数学分析讲义第二册参考答案

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Contents

Chapter 8

空间解析几何

8.1 向量与坐标系

1: Here only prove (1)

Proof

If $\mathbf{a} = \mathbf{0}$,or one of $\lambda, \mu, \lambda + \mu$ is zero, the equation is established.

[1] If $\lambda\mu > 0$,$(\lambda + \mu)\mathbf{a}$ and $\lambda\mathbf{a} + \mu\mathbf{b}$ have the same direction, and $|(\lambda + \mu)\mathbf{a}| = |\lambda + \mu||\mathbf{a}| = (|\lambda| + |\mu|)|\mathbf{a}| = |\lambda\mathbf{a}| + |\mu\mathbf{a}| = |\lambda\mathbf{a} + \mu\mathbf{a}|$ then $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

[2] If $\lambda\mu < 0$, For convenience, we can set $\lambda > 0, \mu < 0$, we only discuss the case $\lambda + \mu > 0$, the case $\lambda + \mu < 0$ is similar. since $(\lambda + \mu)\mathbf{a} + (-\mu)\mathbf{a} = [(\lambda + \mu) + (-\mu)]\mathbf{a} = \lambda\mathbf{a}$, then $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} - (-\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

2: 略

3: 解：(1)不成立。若$\vec{a}$,$\vec{b}$都不为$\vec{0}$，且有$\vec{a}\perp\vec{b}$,则有$\vec{a}\cdot\vec{b}$=0

(2)不成立。如$\vec{a}$,$\vec{b}$大小相等，但$\theta(\vec{a},\vec{b})$与$\theta(\vec{a},\vec{c})$互补

(3)不成立。$\vec{e_1}\cdot\vec{e_2}=|\vec{e_1}|\,|\vec{e_2}|\cos\theta(\vec{e_1},\vec{e_2})=\cos\theta(\vec{e_1},\vec{e_2})$,与两单位向量的夹角有关，大小$\in[-1,1]$

(4)不成立。$(\vec{a}\cdot\vec{b})\vec{c}$与$\vec{c}$共线，$\vec{a}(\vec{b}\cdot\vec{c})$与$\vec{a}$共线,当$\vec{a}$与$\vec{c}$ 不共线时，结论显然不成立。

(5)不成立。$\left|\vec{a}\cdot\vec{b}\right|^2=|\vec{a}|^2\left|\vec{b}\right|^2\cos^2\theta(\vec{a},\vec{b})$，当$\theta=0$或$\pi$ 时， 即$\vec{a}$,$\vec{b}$不共线时，$\left|\vec{a}\cdot\vec{b}\right|^2\neq|\vec{a}|^2\left|\vec{b}\right|^2$

(6)不成立。由向量叉乘的分配律，$(\vec{a}+\vec{b})\times(\vec{a}+\vec{b})=(\vec{a}+\vec{b})\times\vec{a}+(\vec{a}+\vec{b})\times\vec{b}=\vec{a}\times\vec{a}+\vec{b}\times\vec{a}+\vec{a}\times\vec{b}+\vec{b}\times\vec{b}=\vec{0}$

4: 这三个式子的大小同为以$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$为棱的平行六面体体积, 且有序向量组$\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\},\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}\},\{\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}\}$同为左手系或右手系, 从而有

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}=\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a}=\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b}.$$

5:

$$\begin{aligned}
\vec{OM}&=\vec{OA}+\vec{AM}\\
&=\vec{OA}+\frac{1}{2}\vec{AB}\\
&=\vec{OA}+\frac{1}{2}(\vec{OB}-\vec{OA})\\
&=\frac{1}{2}(\vec{OA}+\vec{OB})
\end{aligned}$$

6: 解:

$$\begin{aligned}\because 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) &= (\vec{a}+\vec{c})\cdot\vec{b}+(\vec{b}+\vec{c})\cdot\vec{a}+(\vec{a}+\vec{b})\cdot\vec{c}\\ &= -\vec{b}\cdot\vec{b}-\vec{a}\cdot\vec{a}-\vec{c}\cdot\vec{c}\\ &= -3\\ \therefore \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} &= -\frac{3}{2}\end{aligned}$$

7: 不妨设两者均为非零向量（零向量的情况结论平凡）

$(\overrightarrow{a}+3\overrightarrow{b})\cdot(7\overrightarrow{a}-5\overrightarrow{b})=7|\overrightarrow{a}|^2+16\overrightarrow{a}\cdot\overrightarrow{b}-15|\overrightarrow{b}|^2=0;$

$(\overrightarrow{a}-4\overrightarrow{b})\cdot(7\overrightarrow{a}-2\overrightarrow{b})=7|\overrightarrow{a}|^2-30\overrightarrow{a}\cdot\overrightarrow{b}+8|\overrightarrow{b}|^2=0;$

则$|\overrightarrow{a}|^2=2\overrightarrow{a}\cdot\overrightarrow{b}$，$|\overrightarrow{b}|^2=2\overrightarrow{a}\cdot\overrightarrow{b}$

于是$\overrightarrow{a},\overrightarrow{b}$的夹角$\theta$满足$cos\theta=\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|\cdot|\overrightarrow{b}|}=\frac{1}{2}$, 故$\theta$为$\frac{\pi}{3}$

8: (1)

$$|(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b})|=|\boldsymbol{b}\times\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}|=||\boldsymbol{b}|\,|\boldsymbol{a}|-|\boldsymbol{a}|\,|\boldsymbol{b}||=0$$

(2)

$$|(3\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}-2\boldsymbol{b})|=|-6\boldsymbol{a}\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{a}|=7\,|\boldsymbol{a}|\,|\boldsymbol{b}|=84$$

9: Using the operation law of "$\times$", we can get

(1)$|\mathbf{a}\times\mathbf{b}|^2=(|\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\theta)^2=1\cdot4\cdot\sin^2(\frac{2\pi}{3})=3$

(2)$|(\mathbf{a}+\mathbf{3b})\times(\mathbf{3a}-\mathbf{b})|^2=|10\cdot(\mathbf{a}\times\mathbf{b})|^2=300$

10: $\vec{a}\times\vec{b}=\vec{a}\times(-\vec{a}-\vec{c})=-\vec{a}\times\vec{a}-\vec{a}\times\vec{c}=-\vec{a}\times\vec{c}=\vec{c}\times a$

11: 证明：由题意知，$\vec{a}$fi$\vec{b}$fi$\vec{c}$均不为零向量，$\therefore(\vec{a}+\vec{b}+\vec{c})\times\vec{a}=\vec{a}\times\vec{a}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}=\vec{b}\times\vec{a}+\vec{a}\times\vec{b}=\vec{0}$,同理有$(\vec{a}+\vec{b}+\vec{c})\times\vec{b}=(\vec{a}+\vec{b}+\vec{c})\times\vec{c}=\vec{0}$,故只有$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

12: 将等式两边展开, 都是$(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2$.

13:

$$
\begin{aligned}
V &= |\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}| \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 1 \end{vmatrix} \\
&= 25
\end{aligned}
$$

14: 解:

$$
\begin{aligned}
\left|\vec{a}-\vec{b}\right| &= |(4,-6,12)| \\
&= \sqrt{4^2+(-6)^2+12^2} \\
&= 14
\end{aligned}
$$

方向余弦为:

$$(\cos\alpha, \cos\beta, \cos\gamma) = (\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$$

15: $|\overrightarrow{a}| = \sqrt{3^2+4^2+12^2} = 13$

故$\overrightarrow{a^0} = (\frac{3}{13}, \frac{4}{13}, -\frac{12}{13})$

16: 设x轴和y轴基向量为$\hat{i}$, $\hat{j}$，$\boldsymbol{a} = a_1\hat{i} + a_2\hat{j}$，$a_1, a_2$满足$a_1^2 + a_2^2 = 4$，将$\boldsymbol{a}$与两基向量点乘

$$\boldsymbol{a}\cdot\hat{i} = a_1 + a_2\hat{i}\cdot\hat{j} = |\boldsymbol{a}|\cos\alpha = 1$$
$$\boldsymbol{a}\cdot\hat{j} = a_1\hat{i}\cdot\hat{j} + a_2 = |\boldsymbol{a}|\cos\beta = -1$$

上两式相加：$(a_1 + a_2)(1 + \hat{i}\cdot\hat{j}) = 0$ 因为$\hat{i}\cdot\hat{j} \neq -1$，所以$a_0 + a_1 = 0$ 则$a_1 = \pm\sqrt{2}, a_2 = \pm\mp\sqrt{2}$ 所以$\boldsymbol{a} = (\sqrt{2}, -\sqrt{2})$或$\boldsymbol{a} = (-\sqrt{2}, \sqrt{2})$

17: Using the coordinate operation law of vector, we can get
$\overrightarrow{AB}$=$\overrightarrow{OB} - \overrightarrow{OA}$= $(1, 3, 4)$,$\overrightarrow{AC}$=$\overrightarrow{OC} - \overrightarrow{OA}$= $(2, 6, 8)$
since$\overrightarrow{AB}$ // $\overrightarrow{AC}$ then A, B, C are collinear

18: (1)-6, (2)-61

19: 解：(1)$\vec{a}\cdot\vec{b}=24+6+8=38$

(2)$\sqrt{\vec{b}\cdot\vec{b}}=\left|\vec{b}\right|=\sqrt{36+9+4}=7$

(3)$(2\vec{a}-3\vec{b})\cdot(\vec{a}+2\vec{b})=2\left|\vec{a}\right|^2+\vec{a}\cdot\vec{b}-6\left|\vec{b}\right|^2=2\cdot(16+4+16)+38-6\cdot 7^2=-184$

(4)$\vec{a}-\vec{b}=(-2,1,2),\left|\vec{a}-\vec{b}\right|^2=4+1+4=9$

20: $\cos\theta=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\dfrac{5}{21}.$

21:

$$\boldsymbol{a}\cdot\boldsymbol{e_b}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}=\frac{18}{3}=6$$

22: 解:

(1)

$$\begin{aligned}\vec{a}\times\vec{b}&=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-1&-2\\1&2&-1\end{vmatrix}\\&=5\vec{i}+\vec{j}+7\vec{k}\end{aligned}$$

(2)

$$\begin{aligned}2\vec{a}-\vec{b}&=(5,-4,-3)\\2\vec{a}+\vec{b}&=(7,0,-5)\\\therefore(2\vec{a}-\vec{b})\times(2\vec{a}-\vec{b})&=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\5&-4&-3\\7&0&-5\end{vmatrix}\\&=20\vec{i}+4\vec{j}+28\vec{k}\end{aligned}$$

23: $\overrightarrow{AB}=(2,-2,-3)$， $\overrightarrow{AC}=(4,0,6)$
则$S=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=14$

24: 先求出

$$\boldsymbol{AB}=(3,6,3),\boldsymbol{AC}=(1,3,-2),\boldsymbol{AD}=(2,2,2)$$

四点构成的四面体体积为

$$V=|\boldsymbol{AB}\cdot(\boldsymbol{AC}\times\boldsymbol{AD})=18|$$

25: Using the coordinate operation law of the mixed product of vectors, we can get

(1) $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 3 \\ 1 & 9 & -11 \end{vmatrix}=0$

since $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=0$ then $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar

(2) $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & 2 \\ 3 & -1 & -2 \end{vmatrix}\neq 0$

since $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\neq 0$ then $\mathbf{a},\mathbf{b},\mathbf{c}$ are non coplanar

/

26: 共面

27: (1)关于xOy平面的对称点：(a,b,-c)
关于xOz平面的对称点：(a,-b,c)

关于yOz平面的对称点：(-a,b,c)
(2)关于x轴对称点：(a,-b,-c)
关于y轴对称点：(-a,b,-c)
关于z轴对称点：(-a,-b,c)

28: 到原点距离$5\sqrt{2}$, 到x轴, y轴和z轴距离分别为$\sqrt{34}, \sqrt{41}, 5.$

29: 设所求点为$(0, y, z)$,由条件可列方程

$$9 + (y-1)^2 + (z-2)^2 = 16 + (y+2)^2 + (z+2)^2 = (y-5)^2 + (z-1)^2$$

解得$y = 1, z = -2$,故所求点为$(0, 1, -2)$

8.2 平面与直线

1: 略

2: $\vec{M_1 M_2} = (1, 2, -1)$,平面的法向量$\vec{n} = \vec{M_1 M_2} \times \vec{v} = (7, -7, -7)$.不妨取$\vec{n_0} = (1, -1, -1)$,则$x-2+(-1)*(y+1)-(z-3) = 0$,化简即为$x-y-z = 0$

3: 设平面方程为a(x-5)+b(y+7)+c(z-4)=0
(1)若截距不为零，则由

$$\frac{a}{5a-7b+4c}x + \frac{b}{5a-7b+4c}y + \frac{c}{5a-7b+4c}z = 1$$

可取$a = b = c = 1$则方程为$x + y + z - 2 = 0$

(2)若截距为0则存在无数多解，可用平面束方程。

4: 平行, 相交, 重合, 相交.

5: 两平面法向量分别为$\boldsymbol{n_1}=(2,0,-1)$和$\boldsymbol{n_2}=(0,1,0)$,所求平面的法向量为

$$\begin{aligned}\boldsymbol{n}&=\boldsymbol{n_1}\times\boldsymbol{n_2}\\&=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\2&0&-1\\0&1&0\end{vmatrix}\\&=(1,0,-2)\end{aligned}$$

又平面过$M(3,-1,1)$,平面方程为$x+2z-5=0$

6: 解: $\because$该平面平行于坐标面Oyz,故其一个法向量为$\vec{n}=(1,0,0)$
又$\because$该平面过点M,$\therefore$其平面方程为$x+5=0$.

7: (1).两平面对应的法向量为$\overrightarrow{a}=(2,-1,1)$与$\overrightarrow{b}=(1,1,2)$
则$cos\theta=\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}=\frac{1}{2}$
夹角为$\frac{\pi}{3}$
（2）两平面对应的法向量为$\overrightarrow{a}=(4,2,4)$与$\overrightarrow{b}=(3,-4,0)$
则$cos\theta=\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|}=\frac{2}{15}$
夹角为$arccos(\frac{2}{15})$

8: (1)

$$d=\frac{|16\times 2-12\times(-1)+15\times(-1)-4|}{\sqrt{16^2+12^2+15^2}}=1$$

(2)

$$d=\frac{|12\times 2-5\times(-2)+5|}{\sqrt{12^2+5^2}}=3$$

9: (1)

$$\frac{|14+7|}{\sqrt{9+36+4}}=3. \tag{8.1}$$

(2)

$$\frac{|18+21|}{\sqrt{16+4+16}}=\frac{13}{2} \tag{8.2}$$

10: (1)同侧, (2)异侧

11: 两平面平行，易知所求平面方程为$x+y-2z+1=0$

(1)$x^2+y^2-\frac{z^2}{4}=1$，为单叶双曲面

(2)$\sqrt{(y^2+z^2)}=sinx(0\le x\le\pi)$，名称未知

(3)$4x^2+9y^2+4z^2=36$，为椭球面

12: 两平面的法向量$\vec{v_1}=(2,-1,1)$和$\vec{v_2}=(1,1,2)$模长相等, 于是两个平分面的法向量为$\vec{v_3}=\vec{v_1}+\vec{v_2}=(3,0,3)$与$\vec{v_4}=\vec{v_1}-\vec{v_2}=(1,-2,-1)$. 现在选取两平面的交点$(6,5,0)$, 那么平分面的方程为

$$(x-6,y-5,z)\cdot\vec{v_3}=0,\quad (x-6,y-5,z)\cdot\vec{v_4}=0.$$

也就是$x+z-6=0$与$x-2y-z+4=0$.

13: 由于该点到三个坐标平面的距离相等,因此设该点为$P(x,x,x)$,现在计算该点到平面$x+y+z-1=0$的距离(用书上公式),

$$d=\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|3x-1|}{\sqrt{3}}$$

14: 解:

(1)易知该平面过A,B两点所构成线段的中点$C(\frac{3}{2},\frac{1}{2},\frac{7}{2})$,且该平面的一个法向量$\vec{n}=\vec{AB}=(1,-3,1)$

$\therefore$该平面的平面方程为$x-\frac{3}{2}-3(y-\frac{1}{2})+z-\frac{7}{2}=0$,即

$$x-3y+z-\frac{7}{2}=0$$

(2)$\because$该平面与平面$6x+3y+2z+12=0$平行

$\therefore$可设平面方程为$6x+3y+2z+D=0$

由题意得,$\frac{|6-2+D|}{\sqrt{6^2+3^2+2^2}}=\frac{|6-2+12|}{\sqrt{6^2+3^2+2^2}}$

$\therefore D=-20$

$\therefore$平面方程为

$$6x+3y+2z-20=0$$

(3)通过x轴的平面方程可设为$By+z=0$

$\therefore \frac{|4B+13|}{\sqrt{B^2+1}}=8,\therefore B=-\frac{3}{4}$ 或 $\frac{35}{12}$,

$\therefore$平面方程为

$$-3y+4z=0 \quad 或 \quad 35y+12z=0$$

(4)由题知,可设平面方程为$\frac{x}{3}+\frac{y}{m}+\frac{z}{1}=1$,即$mx+3y+3mz-3m=0$

$\therefore$该平面的一个法向量为$\vec{n_1}=(m,3,3m)$,而Oxy平面的法向量为$\vec{n_2}=$

$(0,0,1)$

$\therefore \cos\frac{\pi}{3} = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$,即$\frac{3m}{\sqrt{m^2+(3m)^2+9}} = \frac{1}{2}$

$\therefore m = \pm\frac{3\sqrt{26}}{26}$

$\therefore$平面方程为

$$x+\sqrt{26}y+3z-3=0 \quad 或 \quad x-\sqrt{26}y+3z-3=0$$

15: (1).直线方向向量为$(1,0,2)\times(0,1,-3) = (-2,3,1)$ 从而直线方程为$\frac{x}{-2} = \frac{y-2}{3} = z-4$

(2).直线方向向量为$(1,1,-2)\times(1,2.,-1) = (3,-1,1)$ 从而直线方程为$\frac{x+1}{3} = -y+2 = z-1$

(3).$\frac{x-2}{2} = \frac{y+3}{-3} = \frac{z-4}{0}$

(4).先计算两条直线的方向向量，分别为$\overrightarrow{a} = (-3,1,10)$与$\overrightarrow{b} = (4,-1,2)$

则l具有方向向量$\overrightarrow{l} = \overrightarrow{a}\times\overrightarrow{b} = (12,46,-1)$

从而方程为$\frac{x+1}{12} = \frac{y+4}{46} = \frac{z-3}{-1}$

16: 两平面的方向量为$\boldsymbol{n}_1 = (2,3,-1), \boldsymbol{n}_2 = (3,-5,2)$,则直线的方向向量为$\boldsymbol{v} = \boldsymbol{n}_1\times\boldsymbol{n}_2 = (1,-7,-19)$.又有$(1,0,-2)$为直线上的一点，可求出此直线的参数方程为

$$\frac{x-1}{1} = \frac{y}{-7} = \frac{z+2}{-19}$$

17: (1)令

$$\frac{x-1}{1}+\frac{y+1}{-2}+\frac{z}{6}=t \tag{8.3}$$

代入得

$$2(t+1)+3(-2t-1)+6t-1=0, \quad \Rightarrow t=1. \tag{8.4}$$

因此$x=2, y=-3, z=6$。

(2)令

$$\frac{x+2}{-2}+\frac{y-1}{3}+\frac{z-3}{2}=t \tag{8.5}$$

代入得

$$-2t-2+2(3t+1)-2(2t+3)+6=0. \tag{8.6}$$

得知t任意都成立，所以直线在平面上。

18: (1)$\frac{\pi}{2}$, (2)$\frac{2\pi}{3}$

19: (1):

$l_1=n_1\times n_2$=(-6,4,-2)

直线l_2=(3,-2,1),两向量平行，所以两直线平行

在两直线上分别取一点A=(-2,1,0),B=(1,-2,-1)

距离d=$\frac{|\vec{AB}\times l_2|}{|l_2|}$=$\sqrt{5}$

(2):

同理可得两直线平行

在两直线上分别取一点A=(-7,5,9),B=(0,-4,-18)

距离d=$\frac{|\vec{AB}\times l_2|}{|l_2|}$=25

20: (1)两直线点向式方程分别为$-x+1=2y=z+1$和$x=y+1=2z+3$. 方向向量$\vec{v_1}=(-1,\frac{1}{2},1)$与$\vec{v_2}=(1,1,\frac{1}{2})$垂直, 交点$(1,0,-1)$.

(2)两直线的方向向量$\vec{v_1}=(1,-4,0)$与$\vec{v_2}=(0,0,1)$垂直, 交点$(-2,-1,5)$.

