 检验 $H_0 : \mu = 7 \leftrightarrow H_1 : \mu \neq 7$ . 方差已知, 检验统计量为 $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim$

$N(0, 1)$ , 所以拒绝域为  $\left\{ \left| \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \right| \geq u_{\frac{\alpha}{2}} \right\}$ . 计算得

$$\left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| = 1.739 < u_{0.025} = 1.96$$

由此可见样本未落入拒绝域内, 所以接受原假设。

(2) 检验  $H_0 : \mu \geq 7 \leftrightarrow H_1 : \mu < 7$ , 检验统计量同上, 拒绝域为  $\left\{ \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq u_{1-\alpha} \right\}$ . 计算得

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = -1.739 < u_{0.95} = -1.645$$

由此可见样本落在拒绝域内, 所以拒绝原假设。

(3) 检验  $H_0 : \mu \leq 7 \leftrightarrow H_1 : \mu > 7$ , 拒绝域为  $\left\{ \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq u_{\alpha} \right\}$ . 计算得

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = 2.2136 > u_{0.05} = 1.645$$

样本落在拒绝域内, 所以拒绝原假设。

8.  $H_0 : \mu > 1550 \leftrightarrow H_1 : \mu \leq 1550$ , 拒绝域为  $\left\{ \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq -t_{\alpha}(n-1) \right\}$ , 查表得  $-t_{\alpha}(15) = -1.753$ , 因为  $\frac{\bar{X} - \mu_0}{S/\sqrt{n}} = 13.33 > -1.753$  未落入拒绝域内, 所以接受原假设。

9.  $H_0 : \mu = 105.2 \leftrightarrow H_1 : \mu \neq 105.2$ , 拒绝域为  $\left\{ \left| \frac{\bar{X} - \mu}{S/\sqrt{n}} \right| \geq t_{\frac{\alpha}{2}}(n-1) \right\}$ , 查表得  $t_{0.025}(7) = 2.365$ , 因为  $\left| \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \right| = 0.01750 < 2.365$  未落入拒绝域内, 所以接受原假设。

13 均值已知  $\mu = 2.5, n = 4, \sigma_0 = 1, \alpha = 0.05$ ,  $H_0 : \sigma^2 = 1 \leftrightarrow H_1 : \sigma^2 \neq 1$ , 检验统计量  $nS_{\mu}^2/\sigma_0^2 = 3.42 \sim \chi_4^2$ , 拒绝域为  $\{nS_{\mu}^2/\sigma_0^2 < \chi_4^2(1-\alpha/2)\}$  或  $\{nS_{\mu}^2/\sigma_0^2 > \chi_4^2(\alpha/2)\}$ ,  $\chi_4^2(1-\alpha/2) = 0.484, \chi_4^2(\alpha/2) = 8.496, 0.484 \leq 3.42 \leq 8.496$ , 接受原假设。

15 均值未知  $\bar{x} = 3.2, n = 25, S = 0.031, \sigma_0 = 0.02, \alpha = 0.05$ ,  $H_0 : \sigma = 0.02 \leftrightarrow H_1 : \sigma \neq 0.02$ , 检验统计量  $(n-1)S^2/\sigma_0^2 = 57.66 \sim \chi_{24}^2$ , 拒绝域为  $\{(n-$

1)  $S^2/\sigma_0^2 < \chi_{24}^2(1-\alpha/2)$  或  $\{(n-1)S^2/\sigma_0^2 > \chi_{24}^2(\alpha/2)\}$ ,  $\chi_{24}^2(1-\alpha/2) = 12.40$ ,  $\chi_{24}^2(\alpha/2) = 34.57$ ,  $57.66 > 34.57$ , 拒绝原假设。

22 设甲方法和乙方法的均值为  $\mu_1, \mu_2$ ,  $H_0: \mu_2 - \mu_1 = 0 \leftrightarrow H_1: \mu_2 - \mu_1 \neq 0$ , 方差未知但方差相同,  $\bar{x} = 31.75, \bar{y} = 21.92, m = n = 12, \alpha = 0.05, S_w^2 = [(m-1)/S_1^2 + (n-1)S_2^2]/(n+m-2) = 12.326$ , 检验统计量  $\sqrt{mn/(m+n)}(\bar{y} - \bar{x})/S_w = -2.572 \sim t_{m+n-2}$ ,  $t_{22}(\alpha/2) = 2.074$ , 因为  $-2.572 < -2.074$ , 拒绝原假设。