21: (1)直线的点向式方程为

$$\frac{x}{2}=\frac{y+12}{3}=\frac{z-4}{6}$$

所以该直线的方向向量为$\boldsymbol{l}=(2,3,6)$,平面法向量为$\boldsymbol{n}=(6,15,-10)$,用书上公式,平面与直线的夹角$\phi$满足

$$sin\phi=\frac{|\boldsymbol{n}\cdot\boldsymbol{l}|}{|\boldsymbol{n}|\,|\boldsymbol{l}|}=\frac{3}{133}$$

(2)直线的点向式方程为

$$\frac{x}{-2}=\frac{y}{-4}=\frac{z}{2}$$

所以该直线的方向向量为$\boldsymbol{l}=(-1,-2,1)$,平面法向量为$\boldsymbol{n}=(1,-1,-1)$,用书上公式,平面与直线的夹角ϕ满足

$$sin\phi=\frac{|\boldsymbol{n}\cdot\boldsymbol{l}|}{|\boldsymbol{n}|\,|\boldsymbol{l}|}=0$$

故直线与平面平行

22: 解:

(1)直线过点$P_1(0,1,0)$,方向向量为$\vec{u}=(1,-2,1)$

$\therefore\ P_1\vec{P}_0=(1,-1,-1)$

$\therefore$距离$d=\frac{\left|P_1\vec{P}_0\times\vec{u}\right|}{|\vec{u}|}=\sqrt{3}$

(2)直线过点$M_1(1,1,1)$,方向向量为$\vec{u}=(1,1,1)\times(2,0,1)=(1,-3,-2)$

$\therefore M_1\vec{M}_0 = (0,1,2)$

$\therefore$距离$d = \frac{\left|M_1\vec{M}_0 \times \vec{u}\right|}{|\vec{u}|} = \frac{\sqrt{6}}{2}$

23: 记P，Q分别为两直线上的一点，$\vec{m}, \vec{n}$分别两直线的方向向量，则两直线的距离$d = \frac{|\vec{m} \times \vec{n} \cdot \vec{PQ}|}{|\vec{m} \times \vec{n}|}$

(1).(9,-2,0)，(0,-7,2)分别为两直线上的一点，$(4,-3,1), (-2,9,2)$分别两直线的方向向量，则两直线的距离$d = \frac{|(4,-3,1)\times(-2,9,2)\cdot(-9,-5,2)|}{|(4,-3,1)\times(-2,9,2)|} = 7$

(2).(1,0,0)，(0,0,-2)分别为两直线上的一点，$(1,1,-1) \times (2,1,-1) = (0,-1,-1), (1,2,-1) \times (1,2,2) = (6,-3,0)$分别两直线的方向向量，则两直线的距离$d = \frac{|(0,-1,-1)\times(6,-3,0)\cdot(-1,0,-2)|}{|(0,1,-1)\times(6,-3,0)|} = 1$

24: 直线方向向量

$$\boldsymbol{v} = (3,2,-1) \times (2,-3,2) = (1,-8,-13)$$

因为平面通过直线，所以$\boldsymbol{n} \perp \boldsymbol{n}$.又因为与法向量为$n_1 = (1,2,3)$的平面垂直，则$\boldsymbol{n} \perp \boldsymbol{n}_1$所以

$$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n}_1 = (2,-16,10)$$

取直线上一点$(0,0,-1)$，则平面也过此点。

设平面方程为$2x - 16y + 10z + d = 0$.代入点$(0,0,-1)$，得$d = 10$.整理后得待求直线方程为

$$x - 8y + 5z + 5 = 0$$

25: 直线$x=3t+1, y=2t+3, z=-t-2$，方向向量为$\vec{n_1}=(3,2,-1)$，其平行于平面。直线

$$\begin{cases} 2x-y+z-3 &= 0 \\ x+2y-z-5 &= 0 \end{cases} \tag{8.7}$$

方向向量为$\vec{n_2}=(-1,3,5)$，其也平行于平面。所以平面的法向量$\vec{n_1}\times\vec{n_2}=(13,-14,-11)$。令$t=0$，则平面经过点(1,3,-2)。所以平面方程为$13x-14y-11z+7=0$。

26: $2+6x+8y-2z=0$,$-x+2y+5z-3=0$

27: 投影：过点(-1,2,0)且与平面垂直的直线与平面的交点
直线的方向向量l=n=(1,2,-1)
将直线的参数方程带入平面方程，可得交点坐标为(-5/3,2/3,2/3)

28: 直线的方向向量$\vec{v}=(1,2,3)$, 设点$A(2,3,1)$, 在直线上的投影$P(t-7,2t-2,3t-2)$, 则由$\overrightarrow{PA}\cdot\vec{v}=0$得$P(-5,2,4)$.

29: 该平面的法向量为$(6,2,-9)$,设对称点为(x,y,z),则有

$$(x,y,z)-(0,0,0)=t(6,2,-9)$$

由于$(0,0,0)$到该平面的距离为

$$d=\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}=\frac{121}{11}=11$$

所求对称点$(6t, 2t, -9t)$到该平面的距离也应为11，故

$$d = \frac{|36t + 4t + 81t + 121|}{11} = 11$$

解得$t = -2$(舍0),故对称点为$(-12, -4, 18)$

30: 解：过点$(1, 2, 3)$且与直线垂直的平面方程为$(x-1) - 3(y-2) - 2(z-3) = 0$,即$x - 3y - 2z + 11 = 0$

联立直线、平面方程得$x = \frac{1}{2}, y = \frac{5}{2}, z = 2$,

$\therefore$根据中点坐标公式，知对称点坐标为$(0, 3, 1)$

31: 两直线上的点可分别设为$(1 + \lambda u, -4 + 5u, 3 - 3u)$和$(-3 + 3v, 9 - 4v, -14 + 7v)$

令$(1 + \lambda u, -4 + 5u, 3 - 3u) = (-3 + 3v, 9 - 4v, -14 + 7v)$

则可解得$\begin{cases} \lambda = 2 \\ u = 1 \\ v = 2 \end{cases}$

从而$\lambda = 2$，交点为(3,1,0)

两直线方向向量分别为(2,5,-3)和(3,-4,7)

则平面法向量为$(2, 5, -3) \times (3, -4, 7) = (23, -23, -23)$

从而平面方程为x-y-z-2=0

32: 设题中给出的两条直线为l_1, l_2，取l_1上一点$A(0, 5, -3)$，取l_2上一点$B(0, -7, 10)$

l_1方向向量：$\boldsymbol{v}_1 = (3, -1, 0) \times (2, 0, -1) = (1, 3, 2)$

l_2方向向量：$\boldsymbol{v}_2 = (4, -1, 0) \times (5, 0, -1) = (1, 4, 5)$

因此l_1上的点可表示为

$$(0,5,-3)+t_1\boldsymbol{v}_1=(t_1,5+3t_1,-3+2t_1)$$

l_2上的点可表示为

$$(0,-7,10)+t_2\boldsymbol{v}_2=(t_2,-7+4t_2,10+5t_2)$$

待求直线l与直线l_1,l_2相交，则t_1,t_2应满足$(0,5,-3)+t_1\boldsymbol{v}_1=(t_1,5+3t_1,-3+2t_1)$,$(0,-7,10)+t_2\boldsymbol{v}_2=(t_2,-7+4t_2,10+5t_2)$,$(-3,5,-9)$三点共线，则

$$\frac{t_1+3}{t_2+3}=\frac{5+3t_1-5}{-7+4t_2-5}=\frac{-3+2t_1-(-9)}{10+5t_2-(-9)}$$

即

$$\frac{t_1+3}{t_2+3}=\frac{3t_1}{4t_2-12}=\frac{2t_1+6}{5t_2+19}$$

得到两组解

$$t_1=-3,t_2=-3$$

或

$$t_1=-\frac{66}{19},t_2=-\frac{13}{3}$$

对于第一组解，直线方程为

$$\begin{cases}x=-3\\y=-9\end{cases}$$

对于第二组解，直线方程为

$$\frac{x+3}{1}=\frac{y-5}{22}=\frac{z+9}{2}$$

33: 先求直线与平面的交点坐标Q，联立方程

$$\begin{cases} x+y-z-1 &= 0 \\ x-y+z+1 &= 0 \\ x+y+z &= 0 \end{cases} \tag{8.8}$$

求得$(0,\frac{1}{2},-\frac{1}{2})$。任取直线上任意一点$A(0,0,-1)$。设在平面上的射影$B(u,v,w)$。由于$AB$垂直于平面，而平面法向量为(1,1,1)，所以$AB$直线方程为

$$\frac{x}{1}=\frac{y}{1}=\frac{z+1}{1} \tag{8.9}$$

那么(u,v,w)满足下面方程

$$u+v+w=0 \tag{8.10}$$

$$\frac{u}{1}=\frac{v}{1}=\frac{w+1}{1} \tag{8.11}$$

所以解得$u=\frac{1}{4},v=\frac{1}{4},w=-\frac{1}{2}$。所以求得的直线方程为

$$\frac{x}{\frac{1}{4}}=\frac{y-\frac{1}{2}}{-\frac{1}{4}}=\frac{z+\frac{1}{2}}{0} \tag{8.12}$$

34: 平面5x-y+3z-2=0与Oxy平面的交线l:5x-y-2=0,z=0.平面的法向量$\vec{n_1}=(5,-1,3)$,直线的方向向量$\vec{n_2}=(\frac{1}{5},1,0)$,所求平面法向量为$\vec{n}=\vec{n_1}\times\vec{n_2}=(-15,3,26)$.因为平面过点(0,2,0),所以方程为$-15x+3y+26z+6=0$

35: 按照题28的思路可得点到直线的垂足的坐标为($\frac{33}{29}$,-$\frac{26}{29}$,$\frac{27}{29}$)
所以垂线方程为$\frac{x}{33}$=$\frac{y}{-26}$=$\frac{z}{27}$

8.3 二次曲面

1: (1)椭圆$\frac{x^2}{4}+\frac{r^2}{9}=1$绕x轴旋转，椭球面

(2)圆绕x,y,z轴其中一轴旋转，是球面

(4)双曲线$r^2-\frac{y^2}{4}=1$绕y轴旋转，单叶双曲面

(6)双曲线$x^2-r^2=1$绕x轴旋转，双叶双曲面

(7)抛物线$r^2=4z$绕z轴旋转，抛物面

2: (1)都是直线(2)都是直线(3)Oxy坐标系为圆，Oxyz坐标系为圆柱(4)Oxy坐标系为双曲线，Oxyz坐标系为双曲柱面(5)Oxy坐标系为抛物线，Oxyz坐标系为抛物柱面(6)Oxy坐标系为两个点，Oxyz坐标系为两条直线(7)Oxy坐标系为一个点，Oxyz坐标系为一条直线(8)Oxy坐标系为一个点，Oxyz坐标系为一条直线

3: (1)$x^2+y^2-\frac{z^2}{4}=1$，为单叶双曲面

(2)$\sqrt{(y^2+z^2)}=sinx(0\le x\le\pi)$，名称未知

(3)$4x^2+9y^2+4z^2=36$，为椭球面

4: Oyz平面上的直线$y=z$的方向向量为$\vec{v}=(0,1,1)$, 与这条直线垂直的平面为$y+z-t=0, t\in\mathbb{R}$. 平面与Oyz平面上的直线$y-2z+1=0$的交点为$A(0,\frac{2t-1}{3},\frac{t+1}{3})$, 平面上的点$P(x,y,z)$是旋转面上的点意味着$|\overrightarrow{OA}\times\vec{v}|=|\overrightarrow{OP}\times\vec{v}|$, 即$x^2+4y^2+4z^2-10yz+2y+2z-2=0$.

5: (1)平面π的法向量为$\boldsymbol{n}=(1,-1,2)$,直线L的方向向量为$\boldsymbol{l}=(1,1,-1)$,因此过L且与π垂直的平面的法向量$\boldsymbol{n'}=\boldsymbol{n}\times\boldsymbol{l}=(1,-3,-2)$,
投影直线L_0的方向向量$\boldsymbol{l_0}=\boldsymbol{n'}\times\boldsymbol{n}=(-4,-2,1)$,联立$L$与$\pi$的方程,可解得交点为$M(2,1,0)$,$M$在$L_0$上,故$L_0$方程为

$$\frac{x-2}{-4}=\frac{y-1}{-2}=\frac{z}{1}$$

(2)L_0绕y轴旋转,所得曲面上点$P(x,y,z)$对应子午线上$(2y,y,\frac{1-y}{2})$,由距离关系

$$x^2+y^2+z^2=(2y)^2+y^2+\left(\frac{1-y}{2}\right)^2$$

整理得曲面方程

$$4x^2-17y^2+4z^2+2y-1=0$$

6: 略

7: (1).将z=0代入得$x^2-y^2=9$
从而截痕为双曲线
(2).将x=0代入得$y^2+z^2=-9$
从而无截痕
(3).将y=0代入得$x^2-z^2=9$
从而截痕为双曲线
(4).将x=5代入得$y^2+z^2=16$
从而截痕为圆

8: 设动点坐标为(x,y,z),由题意有

$$\sqrt{x^2+y^2+z^2}=|z-4|$$

两边平方，整理得

$$x^2+y^2+8z-16=0$$

可看出是抛物面

9: 联立两曲面方程，消去z即得母线平行于z轴的柱面方程$5x^2-3y^2=1$

10: 设球心$(0,0,z_0)$,半径R。构造方程：$3^2+(z_0-1)^2=z_0^2+16$得$z_0=-3, R=5$。球的方程为$x^2+y^2+(z+3)^2=25$

11: 利用两个方程消去z有$\frac{x^2}{16}+\frac{y^2}{4}-\frac{(x+3)^2}{20}=1$,即$\frac{(x-12)^2}{260}+\frac{y^2}{13}=1$

12: 在平面中$xy=h$在$h>0$时表示位于一, 三象限的双曲线; 在$h<0$时表示位于二, 四象限的双曲线, 在$h=0$时表示两条坐标轴. 在空间坐标系中$xy=z$是马鞍面, 用平面$z=h$去截这个曲面, 在$h>0$和$h<0$时得到不同方向的双曲线，在$h=0$时得到x,y坐标轴.

8.4 坐标变换和其他常用坐标系

1: $(x-1)^2-(y-1)^2-(z-1/2)^2=3/4$，双叶双曲面

2: 平移、旋转变换可得$z=\frac{y^2}{2}-\frac{x^2}{2}$，双曲抛物面。

3: 解：此题需要对二次曲面进行化简，参考文献见链接。为消去yz项，令$\cot 2\theta=\frac{-3-3}{8}=\frac{-3}{4}$,取$\tan\theta=2$,且$\theta$为锐角,做变换

$$\begin{cases} x=x' \\ y=\frac{1}{\sqrt{5}}y'-\frac{2}{\sqrt{5}}z' \\ z=\frac{2}{\sqrt{5}}y'+\frac{1}{\sqrt{5}}z' \end{cases}$$

代入原方程并化简得：$x'^2+y'^2-z'^2=1$，表示一个单叶双曲面。

4: (1)$r^2\cos 2\theta=25, r^2\sin^2\theta\cos 2\phi=25$;

(2)$r^2+4z^2=10, r^2\cos^2\theta=3$;

(3)$\sin^2\theta=2\cos^2\theta$;

(4)$r^2\sin^2\theta\cos 2\phi-r^2\cos^2\theta=1$;

(5)$r^2\cos^2\theta=3$;

(6)$r^2+z^2=2z$;

(7)$r(\cos\theta+\sin\theta)=4$;

(8)$r(\sin\theta\cos\phi+\sin\theta\sin\phi+\cos\theta)=1$;

(9)$x^2+y^2=2y$;

(10)$x^2-y^2=z$;

(11$x^2+y^2=1$.

5: 旋转后的方程为$z=2(x^2+y^2)$,柱面坐标系中$x^2+y^2=r^2$,故方程为$z=2r^2$

6: 设曲面为S, 双曲线为Γ, $\forall(r,\theta,z)\in S$, $\exists(r,0,z)\in\Gamma \Leftrightarrow 2r^2-z^2=2$. 故曲面方程为$S: 2r^2-z^2=2$

7: $\begin{cases} x = x_0 + a\cos t \\ y = y_0 + b\sin t \end{cases} \quad t\in[0,2\pi)$

8:

$$\begin{cases} x = x_0 + a\sin u\cos v \\ y = y_0 + \sin u\sin v \\ z = z_0 + a\cos u \end{cases}$$

9: 类比球面的参数表示

$x = a\sin\theta\cos\phi; y = b\sin\theta\sin\phi; z = c\cos\theta$

$0\le\theta\le\pi; 0\le\phi\le 2\pi$

10: $\begin{cases} x = a\cosh t \\ y = b\sinh t \end{cases}$

11: 解:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\begin{cases} x = a\cosh\theta\cos\varphi \\ y = b\cosh\theta\sin\varphi, \theta\in(-\infty,+\infty), \varphi\in[0,2\pi) \\ z = c\sinh\theta \end{cases}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

$$\begin{cases} x = a\sinh\theta\cos\varphi \\ y = b\sinh\theta\sin\varphi, \theta\in[0,+\infty), \varphi\in[0,2\pi) \\ z = c\pm\cosh\theta \end{cases}$$

用三角函数也可$\sec^2\theta-\tan^2\theta=1$,双曲函数则是$\cosh^2\theta-\sinh^2\theta=1$

8.5 综合习题

1: It's easy to check "M on a plane ABC" $\Leftrightarrow\overrightarrow{AM}=\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AC}\Leftrightarrow$ The conclusion

2: 略

3: 任取三维空间中的四个向量

A=(a1,a2,a3) B=(b1,b2,b3)

C=(c1,c2,c3) D=(d1,d2,d3)

可得矩阵

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \tag{8.13}$$

矩阵的秩≤3¡4=向量的个数

由线性代数知识可得，这四个向量线性相关

所以三维空间中的任意四个向量必定线性相关

4: (1)系数行列式$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 3$. 非退化.

(2) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & x & 2 \end{vmatrix} = 0 \implies x = 1.$

5: (1)

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\, sin\theta = 1 \times 2 \times \left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

所以,

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = 3$$

(2)

$$|(\boldsymbol{a}+\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{b})| = |(\boldsymbol{a}+\boldsymbol{b})|\,|(\boldsymbol{a}-\boldsymbol{b})|\, sin\phi$$

分别计算

$$|(\boldsymbol{a}+\boldsymbol{b})| = \sqrt{(\boldsymbol{a}+\boldsymbol{b})^2} = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{ab} = 7$$

$$|(\boldsymbol{a}-\boldsymbol{b})| = \sqrt{(\boldsymbol{a}-\boldsymbol{b})^2} = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab} = 3$$

$$cos\phi = \frac{(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})}{|\boldsymbol{a}+\boldsymbol{b}|\,|\boldsymbol{a}-\boldsymbol{b}|} = \frac{\boldsymbol{a}^2 - \boldsymbol{b}^2}{|\boldsymbol{a}+\boldsymbol{b}|\,|\boldsymbol{a}-\boldsymbol{b}|} = -\frac{\sqrt{21}}{7}$$

故

$$|(\boldsymbol{a}+\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{b})| = |(\boldsymbol{a}+\boldsymbol{b})|\,|(\boldsymbol{a}-\boldsymbol{b})|\, sin\phi = \sqrt{7} \times \sqrt{3} \times \frac{2\sqrt{7}}{7} = 2\sqrt{3}$$

6: 若$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$共面，则等式显然成立.若三者不共面，则构成空间中的一组基.利用混合积的轮换性，有:

$$\begin{aligned}&\boldsymbol{a}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})\\=&(\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b}))\cdot\boldsymbol{c}+\boldsymbol{0}+(\boldsymbol{a}\times(\boldsymbol{c}\times\boldsymbol{a}))\cdot\boldsymbol{b}\\=&\boldsymbol{0}\end{aligned}$$

同理有:

$$\boldsymbol{b}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})=\boldsymbol{0}$$

$$\boldsymbol{c}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})=\boldsymbol{0}$$

由$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$构成空间中的一组基可知，原式为$\boldsymbol{0}$.

7: 在准线上任取一点A(x_0,y_0,z_0)

在柱面上任取一点$B(x,y,z)$且B为A沿母线移动后得到

则有$\begin{cases}x=x_0+2t\\y=y_0+t\\z=z_0+t\end{cases}$

由于$\begin{cases}y_0^2+z_0^2=1\\x_0=1\end{cases}$

则$\begin{cases}(y-t)^2+(z-t)^2=1\\x-2t=1\end{cases}$

消去t得$(y-\frac{x-1}{2})^2+(z-\frac{x-1}{2})^2=1$

亦即$x^2+2y^2+2z^2-2xy-2xz-2x+2y+2z-1=0$

8: 设顶点为$P_0(2,1,1)$，任取锥面上一点$P(x,y,z)$，直线PP_0与准线相交于$P_1(x_1,y_1,z_1)$，满足

$$\boldsymbol{P_0P_1} = t\boldsymbol{P_0P}$$

即

$$\begin{cases} x_1 = 2+(x-2)t \\ y_1 = 1+(y-1)t \\ z_1 = 1+(z-1)t \end{cases}$$

将准线方程表达式代入此方程组，得到

$$\begin{cases} x\left[1+(y-1)t\right]^2 + \left[1+(z-1)t\right]^2 = 1 \\ 2+(x-2)t = 1 \end{cases}$$

消去t，得

$$2y^2+z^2-2xy-2yz+2x+4y-2z-2=0$$

此为待求锥面的一般方程。

9: Select a point (x_1,y_1,z_1) on the line $\frac{x-1}{1}=\frac{y}{1}=\frac{z}{1}$ It's easy to select a point $(1,1,0)$ on $\frac{x-1}{0}=\frac{y-1}{0}=\frac{z}{1}$, then latitude circle can be represented by a sphere whose centre is $(1,1,0)$ intersect a plane.

The equation of sphere is $(x-1)^2+(y-1)^2+z^2=(x_1-1)^2+(y_1-1)^2+z_1^2$ and the plane is $z-z_1=0$, then the equation of latitude circle is

$$\begin{cases} (x-1)^2+(y-1)^2+z^2=(x_1-1)^2+(y_1-1)^2+z_1^2 \\ z-z_1=0 \end{cases}$$

since (x_1,y_1,z_1) on the line $\frac{x-1}{1}=\frac{y}{1}=\frac{z}{1}$, then $\begin{cases} x_1-1=y_1 \\ y_1=z_1 \end{cases}$

The above four equations eliminate x_1, y_1, z_1, we get the equation of revolution surface

$$(x-1)^2+(y-1)^2-2(z-\frac{1}{2})^2=\frac{1}{2}$$

The parametric equation is $\begin{cases} x=\frac{\sqrt{2}}{2}\sec\theta\cos\varphi+1 \\ y=\frac{\sqrt{2}}{2}\sec\theta\sin\varphi+1 \\ z=\frac{1}{2}\tan\theta+\frac{1}{2} \end{cases}$

10: 参数方程$\begin{cases} x=(cos\theta+2)\cos\varphi \\ y=sin\theta \\ z=(cos\theta+2)\sin\varphi \end{cases}$ 一般方程$(\sqrt{x^2+z^2}-2)^2+y^2=1$

11: 根据题意可得如下表格(坐标系绕e1旋转)

	e1	e2	e3
e11	0	$\pi/2$	$\pi/2$
e22	$\pi/2$	α	$\pi/2$-α
e33	$\pi/2$	$\pi/2$+α	α

坐标(e1,e2,e3)与坐标(e11,e22,e33)的变换公式如下

$$\begin{pmatrix} e11 \\ e22 \\ e33 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{pmatrix}\begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.14}$$

坐标系绕e22旋转时可得类似表格

	e1	e2	e3
e11	β	$\pi/2$	$\pi/2+\beta$
e22	$\pi/2$	0	$\pi/2$
e33	$\pi/2$-β	$\pi/2$	β

坐标(e11,e22,e33)与坐标(e111,e222,e333)的变换公式如下

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{pmatrix} \begin{pmatrix} e11 \\ e22 \\ e33 \end{pmatrix} \tag{8.15}$$

综上坐标(e1,e2,e3)与坐标(e111,e222,e333)的变换公式如下

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & sin\alpha sin\beta & -cos\alpha sin\beta \\ 0 & cos\alpha & sin\alpha \\ sin\beta & -sin\alpha cos\beta & cos\alpha cos\beta \end{pmatrix} \begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.16}$$

注：若第二步是绕e2旋转，则坐标变换公式为

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ sin\alpha sin\beta & cos\alpha & sin\alpha cos\beta \\ cos\alpha sin\beta & -sin\alpha & cos\alpha cos\beta \end{pmatrix} \begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.17}$$

坐标变换矩阵的逆为旋转变换，可通过对应坐标轴的坐标改变求出坐标系的变换矩阵

12: 对称中心为$P(2,2,2)$, 三条直线的单位方向向量分别为$\vec{v_1}=(-\frac{1}{3},\frac{2}{3},\frac{2}{3})$,$\vec{v_2}=(\frac{2}{3},-\frac{1}{3},\frac{2}{3})$,$\vec{v_3}=(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$. 那么椭球面上的点$A(x,y,z)$满足

$$\frac{(\overrightarrow{PA}\cdot\vec{v_1})^2}{a^2}+\frac{(\overrightarrow{PA}\cdot\vec{v_2})^2}{b^2}+\frac{(\overrightarrow{PA}\cdot\vec{v_3})^2}{c^2}=1.$$

即

$$\frac{(-x+2y+2z-6)^2}{9a^2}+\frac{(-y+2z+2x-6)^2}{9b^2}+\frac{(-z+2x+2y-6)^2}{9c^2}=1.$$

13:

$$\begin{aligned}
&\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\\
=&\sqrt{(\rho_2 sin\theta_2 cos\phi_2-\rho_1 sin\theta_1 cos\phi_1)^2+(\rho_2 sin\theta_2 sin\phi_2-\rho_1 sin\theta_1 sin\phi_1)^2+(\rho_2 cos\theta_2-\rho_1 cos\theta_1)^2}\\
=&\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\left(sin\theta_1 sin\theta_2 cos\phi_1 cos\phi_2+sin\theta_1 sin\theta_2 sin\phi_1 sin\phi_2+cos\theta_1 cos\theta_2\right)}\\
=&\sqrt{\left(\rho_1-\rho_2\right)^2+2\rho_1\rho_2\left[1-cos\left(\phi_1-\phi_2\right)sin\theta_1 sin\theta_2-cos\theta_1 cos\theta_2\right]}
\end{aligned}$$

注：此题题中θ与ϕ记号应是弄反

14: 两点之间的球面距离即为连接两点的大圆的劣弧长，由大圆的半径等于球面半径a以及弧长公式知，只需验证γ为两点的位置向量的夹角即可.

$$\begin{aligned}
\cos\gamma=&\frac{(a\sin\theta_1\cos\phi_1,a\sin\theta_1\sin\phi_1,a\cos\theta_1)\cdot(a\sin\theta_2\cos\phi_2,a\sin\theta_2\sin\phi_2,a\cos\theta_2)}{a\cdot a}\\
=&(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2)\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2\\
=&\cos(\phi_1-\phi_2)\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2
\end{aligned}$$

因此书中题目可能有误.

Chapter 9

多变量函数的微分学

9.1 多变量函数及其连续性

1:

$$\forall x \in (A \cap B)^c$$
$$\Leftrightarrow x \notin A \cap B$$
$$\Leftrightarrow x \notin A \text{或} x \notin B$$
$$\Leftrightarrow x \in A^c \text{或} x \in B^c$$
$$x \in (A \cup B)^c$$
$$\Leftrightarrow x \notin A \cup B$$
$$\Leftrightarrow x \notin A \text{且} x \notin B$$
$$\Leftrightarrow x \in A^c \text{且} x \in B^c$$
$$\Leftrightarrow x \in A^c \cap B^c$$

2: 略

3: 记满足y¿ax+b的所有点(x,y)组成的集合为E，任取E中一点M
显然,存在正数r, 使得B(M,r)⊆E
所以M是E的内点，由M的任意性可得，E为开集
绘图略，边界点满足的关系是y=ax+b

4: 由三角不等式

$$\rho(M_0,M_0')-\rho(M_n,M_0)-\rho(M_n',M_0') \le \rho(M_n,M_n') \le \rho(M_0,M_0')+\rho(M_n,M_0)+\rho(M_n',M_0').$$

从而根据夹逼准则得到$\lim \rho(M_n,M_n') = \rho(M_0,M_0')$.

5: 假设点列$\{M_n\}$是平面上的点列,收敛到点M_0,由点列收敛的定义,$\forall \epsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, \rho(M_n,M_0) < \epsilon$,即$M_n \subset B(0,\rho(M_0,0)+\epsilon), n > N$,取$R_1 = \rho(M_0,0)+\epsilon, R_2 = max\{\rho(M_i,0), i \le N\}$,令$R = max\{R_1,R_2\}$,则$M_n \subset B(0,R), \forall n$,故$\{M_n\}$为有界数列.

6: 若$\exists \gamma : [a,b] \to E$为连续函数s.t. $\gamma(a) = (0,0), \gamma(b) = (\frac{2}{\pi},1)$.取$c = \sup\{a \le t \le b : x(\gamma(t)) = 0\}$, 则$a \le c < b$. 取$t \to c^+$, 由$\gamma$的连续性知$\lim_{t\to c+} \sin\frac{1}{x(t)} = y(c)$.由$x(t) \to x(c) = 0$知前者极限是不存在的，得到矛盾.因而$E$不是道路连通的.
记$E_1 = \{a \in E : x(a) = 0\}, E_2 = \{a \in E : x(a) > 0\}$,$E_1, E_2$均为道路连通从而为连通集.若存在$U_1, U_2 \subset E, U_1 \ne \emptyset, U_2 \ne \emptyset, E = U_1 \cup U_2, U_1 \cap \overline{U_2} =$

$\overline{U_1} \cap U_2 = \emptyset$, 由$E_1, E_2$各自的连通性知，只可能为$U_1 = E_1, U_2 = E_2$或$U_1 = E_2, U_2 = E_1$.但$(0,0) \in E_1 \cap \overline{E_2}$,矛盾.因而$E$是连通集.

7: 以下画图均略

(1)$\{(x,y)\big|x+y \geqslant 0\}$是区域，是闭区域

(2)$\{(x,y)\big|x-2y^2 \geqslant 0\}$是区域，是闭区域

(3)$\{(x,y)\big|x^2+y^2+2x \geqslant 0$且$2x-x^2-y^2 > 0\} = (x,y)\big|2x-x^2-y^2 > 0$
是区域，不是闭区域

(4)$\{(x,y)\big|2n\pi \leqslant x^2+y^2 \leqslant (2n+1)\pi, n \in Z\}$ 不是区域

(5)$\{(x,y)\big|\frac{x^2}{a^2}+\frac{y^2}{b^2} < 1\}$是区域，不是闭区域

(6)$\{(x,y)\big| -\frac{\pi}{2}+2k_1\pi \leqslant x \leqslant \frac{\pi}{2}+2k_1\pi,$且$2k_2\pi \leqslant y \leqslant (2k_2+1)\pi, k_1 \in Z, k_2 \in Z\} \cup$

$\{(x,y)\big|\frac{\pi}{2}+2k_3\pi \leqslant x \leqslant \frac{3\pi}{2}+2k_3\pi,$且$(2k_4-1)\pi \leqslant y \leqslant 2k_4\pi, k_3 \in Z, k_4 \in Z\}$

不是区域

(7)$(x,y,z)\big|x^2+y^2 \leqslant z^2$且z$\neq 0$不是区域

(8)$(x,y,z)\big|x^2+y^2+(z-a)^2 \leqslant a^2$是区域，是闭区域

8: 图略.

等高线: $z=0: 2x+y=(k+\frac{1}{2})\pi, k \in \boldsymbol{Z}$

$z=1: 2x+y=2k\pi, k \in \boldsymbol{Z}$

$z=-1: 2x+y=(2k+1)\pi, k \in \boldsymbol{Z}$

$z=\frac{1}{2}: 2x+y=(2k\pi \pm \frac{\pi}{3}), k \in \boldsymbol{Z}$

$z = -\frac{1}{2} : 2x + y = (2k\pi \pm \frac{2\pi}{3}), k \in \boldsymbol{Z}$

9: (1) Consider the bivariate n-degree polynomials $x^r y^t$, where $r + t \leq n$

if $r = 0$ then $t \leq n$ there are $n+1$ polynomials.

if $r = 1$ then $t \leq n-1$ there are n polynomials.

...

if $r = n$ then $t \leq 0$ there is 1 polynomial.

then there are totally $\sum\limits_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

(2) For the polynomial of degree n of k variable, let $p_k^{(n)}$ be the number of polynomials

then similar to the above method we have $p_k^{(n)} = \sum\limits_{i=0}^{n} p_{k-1}^{(i)}$ and $p_2^{(n)} = \frac{(n+1)(n+2)}{2} = C_{n+2}^2$

Lemma

$$\sum_{i=0}^{n} C_{m+i}^{m} = C_{m+n+1}^{m+1}$$

Proof

$\sum\limits_{i=0}^{n} C_{m+i}^{m} = C_m^m + C_{m+1}^m + ... + C_{m+n}^m = C_{m+1}^{m+1} + C_{m+1}^m + ... + C_{m+n}^m$

by the formula $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$

we have $(C_{m+1}^{m+1} + C_{m+1}^m) + C_{m+2}^m ... + C_{m+n}^m = C_{m+2}^{m+1} + C_{m+2}^m ... + C_{m+n}^m = ... = C_{m+n}^{m+1} + C_{m+n}^m = C_{m+n+1}^{m+1}$

$\square$

then $p_3^{(n)} = \sum\limits_{i=0}^{n} p_2^{(i)} = \sum\limits_{i=0}^{n} C_{i+2}^2 = C_{n+3}^3$

By the lemma, we can get $p_k^{(n)} = C_{n+k}^k$

10: $f(1,1)=1$，$f(y,x)=\frac{2xy}{x^2+y^2}$，$f(1,\frac{y}{x})=\frac{2xy}{x^2+y^2}$，$f(u,v)=\frac{2uv}{u^2+v^2}$，$f(\cos t,\sin t)=\sin 2t$

11: 根据题意可得

$$F(t)=\begin{cases}1 & 2k\pi+\pi/4\le t\le 2k\pi+5\pi/4, k\in Z,\\ 0 & else.\end{cases}$$

12: $f(2,3)=-2, f(x,y)=\frac{x^2(1-y)}{y+1}$.

13:

$$(x+y)^{x-y},\quad x^y+(x-y),\quad x+y-x^y$$

14: 1、

$$\frac{x^2+y^2}{|x|+|y|}\le\frac{(|x|+|y|)^2}{|x|+|y|}=|x|+|y|\le 2r\to 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{|x|+|y|}=0$$

2、

$$\lim_{\substack{x\to 0\\ y\to a}}\frac{\sin(xy)}{x}=\lim_{\substack{x\to 0\\ y\to a}}\frac{\sin(xy)}{xy}\cdot\lim_{\substack{x\to 0\\ y\to a}}y=1\cdot a=a$$

3、

$$(\frac{xy}{x^2+y^2})^{x^2}\le(\frac{1}{2})^{x^2}\to 0$$

$$\Rightarrow \lim_{\substack{x\to +\infty\\ y\to +\infty}}(\frac{xy}{x^2+y^2})^{x^2}=0$$

4、

$$\ln((1+\frac{1}{x})^{\frac{x^2}{x+y}}) = \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\frac{x}{x+y} \to 1$$

$$\Rightarrow \lim_{\substack{x\to\infty\\ y\to a}}(1+\frac{1}{x})^{\frac{x^2}{x+y}} = e$$

5、

$$|\frac{x^3+y^3}{x^2+y^2}| = |\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2}| \le 2r \to 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^3}{x^2+y^2} = 0$$

6、

$$|\frac{x^2+y^2}{x^4+y^4}| = |\frac{r^2}{r^4(\cos^4\theta+\sin^4\theta)}| \le \frac{1}{2r^2} \to 0$$

$$\Rightarrow \lim_{\substack{x\to\infty\\ y\to\infty}}\frac{x^2+y^2}{x^4+y^4} = 0$$

7、

$$(x^2+y^2)e^{-(x+y)} = r^2e^{-r(\cos\theta+\sin\theta)} \le r^2e^{-r} \to 0$$

$$\Rightarrow \lim_{\substack{x\to+\infty\\ y\to+\infty}}(x^2+y^2)e^{-(x+y)} = 0$$

8、

$$\lim_{\substack{x\to 1\\ y\to 0}}\frac{\ln(x+e^y)}{\sqrt{x^2+y^2}} = \frac{\ln(1+e^0)}{\sqrt{1+0}} = \ln 2$$

9、

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{\sqrt{xy+1}-1} = \lim_{t\to 0}\frac{t}{\sqrt{t+1}-1} = 2$$

10、

$$\frac{\sqrt{xy+1}-1}{x+y} = \frac{\sqrt{xy+1}-1}{xy}\frac{xy}{x+y}$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+1}-1}{xy} = \frac{1}{2}$$

取$(x,y)=(t,-t+kt^2)$, 则当$t\to 0$时，$(x,y)\to(0,0)$,此时

$$\frac{xy}{x+y}=t-\frac{1}{k}\to-\frac{1}{k}$$

极限值与参数选取有关，因此$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+1}-1}{x+y}$极限不存在

11、

$$\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}=\frac{1-\cos(r^2)}{r^4}\frac{r^2}{r^4\sin^2\theta\cos^2\theta}\geq\frac{1-\cos(r^2)}{r^4}\frac{1}{r^2}\to\infty$$

因此$\lim_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$极限不存在

12、

$$(1+xy)^{\frac{1}{x+y}}=(1+xy)^{\frac{1}{xy}\frac{xy}{x+y}}$$

由$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x+y}$极限不存在知$\lim_{\substack{x\to 0\\ y\to 0}}(1+xy)^{\frac{1}{x+y}}=0$极限不存在

15: (1)

$$\lim_{\rho\to 0^+}e^{\frac{1}{x^2-y^2}}=\lim_{\rho\to 0^+}e^{\frac{1}{\rho^2 cos2\varphi}}$$

若极限存在,则$cos2\varphi<0$

又因为$0\leqslant\varphi\leqslant 2\pi$

则$\varphi\in(\frac{\pi}{4},\frac{3\pi}{4})\cup(\frac{5\pi}{4},\frac{7\pi}{4})$

(2)

$$\lim_{\rho\to+\infty}e^{x^2-y^2}sin(2xy)=\lim_{\rho\to+\infty}e^{\rho^2 cos2\varphi}sin(r^2 sin2\varphi)$$

若极限存在,则$sin2\varphi=0$或$cos2\varphi<0$

从而$\varphi\in(\frac{\pi}{4},\frac{3\pi}{4})\cup(\frac{5\pi}{4},\frac{7\pi}{4})\cup\{0,\pi,2\pi\}$

16: (1)记 $\lim_{x\to x_0} f(x,y) = \phi(y)(y_1 \neq y_0)$. 由题设知，对任意$\epsilon > 0$, 存在$\delta > 0$, 只要 $|x - x_0| < \delta, |y_1 - y_0| < \delta, |y_2 - y_0| < \delta$, 便有

$$|f(x, y_1) - f(x, y_2)| < \epsilon$$

令 $x \to x_0$，则有$|\phi(y_1) - \phi(y_2)| < \epsilon$, 故 $\lim_{y\to y_0} \phi(y)$ 存在.

再证明 $\lim_{y\to y_0} \phi(y) = A$. 对上述的$\epsilon, \delta$, 当 $0 < |x - x_0|, |y_1 - y_0| < \delta$ 时，有

$$|f(x, y_1) - A| < \epsilon, \quad |f(x, y_1) - \phi(y_1)| < \epsilon$$

于是

$$|\phi(y_1) - A| = |\phi(y_1) - f(x, y_1) + f(x, y_1) - A| < 2\epsilon,$$

所以 $\lim_{y\to y_0} \phi(y) = A$, 命题得证. 同理可证(2).

17: (1)容易知道在$y \neq x$时整个函数是连续的。我们主要讨论$y = x$时的连续性。则当x和y趋于相等时。第一种情况x和y趋于相等且不等于0，那么分子不为零，分母趋于零，整个趋于无穷，所以在$x = y$的且不等于零的地方肯定不连续。下面再讨论在$(0,0)$处的连续性。分别取$y = 2x$和$y = x - x^2$所得极限不一样，所以在原点处也没有极限。因此整个函数在$y = x$处不连续，在其他地方连续。

(2).Consider the point on the line $y = 0$, we set it $(x_0, 0)$

[1].If $x_0 = 0$, that is to say $(0,0)$, then $f(0,0) = 0$.

since $|x \sin \frac{1}{y}| \leq |x|$, by the definition of limit we know $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} x\sin(\frac{1}{y}) = 0 = f(0,0)$ and along the y axis $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

[2].If $x_0 \neq 0$, we know $f(x_0, 0) = 0$

along the line $x = x_0$, $f(x, y) = x_0 \sin(\frac{1}{y})$ since the limit $\lim\limits_{(x,y)\to(x_0,0)} f(x, y) = \lim\limits_{(x,y)\to(x_0,0)} x_0 \sin(\frac{1}{y})$ is not exist, thus $\lim\limits_{(x,y)\to(x_0,0)} f(x, y)$ is not exist.
In short, $f(x, y)$ is continuous on the set $\{(x, y) \mid y \neq 0\} \cup \{(0, 0)\}$

(3). 令$x = \rho\cos\theta, y = \rho\sin\theta$，那么

$$|\frac{x^2y}{x^2 + y^2}| = |\rho\sin\theta\cos^2\theta| \leq |\rho| \to 0.$$

所以函数在整个平面是连续的。

(4). 函数在$x + y \neq 0$的地方肯定是连续的。下面我们讨论在$x + y = 0$地方的连续性。首先在$(0, 0)$处，令$y = kx$发现极限的结果与k的取值有关，所以在$(0, 0)$处不连续。在其他$x + y = 0$的地方，分母会趋于0，但是分子是有限非零数，这个时候整个函数会趋于无穷。所以在$x + y = 0$上是不连续的。

18: 略

19: 不妨设f(x,y)关于y是单调递增的

$\forall$取D中一点(x$_0$,y$_0$),因为f(x,y)关于y连续

$\therefore$ $\forall$ ϵ¿0,$\exists$ δ_1 当—y-y$_0$—$\leq$ δ_1时，—f(x$_0$,y)-f(x$_0$,y$_0$)—¡ϵ/2

对于点(x$_0$,y$_0$-δ_1),(x$_0$,y$_0$+δ_1)

$\because$f(x,y)关于x连续

$\therefore$对上述ϵ,$\exists\delta_2$¿0,当—x-x$_0$—¡δ_2时

—f(x,y$_0$-δ_1)-f(x$_0$,y$_0$-δ_1)—¡ϵ/2

—f(x,y$_0$+δ_1)-f(x$_0$,y$_0$+δ_1)—¡ϵ/2

令δ=min{δ_1,δ_2},则当—x-x_0—¡δ,—y-y_0—¡δ时

—f(x,y)-f(x_0,y_0)—≤max{—f(x,y_0-δ_1)-f(x_0,y_0)—,—f(x,y_0+δ_1)-f(x_0,y_0)—}

—f(x,y_0-δ_1)-f(x_0,y_0)—≤—f(x,y_0-δ_1)-f(x_0,y_0-δ_1)—+—f(x_0,y_0-δ_1)-f(x_0,y_0)—¡ϵ/2+ϵ/2=ϵ

—f(x,y_0+δ_1)-f(x_0,y_0)—≤—f(x,y_0+δ_1)-f(x_0,y_0+δ_1)—+—f(x_0,y_0+δ_1)-f(x_0,y_0)—¡ϵ/2+ϵ/2=ϵ

∴—f(x,y)-f(x_0,y_0)—¡ϵ,f(x,y)在点(x_0,y_0)处连续

由点(x_0,y_0)的任意性知，f(x,y)在D连续，证毕

20: 反例：$\{(x,y)|y \geq \frac{1}{x} > 0\}$.

21: 二元函数Cauchy收敛准则：设$f(x,y)$是定义在$D \in \mathbb{R}^2$上的二元函数，则$f(x,y)$在$M_0(x_0,y_0)$处收敛等价于$\forall\epsilon > 0, \exists\delta > 0, s.t. \forall M_1, M_2 \in B(M_0,\delta) \cap D$,有$|f(M_1) - f(M_2)| < \epsilon$

证明略.

22: 证：$f(x,y)$在(x_0,y_0)处连续

$\Rightarrow \forall\epsilon > 0\ \exists\delta > 0$, s.t. $\forall|x - x_0| < \delta, |y - y_0| < \delta$, 有$|f(x,y) - f(x_0,y_0)| < \epsilon$

$x(u,v), y(u,v)$在(u_0,v_0)处连续

$\Rightarrow \exists\delta' > 0$, s.t. $\forall|u - u_0| < \delta', |v - v_0| < \delta'$，有$|x(u,v) - x_0| < \delta, |y(u,v) - y_0| < \delta$

$\Rightarrow |f(x(u,v), y(u,v)) - f(x_0,y_0)| < \epsilon, \forall|u - u_0| < \delta', |v - v_0| < \delta'$

$\Rightarrow f(x(u,v), y(u,v))$在$(u_0,v_0)$处连续.