27(1) 均值未知的两样本方差比检验:  $H_0: \sigma_2^2/\sigma_1^2 = 1 \leftrightarrow H_1: \sigma_2^2/\sigma_1^2 \neq 1$ ,  $m = 13, n = 10, \alpha = 0.1, S_2^2/S_1^2 = 0.776 \sim F_{n-1, m-1}$ ,  $F_{9,12}(1-\alpha/2) = 0.0325$ ,  $F_{9,12}(\alpha/2) = 2.796$ , 接受原假设。

(2)  $H_0: \mu_2 - \mu_1 = 0 \leftrightarrow H_1: \mu_2 - \mu_1 \neq 0$ , 方差未知但方差相同,  $\bar{x} = 1.752, \bar{y} = 2.507, m = 13, n = 10, \alpha = 0.1, S_w^2 = [(m-1)/S_1^2 + (n-1)S_2^2]/(n+m-2) = 0.031$ , 检验统计量  $\sqrt{mn/(m+n)}(\bar{y} - \bar{x})/S_w = 10.242 \sim t_{m+n-2}$ ,  $t_{21}(\alpha/2) = 1.721$ , 因为  $10.242 > 1.721$ , 拒绝原假设。

28. 成对数据检验: 令  $Y_i = X_{1i} - X_{2i}$ , 其中  $X_{1i}$  为训练前体重,  $X_{2i}$  为训练后体. 检验  $H_0: \mu \leq 8 \leftrightarrow H_1: \mu > 8$ . 方差未知, 在  $\mu = \mu_0 = 8$  时, 检验统计量  $\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t(n-1)$ , 拒绝域为  $\left\{ \frac{\bar{Y} - \mu}{S/\sqrt{n}} \geq t_{\alpha}(n-1) \right\}$ . 经计算,

$$\frac{\bar{Y} - \mu}{S/\sqrt{n}} = 0.1457 < t_{0.05}(8) = 1.860.$$

由此可见, 样本未落入拒绝域, 接受原假设  $H_0$ , 即认为该乐部的宣传不可信. 注: 根据原假设的提法原则, 此题应站在保护消费者的角度考虑原假设的取法. 若取原假设与备择假设为  $H_0: \mu \geq 8 \leftrightarrow H_1: \mu < 8$ , 则得到该乐部的宣传可信的结论. 所以上面的检验原假设的提法更可取.

8.(1)由题意可知,  $X$ 服从对数似然分布, 所以

$$EX = \int_0^{\infty} \frac{x}{\sqrt{2\pi}x} e^{-\frac{(\ln x - \mu)^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^t e^{-\frac{(t-\mu)^2}{2}} dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-\mu-1)^2 - 2\mu - 1}{2}} dt = e^{\mu + \frac{1}{2}}.$$

所以  $a = e^{\mu + \frac{1}{2}}$ 。

(2)因为  $(\bar{Y} - \mu)/2 \sim N(0, 1)$ , 所以  $\mu$  的置信区间为  $[\bar{Y} - u_{\alpha/2}/2, \bar{Y} + u_{\alpha/2}/2]$ , 代入样本数据, 则:

$\mu$  的95%置信区间为  $[-0.9222, 1.0378]$ , 90%置信区间为  $[-0.7646, 0.8802]$ 。

(3)由(1)(2)可知,  $a$  的置信区间为  $[\exp(\bar{Y} - u_{\alpha/2}/2 + 1/2), \exp(\bar{Y} + u_{\alpha/2}/2 + 1/2)]$ , 所以:

$a$  的95%置信区间为  $[0.6556, 4.6542]$ , 90%置信区间为  $[0.7675, 3.9757]$ 。

9.(1)设包月用户的使用时间为  $X$ , 由题意可知  $n = 900, \bar{X} = 220, S_X = 90$ , 所以  $(\bar{X} - \mu_X)/(S_X/\sqrt{n}) \sim t(n-1)$ , 所以平均使用时间的置信区间为

$$\left[ \bar{X} - \frac{S_X}{\sqrt{n}} t_{\alpha/2}(n-1), \bar{X} + \frac{S_X}{\sqrt{n}} t_{\alpha/2}(n-1) \right],$$

所以95%置信区间为  $[214.1122, 225.8878]$ 。(2)同理可求按流量收费用户的平均使用时间为  $[154.448, 165.552]$ 。

13.由题意可知  $(X - \mu)/\sigma \sim N(0, 1)$ , 所以  $(X - \mu)^2/\sigma^2 \sim \chi^2(1)$ , 所以  $\sum_{i=1}^4 (X_i - 2.5)^2/\sigma^2 \sim \chi^2(4)$ 。所以置信区间为

$$\left[ \frac{\sum_{i=1}^4 (X_i - 2.5)^2}{\chi_{\alpha/2}^2(4)}, \frac{\sum_{i=1}^4 (X_i - 2.5)^2}{\chi_{1-\alpha/2}^2(4)} \right].$$

代入样本数据并查表可得,  $\sigma^2$  的95%置信区间为  $[0.3069, 7.0600]$ 。

16.(1) $\mu$  已知时, 有  $\frac{X_i - \mu}{\sigma} \sim N(0, 1)$ ,

所以  $\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$ ,

置信区间为

$$\left[ \frac{\sum_{i=1}^n (X_i - \mu)^2}{\chi_{0.025}^2(n)}, \frac{\sum_{i=1}^n (X_i - \mu)^2}{\chi_{0.975}^2(n)} \right],$$

置信区间为  $[0.14, 0.89]$

(2) $\mu$  未知时, 有  $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ ,

置信区间为

$$\left[ \frac{(n-1)S^2}{\chi_{0.025}^2(n-1)}, \frac{(n-1)S^2}{\chi_{0.975}^2(n-1)} \right],$$

置信区间为[0.15, 1.05]。

22.(1)由题意可知, 一件产品是否为次品服从参数为 $p$ 的0-1分布, 所以 $\hat{p} = \bar{X}$ 。由中心极限定理可知,  $(\bar{X} - p)/(p(1-p)/\sqrt{n}) \sim N(0, 1)$ 。因为 $n$ 足够大, 所以在这里我们忽略 $u_{\alpha/2}^2/n$ , 置信区间为

$$\left[ \hat{p} - u_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + u_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right],$$

所以95%的置信区间为[0.0279, 0.1221]。

(2)置信上限为 $\hat{p} + u_{\alpha} \sqrt{\hat{p}(1-\hat{p})/n}$ , 代入可得95%置信上限为0.1145。

29.总体分布为 $N(\mu, \sigma^2)$ , 对于样本均值 $\bar{X}$ ,  $\sqrt{n}(\bar{X} - \mu)/\sigma \sim N(0, 1)$  且 $\mu = 80$ .

$$P(\bar{X} > 75) = P\left(\frac{\bar{X} - \mu}{\sqrt{\sigma/n}} \geq \frac{75 - \mu}{\sqrt{\sigma/n}}\right) = 0.99,$$

得 $\sqrt{n}(75 - 80)/5 = -2.3263$ ,  $n = 5.4119$ , 所以至少6 块试验田。

33.因为 $\sqrt{n}(\bar{X} - \mu)/S \sim t(n-1)$ , 所以

$$P(\mu \geq \bar{X} - \frac{S}{\sqrt{n}} t_{\alpha}(n-1)) = 1 - \alpha,$$

置信下限为41147.53。

34.(1) 由题意可得

$$P\left(\mu \geq \bar{X} - \frac{S}{\sqrt{n}} t_{\alpha}(n-1)\right) = 1 - \alpha,$$

所以置信下限为1593.4262。

(2)同理

$$P\left(\sigma^2 \leq \frac{(n-1)S^2}{\chi_{1-\alpha}^2(n-1)}\right) = 1 - \alpha,$$

置信上限为464.8120。