23: 证明：$f(x) = \frac{1}{1-xy}$在$[0,1] \times [0,1] \cap \{(x,y)\big|(x,y) \neq (1,1)\}$上由初等函数的四则运算产生，显然连续

下证其不一致连续:

取$\epsilon = \frac{1}{8}$

则对$\forall\delta > 0$(不妨设$\delta < 1$),

取$A = (x_1, y_1) = (1-\delta, 1-\delta), B = (x, y) = (1-\frac{\delta}{2}, 1-\frac{\delta}{2})$

则$|AB| < \delta$

且$|f(A) - f(B)| = |\frac{1}{1-x_1y_1} - \frac{1}{1-x_2y_2}| = \frac{4-3\delta}{\delta(4-\delta)(2-\delta)} > \frac{1}{8}$

从而f不一致连续

9.2 多变量函数的微分

1: (1)$f'_x(x,y) = 1 - \frac{x}{\sqrt{x^2+y^2}}$,$f'_x(3,4) = \frac{2}{5}$

(2)$f'_x(x,y) = 2xy\cos x^2y$,$f'_x(1,\pi) = -2\pi$

(3)$f'_x(x,y) = \frac{y^2+2xy+\frac{(xy^2+x^2y)(y^2+2xy)}{\sqrt{1+(xy^2+x^2y)^2}}}{xy^2+x^2y+\sqrt{1+(xy^2+x^2y)^2}}$

由对称性，$f'_y(x,y) = \frac{x^2+2xy+\frac{(xy^2+x^2y)(x^2+2xy)}{\sqrt{1+(xy^2+x^2y)^2}}}{xy^2+x^2y+\sqrt{1+(xy^2+x^2y)^2}}$

$f'_x(1,y) = \frac{y^2+2y+\frac{(y^2+y)(y^2+2y)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}} = \frac{y^2+2y}{\sqrt{1+(y^2+y)^2}}$

$f'_y(1,y) = \frac{1+2y+\frac{(y^2+y)(1+2y)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}} = \frac{1+2y}{\sqrt{1+(y^2+y)^2}}$

2: (1) $\frac{\partial z}{\partial x} = \frac{e^y}{y^2}$, $\frac{\partial z}{\partial y} = \frac{xe^y(y-2)}{y^3}$

(2) $\frac{\partial z}{\partial x} = \frac{ln3y3^{-\frac{y}{x}}}{x^2}$, $\frac{\partial z}{\partial y} = -\frac{3^{-\frac{y}{x}}ln3}{x}$

(3) $\frac{\partial z}{\partial x} = \frac{cos(\frac{x}{y})cos(\frac{y}{x})}{y} + \frac{ysin(\frac{x}{y})sin(\frac{y}{x})}{x^2}$, $\frac{\partial z}{\partial y} = -\frac{-xcos(\frac{x}{y})cos(\frac{y}{x})}{y^2} - \frac{sin(\frac{x}{y})sin(\frac{y}{x})}{x}$

(4) $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}$

(5) $\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}$, $\frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$

(6) $\frac{\partial u}{\partial x} = (3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$, $\frac{\partial u}{\partial y} = 2xye^{x(x^2+y^2+z^2)}$, $\frac{\partial u}{\partial x} = 2xze^{x(x^2+y^2+z^2)}$

(7) $\frac{\partial u}{\partial x}=x^{-1+yz}y^z$, $\frac{\partial u}{\partial y}=ln(x)zx^{y^z}y^{-1+z}$, $\frac{\partial u}{\partial z}=ln(x)ln(y)x^{y^z}y^z$

(8) $\frac{\partial u}{\partial x}=e^{-z}+\frac{1}{x+ln(y)}$, $\frac{\partial u}{\partial y}=\frac{1}{y(x+ln(y))}$, $\frac{\partial u}{\partial z}=1-xe^{-z}$

3:

$$\frac{\partial f}{\partial x}=\frac{\sin x^2y}{x^2y}2xy=\frac{2\sin x^2y}{x},\frac{\partial f}{\partial y}=\frac{\sin x^2y}{x^2y}x^2=\frac{\sin x^2y}{y}$$

4: 直接计算:

$$\frac{\partial f}{\partial x}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0,$$

$$\frac{\partial f}{\partial y}=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{y\sin\frac{1}{y^2}}{y}=\lim_{y\to 0}\sin\frac{1}{y^2}\text{不存在}.$$

5: 点(x,y)与$(0,0)$的距离$\rho=\sqrt{x^2+y^2}$.只需令$\delta=\epsilon$,那么当$\rho<\delta$时,有$|z(x,y)-z(0,0)|=\rho<\delta=\epsilon$,即$z(x,y)$在$(0,0)$处连续.

由于

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\frac{|x|}{x}$$

极限不存在,故$z(x,y)$在$(0,0)$处对x偏导数不存在，对y偏导数同理

6: 解:

由题意可设切线的方向向量为$\vec{\tau}=(x_0,0,z_0)$，且有$\frac{\partial z}{\partial x}=\frac{1}{2}x$，故在$(2,4,5)$点取方向向量为$\vec{\tau}=(1,0,1)$。而$Ox$轴正向单位方向向量为$\vec{n}=(1,0,0)$，则有:

$$\theta=\arccos\frac{\vec{\tau}\cdot\vec{n}}{|\vec{\tau}|\,|\vec{n}|}=\frac{\pi}{4}$$

故所成角为$\frac{\pi}{4}$。

7: $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2+1}}$

$\left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,1)}=\frac{\sqrt{3}}{3}$

则曲线在$(1,1,\sqrt{3})$处切线与x轴，y轴，z轴夹角分别为$\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{3}$

8: 考虑取对数 $\ln u=-\frac{1}{2}\ln t-\frac{x^2}{4t}$, 则有

$$\frac{\partial \ln u}{\partial t}=-\frac{1}{2t}+\frac{x^2}{4t^2}=\frac{1}{u}\frac{\partial u}{\partial t}$$

$$\frac{\partial \ln u}{\partial x}=-\frac{x}{2t}=\frac{1}{u}\frac{\partial u}{\partial x}$$

于是

$$\frac{\partial u}{\partial t}=u\left(-\frac{1}{2t}+\frac{x^2}{4t^2}\right),\frac{\partial u}{\partial x}=u\left(-\frac{x}{2t}\right),$$

因此

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial x}\left(-\frac{x}{2t}\right)+u(-\frac{1}{2t})=u\left(-\frac{1}{2t}+\frac{x}{2t^2}\right)=\frac{\partial u}{\partial t}.$$

9: $(1)z_{xx}=-\frac{4y}{(x+y)^3},z_{yy}=\frac{4x}{(x+y)^3},z_{xy}=\frac{2(x-y)}{(x+y)^3}$

$(2)z_{xx}=-\frac{2x}{(1+x^2)},z_{yy}=-\frac{2y}{(1+y^2)},z_{xy}=0$

$(3)z_{xx}=-\frac{x}{(x^2+y^2)^{3/2}},z_{xy}=-\frac{y}{(x^2+y^2)^{3/2}}$

$z_{yy}=\frac{x(x^2+y^2)-xy^2-2y^2\sqrt{x^2+y^2}}{(x\sqrt{x^2+y^2}+x^2+y^2)^2}$

$(4)z_{xx}=a^2\sin 4(ax+by),z_{yy}=b^2\sin 4(ax+by)\ ,z_{xy}=ab\sin 4(ax+by)$

(5)$z_{xx} = (-\frac{lny}{x^2} + (\frac{lny}{x})^2)e^{lnxlny}$

$z_{yy} = (-\frac{lnx}{y^2} + (\frac{lnx}{y})^2)e^{lnxlny}$

$z_{xy} = \frac{1+lnxlny}{xy}e^{lnxlny}$

(6)$z_{xx} = \frac{y^3x}{(1-x^2y^2)^{3/2}}, z_{yy} = \frac{x^3y}{(1-x^2y^2)^{3/2}}$

$z_{xy} = \frac{1}{(1-x^2y^2)^{3/2}}$

10: $\frac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$, $\frac{\partial^3 u}{\partial x\partial y^2} = (2+xyz)xz^2e^{xyz}$

11: (1)

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}}\\ \frac{\partial^2 r}{\partial x^2} &= \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\\ \text{同理，}\frac{\partial^2 r}{\partial y^2} &= \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\\ \frac{\partial^2 r}{\partial z^2} &= \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\\ \text{故}\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2} &= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{r}\end{aligned}$$

(2)

$$\begin{aligned}\ln r &= \frac{1}{2}\ln(x^2+y^2+z^2)\\ \frac{\partial \ln r}{\partial x} &= \frac{x}{x^2+y^2+z^2}\\ \frac{\partial^2 \ln r}{\partial x^2} &= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}\\ \therefore \frac{\partial^2 \ln r}{\partial x^2}+\frac{\partial^2 \ln r}{\partial y^2}+\frac{\partial^2 \ln r}{\partial z^2} &= \frac{y^2+z^2+x^2}{(x^2+y^2+z^2)^2} = \frac{1}{r^2}\end{aligned}$$

(3)

$$
\begin{aligned}
\frac{1}{r} &= (x^2+y^2+z^2)^{-\frac{1}{2}} \\
\frac{\partial}{\partial x}\frac{1}{r} &= \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial^2}{\partial x^2}\frac{1}{r} &= \frac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
\therefore \frac{\partial^2}{\partial x^2}\frac{1}{r}+\frac{\partial^2}{\partial y^2}\frac{1}{r}+\frac{\partial^2}{\partial z^2}\frac{1}{r} &= \frac{-3(x^2+y^2+z^2)+3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0
\end{aligned}
$$

12: 函数的二阶偏导数

$$
f''_{xx} = \begin{cases} \frac{-4x^3y^3+12xy^5}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 0. & x^2+y^2\neq 0. \end{cases} \qquad f''_{xy} = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, & x^2+y^2=0; \\ -1. & x^2+y^2\neq 0. \end{cases}
$$

$$
f''_{yy} = \begin{cases} \frac{4x^3y^3-12x^5y}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 0. & x^2+y^2\neq 0. \end{cases} \qquad f''_{yx} = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 1. & x^2+y^2\neq 0. \end{cases}
$$

沿着$y=kx$容易看出它们在$(0,0)$处都不连续, 且$f''_{xy}\neq f''_{yx}$.

13: (1)

$$
\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2}, \quad \frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}, \quad dz=\frac{2}{x^2+y^2}(xdx+ydy)
$$

(2)

$$
\frac{\partial z}{\partial x}=\frac{y(y^2-x^2)}{(x^2+y^2)^2}, \quad \frac{\partial z}{\partial y}=\frac{x(x^2-y^2)}{(x^2+y^2)^2}, \quad dz=\frac{y^2-x^2}{(x^2+y^2)^2}(ydx-xdy)
$$

(3)

$$\frac{\partial u}{\partial s}=\frac{-2t}{(s-t)^2},\quad \frac{\partial u}{\partial t}=\frac{2s}{(s-t)^2},\quad du=\frac{2}{(s-t)^2}(sdt-tds)$$

(4)

$$dz=\frac{1}{x^2+y^2}(xdy-ydx)$$

(5)

$$\frac{\partial z}{\partial x}=y\cos xy,\quad \frac{\partial z}{\partial y}=x\cos xy$$

故在$(0,0)$处,有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$$

因此

$$dz=0$$

(6)

$$\frac{\partial z}{\partial x}=4x^3-8xy^2,\quad \frac{\partial z}{\partial y}=4y^3-8x^2y$$

在$(0,0)$处,有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$$

因此

$$dz=0$$

在$(1,1)$处,有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=-4$$

因此

$$dz=-4dx-4dy$$

14: 证明：

由定理9.14知：

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy$$

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial x}dy$$

那么可以得到：

$$d\left(fg\right) = \frac{\partial fg}{\partial x}dx + \frac{\partial fg}{\partial x}dy = \left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f\right)dx + \left(\frac{\partial f}{\partial y}g + \frac{\partial g}{\partial y}f\right)dy = fdg + gdf$$

证毕。

15: $f(x,y) = \sqrt{|xy|}$

若$f(x,y)$在(0,0)处可微

则在(0,0)的邻域有$f(x,y) - f(0,0) = ax + by + o(\rho)$

故$f(x,y) = ax + by + o(\rho)$

而$f(x,y) = f(x,-y) = f(-x,y)$

则必有$a = b = 0$

则$f(x,y) = o(\rho)$

而沿着直线y=kx

$$\lim_{x\to 0, y=kx} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = \frac{\sqrt{|k|}}{\sqrt{1+k^2}} \neq 0$$

从而在(0,0)的邻域上,$f(x,y) \neq o(\rho)$

f在(0，0)处不可微

16: 对任意 $\epsilon > 0$, 取 $\delta = \epsilon$, 则当$0 < |x|, |y| < \delta$ 时有

$$|f(x,y)| = \frac{x^2|y|}{x^2+y^2} \le |y| < \epsilon$$

由定义知$f(x,y)$ 在点$(0,0)$ 处连续. 由定义易知f 在$(0,0)$ 处的偏导数均为0. 若f 在点$(0,0)$ 处可微，则

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + o(\rho) \quad (\rho = \sqrt{x^2+y^2} \to 0)$$

因此 $f(x,y) = o(\rho)(\rho \to 0)$, 又若令$y = kx(k \ne 0)$, 则

$$\frac{f(x,y)}{\rho} = \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{k}{(k^2+1)^{\frac{3}{2}}}$$

于是矛盾，故f 在$(0,0)$ 处不可微.

17: $\frac{\sin r}{r^2} \to 0, r \to \infty$故连续

$x^2+y^2 \ne 0, f_x = 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}$

$f_y = 2y \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}$

$x = y = 0, f_x(0,0) = \lim_{x\to 0} x \sin \frac{1}{x} = 0, f_y(0,0) = 0$

$lim_{x\to 0} f_x(x,0) = lim_{x\to 0} \cos \frac{1}{x}\ not\ exist$，偏导数不连续

$x, y \to 0, count, \frac{(x^2+y^2)\sin \frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

$= \frac{\sin r}{r} \to 0, r \to \infty$

18: $(1)\frac{\partial z}{\partial x} = 2xln(\frac{1}{1+y}), \frac{\partial z}{\partial y} = -\frac{x^2}{1+y}$

$\frac{\partial^2 z}{\partial x^2} = 2ln(\frac{1}{1+y}), \frac{\partial^2 z}{\partial x \partial y} = -\frac{2x}{1+y}, \frac{\partial^2 z}{\partial x^2} = \frac{x^2}{(1+y)^2}$

$(2)\frac{\partial z}{\partial x} = -\frac{xy(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)} + \frac{xyarctan(x-y-x^2y)}{(x-y)^2} - \frac{yarctan(x-y-x^2y)}{x-y}$

$\frac{\partial z}{\partial y} = -\frac{x(-1-x^2)y}{(x-y)(1+(x-y-x^2y)^2)} - \frac{xarctan(x-y-x^2y)}{x-y} - \frac{xyarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{2xy(1-2xy)^2(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2} + \frac{2xy^2}{(x-y)(1+(x-y-x^2y)^2)} + \frac{2xy(1-2xy)}{(x-y)^2(1+(x-y-x^2y)^2)} - \frac{2y(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)} -$

$\frac{2xyarctan(x-y-x^2y)}{(x-y)^3} + \frac{2yarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{2x(-1-x^2)y(1-2xy)(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2} + \frac{x(-1-x^2)y}{(x-y)^2(1+(x-y-x^2y)^2)} + \frac{2x^2y}{(x-y)(1+(x-y-x^2y)^2)} - \frac{(-1-x^2)y}{(x-y)(1+(x-y-x^2y)^2)} - \frac{x(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)} - \frac{xy(1-2xy)}{(x-y)^2(1+(x-y-x^2y)^2)} + \frac{xarctan(x-y-x^2y)}{(x-y)^2} - \frac{arctan(x-y-x^2y)}{x-y} + \frac{2xyarctan(x-y-x^2y)}{(x-y)^3} - \frac{yarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{2x(-1-x^2)^2y(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2} - \frac{2x(-1-x^2)}{(x-y)(1+(x-y-x^2y)^2)} - \frac{2x(-1-x^2)y}{(x-y)^2(1+(x-y-x^2y)^2)} - \frac{2xarctan(x-y-x^2y)}{(x-y)^2} - \frac{2xyarctan(x-y-x^2y)}{(x-y)^3}$

19: (1)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = e^t yx^{y-1} + \frac{2t}{1+t^2}yx^{y-1} = e^{x^y}yx^{y-1} + \frac{2yx^{2y-1}}{1+x^{2y}}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = e^t x^y \ln x + \frac{2t}{1+t^2}x^y \ln x = e^{x^y}x^y \ln x + \frac{2\ln x x^{2y}}{1+x^{2y}}$$

(2)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = yze^{xyz}s + xze^{xyz}/s + xye^{xyz}sr^{s-1}$$
$$= 2(r^{s+1}e^{r^{s+2}}) + r^{s+1}se^{r^{s+2}}$$
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = yze^{xyz}r + xze^{xyz}(-\frac{r}{s^2}) + xye^{xyz}r^s$$
$$= r^{s+2}e^{r^{s+2}}\ln r$$

(3)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{2x}{x^2+y^2}e^{t+s+r} + \frac{2y}{x^2+y^2} * 0 = \frac{2e^{2(t+s+r)}}{e^{2(t+s+r)} + 16(s^2+t^2)^2}$$
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{2x}{x^2+y^2}e^{t+s+r} + \frac{2y}{x^2+y^2}8s = \frac{2e^{2(t+s+r)} + 64s(s^2+t^2)}{e^{2(t+s+r)} + 16(s^2+t^2)^2}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{2x}{x^2+y^2}e^{t+s+r} + \frac{2y}{x^2+y^2}8t = \frac{2e^{2(t+s+r)} + 64t(s^2+t^2)}{e^{2(t+s+r)} + 16(s^2+t^2)^2}$$

(4)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{e^{ax}}{a^2+1}a\cos x + \frac{e^{ax}}{a^2+1}\sin x = \frac{e^{ax}}{a^2+1}(\sin x + a\cos x)$$

20: (1)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 3t^2 f_1' + 4t f_2'$$

.

(2)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \cos t f_1' - \sin t f_2' + \mathrm{e}^t f_3'$$

.

(3)

$$\frac{\partial u}{\partial x} = 2x f_1' + y\mathrm{e}^{xy} f_2',$$
$$\frac{\partial^2 u}{\partial x \partial y} = -4xy f_{11}'' + (2x^2 - 2y^2)\mathrm{e}^{xy} f_{12}'' + xy\mathrm{e}^{2xy} f_{22}'' + (1+xy)\mathrm{e}^{xy} f_2'.$$

(4)

$$\frac{\partial u}{\partial x} = f_1' + 2x f_2', \qquad \frac{\partial^2 u}{\partial x^2} = f_{11}'' + 4x f_{12}'' + 2f_2' + 4x^2 f_{22}'',$$
$$\frac{\partial^2 u}{\partial x \partial y} = f_{11}'' + (2x+2y) f_{12}'' + 4xy f_{22}''.$$

21: 根据方向微商的计算公式

$$\frac{\partial u}{\partial \boldsymbol{l}} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \cdot \frac{\boldsymbol{l}}{|\boldsymbol{l}|} = (-2, -1, 2) \cdot \frac{(3, -1, 1)}{\sqrt{11}} = -\frac{3\sqrt{11}}{11}$$

22: 解:

设$\vec{l}$为P点沿圆周逆时针方向的单位方向向量，易知:

$$\vec{l}=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

由定理知:

$$grad\,(z)=\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}$$

且:

$$\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2},\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$$

则在$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$点的方向微商为:

$$grad(z)\cdot\vec{l}_{x=\frac{1}{2},y=\frac{\sqrt{3}}{2}}=\frac{1}{2}$$

23: $u'_x=2x+y+3,u'_y=x+4y-2,u'_z=6z-2$

从而.$u'_x(1,1,-1)=6,u'_y(1,1,-1)=4,u'_z(1,1,-1)=-12$

$gradu\big|_{(1,1,-1)}=(6,3,-12)$

$(\frac{\partial f}{\partial\vec{e}})_{max}=|gradu\big|_{(1,1,-1)}|=3\sqrt{21}$

24: (1) 由$\frac{1}{r^2}=\frac{1}{x^2+y^2+z^2}$ 有

$$\begin{aligned}\frac{\partial\frac{1}{r^2}}{\partial x}&=-\frac{2x}{(x^2+y^2+z^2)^2}=-\frac{2x}{r^4},\\ \frac{\partial\frac{1}{r^2}}{\partial y}&=-\frac{2y}{(x^2+y^2+z^2)^2}=-\frac{2y}{r^4},\\ \frac{\partial\frac{1}{r^2}}{\partial z}&=-\frac{2z}{(x^2+y^2+z^2)^2}=-\frac{2z}{r^4},\end{aligned}\tag{9.1}$$

所以 $\mathbf{grad}\frac{1}{r^2} = -\frac{2}{r^4}\boldsymbol{r}$

(2) 由$\ln r = \frac{1}{2}\ln(x^2+y^2+z^2)$ 易有$\mathbf{grad}\ln r = \frac{1}{r^2}\boldsymbol{r}$.

25: $u_x = f'(\phi_1 y + \phi 2), u_y = f'(\phi_1 x + \phi_2)$

$u_{xy} = f''(\phi_1 x + \phi_2)(\phi_1 y + \phi_2) + f'(\phi_{11}xy + \phi_{12}y + \phi_{21}x + \phi_{22} + \phi_1)$

26: 略

27: 证明：可设中间变量$u = xy$,则$z = f(u)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = y\frac{\partial z}{\partial u}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = x\frac{\partial z}{\partial u}$$
$$\therefore x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xy\frac{\partial z}{\partial u} - xy\frac{\partial z}{\partial u} = 0$$

28:

$$\frac{\partial^2 u}{\partial t^2} = a^2\phi''(x-at) + a^2\psi''(x+at).$$

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x-at) + \psi''(x+at).$$

从而$\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$.

29: 证明：有

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\varphi\frac{\partial u}{\partial x} + \sin\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}=-r\sin\varphi\frac{\partial u}{\partial x}+r\cos\varphi\frac{\partial u}{\partial y}$$

因此,有

$$\begin{aligned} left &= \left(\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}\right)^2+\left(-\sin\varphi\frac{\partial u}{\partial x}+\cos\varphi\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 \\ &= right \end{aligned}$$

30: 证明:

由:

$$\begin{cases} \xi = x+y \\ \eta = 3x-y \end{cases}$$

可知:

$$\begin{cases} x=\frac{1}{4}\left(\xi+\eta\right) \\ y=\frac{1}{4}\left(3\xi-\eta\right) \end{cases}$$

得到:

$$\frac{\partial x}{\partial \xi}=\frac{\partial x}{\partial \eta}=\frac{1}{3}\frac{\partial y}{\partial \xi}=-\frac{\partial y}{\partial \eta}=\frac{1}{4}$$

求得:

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}=\frac{1}{4}\left(\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \eta\partial \xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}\right)$$

代入得:

$$\frac{\partial^2 u}{\partial \eta\partial \xi}+\frac{1}{2}\frac{\partial u}{\partial \xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial \xi}+6\frac{\partial u}{\partial \xi}\right)=0$$

证毕。

31: $\xi = x - sinx + y, \eta = x + sinx - y$

则$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}(1 - cosx) + \frac{\partial u}{\partial \eta}(1 + cosx)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}(1-cosx)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}(1-cos^2x) + \frac{\partial u}{\partial \xi}sinx + \frac{\partial^2 u}{\partial \eta^2}(1+cosx)^2 + \frac{\partial^2 u}{\partial \xi \partial \eta}(1-cos^2x) + \frac{\partial u}{\partial \eta}(-sinx)$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}(1 - cosx) + \frac{\partial^2 u}{\partial \eta \partial \xi}(1 + cosx) - \frac{\partial^2 u}{\partial \eta^2}(1 + cosx) - \frac{\partial^2 u}{\partial \xi \partial \eta}(1 - cosx)$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$

再由$\frac{\partial^2 u}{\partial \eta \partial \xi} = \frac{\partial^2 u}{\partial \xi \partial \eta}$

则$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}cosx - \frac{\partial^2 u}{\partial y^2}sin^2x - \frac{\partial u}{\partial y}sinx = 4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

故$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

32: 由求导的链式法则有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} \tag{9.2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = -2\frac{\partial z}{\partial u} \tag{9.3}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u \partial v} \tag{9.4}$$

同理可有 $\frac{\partial^2 z}{\partial x \partial y} = a\frac{\partial^2 z}{\partial u \partial v}, \frac{\partial^2 z}{\partial y^2} = -2a\frac{\partial^2 z}{\partial u \partial v}$, 于是可得$a = 6$

33: $\int z_y dy = \int (x^2 + 2y)dy = yx^2 + y^2 + c(x)$

$z(x, x^2) = 1 = x^4 + x^4 + c(x) = 1$

$z(x, y) = yx^2 + y^2 + 1 - 2x^4$

34: $u(x, x^2) = 1$两边对x求导,得$u_x'(x, x^2) + 2xu_y'(x, x^2) = 0$

从而$u_y'(x, x^2) = -u_x'(x, x^2)/2x = -\frac{1}{2}$

35: 解：对$u(x,2x)=x$两边关于x求导可得：

$$u_x^{'}(x,2x)+2u_y^{'}(x,2x)=1,$$

再由已知

$$u_x^{'}(x,2x)=x^2,$$

则

$$u_y^{'}(x,2x)=\frac{1-x^2}{2},$$

以上两式关于x求导可得：

$$\begin{cases} u_{xx}^{''}(x,2x)+2u_{xy}^{''}(x,2x)=2x \\ u_{yx}^{''}(x,2x)+2u_{yy}^{''}(x,2x)=-x \end{cases}$$

由题设条件知

$$u_{xx}^{''}=u_{yy}^{''},u_{xy}^{''}=u_{yx}^{''}$$

联立解得

$$u_{xx}^{''}(x,2x)=u_{yy}^{''}(x,2x)=-\frac{4x}{3},u_{xy}^{''}(x,2x)=\frac{5x}{3}$$

36: (1)$\mathrm{d}u=f'\mathrm{d}x+f'\mathrm{d}y$.

(2)$\mathrm{d}u=(f_1'y+\frac{f_2'}{y})\mathrm{d}x+(f_1'x-\frac{xf_2'}{y^2})\mathrm{d}y$.

(3)$\mathrm{d}u=(f_1'+2tf_2'+3t^2f_3')\mathrm{d}t$.

(4)$\mathrm{d}u=(f_1'+2xf_2'+2xf_3')\mathrm{d}x+(2yf_2'+2yf_3')\mathrm{d}y+2zf_3'\mathrm{d}z$.

(5)$\mathrm{d}u=(2xf_1'+2xf_2'+2yf_3')\mathrm{d}x+(2yf_1'-2yf_2'+2xf_3')\mathrm{d}y$.

37: 球坐标下的Laplace方程的形式:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0$$

或:

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0$$

38: 解:

由题意得:

$$\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}$$

又知:

$$\frac{\partial x}{\partial r}=\cos\theta,\frac{\partial x}{\partial\theta}=-r\sin\theta,\frac{\partial y}{\partial r}=\sin\theta,\frac{\partial y}{\partial\theta}=r\cos\theta$$

代入得:

$$\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)}=r\cos^2\theta+r\sin^2\theta=r$$

9.3 隐函数定理和逆映射定理

1: (1) Let $F(x,y)=x^2+xy+y^2-7$, then it's easy to check

[1] $F(x,y)\in C^1$

[2] $F(2,1)=2^2+2\cdot1+1^2-7=0$

[3] $F_y(2,1)=2+2\cdot1\neq0$

By implicit function theorem, there exist $y=y(x)$ who is determined by $x^2+xy+y^2-7=0$ near point $(2,1)$.

derivation of x on both sides of the equation, we get $2x+y+xy'+2yy'=0$, then $y'(2,1)=-\frac{5}{4}$

derivation of x on both sides of the equation, $2+y'+y'+xy''+2y'y'+2yy''=0$, then $y''(2,1) = -\frac{21}{32}$

(2)Similar to (1). $y'(1, \frac{\pi}{2}) = -\frac{\pi}{2}$, $y''(1, \frac{\pi}{2}) = \pi$

2: (1)$\frac{dy}{dx} = \frac{2xy + ye^{xy} - ycos(xy)}{xcos(xy) - xe^{xy} - x^2}$

(2)$\frac{dy}{dx} = \frac{x+y}{x-y}, \frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}$

(3)$\frac{dy}{dx} = \frac{ln(y) - \frac{y}{x}}{ln(x) - \frac{x}{y}}$

$\frac{d^2y}{dx^2} = \frac{2(\frac{1}{y} - \frac{1}{x})(ln(y) - \frac{y}{x})(ln(x) - \frac{x}{y}) + \frac{y}{x^2}(ln(x) - \frac{x}{y})^2 - \frac{x}{y^2}(ln(y) - \frac{y}{x})^2}{(ln(x) - \frac{x}{y})^3}$

(4)$\frac{\partial z}{\partial x} = \frac{ye^{-xy}}{e^z - 2}$, $\frac{\partial z}{\partial y} = \frac{xe^{-xy}}{e^z - 2}$

$\frac{\partial x}{\partial y} = -\frac{y}{x}$, $\frac{\partial^2 z}{\partial x^2} = \frac{-y^2e^{-xy}((e^z-2)^2 + ez - xy)}{(e^z-2)^3}$

(5)$\frac{\partial z}{\partial x} = \frac{xz}{x^2+z^2}$, $\frac{\partial z}{\partial y} = \frac{z^3}{y(x^2+z^2)}$

(6)$\frac{\partial z}{\partial x} = -\frac{F_1' + F_2' + F_3'}{F_3'}$, $\frac{\partial z}{\partial y} = -\frac{F_2' + F_3'}{F_3'}$

(7)$\frac{\partial z}{\partial x} = -\frac{zF_1'}{xF_1' + yF_2'}$, $\frac{\partial z}{\partial y} = -\frac{zF_2'}{xF_1' + yF_2'}$

3: 令$F(x,y) = x^2 + xy + y^2 - 27$，易知道(3,3)和(-3,-3)为F的零点。对F求偏导有$F_x' = 2x + y, F_y' = x + 2y$，因此

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x'(x,y)}{F_y'(x,y)} = -\frac{2x+y}{x+2y}$$

令$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$有$y = -2x$，代入原方程得到$x^2 + x(-2x) + (-2x)^2 = 0$，得到$x = 3, y = -6$或$x = -3, y = 6$。由于$F(x,y)$为椭圆，可知道y=6为极大值，y=-6为极小值

也可用二阶导确定是极大值还是极小值。对$y'(x+2y) = -(2x+y)$两边对x求导有

$$y''(x+2y) + y'(1+2y') = -2 - y'$$

化简有$y'' = -2\frac{1+y'+(y')^2}{x+2y} = -6\frac{x^2+xy+y^2}{(x+2y)^3}$,$x = 3, y = -6$时$y'' = \frac{2}{9}$因此y=-6极小值，而$x = -3, y = 6$时$y'' = -\frac{2}{9}$因此y=6为极大值点。

4: (1) 两边求微分有

$$-2\cos x\sin x\mathrm{d}x-2\cos y\sin y\mathrm{d}y-2\cos z\sin z\mathrm{d}z=0.$$

于是$\mathrm{d}z=-\dfrac{\cos x\sin x\mathrm{d}x+\cos y\sin y\mathrm{d}y}{\cos z\sin z}$.
(2) 两边求微分有

$$yz\mathrm{d}x+xz\mathrm{d}y+xy\mathrm{d}z=\mathrm{d}x+\mathrm{d}y+\mathrm{d}z.$$

于是$\mathrm{d}z=-\dfrac{(yz-1)\mathrm{d}x+(xz-1)\mathrm{d}y}{xy-1}$.
(3) 两边求微分

$$3u^2\mathrm{d}u-(3\mathrm{d}x+3\mathrm{d}y)u^2-6(x+y)u\mathrm{d}u+3z^2\mathrm{d}z=0.$$

于是$\mathrm{d}u=\dfrac{u^2\mathrm{d}x+u^2\mathrm{d}y-z^2\mathrm{d}z}{u^2-2(x+y)u}$.
(4) 两边求微分

$$F_1'(\mathrm{d}x-\mathrm{d}y)+F_2'(\mathrm{d}y-\mathrm{d}z)+F_3'(\mathrm{d}z-\mathrm{d}x)=0.$$

于是$\mathrm{d}z=\dfrac{(F_1'-F_2')\mathrm{d}y+(F_3'-F_1')\mathrm{d}x}{F_3'-F_2'}$.

5: 等式$1+xy=k(x-y)$两边同时求微分，有

$$xdy+ydx=k(dx-dy)$$

两边同除dx,得

$$\frac{dy}{dx}=\frac{k-y}{k-x}$$

把$k=\frac{1+xy}{x-y}$代入上式,消去k,得到

$$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

得证.

6: 证明:

令

$$F\left(x,y,z\right)=2\sin\left(x+2y-3z\right)-x-2y+3z$$

可得:

$$F_x^{'}=2\cos\left(x+2y-3z\right)-1$$

$$F_y^{'}=4\cos\left(x+2y-3z\right)-2$$

$$F_z^{'}=-6\cos\left(x+2y-3z\right)+3$$

则有

$$\frac{\partial z}{\partial x}=-\frac{F_x^{'}}{F_z^{'}}=3$$

$$\frac{\partial z}{\partial y}=-\frac{F_y^{'}}{F_z^{'}}=-2$$

代入得

$$\frac{\partial z}{\partial y}+\frac{\partial z}{\partial x}=1$$

证毕。

7: $\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = \frac{c\varphi_1}{a\varphi_1+b\varphi_2}$

$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = \frac{c\varphi_2}{a\varphi_1+b\varphi_2}$

从而$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c$

8: 由$\mathrm{d}(x^2-xy-y^2) = 2x\mathrm{d}x - y\mathrm{d}x - x\mathrm{d}y + 2y\mathrm{d}y = 0$有$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-y}{x-2y}(x \neq 2y)$,于是

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{dx} = 2x + y\frac{2x-y}{x-2y}, \quad x \neq 2y$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} = 2 + 2\left(\frac{\mathrm{d}y}{dx}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 + \left(\frac{2x-y}{x-2y}\right)^2 + 6y\frac{x-y}{(x-2y)^2}$$

9: 对下面两个式子同时做全微分

$$\begin{cases} y = f(x+t) \\ y + g(x,t) = 0 \end{cases}$$

得到

$$\begin{cases} dy = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial t}dt = 0 \\ dy = f'dx + f'dt \end{cases}$$

联立两个方程,消去dt得到

$$\frac{dy}{dx} = \frac{1 - \frac{\partial g}{\partial x}}{\frac{\frac{\partial g}{\partial t}}{f'} + 1}$$

10: 在两个方程两端对z求导得到$\begin{cases}\frac{dx}{dz}+\frac{dy}{dz}+1=0\\ 2\frac{dx}{dz}x+2\frac{dy}{dz}y+2z=0\end{cases}$ 从而解得$\begin{cases}\frac{dx}{dz}=\frac{y-z}{x-y}\\ \frac{dy}{dz}=\frac{x-z}{y-x}\end{cases}$

12: (1)求微分得到

$$\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}=\begin{pmatrix}f_1' & f_2'\\ g_1' & g_2'\end{pmatrix}\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}v\end{pmatrix}.$$

从而

$$\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}v\end{pmatrix}=\begin{pmatrix}f_1' & f_2'\\ g_1' & g_2'\end{pmatrix}^{-1}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}.$$

也就是

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}.$$

(2) 取$f(u,v)=\mathrm{e}^u+u\sin v, g(u,v)=\mathrm{e}^u-u\cos v$代入(1)得到

$$\begin{aligned}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}&=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}.\\ &=\frac{1}{u\mathrm{e}^u(\sin v-\cos v)+u}\begin{pmatrix}u\sin v & -u\cos v\\ -\mathrm{e}^u+\cos v & \mathrm{e}^u+\sin v\end{pmatrix}.\end{aligned}$$

13: 方程$\varphi\left(x^2,e^y,z\right)=0$两边同时对$x$求导，得

$$2x\varphi_1'+\cos xe^{sinx}\varphi_2'+\frac{dz}{dx}\varphi_3'=0$$

从中解出
$$\frac{dz}{dx}=-\frac{2x\varphi_1'+\cos xe^{sinx}\varphi_2'}{\varphi_3'}$$
再将方程$u=f(x,y,z)$两边同时对x求导,有
$$\frac{du}{dx}=f_1'+f_2'\frac{dy}{dx}+f_3'\frac{dz}{dx}=f_1'+f_2'\cos x+f_3'\frac{dz}{dx}$$
将$\frac{dz}{dx}$的表达式代入上式,得
$$\frac{du}{dx}=f_1'+f_2'\cos x-f_3'\frac{2x\varphi_1'+\cos xe^{sinx}\varphi_2'}{\varphi_3'}$$

14: 解：令
$$G\left(x,y,z\right)=xf\left(x+y\right)-z$$
则有：
$$G_x^{'}=f\left(x+y\right)+xf^{'}\left(x+y\right)$$
$$G_y^{'}=xf^{'}\left(x+y\right)$$
$$G_z^{'}=-1$$
由隐函数定理可知：
$$\frac{dz}{dx}=-\frac{F_x^{'}G_y^{'}-F_y^{'}G_x^{'}}{F_y^{'}G_z^{'}-F_z^{'}G_y^{'}}$$
代入得：
$$\frac{dz}{dx}=\frac{F_x^{'}f^{'}\left(x+y\right)x-F_y^{'}f\left(x+y\right)-F_y^{'}f^{'}\left(x+y\right)x}{F_y^{'}+F_z^{'}f^{'}\left(x+y\right)x}$$

15: $F(x,y,u,v)=0,G(x,y,u,v)=0,u=u(x,y),v=v(x,y)$
$$\text{则}\begin{cases}\frac{\partial F}{\partial x}=F_1'+F_3'u_x'+F_4'v_x'=0\\ \frac{\partial F}{\partial y}=F_2'+F_3'u_y'+F_4'v_y'=0\\ \frac{\partial G}{\partial x}=G_1'+G_3'u_x'+G_4'v_x'=0\\ \frac{\partial G}{\partial y}=G_2'+G_3'u_y'+G_4'v_y'=0\end{cases}$$

从而$u'_x=-\frac{\partial(F,G)}{\partial(x,v)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$u'_y=-\frac{\partial(F,G)}{\partial(y,v)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$v'_x=-\frac{\partial(F,G)}{\partial(u,x)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$v'_y=-\frac{\partial(F,G)}{\partial(u,y)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

故$du=u'_xdx+u'_ydy$

$dv=v'_xdx+v'_ydy$

将u'_x,u'_y,v'_x,v'_x代入即可

16: 由$u(x,y)=f(x,y,z,t)$知$z=z(x,y),t=t(x,y)$. 由方程 $g(y,z,t)=0,h(z,t)=0$ 有

$$g_zz_y+g_tt_y=-g_y$$

$$h_zz_y+h_tt_y=0$$

联立解得

$$\begin{pmatrix} z_y \\ t_y \end{pmatrix}=\begin{pmatrix} g_z & g_t \\ h_z & h_t \end{pmatrix}^{-1}\begin{pmatrix} -g_y \\ 0 \end{pmatrix} \tag{9.5}$$

$$u_y=f_y+f_zz_y+f_tt_y=f_y+(f_z,f_t)\begin{pmatrix} z_y \\ t_y \end{pmatrix} \tag{9.6}$$

$$=f_y+(f_z,f_t)\begin{pmatrix} g_z & g_t \\ h_z & h_t \end{pmatrix}^{-1}\begin{pmatrix} -g_y \\ 0 \end{pmatrix} \tag{9.7}$$

类似的有

$$g_zz_x+g_tt_x=0 \tag{9.8}$$

$$h_zz_x+h_tt_x=0 \tag{9.9}$$

易得$z_x=t_x=0$. 于是 $u_x=f_x+f_zz_x+f_tt_x=f_x$.

9.4 空间曲线与曲面

1:

$$\vec{r'} = (a\cos t, a\sin t, 2bt)$$
$$\vec{r''} = (-a\sin t, a\cos t, 2b)$$

2: 设$\vec{r}(t) = (r_1(t), \ldots, r_n(t))$,则$\vec{r'}(t) = (r_1'(t), \ldots, r_n'(t))$.
由$r_1^2(t)+\cdots+r_n^2(t) = 1$两边对t求导得到$2(r_1(t)r_1'(t)+\cdots+r_n(t)r_n'(t)) = 0$,即得结论.
几何意义:长度不变的向量函数在其上每一点与其切向量正交.

3: 由题意可得曲线的切向量
r$'$(t)=(-asint,acost,b)
z轴的方向向量是k=(0,0,1)
所以切线与z轴的夹角余弦为
cosθ=$\frac{r'\cdot k}{|r'|\cdot|k|}$=$\frac{b}{\sqrt{a^2+b^2}}$为常数
$\therefore$曲线的切线与Oz轴夹角为常值

4: 是简单曲线也是光滑曲线. $\boldsymbol{r}'(t) = (\frac{1}{(1+t)^2}, -\frac{1}{t^2}, 2t)$,
将$t = 1$代入得切线的方向向量$\vec{v} = (\frac{1}{4}, -1, 2)$, 又$\boldsymbol{r}(1) = (\frac{1}{2}, 2, 1)$.
从而切线方程: $\dfrac{4x-2}{1} = \dfrac{y-2}{-1} = \dfrac{z-1}{2}$.
法平面方程: $\dfrac{1}{4}x - y + 2z - \dfrac{1}{8} = 0$.

5: (1)曲线切向量为$\left(2a\sin t\cos t, -b\sin^2 t + b\cos^2 t, -2c\sin t\cos t\right)$
在$t_0 = \pi/4$处切向量为$(a, 0, -c)$,且$t_0 = \pi/4$对应曲线上点$\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$
故切线方程为

$$\frac{x - \frac{a}{2}}{a} = \frac{y - \frac{b}{2}}{0} = -\frac{z - \frac{c}{2}}{c}$$

法平面方程为

$$a\left(x - \frac{a}{2}\right) - c\left(z - \frac{c}{2}\right) = 0$$

(2)曲线切向量为$(1 + \sin t, 2\sin t\cos t, -3\sin 3t)$
在$t_0 = \pi/2$处切向量为$(2, 0, 3)$,且$t_0 = \pi/2$对应曲线上点$\left(\frac{\pi}{2}, 4, 1\right)$
故切线方程为

$$\frac{x - \frac{\pi}{2}}{2} = \frac{y - 4}{0} = \frac{z - 1}{3}$$

法平面方程为

$$2\left(x - \frac{\pi}{2}\right) + 3(z - 1) = 0$$

6: 解: (1)

$$x_u' = \cos v, y_u' = \sin v, z_u' = 0$$
$$x_v' = -u\sin v, y_v' = u\cos v, z_v' = a$$

所以法向量为$\vec{n} = (x_u', y_u', z_u') \times (x_v', y_v', z_v') = (a\sin v, -a\cos v, u)$,则在$(u_0, v_0)$处的切平面方程为:

$$a\sin v_0(x - u_0\cos v_0) - a\cos v_0(y - u_0\sin v_0) + u_0(z - av_0) = 0$$

法线方程为:

$$\frac{x - u_0\cos v_0}{a\sin v_0} = \frac{y - u_0\sin v_0}{-a\cos v_0} = \frac{z - av_0}{u_0}$$

(2)

$$x_\theta^{'} = a\cos\theta\cos\varphi, y_\theta^{'} = b\cos\theta\sin\varphi, z_\theta^{'} = -c\sin\theta$$
$$x_\varphi^{'} = -a\sin\theta\sin\varphi, y_\varphi^{'} = b\sin\theta\cos\varphi, z_\varphi^{'} = 0$$

所以法向量为$\vec{n} = (x_\theta^{'}, y_\theta^{'}, z_\theta^{'}) \times (x_\varphi^{'}, y_\varphi^{'}, z_\varphi^{'}) = (bc\sin^2\theta\cos\varphi, ac\sin^2\theta\sin\varphi, ab\sin\theta\cos\theta)$,则在$(u_0, v_0)$处的切平面方程为:

$$bc\sin^2\theta_0\cos\varphi_0(x - a\sin\theta_0\cos\varphi_0) + ac\sin^2\theta_0\sin\varphi_0(y - b\sin\theta_0\sin\varphi_0) +$$
$$ab\sin\theta_0\cos\theta_0(z - c\cos\theta_0) = 0$$

法线方程为:

$$\frac{x - a\sin\theta_0\cos\varphi_0}{bc\sin^2\theta_0\cos\varphi_0} = \frac{y - b\sin\theta_0\sin\varphi_0}{ac\sin^2\theta_0\sin\varphi_0} = \frac{z - c\cos\theta_0}{ab\sin\theta_0\cos\theta_0}$$

7: $F(x,y) = F(x, F(x)) = 0$

$\frac{dF}{dx} = F_1' + F_2' f'(x) = 0$

则$f'(x) = -\frac{F_1'}{F_2'}, f'(x_0) = -\frac{F_1'(x_0,y_0)}{F_2'(x_0,y_0)}$

故$\overrightarrow{n_1} = (1, -\frac{F_1'(x_0,y_0)}{F_2'(x_0,y_0)})$

同理$\overrightarrow{n_2} = (1, -\frac{G_1'(x_0,y_0)}{G_2'(x_0,y_0)})$

$<\overrightarrow{n_1}, \overrightarrow{n_2}> = arccos\frac{\overrightarrow{n_1}\cdot\overrightarrow{n_1}}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|} = arccos\frac{F_1'(x_0,y_0)G_1'(x_0,y_0) + F_2'(x_0,y_0)G_2'(x_0,y_0)}{\sqrt{(F_1'(x_0,y_0))^2 + (F_2'(x_0,y_0))^2)^2}\cdot\sqrt{(G_1'(x_0,y_0))^2 + (G_2'(x_0,y_0))^2)^2}}$

8: (1) $\boldsymbol{n} = (17, 11, 5)$, $\boldsymbol{\pi} : 17x + 11y + 5z - 60 = 0$

(2) $\boldsymbol{n} = (1, -1, 2)$, $\boldsymbol{\pi} : x - y + 2z - \frac{\pi}{2} = 0$

(3) $\boldsymbol{n} = (1, 2, 0)$, $\boldsymbol{\pi} : x + 2y - 4 = 0$

(4) $\boldsymbol{n}=(5,4,1)$, $\boldsymbol{\pi}: 5x+4y+z-28=0$

9: 椭球面在(x_0,y_0,z_0)处的切平面为

$$xx_0+2yy_0+zz_0=1$$
$$\frac{x_0}{1}=\frac{2y_0}{-1}=\frac{z_0}{2}$$
$$x_0^2+2y_0^2+z_0^2=1$$

解得$(\frac{\sqrt{22}}{2},-\frac{\sqrt{22}}{4},\sqrt{22})$和$(-\frac{\sqrt{22}}{2},\frac{\sqrt{22}}{4},-\sqrt{22})$，再根据法向量即可得到平面方程。

10: 显然平面$x+3y+z=0$的法向量为$\vec{n}=(1,3,1)$,而曲面上(x,y,z)点处的法向量为$\vec{n}_s=(z'_x,z'_y,-1)=(y,x,-1)$,由两法向量平行即可解出$\begin{cases}x=-3\\ y=-1\end{cases}$,故所求曲面上的点为$(-3,-1,3)$,法线方程为$\frac{x+3}{1}=\frac{y+1}{3}=\frac{z-3}{1}$

11: 记点M的坐标为(x_0,y_0,z_0)

椭球面在M点的梯度gradF(M)=$(x_0,2y_0,3z_0)$

∴过点M的切平面的法向量为$(x_0,2y_0,3z_0)$

直线的方向向量为(2,1,-1)，过点(6,3,1/2),联立可得如下方程

$2x_0+2y_0-3z_0=0$

$x_0(6-x_0)+2y_0(3-y_0)+3z_0(1/2-z_0)=0$

${x_0}^2+2{y_0}^2+3{z_0}^2=21$

求解可得$x_0=1,y_0=2,z_0=2$或者$x_0=3,y_0=0,z_0=2$

当M为(1,2,2)时，切平面方程为x+4y+6z-21=0

当M为(3,0,2)时，切平面方程为3x+6z-21=0

12: 曲面于点$(1,-2,5)$处的法向量为$(2,-4,-1)$,
因此平面π: $2x-4y-z-5=0$. 任意选取直线上两点代入平面π, 得$a=-5, b=-2$.

13: 两个曲面在(x,y,z)处的法向量分别为

$$\boldsymbol{n_1}=(2x-a,2y,2z), \boldsymbol{n_2}=(2x,2y-b,2z)$$

$$\begin{aligned}\boldsymbol{n_1}\cdot\boldsymbol{n_2}&=4\left(x^2+y^2+z^2\right)-2ax-2by\\&=2\left(x^2+y^2+z^2-ax\right)+2\left(x^2+y^2+z^2-by\right)\\&=0\end{aligned}$$

因此两曲面正交.

14: 解：曲面$x+2y-\ln z+4=0$在点(x,y,z)处的法向量为$(1,2,-\frac{1}{z})$,曲面$x^2-xy-8x+z+5=0$在点(x,y,z)处的法向量为$(2x-y-8,-x,1)$,所以将点$(2,-3,1)$分别代入上面的两个法向量，得到$\vec{n_1}=(1,2,-1), \vec{n_2}=(-1,-2,1)$,即$\vec{n_1}\parallel\vec{n_2}$,则两曲面在该点有公共的切平面：

$$(x-2)+2(y+3)-(z-1)=0$$

15: 在$z=xe^{x/y}$上任取一点$(x_0,y_0,x_0e^{x_0/y_0})$

$\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}=(\frac{x_0}{y_0}+1)e^{x_0/y_0}$

$\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}=-\frac{x_0^2}{y_0^2}e^{x_0/y_0}$

则$\overrightarrow{n_1}=(1,0,(\frac{x_0}{y_0}+1)e^{x_0/y_0})$

$\overrightarrow{n_2}=(0,1,-\frac{x_0^2}{y_0^2}e^{x_0/y_0})$

$\overrightarrow{n_1}\times\overrightarrow{n_2}=(-(\frac{x_0}{y_0}+1)e^{x_0/y_0},\frac{x_0^2}{y_0^2}e^{x_0/y_0},1)$

在该点的切平面为$-(\frac{x_0}{y_0}+1)e^{x_0/y_0}(x-x_0)+\frac{x_0^2}{y_0^2}e^{x_0/y_0}(y-y_0)+(z-x_0e^{x_0/y_0})=0$

将(x,y,z) =(0,0,0)代入得

$(\frac{x_0}{y_0}+1)e^{x_0/y_0}x_0-\frac{x_0^2}{y_0^2}e^{x_0/y_0}y_0-x_0e^{x_0/y_0}=0$

该式恒成立，从而命题得证

16: (1) $l:x+y-2=0,\quad l_n:x-y=0$

(2) $l:x+2y-1=0,\quad l_n:2x-y-2=0$

17: (1)Let $F_1(x,y,z)=y^2+z^2-25,F_2(x,y,z)=x^2+y^2-10$, then for $F_1(x,y,z)=0$ the point $(1,3,4)$ follows the normal vector is $\mathbf{n}_1=(0,6,8)$, for $F_2(x,y,z)=0$ the point $(1,3,4)$ follows the normal vector is $\mathbf{n}_2=(2,6,0)$ then the tangent direction is $\mathbf{n}_1\times\mathbf{n}_2=(-48,16,-12)$ it can be instead of $(-12,4,-3)$

the equation of tangent line is $\frac{x-1}{-12}=\frac{y-3}{4}=\frac{z-4}{-3}$,

the equation of normal plane is $-12(x-1)+4(y-3)-3(z-4)=0$

(2)Similar to (1).the tangent direction is $(27,28,4)$, the equation of tangent line is $\frac{x+2}{27}=\frac{y-1}{28}=\frac{z-6}{4}$

the equation of normal plane is $27(x+2)+28(y-1)+4(z-6)=0$

18: 略

9.5 多变量函数的Taylor公式与极值

1: (1)$F(t) = \sin((x+th)^2+y+tk), F'(t) = \cos((x+th)^2+y+tk)(2(x+th)h+k), F'(1) = \cos((x+h)^2+y+k)(2(x+h)h+k)$

(2)$F(t) = (x+th)^2+2(x+th)(y+tk)^2-(y+tk)^4, F'(t) = 2h(x+th)+2h(y+tk)^2+4k(x+th)(y+tk)-3k(y+tk)^3$

2: (1)$f(x_0+h,y_0+k)-f(x_0,y_0) = 106-39(5+h)+(5+h)^3+18(6+k)-6(5+h)(6+k)+(6+k)^2 = 15h^2+h^3-6hk+k^2$

(2)$f(x_0+h,y_0+k)-f(x_0,y_0) = -2-2(1+h)(-1+k)+(1+h)^2(-1+k)+(1+h)(-1+k)^2 = h^2(-1+k)+h(-1+k)^2+(-3+k)k$

4: (1)成立区域：$\{(x,y)|y>-1\}$.

$$\begin{aligned}f(x,y) &= (1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3))(y-\frac{1}{2}y^2+\frac{1}{3}y^3+o(y^3))\\ &= y+xy-\frac{1}{2}y^2+\frac{1}{2}x^2y-\frac{1}{2}xy^2+\frac{1}{3}y^3+o(\rho^3).\end{aligned}$$

(2)成立区域：$\{(x,y)|x^2+y^2<1\}$.

$$\begin{aligned}f(x,y) = \sqrt{1-\rho^2} &= 1-\frac{1}{2}\rho^2-\frac{1}{8}\rho^4+o(\rho^4)\\ &= 1-\frac{1}{2}(x^2+y^2)-\frac{1}{8}(x^2+y^2)^2+o(\rho^4).\end{aligned}$$

(3)成立区域：$\{(x,y)|x<-1,y<-1\}$.

$$f(x,y)=\frac{1}{(1-x)(1-y)}=(\sum_{i=0}^{n}x^i+o(x^n))(\sum_{i=0}^{n}y^i+o(y^n))$$
$$=\sum_{k=0}^{n}\sum_{i=0}^{k}x^iy^{k-i}+o(\rho).$$

(4)成立区域：$\{(x,y)|1-x+y>0\}$.

$$f(0,0)=\frac{\pi}{4},\quad \frac{\partial f}{\partial x}(0,0)=1,\quad \frac{\partial f}{\partial y}=0,$$
$$\frac{\partial^2 f}{\partial x^2}(0,0)=0,\quad \frac{\partial^2 f}{\partial x\partial y}(0,0)=-1,\quad \frac{\partial^2 f}{\partial y^2}(0,0)=0.$$

从而

$$f(x,y)=\frac{\pi}{4}+x-xy+o(\rho^2).$$

(5)成立区域：$\mathbb{R}^2$. 下面$\rho=\sqrt{x^2+y^2}$, $m\in\mathbb{Z}$.

$$f(x,y)=\sin\rho^2=\begin{cases}\sum\limits_{k=0}^{m}\dfrac{\rho^{4k+2}}{(2k+1)!}+o(\rho^{4m+2}), & n=4m+2,4m+3,4m+4.\\ \sum\limits_{k=0}^{m}\dfrac{\rho^{4k-2}}{(2k-1)!}+o(\rho^{4m-2}), & n=4m+1.\end{cases}$$

(6)成立区域：$\mathbb{R}\times(-\frac{\pi}{2},\frac{\pi}{2})$.

$$f(0,0)=1,\quad \frac{\partial f}{\partial x}(0,0)=0,\quad \frac{\partial f}{\partial y}=0,$$
$$\frac{\partial^2 f}{\partial x^2}(0,0)=-1,\quad \frac{\partial^2 f}{\partial x\partial y}(0,0)=0,\quad \frac{\partial^2 f}{\partial y^2}(0,0)=1.$$

从而

$$f(x,y)=1-\frac{1}{2}x^2+\frac{1}{2}y^2+o(\rho^2).$$

(7)成立区域：$\mathbb{R}^2$. 配方得：

$$f(x,y)=2(x-1)^2-(y+2)^2-(x-1)(y+2)+5.$$

5: 用多元函数Taylor公式将$z = z(x,y)$展开至二阶:

$$z = z_0+\frac{\partial z}{\partial x}(x-x_0)+\frac{\partial z}{\partial y}(y-y_0)+\frac{1}{2}\frac{\partial^2 z}{\partial x^2}(x-x_0)^2+\frac{1}{2}\frac{\partial^2 z}{\partial y^2}(y-y_0)^2+\frac{\partial^2 z}{\partial x\partial y}(x-x_0)(y-y_0)+o\left(\rho^2\right)$$

方程$z^3 - 2xz + y = 0$两边同时对x求导,得

$$3z^2\frac{\partial z}{\partial x} - 2x\frac{\partial z}{\partial x} - 2z = 0$$

再将此式对x求导,得

$$\left(6z\frac{\partial z}{\partial x} - 2\right)\frac{\partial z}{\partial x} + (3z^2 - 2x)\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} = 0$$

从上两式中解得

$$\frac{\partial z}{\partial x} = \frac{2z}{3z^2 - 2x}, \quad \frac{\partial^2 z}{\partial x^2} = \frac{2\frac{\partial z}{\partial x} - \left(6z\frac{\partial z}{\partial x} - 2\right)\frac{\partial z}{\partial x}}{3z^2 - 2x}$$

再将方程$z^3 - 2xz + y = 0$两边同时对y求导,得

$$3z^2\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} + 1 = 0$$

再将此式分别对x, y求导,得

$$\left(6z\frac{\partial z}{\partial x} - 2\right)\frac{\partial z}{\partial y} + (3z^2 - 2x)\frac{\partial^2 z}{\partial x\partial y} = 0, \quad 6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial y} + (3z^2 - 2x)\frac{\partial^2 z}{\partial y^2} = 0$$

从上三式中解得

$$\frac{\partial z}{\partial y} = -\frac{1}{3z^2 - 2x}, \ \frac{\partial^2 z}{\partial x\partial y} = -\frac{\left(6z\frac{\partial z}{\partial x} - 2\right)\frac{\partial z}{\partial y}}{3z^2 - 2x}, \ \frac{\partial^2 z}{\partial y^2} = -\frac{6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial y}}{3z^2 - 2x}$$

代入点$(1,1,1)$,得

$$\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = -1, \quad \frac{\partial^2 z}{\partial x^2} = -16, \quad \frac{\partial^2 z}{\partial x\partial y} = 10, \quad \frac{\partial^2 z}{\partial y^2} = -6$$

因此,

$$z(x,y) = 1+2(x-1)-(y-1)-8(x-1)^2-3(y-1)^2+10(x-1)(y-1)+o\left(\rho^2\right)$$

6: 证明:$\cos x, \cos y, \cos z$在(0,0)点的二阶泰勒展开式为:

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$
$$\cos y = 1 - \frac{y^2}{2} + o(y^2)$$
$$\cos z = 1 - \frac{z^2}{2} + o(z^2)$$

$\sin x, \sin y$的二阶泰勒展开式为:

$$\sin x = x + o(x^2)$$
$$\sin y = x + o(y^2)$$

$\therefore \cos x \cos y + \sin x \sin y \cos\theta$在(0,0)处的二阶泰勒展开式为:

$$\cos x \cos y + \sin x \sin y \cos\theta = 1 - \frac{x^2}{2} - \frac{y^2}{2} + xy\cos\theta + o(x^2 + y^2)$$

又$\because \cos z = \cos x \cos y + \sin x \sin y \cos\theta$,$\therefore$在原点的邻域内有:

$$1 - \frac{z^2}{2} = 1 - \frac{x^2}{2} - \frac{y^2}{2} + xy\cos\theta$$

即$z^2 = x^2 + y^2 - 2xycos\theta$.

7:

(1) $\frac{\partial f}{\partial x} = y - \frac{50}{x^2}, \frac{\partial f}{\partial y} = x - \frac{20}{y^2}, \frac{\partial^2 f}{\partial x^2} = \frac{100}{x^3}, \frac{\partial^2 f}{\partial x \partial y} = 1, \frac{\partial^2 f}{\partial y^2} = \frac{40}{y^3}$.

令$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$,

可解得$x = 5, y = 2$

当$x > 0, y > 0$时，$Q(h,k) = 0.8h^2 + 2hk + 5k^2$ 是正定的,

因此$(x,y) = (5,2)$ 是小极值点，极小值为30.

(2) $\frac{\partial f}{\partial x}=4-2x, \frac{\partial f}{\partial y}=-4-2y, \frac{\partial^2 f}{\partial x^2}=-2, \frac{\partial^2 f}{\partial x\partial y}=0, \frac{\partial^2 f}{\partial y^2}=-2.$

令$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0,$

可解得$x=2, y=-2.$

由于$Q(h,k)=-2h^2-2k^2$是负定的. 因此$(x,y)=(2,-2)$ 是极大值点，极大值为8

(3) $\frac{\partial f}{\partial x}=2e^{2x}(x+2y+y^2)+e^{2x}, \frac{\partial f}{\partial y}=e^{2x}(2+2y), \frac{\partial^2 f}{\partial x^2}=e^{2x}(4x+8y+4y^2+4), \frac{\partial^2 f}{\partial x\partial y}=e^{2x}(4+4y), \frac{\partial^2 f}{\partial y^2}=2e^{2x}.$

令$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$, 可解得$x=0.5, y=-1.$

由于$Q(h,k)=e(2h^2+2k^2)$ 是正定的, 因此$(x,y)=(0.5,-1)$ 是极小值点，极小值为$-frace2$

(4) 记$f(x,y)=(x^2+y^2)^2-a^2(x^2-y^2),$

则$\frac{\partial f}{\partial x}=4x(x^2+y^2)-2a^2x, \frac{\partial f}{\partial y}=4y(x^2+y^2)+2a^2y.$

因此$\frac{dy}{dx}=-\frac{2x(x^2+y^2)-a^2x}{2y(x^2+y^2)+a^2y}=0 \Leftrightarrow x=0$, 或$2(x^2+y^2)=a^2.$

若$x=0$, 那么$f(x,y)=0\rightarrow y=0$, 从而$\frac{\partial f}{\partial y}=0$, 这说明$y(x)$ 不存在.

若$2(x^2+y^2)=a^2$, 那么$f(x,y)=0\rightarrow x^2=\frac{3}{8}a^2, y^2=\frac{1}{8}a^2, a\neq 0$. 再通过计算$\frac{d^2y}{dx^2}$可知, $(\pm\sqrt{\frac{3}{8}}|a|, \pm\sqrt{\frac{1}{8}}|a|)$是极值点，$y$极大值为$\sqrt{\frac{1}{8}}|a|)$,极小值为$-\sqrt{\frac{1}{8}}|a|)$

(5) 原方程可化为$(x-1)^2+(y+1)^2+(z-2)^2=16.$

这是以$(1,-1,2)$ 为圆心半径为4的圆.

故$z(x,y)$ 再$(1,-1)$.处取得极值，极大值为6，极小值为-2

8: 设该三角形的两个内角分别为 $x, y, z, 0 < x, y, z < \pi, x + y + z = \pi$. 记 $f(x, y, z) = \sin x \sin y \sin z, F(x, y, z) = f(x, y, z) - \lambda(x + y + z - \pi)$, 对$F$分别关于$x, y, z, \lambda$ 求导并令其为0有

$$\frac{\partial F}{\partial x} = \cos x \sin y \sin z - \lambda = 0 \tag{9.10}$$
$$\frac{\partial F}{\partial y} = \sin x \cos y \sin z - \lambda = 0 \tag{9.11}$$
$$\frac{\partial F}{\partial z} = \sin x \sin y \cos z - \lambda = 0 \tag{9.12}$$
$$\frac{\partial F}{\partial \lambda} = x + y + z - \pi = 0 \tag{9.13}$$

解上述方程得$x = y = z = \frac{\pi}{3}$，显而易见，此即为$f$ 得条件最值点. 故正三角形的三个内角的正弦乘积最大，为$(\frac{\sqrt{3}}{2})^3$.

9: 按照$xy = 1, 2, ...$作图，易见和圆的切点处是最大最小值

10: (1)极小值$f(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}) = \frac{a^2b^2}{a^2+b^2}$
(2)极小值$f(3, 3, 3) = 9$
(3)极小值$f(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = \frac{1}{8}$
(4)极大值$\frac{\sqrt{6}}{18}$,极小值$-\frac{\sqrt{6}}{18}$

12: 只需求出函数

$$f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2 + (x-2)^2 + (y-3)^2 + (z-4)^2,$$

在约束条件$3x-2z=0$下的最小值即可.

取Lagrange函数为$F(x,y,z,\lambda)=f(x,y,z)+\lambda(3x-2z)$. 那么

$$\begin{cases}\dfrac{\partial F}{\partial x}=4x-6+3\lambda=0,\\ \dfrac{\partial F}{\partial y}=4y-8=0,\\ \dfrac{\partial F}{\partial z}=4z-10-2\lambda=0,\\ \dfrac{\partial F}{\partial \lambda}=3x-2z=0.\end{cases}\qquad\Longrightarrow\qquad\begin{cases}x=2,\\ y=2,\\ z=3.\end{cases}$$

依题意知最小值一定存在，从而最小距离为$f(2,2,3)=8$.

13: 联立二式得$x^2+y^2-2=0$,此为约束条件,为求$z=x^2+2y^2$的最值,令

$$F(x,y,\lambda)=x^2+2y^2+\lambda(x^2+y^2-2)$$

求得驻点方程组

$$\begin{cases}F'_x=2x(1+\lambda)=0\\ F'_y=2y(2+\lambda)=0\\ F'_\lambda=x^2+y^2-2=0\end{cases}$$

解得驻点有4个:$(0,\pm\sqrt{2}),(\pm\sqrt{2},0)$.一一代入,可得到使$z$为最大值的点为$(0,\pm\sqrt{2})$,最大值为4;使$z$为最小值的点为$(\pm\sqrt{2},0)$,最小值为2.

14: 证明:

对点$O(0,0)$的任意$B(0,\epsilon)$邻域，取$x=0,0<y<\epsilon$，则有:

$$f(x,y)=-2y^2<f(0,0)=0$$

而取$0 < x < \epsilon, 0 < y < \epsilon$，由基本不等式可知：

$$f(x,y) = 3x^2y - x^4 - 2y^2 \leqslant 3x^2y - 2\sqrt{2}x^2y = (3-2\sqrt{2})x^2y > f(0,0) = 0$$

因为原点的任意邻域内既存在$f(x,y) > 0$的点，亦存在$f(x,y) < 0$的点，所以原点不是极值点。
过原点的直线的参数式方程为：

$$x = t\cos\alpha, y = t\sin\alpha$$

易知：

$$f(x,y) = f(t\cos\alpha, t\sin\alpha) = 3t^3\cos^2\alpha\sin\alpha - t^4\cos^4\alpha - 2t^2\sin^2\alpha$$

求其二阶导数为：

$$f''(t) = 18t\cos^2\alpha\sin\alpha - 12t^2\cos^4\alpha - 4\sin^2\alpha$$

由：

$$f''(0) = -4\sin^2\alpha < 0(\alpha \neq 0)$$

和：

$$f(t) = -t^4(\alpha = 0), Max(f) = f(0) = 0$$

（最大值必为极大值）可知：沿过原点的每条直线，原点均是其极大值点。证毕。

15: 帐篷的表面积$S(H,R,h) = \pi R(2H + \sqrt{R^2+h^2})$, 体积是$V(H,R,h) = \pi R^2(H + \frac{1}{3}h) = V_0$.

记$f(H,R,h) = S - \lambda(V - V_0)$,

令$\begin{cases} f_H = 2\pi R - \lambda\pi R^2 = 0, \\ f_R = 2\pi H + \pi\sqrt{R^2+h^2} + \pi R^2\frac{1}{\sqrt{R^2+h^2}} - \lambda(2\pi R(H+\frac{1}{3}h)) = 0, \\ f_h = \pi Rh\frac{1}{\sqrt{R^2+h^2}} - \lambda(\frac{1}{3}\pi R^2) = 0, \\ \pi R^2(H+\frac{1}{3}h) - V_0 = 0, \end{cases}$

因此$\lambda R = 2\pi, \frac{\pi h}{\sqrt{h^2+R^2}} = \frac{\lambda R}{3} = \frac{2\pi}{3} \Rightarrow h = \frac{2R}{\sqrt{5}}$,

故$R = \sqrt{5}H, h = 2H$.

由于极值点是唯一的，且该问题是中最小值存在，则$R = \sqrt{5}H, h = 2H$即为最小值

16: 设该平行六面体的底面平行四边形的边长分别为x, y, 侧棱长为z,则有

$$4(x+y) + 4z = 12a, \quad i.e. \;\; x+y+z-3a = 0.$$

易见，同底面时，四棱柱比平行六面体的体积大, 因此只需考虑四棱柱的体积. 更进一步, 边长对应相等的长方形比平行四边形面积更大，因此只需考虑长方体的情形. 设体积$v(x,y,z) = xyz$, 问题等价于求V 在条件$x+y+z-3a = 0$ 下的最大值. 记

$$F(x,y,z) = V(x,y,z) + \lambda(x+y+z-3a)$$

F 分别对x, y, z, λ 求导并令其等于0有

$$yz + \lambda = 0 \tag{9.14}$$

$$xz + \lambda = 0 \tag{9.15}$$

$$xy + \lambda = 0 \tag{9.16}$$

$$x+y+z-3a = 0 \tag{9.17}$$

于是得$x=y=z=a$, 由题知，V 必存在最大值，于是$V_{max}=V(a,a,a)=a^3$, 此时该平行六面体为棱长为a 的正方体, 体积为a^3.

17: 不妨在第一象限讨论

$\frac{x_0dx}{a^2}+\frac{y_0dy}{b^2}=0, dy/dx=-\frac{x_0b^2}{y_0a^2}$

$x=0, y_0+\frac{x_0^2b^2}{y_0a^2}; y=0, x_0+\frac{y_0^2a^2}{x_0b^2}$

$S(x,y)=(y+\frac{x^2b^2}{ya^2})(x+\frac{y^2a^2}{xb^2})$

18: $(\frac{\sum_{i=1}^n x_i}{n}, \frac{\sum_{i=1}^n y_i}{n})$

20. 点(x,y,z)到平面$x+2y+z=9$的距离为$\dfrac{|x+y+2z-9|}{\sqrt{6}}$.
因此只需求出函数$f(x,y,z)=d^2=\dfrac{(x+y+2z-9)^2}{6}$
在约束条件$\dfrac{x^2}{4}+y^2+z^2-1=0$下取最大值、最小值的点.
取Lagrange函数为$F(x,y,z,\lambda)=f(x,y,z)+\lambda(\frac{x^2}{4}+y^2+z^2-1)$. 那么

$$\begin{cases}\dfrac{\partial F}{\partial x}=\dfrac{x+y+2z-9}{3}+\dfrac{\lambda}{2}x=0,\\ \dfrac{\partial F}{\partial y}=\dfrac{x+y+2z-9}{3}+2\lambda y=0,\\ \dfrac{\partial F}{\partial z}=\dfrac{2(x+y+2z-9)}{3}+2\lambda z=0,\\ \dfrac{\partial F}{\partial \lambda}=\dfrac{x^2}{4}+y^2+z^2-1=0.\end{cases} \Longrightarrow \begin{cases}x=\dfrac{4}{3},\\ y=\dfrac{1}{3},\\ z=\dfrac{2}{3}.\end{cases} \quad \text{或} \quad \begin{cases}x=-\dfrac{4}{3},\\ y=-\dfrac{1}{3},\\ z=-\dfrac{2}{3}.\end{cases}$$

代入得$f\big(\dfrac{4}{3},\dfrac{1}{3},\dfrac{2}{3}\big)=6, f\big(-\dfrac{4}{3},-\dfrac{1}{3},-\dfrac{2}{3}\big)=24.$
依题意知f一定有最大值、最小值. 因此距离平面最近、最远的点分别是$\big(\dfrac{4}{3},\dfrac{1}{3},\dfrac{2}{3}\big)$和$\big(-\dfrac{4}{3},-\dfrac{1}{3},-\dfrac{2}{3}\big)$.

21: (1)曲面S在(x_0, y_0, z_0)处的切平面方程为

$$\frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0$$

即

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{a}$$

与x, y, z轴截距分别为$\sqrt{ax_0}, \sqrt{ay_0}, \sqrt{az_0}$,截距之和为$\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) = \sqrt{a} \cdot \sqrt{a} = a$.

(2)记切平面与各坐标轴截距分别为r, s, t,不妨考虑$r, s, t > 0$的情形.围成四面体体积为$V(r, s, t) = \frac{1}{6}rst$.由(1)知$r + s + t - a = 0$.令

$$F(r, s, t, \lambda) = \frac{1}{6}rst + \lambda(r + s + t - a)$$

求得驻点方程组

$$\begin{cases} F'_r = \dfrac{1}{6}st + \lambda = 0 \\ F'_s = \dfrac{1}{6}rt + \lambda = 0 \\ F'_t = \dfrac{1}{6}rs + \lambda = 0 \\ F'_\lambda = r + s + t - a = 0 \end{cases}$$

解得

$$r = s = t = \frac{a}{3}$$

最大体积为

$$V = \frac{1}{6}\left(\frac{a}{3}\right)^3 = \frac{a^3}{162}$$

由(1)结果可知切点为$\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right)$,从而切平面方程为$x + y + z = \frac{a}{3}$.

注:或用基本不等式求解,简单快捷

9.6 向量场的微商

4: (1)

$$\operatorname{div}[(\boldsymbol{r}\cdot\boldsymbol{w})\boldsymbol{w}] = (\boldsymbol{r}\cdot\boldsymbol{w})(\nabla\cdot\boldsymbol{w}) + \boldsymbol{w}\cdot\nabla(\boldsymbol{r}\cdot\boldsymbol{w}) = \boldsymbol{w}\cdot\boldsymbol{w}.$$

(2)

$$\operatorname{div}\frac{\boldsymbol{r}}{r} = \frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{y^2+x^2}{r^3} = \frac{2}{r}.$$

(3)

$$\operatorname{div}(\boldsymbol{w}\times\boldsymbol{r}) = \boldsymbol{r}\cdot\nabla\times\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\times\boldsymbol{r} = 0.$$

(4)

$$\operatorname{div}(r^2\boldsymbol{w}) = r^2\nabla\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\nabla r^2 = 2\boldsymbol{w}\cdot\boldsymbol{r}.$$

5: (1)

$$\operatorname{rot}\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\boldsymbol{i} - 2x\boldsymbol{j} - 2y\boldsymbol{k}$$

(2)

$$\operatorname{rot}\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y + y & z + e^y & y + 2ze^y \end{vmatrix} = (2ze^y)\boldsymbol{i} - (xe^y + 1)\boldsymbol{k}$$

6: 1、$\mathbf{rot}(\mathbf{w}\times\mathbf{r}) = 2\mathbf{w}$

2、$\mathbf{rot}[f(r)\mathbf{r}] = \mathbf{0}$

3、$\mathbf{rot}[f(r)\mathbf{w}] = \frac{f'(r)}{r}\mathbf{r}\times\mathbf{w}$

4、$div[\mathbf{r}\times f(r)\mathbf{w}] = 0$

7: (1)

$$\begin{aligned}\nabla(\overrightarrow{\omega}\cdot f(r)\overrightarrow{r}) &= \overrightarrow{\omega}\cdot\overrightarrow{r}\nabla f(r)+f(r)\overrightarrow{\omega}\\ &= (\overrightarrow{\omega}\cdot\overrightarrow{r})f'(r)\frac{\overrightarrow{r}}{|\overrightarrow{r}|}+f(r)\overrightarrow{\omega}\end{aligned}$$

(2)

$$\begin{aligned}\nabla\cdot(\overrightarrow{\omega}\times f(r)\overrightarrow{r}) &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}-\overrightarrow{\omega}\cdot\nabla\times[f(r)\overrightarrow{r}]\\ &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}-\overrightarrow{\omega}\cdot(\nabla f(r)\times\overrightarrow{r}+f(r)\nabla\times\overrightarrow{r})\\ &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}\\ &= 0\end{aligned}$$

(3).

$$\begin{aligned}\overrightarrow{\nabla}\times(\overrightarrow{\omega}\times f(r)\overrightarrow{r}) &= \nabla f(r)\times(\overrightarrow{\omega}\times\overrightarrow{r})+f(r)\nabla\times(\overrightarrow{\omega}\times\overrightarrow{r})\\ &= f'(r)\frac{\overrightarrow{r}}{|\overrightarrow{r}|}\times(\overrightarrow{\omega}\times\overrightarrow{r})+f(r)\nabla\times(\omega_2 z-\omega_3 y,\omega_3 x-\omega_1 z,\omega_1 y-\omega_2 x)\\ &= f'(r)r\overrightarrow{\omega}-f'(r)(\frac{\overrightarrow{r}}{r}\cdot\overrightarrow{\omega})\overrightarrow{r}+2f(r)\overrightarrow{\omega}\end{aligned}$$

8: 设$\phi=\phi(x,y,z),\psi=\psi=\psi(x,y,z),\boldsymbol{a}=P_1\boldsymbol{i}+Q_1\boldsymbol{j}+Z_1\boldsymbol{k},\boldsymbol{b}=P_2\boldsymbol{i}+Q_2\boldsymbol{j}+Z_2\boldsymbol{k}$.

(1) 按定义计算可得.

(2) 按定义计算可得.

(3) 由行列式的知识知

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{x} & \frac{\partial}{y} & \frac{\partial}{z}\\ P_1+P_2 & Q_1+Q_2 & R_1+R_2\end{vmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_1 & Q_1 & R_1\end{vmatrix}+\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_2 & Q_2 & R_2\end{vmatrix} \tag{9.18}$$

于是$\nabla\times(\boldsymbol{a}+\boldsymbol{b})=\nabla\times\boldsymbol{a}+\nabla\times\boldsymbol{b}$

(4) 由乘积的求导法则易得.

(5) 由乘积的求导法则易得.

(6) 直接计算得.

(7) 直接计算得.

9.7 微分形式*

9.8 综合习题

4: 由齐次函数的Euler定理, f是n次其次函数等价于

$$xf'_x+yf'_y+zf'_z=nf.$$

当$f(x,y,z)=0$时, 有

$$z=z-nf=-\frac{xf;_x+yf'_y}{f'_z}=x\phi'_x+y\phi'_y.$$

这意味着$\phi(x,y)$是一次齐次函数.

6: 证明:

令:

$$f\left(x,y\right)=\frac{1}{4}\left(x^2+y^2\right)-e^{x+y-2}$$

易知:

$$f_x^{'}=\frac{1}{2}x-e^{x+y-2},f_y^{'}=\frac{1}{2}y-e^{x+y-2}$$

联立其偏导均为0，可以得到其在定义域上无驻点，且在边界有：

$$f(x,0)=\frac{1}{4}x^2-e^{x-2}\leqslant 0, f(0,y)=\frac{1}{4}y^2-e^{y-2}\leqslant 0$$

$$\lim_{x\to\infty,y\to\infty}f(x,y)=-\infty$$

所以$f(x,y)$在定义域上不大于0，由此：

$$\frac{x^2+y^2}{4}\leq e^{x+y-2}$$

证毕

7: 任取$(x_0,y_0,z_0)\in R^2$

记$D=\{(x,y,z)\big||x-x_0|\leqslant 1,|y-y_0|\leqslant 1,|z-z_0|\leqslant 1\}$

则$\exists M,s.t.((x,y,z)\in D \implies |\frac{\partial f(x,y,z)}{\partial x}|\leqslant M,|\frac{\partial f(x,y,z)}{\partial y}|\leqslant M)$

$\forall\epsilon>0,$

由f关于z的连续性,$\exists\delta_0>0,s.t.(|z-z_0|<\delta_0 \implies |f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)|<\frac{\epsilon}{3})$

取$\delta=min\{\delta_0,\frac{\epsilon}{3M},1\}$

则当$|(x,y,z)-(x_0,y_0,z_0)|<\delta$时,$(x,y,z)\in D$

记$\Delta x=x-x_0,\Delta y=y-y_0,\Delta z=z-z_0$

则$|\Delta x|<\frac{\epsilon}{3M},|\Delta y|<\frac{\epsilon}{3M},|\Delta z|<\delta_0$

故

$$
\begin{aligned}
&|f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0,y_0,z_0)| \\
&\leqslant |f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0, y_0+\Delta x, z_0+\Delta z)|+|f(x_0, y_0+\Delta y, z_0+\Delta z) \\
&\quad -f(x_0,y_0,z_0+\Delta z)|+|f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)| \\
&=|\frac{\partial f(x+p\Delta x, y+\Delta y, z+\Delta z)}{\partial x}\Delta x|+|\frac{\partial f(x, y+q\Delta y, z+\Delta z)}{\partial x}\Delta y| \\
&\quad +|f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)(\text{其中}(0\leqslant p,q\leqslant 1)) \\
&<\epsilon
\end{aligned}
$$

由此即说明了f的连续性

8: 对任意$(x,y)\in\mathcal{D}$, 由二元函数的中值定理有

$$f(x,y)-f(0,0)=xf'_x(\theta x,\theta y)+yf'_y(\theta x,\theta y),\quad \theta\in[0,1],$$

由题设知$f(x,y)=f(0,0)$. 由(x,y) 的任意性知f为常值函数.

12: 令$\phi(t)=(x(b)-x(a))x(t)+(y(b)-y(a))y(t)$, 则由微分中值定理, 存在$\theta\in(a,b)$, 使得

$$\phi(b)-\phi(a)=\phi'(\theta)(b-a).$$

又因为

$$\begin{aligned}|\boldsymbol{r}(b)-\boldsymbol{r}(a)|^2 &= (x(b)-x(a))^2+(y(b)-y(a))^2\\ &= \phi(b)-\phi(a)=\phi'(\theta)(b-a)\\ &= [(x(b)-x(a))x'(\theta)+(y(b)-y(a))y'(\theta)](b-a)\\ &\le \sqrt{(x(b)-x(a))^2+(y(b)-y(a))^2}\cdot\sqrt{x'(\theta)+y'(\theta)}(b-a)\\ &= |\boldsymbol{r}(b)-\boldsymbol{r}(a)||\boldsymbol{r}'(\theta)|(b-a).\end{aligned}$$

移项就有

$$|\boldsymbol{r}(b)-\boldsymbol{r}(a)|\le|\boldsymbol{r}'(\theta)|(b-a).$$

14: 证明：对于：

$$f(x,y)=x^2+xy^2-x$$

分别对x,y求偏导为：

$$f_x^{'}=2x+y^2-1,f_y^{'}=2xy$$

由此，求得驻点为：

$$(0,\pm1),\left(\frac{1}{2},0\right)$$

函数在各个驻点取值分别为：

$$f(0,\pm1)=0,f\left(\frac{1}{2},0\right)=-\frac{1}{4}$$

而在边界$x^2+y^2=2$上，令：

$$g(x,y)=f(x,y)-\lambda\left(x^2+y^2-2\right)$$

求其偏导为：

$$g_x^{'}=f_x^{'}-2\lambda x,g_y^{'}=f_y^{'}-2\lambda y,g_\lambda^{'}=x^2+y^2-2$$

得到驻点为:

$$\left(-\frac{1}{3},\pm\frac{\sqrt{17}}{3}\right),(1,\pm1),\left(\sqrt{2},0\right),\left(-\sqrt{2},0\right)$$

求得在边界的最大值和最小值分别为:

$$Min\left(f,x^2+y^2=2\right)=\frac{11}{27},Max\left(f,x^2+y^2=2\right)=2+\sqrt{2}$$

综上，可以得到函数在定义域上的最大值和最小值分别为:

$$Min\,(f)=-\frac{1}{4},Max\,(f)=2+\sqrt{2}$$

15: $f(x_1,x_2,\dots x_n)=\prod_{i=1}^{n}x_i\sum_{j=1}^{n}\frac{1}{x_j}$

$\varphi(x_1,x_2,\dots x_n)=\sum_{j=1}^{n}x_j-n$

$F(x_1,x_2,\dots x_n)=f(x_1,x_2,\dots x_n)+\lambda\varphi(x_1,x_2,\dots x_n)$

令

$$\begin{aligned}\frac{\partial F}{\partial x_i}&=\prod_{k=1,k\neq i}^{n}x_k\sum_{j=1}^{n}\frac{1}{x_j}+\prod_{j=1}^{n}x_j(-\frac{1}{x_i^2})+\lambda\\&=\prod_{k=1,k\neq i}^{n}x_k\sum_{j=1,j\neq i}^{n}\frac{1}{x_j}=0\\\varphi(x_1,x_2,&\dots x_n)=0\end{aligned}$$

从而可解得$x_1=x_2=\cdots=x_n=1$为唯一极值点，从而为极大值

则$\prod_{i=1}^{n}x_i\sum_{j=1}^{n}\frac{1}{x_j}\leqslant n$

等号成立当且仅当$x_1=x_2=\cdots=x_n=1$

16: 考虑函数

$$f(x_1,x_2,...,x_n)=\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}$$

在条件$\frac{x_1+x_2+\cdots+x_n}{n} = A \geq 0$ 下的极值. 不妨设$A > 0$, 令

$$F(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \lambda \left(\frac{x_1 + x_2 + \cdots + x_n}{n} - A \right)$$

分别对$x_1, x_2, ..., x_n, \lambda$ 求导有

$$\frac{1}{n}(px_i^{p-1} + \lambda) = 0, \quad i = 1, ..., n \tag{9.19}$$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} - A = 0 \tag{9.20}$$

解的$x_1 = x_2 = \cdots = x_n = A$. 由题意知, f 的最小值一定存在, 故$(A, A, ..., A)$为最小值点, 且

$$f(x_1, ..., x_n) \geq f(A, \cdots, A) = A^p = \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^p$$

且等号当且仅当

$$x_i = \begin{cases} \dfrac{x_1 + x_2 + \cdots + x_n}{1(x_1 > 0) + 1(x_2 > 0) + \cdots + 1(x_n > 0)}, & x_i \neq 0 \\ 0 & \end{cases}$$

或者考虑$Hessian$ 矩阵的正定性: $H(A, A, ..., A) = \frac{p(p-1)}{n} A^{p-2} \boldsymbol{I} > 0 \quad (p > 1, A > 0)$.

Chapter 10

多变量函数的重积分

10.1　二重积分

1:　通过积分的边界确定积分区域，然后换序，一定要画图！

(1)积分区域是一个半圆，边界方程是$x^2+y^2=1$

$$\int_0^1 \mathrm{dy}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\mathrm{dx}$$

(2)积分区域是三角形，边界为$x=0,y=2x,x+y=6$

$$\int_0^4 \mathrm{dy}\int_0^{\frac{y}{2}} f(x,y)\mathrm{dx}+\int_4^6 \mathrm{dy}\int_0^{6-y} f(x,y)\mathrm{dx}$$

(3)积分区域是半圆，边界是$(x-a)^2+y^2=a^2$

$$\int_0^{2a} \mathrm{dx}\int_0^{\sqrt{a^2-(x-a)^2}} f(x,y)\mathrm{dy}$$

(4)积分区域是三角形，边界为$x=b,y=a,y=x$

$$\int_a^b \mathrm{dy}\int_a^x f(x,y)\mathrm{dy}$$

(5)积分区域是三角形，边界为$y=0,y=x,x+y=2$

$$\int_0^1 \mathrm{dy}\int_y^{2-y} f(x,y)\mathrm{dx}$$

(6)积分区域是曲边三角形，边界为$x=\frac{1}{2},x=1,xy=1$

$$\int_{\frac{1}{2}}^1 \mathrm{dx}\int_0^{\frac{1}{x}} f(x,y)\mathrm{dy}$$

2: (1)$\iint_D \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}dxdy=\int_0^1 dx\int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}dy=\int_0^1(\frac{-1}{\sqrt{x^2+2}}+\frac{1}{\sqrt{x^2+1}})dx=ln(\frac{2+\sqrt{2}}{\sqrt{3}+1})$

(2)$\iint_D sin(x+y)dxdy=\int_0^\pi dx\int_0^\pi sin(x+y)dy=\int_0^\pi 2cosxdx=0$

(3)$\iint_D cos(x+y)dxdy=\int_0^\pi dx\int_x^\pi cos(x+y)dy=\int_0^\pi(-sinx-sin(2x))dx=-2$

(4)$\iint_D(x+y)dxdy=\int_0^a dx\int_0^{\sqrt{a^2-x^2}}(x+y)dy=\int_0^a(x\sqrt{a^2-x^2}+\frac{1}{2}(a^2-x^2))dx=\frac{2}{3}a^3$

(5)$\iint_D(x+y-1)dxdy=\int_a^{3a} dy\int_{y-a}^{y}(x+y-1)dx=\int_a^{3a}(2ay-\frac{a^2}{2}-a)dy=7a^3-2a^2$

(6)$\iint_D \frac{siny}{y}dxdy=\int_0^1 dy\int_{y^2}^{y}\frac{siny}{y}dx=\int_0^1(siny-ysiny)dx=1-sin1$

(7)$\iint_D \frac{x^2}{y^2}dxdy=\int_1^2 dx\int_{\frac{1}{x}}^{x}\frac{x^2}{y^2}dy=\int_1^2(x^3-x)dx=\frac{9}{4}$

(8)$\iint_D |cos(x+y)|dxdy = \int_0^{\frac{\pi}{4}} dy \int_y^{\frac{\pi}{2}-y} cos(x+y)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^{x} -cos(x+y)dy = \int_0^{\frac{\pi}{4}}(1-sin2y)dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-sin2x)dx = \frac{\pi}{2}-1$

3: 解:

(1)由积分区域的对称性和被积函数对x,y都是偶函数，可得:

$$\iint_D (x^2+y^2)dxdy = 4\int_0^1 dx \int_0^1 (x^2+y^2)dy = \frac{8}{3}$$

(2)由积分区域的对称性和被积函数对x,y都是奇函数，可得:

$$\iint_D (\sin x + \sin y)dxdy = 0$$

6: 解:

$\because f(x)$有二阶连续偏导数,$\therefore \frac{\partial^2 f(x,y)}{\partial x \partial y}$在$D$上可积,且

$$\begin{aligned}
\iint_D \frac{\partial^2 f(x,y)}{\partial x \partial y}dxdy &= \int_a^b dx \int_c^d \frac{\partial^2 f(x,y)}{\partial x \partial y}dy \\
&= \int_a^b dx \int_c^d \frac{\partial^2 f(x,y)}{\partial y \partial x} \\
&= \int_a^b \frac{\partial f(x,y)}{\partial x}|_{y=c}^{y=d}dx \\
&= \int_a^b (\frac{\partial f(x,d)}{\partial x} - \frac{\partial f(x,c)}{\partial x})dx \\
&= [f(x,d)-f(x,c)]|_{x=a}^{x=b} \\
&= f(b,d)+f(a,c)-f(b,c)-f(a,d)
\end{aligned}$$

7: 由积分中值定理

$\exists(x_0,y_0) \in D = \{(x,y)|x^2+y^2 \leqslant r^2\} s.t. \iint_{x^2+y^2\leqslant r^2} f(x,y)dxdy = f(x_0,y_0)\pi r^2$

故

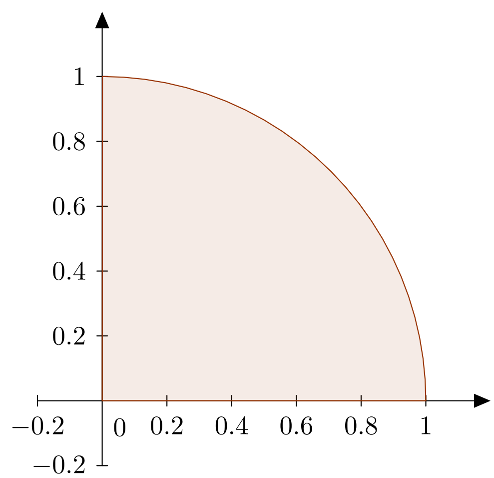
$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint_{x^2+y^2\leqslant r^2} f(x,y)dxdy=\lim_{r\to 0} f(x_0,y_0)=f(0,0)$$

10.2 二重积分的换元

1: (1)

做极坐标换元，令$x=r\cos\theta$，$y=r\sin\theta$，其中$r\in[0,R]$，$\theta\in[0,\frac{\pi}{2}]$。那么则有

$$\begin{aligned}&\int_0^R dx\int_0^{\sqrt{R^2-x^2}}\ln(1+x^2+y^2)dy\\&=\int_0^{\frac{\pi}{2}}d\theta\int_0^R\ln(1+r^2)rdr\\&=\frac{\pi}{2}\times\frac{1}{2}\left[(1+R^2)\ln(1+R^2)-(1+R^2)+1\right]\\&=\frac{\pi}{4}\left[(1+R^2)\ln(1+R^2)-R^2\right]\end{aligned}$$



(2)

$$\begin{aligned}&\int_0^a dx\int_0^b xy(x^2-y^2)dy\\&\int_0^a \frac{1}{2}b^2x^3-\frac{1}{4}b^4xdx\\&=\frac{1}{8}b^2a^4-\frac{1}{8}b^4a^2\end{aligned}$$

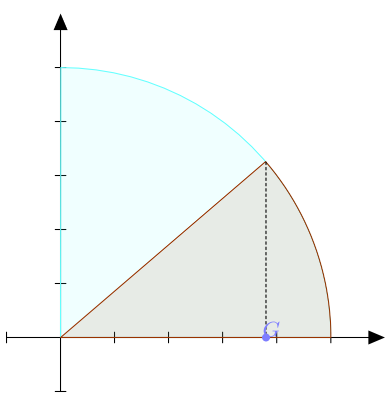
(3)

$$
\begin{aligned}
&\int_0^\pi \int_0^\pi \cos(x+y)dxdy \\
&= \int_0^\pi \sin(x+\pi) - \sin(x)dx \\
&= -2\int_0^\pi \sin x dx = -4
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\int_0^{\frac{1}{\sqrt{2}}} dx \int_x^{\sqrt{1-x^2}} xy(1+y)dy \\
&= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{2}x^2(1-2x^2) + \frac{1}{3}x[(1-x^2)^{\frac{3}{2}} - x^3]dx \\
&= \frac{1}{6}(\frac{1}{\sqrt{2}})^3 - \frac{1}{5}(\frac{1}{\sqrt{2}})^5 - \frac{1}{15}(\frac{1}{\sqrt{2}})^5 + \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{3}x(1-x^2)^{\frac{3}{2}}dx = \frac{1}{15}
\end{aligned}
$$

(5)



$$
\begin{aligned}
&\int_0^{\frac{R^2}{\sqrt{1+R^2}}} dy \int_{\frac{R}{y}}^{\sqrt{R^2-y^2}} (1+\frac{y^2}{x^2})dx \\
&= \int_0^{\frac{R^2}{\sqrt{1+R^2}}} \sqrt{R^2-y^2} - \frac{y}{R} + Ry - \frac{y^2}{\sqrt{R^2-y^2}}dy \\
&= \frac{1}{2}\frac{R^3(R^2-1)}{R^2+1} + \int_0^{\frac{R^2}{\sqrt{1+R^2}}} \frac{R^2-2y^2}{\sqrt{R^2-y^2}}dy
\end{aligned}
$$

使用极坐标换元$x = r\cos\theta$，$y = r\sin\theta$，此时$\theta \in [0, \arcsin\frac{R}{\sqrt{1+R^2}}]$。不妨

设$\theta_0 = \arcsin \frac{R}{\sqrt{1+R^2}}$。那么则有

$$\frac{R^2-2y^2}{\sqrt{R^2-y^2}}dy = \int_0^{\arcsin \frac{R}{\sqrt{1+R^2}}} R^2(1-2\sin^2\theta)d\theta$$
$$= R^2\sin\theta_0\cos\theta_0$$
$$= \frac{R^3}{1+R^2}$$

所以原结果即为

$$\frac{1}{2}\frac{R^3(R^2-1)}{1+R^2} + \frac{R^3}{1+R^2} = \frac{R^3(1+R^2)}{2(1+R^2)} = \frac{1}{2}R^3.$$

2: (1)做极坐标变换$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

则$D = \{(x,y)|x^2+y^2 < x+y\} = \{(r,\theta)|0 \le r \le sin\theta + cos\theta, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

故$\iint_D \sqrt{x^2+y^2}dxdy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \frac{0}{sin\theta+cos\theta} r^2 d\theta = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{3}(sin\theta+cos\theta)^3 d\theta = \frac{8\sqrt{2}}{9}$

(2)做类极坐标变换$\begin{cases} x = arcos\theta \\ y = brsin\theta \end{cases}$

则$D = \{(x,y)|\dots\} = \{(r,\theta)|a \le r \le 2, 0 \le \theta \le arctan\frac{a}{b}\}$

故$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}dxdy = ab\int_0^{arctan\frac{a}{b}} d\theta \int_0^2 r^2 dr = \frac{8ab}{3}\int_0^{arctan\frac{a}{b}} d\theta = \frac{8ab}{3}arctan\frac{a}{b}$

(3)做如下变换$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$

则$D = \{(x,y)|\dots\} = \{(u,v)|1 \le u \le 2, 1 \le v \le 2\}$

故$\iint_D (x^2+y^2)dxdy = \int_1^2 dv \int_1^2 (\frac{u}{v}+uv)\frac{1}{2v}du = \frac{3}{4}\int_1^2 (1+\frac{1}{v^2})dv = \frac{9}{8}$

(4)做如下变换$\begin{cases} u = \frac{y^2}{x} \\ v = \frac{x^2}{y} \end{cases}$

则$D = \{(x,y)|\dots\} = \{(u,v)|b \le u \le a, n \le v \le m\}$

故$\iint_D dxdy = \int_a^b du \int_n^m \frac{1}{3}dv = \frac{1}{3}(a-b)(m-n)$

(5)做如下变换$\begin{cases} u = xy \\ v = \frac{y^2}{x} \end{cases}$

则$D = \{(x,y)| \dots\} = \{(u,v)|a \le u \le b, c \le v \le d\}$

故$\iint_D dxdy = \int_a^b du \int_c^d \frac{u}{3v}dv = \frac{1}{3}ln\frac{d}{c}\int_a^b udu = \frac{1}{6}ln\frac{d}{c}(b^2-a^2)$

(6)做极坐标变换$\begin{cases} x = \sqrt{rcos\theta} \\ y = \sqrt{rsin\theta} \end{cases}$

则$D = \{(x,y)|x^4+y^4<1, x \ge 0, y \ge 0\} = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\}$

故$\iint_D 4xydxdy = \int_0^{\pi/2} d\theta \int_0^1 rdr = \frac{\pi}{4}$

(7)做如下变换$\begin{cases} u = x+y \\ v = x-y \end{cases}$

则$D = \{(x,y)||x|+|y| \le 1\} = \{(u,v)|-1 \le u \le 1, -1 \le v \le 1\}$

故$\iint_D \frac{x^2-y^2}{\sqrt{x+y+3}}dxdy = \int_{-1}^1 du \int_{-1}^1 \frac{uv}{2\sqrt{u+3}}dv = 0$

(8)做如下变换$\begin{cases} u = x+y \\ v = y \end{cases}$

则$D = \{(x,y)| \dots\} = \{(u,v)|0 \le u \le 1, 0 \le v \le u\}$

故$\iint_D sin\frac{y}{x+y}dxdy = \int_0^1 du \int_0^u sin\frac{v}{u}dv = (1-cos1)\int_0^1 udu = \frac{1}{2}(1-cos1)$

(9)做极坐标变换$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

则$D = \{(x,y)|x^2+y^2<a^2\} = \{(r,\theta)|0 \le r \le a, 0 \le \theta \le 2\pi\}$

故$\iint_D |xy|dxdy = \int_0^{2\pi} d\theta \int_0^a |sin\theta cos\theta|r^3dr = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^a |sin\theta cos\theta|r^3dr = \frac{a^4}{2}$

3: (1)所求的区域D是由关于原点对称的两部分组成，为第一象限D1面积的两倍。对于第一象限做变量代换$x = rcos\theta, y = \frac{1}{\sqrt{2}}rsin\theta$，由所围成的区

域可表示为

$$\begin{cases} 0 \le r^2 \le 3 \\ \dfrac{1}{\sqrt{2}} r^2 cos\theta sin\theta \ge 1 \end{cases}$$

$$\Rightarrow \begin{cases} 0 \le r^2 \le 3 \\ \dfrac{2\sqrt{2}}{r^2} \le sin2\theta \le 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2\sqrt{2} \le r^2 \le 3 \\ \dfrac{arcsin(\frac{2\sqrt{2}}{r^2})}{2} \le \theta \le \dfrac{\pi}{2} - \dfrac{arcsin(\frac{2\sqrt{2}}{r^2})}{2} \end{cases}$$

并且

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} cos\theta & \frac{1}{\sqrt{2}} sin\theta \\ -rsin\theta & \frac{1}{\sqrt{2}} rcos\theta \end{vmatrix} = \frac{1}{\sqrt{2}} r$$

因此我们有

$$\begin{aligned} \iint_D 1dxdy &= 2\int_{\sqrt{2\sqrt{2}}}^{\sqrt{3}} \int_{\frac{arcsin(\frac{2\sqrt{2}}{r^2})}{2}}^{\frac{\pi}{2}-\frac{arcsin(\frac{2\sqrt{2}}{r^2})}{2}} \frac{1}{\sqrt{2}} rd\theta dr = \sqrt{2}\int_{\sqrt{2\sqrt{2}}}^{\sqrt{3}} (\frac{\pi}{2} - arcsin(\frac{2\sqrt{2}}{r^2}))rdr \\ &\overset{t=\frac{r^2}{2\sqrt{2}}}{=} \frac{(3\sqrt{2}-4)\pi}{4} - 2\int_1^{\frac{3}{2\sqrt{2}}} arcsin(\frac{1}{t})dt \end{aligned}$$

计算有

$$\begin{aligned} \int arcsin(\frac{1}{t})dt &= tarcsin(\frac{1}{t}) + \int \frac{1}{t\sqrt{1-t^2}} dt \\ &= \ln(\sqrt{t^2-1}+t) + tarcsin\left(\frac{1}{t}\right) + \text{ constant} \end{aligned}$$

所以

$$\begin{aligned}
\iint_D 1dxdy &= \frac{(3\sqrt{2}-4)\pi}{4} - 2(\ln(\sqrt{t^2-1}+t)+tarcsin\left(\frac{1}{t}\right)))\Big|_1^{\frac{3}{2\sqrt{2}}} \\
&= \frac{(3\sqrt{2}-4)\pi}{4} - 2(\ln\sqrt{2}+\frac{3}{2\sqrt{2}}arcsin(\frac{2\sqrt{2}}{3})-arcsin(1)) \\
&= -\ln 2+\frac{3\sqrt{2}}{2}(\frac{\pi}{2}-arcsin(\frac{2\sqrt{2}}{3})) \\
&= -\ln 2+\frac{3\sqrt{2}}{2}(arcsin\frac{1}{3}))
\end{aligned}$$

(2) 做变量代换$x-y=rcos\theta, x=rsin\theta$ 就把区域D' : $0\le r\le a, 0\le\theta\le 2\pi$映成$D:(x-y)^2+x^2\le a^2$ 可知

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} sin\theta & sin\theta-cos\theta \\ rcos\theta & r(cos\theta+sin\theta) \end{vmatrix} = r$$

所以

$$\iint_{(x-y)^2+x^2\le a^2} 1dxdy = \int_0^{2\pi} d\theta\int_0^a rdr = \pi a^2$$

(3) 变量代换$x+y=u, y=vx$就把$O'uv$平面上的区域D' : $a\le u\le b, k\le v\le m$映成Oxy平面上的区域D，解出

$$x=\frac{u}{1+v}, y=\frac{uv}{1+v}$$

可知

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{1+v} & \frac{v}{1+v} \\ -\frac{u}{(1+v)^2} & \frac{u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^2}$$

故

$$\iint_D 1dxdy = \int_k^m \frac{1}{(1+v)^2}dv\int_a^b udu = (\frac{1}{1+k}-\frac{1}{1+m})\frac{b^2-a^2}{2}$$

6: 证明:

因为区域$D:|x|+|y|\leq 1$关于原点对称，所以有

$$\begin{aligned}\iint_{|x|+|y|\leq 1} e^{f(x+y)}dxdy &= \iint_{D_1\cup D_2} e^{f(x+y)}dxdy \\ &= \iint_{D_1} e^{f(x+y)}dxdy + \iint_{D_2} e^{f(x+y)}dxdy \\ &= \iint_{D_1} e^{f(x+y)}dxdy + \iint_{D_1} e^{f(-x-y)}dxdy \\ &= \iint_{D_1} [e^{f(x+y)} + e^{f(-x-y)}]dxdy \\ &= \iint_{D_1} [e^{f(x+y)} + e^{-f(x+y)}]dxdy \\ &\geq \iint_{D_1} 2dxdy \\ &= 2\times\frac{1}{2}\times 2\times 1 \\ &= 2\end{aligned}$$

其中,$D_1:|x|+|y|\leq 1, x\geq 0$; $D_2:|x|+|y|\leq 1, x\leq 0$.

7: 令t=x-y,m=x

则$\iint_D f(x-y)dxdy = \iint_{D'} f(t)dtdm = \iint_{D'} f(x)dxdy$

其中$D' = \{(t,m)||t-m|<\frac{A}{2}, |m|<\frac{A}{2}\}$

$$= \{(x,y)|0\leq x<A, x-\frac{A}{2}\leq y<\frac{A}{2}\}\cup\{(x,y)|-A<x\leq 0, -\frac{A}{2}\leq y<x+\frac{A}{2}\}$$

故 $$\begin{aligned}\iint_{D'} f(x)dxdy &= \int_0^A f(x)dx\int_{x-\frac{A}{2}}^{\frac{A}{2}} dy + \int_{-A}^0 f(x)dx\int_{-\frac{A}{2}}^{x+\frac{A}{2}} dy \\ &= \int_0^A (A-x)f(x)dx + \int_{-A}^0 (A+x)f(x)dx \\ &= \int_{-A}^A (A-|x|)f(x)dx\end{aligned}$$

10.3 三重积分

1: 此类题目被积函数不是关键之处，重要的是确定积分区域

把题目给出的积分区域转成$\int_?^? \int_?^? \int_?^?$的形式

(1)直接了当，x，y，z之间没有纠缠关系

$\int_0^{1/2} \int_{-2}^1 \int_1^2 xydxdydz$

(2)先粗略画图，投影在xy平面上是一个三角区域，如果是投影在z轴相关平面z=xy的投影不好观察出

$\int_0^1 \int_0^x \int_0^{xy} (xy^2z^3)dzdydx$

(3)观察图像，和式子，先定x，再由x定y和z

$\int_0^{\pi/2} \int_0^{\pi/2-x} \int_0^{\sqrt{x}} y\cos(x+z)dydzdx$

(4)画图注意到，如果先定x的话，xy平面的图像是两部分，要拆成两部分去算

所以这里我们先定y，可以进一步定出x和z

$\int_0^a \int_{(a-y)/2}^{a-y} \int_0^{a-y} (a-y)dzdxdy$

2: (1)柱坐标变换，$\frac{16}{9}$

(2)$\frac{4\pi R^5}{15}$

(3)π

(4)$\frac{2}{5}(2^{3/2}-1)\pi$

3: (1) $\iiint_V (x^2+y^2)\,dx\,dy\,dz$=$\iint_{x^2+y^2\le 4}\,dx\,dy\int_{\frac{x^2+y^2}{2}}^{2}\,x^2+y^2\,dz$

使用参数变换x=rcosθ y=rsinθ

0≤r≤2 0≤θ≤2π dxdy=rdrdθ

原式=$\int_0^{2\pi}\,d\theta\int_0^2 2r^3-\frac{r^5}{2}\,dr$=$\frac{16\pi}{3}$ (2) $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$=$\iint_{x^2+y^2\le 1}\,dx\,dy$

$\int_{\sqrt{x^2+y^2}}^{1}\sqrt{x^2+y^2}\,dz$
使用参数变换x=rcosθ y=rsinθ

0≤r≤1 0≤θ≤2π dxdy=rdrdθ
原式=$\int_0^{2\pi}\,d\theta\int_0^1 r^2{-}r^3\,dr{=}\frac{\pi}{6}$ (3) $\iiint_V z\,dx\,dy\,dz{=}\iint_{x^2+y^2\leq 3}\,dx\,dy\int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}}\,z\,dz$
使用参数变换x=rcosθ y=rsinθ

0≤r≤$\sqrt{3}$ 0≤θ≤2π dxdy=rdrdθ
原式=$\int_0^{2\pi}\,d\theta\int_0^{\sqrt{3}} 2r{-}\frac{r^3}{2}{-}\frac{r^5}{18}\,dr{=}\frac{13\pi}{4}$ (4) $\iiint_V xyz\,dx\,dy\,dz{=}\iint_{x^2+y^2\leq 1}\,dx\,dy\int_0^{\sqrt{1-x^2-y^2}}\,xyz\,dz$
使用参数变换x=rcosθ y=rsinθ

0≤r≤1 0≤θ≤π/2 dxdy=rdrdθ
原式=$\int_0^{\frac{\pi}{2}}\,sin\theta cos\theta\,d\theta\int_0^1\,\frac{r^3}{2}{-}\frac{r^5}{2}\,dr{=}\frac{1}{48}$ (5) V关于z轴的截面是由y=$\sqrt{z}$,y=$\frac{\sqrt{z}}{2}$,x=z,x=z/2围成

先xy后z的累次积分是
$\int_0^1\,dz\int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}}\,dy\int_{z/2}^{z} x^2\,dx{=}\frac{7}{216}$ (6) 作球坐标变换
原式=$\int_0^{2\pi}\,d\varphi\int_0^{\pi}\,sin\theta\,d\theta(\int_0^1 r^2{-}r^4\,dr{+}\int_1^2 r^4{-}r^2\,dr){=}16\pi$ (7） 由对称性,先xy后z得
原式=$2\int_0^1\,dz\iint_{D_z}\,e^z\,dx\,dy{=}2\int_0^1\,e^z\pi(1-z^2)\,dz{=}2\pi$ (8) 由对称性得
原式=$\iiint_V |x|e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$,作球坐标变换
$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\,cos\varphi\,d\varphi\int_0^{\pi}\,sin^2\theta\,d\theta\int_1^2\,r^3e^{-r^2}\,dr{=}\pi(\frac{2}{e}\text{-}\frac{5}{e^4})$

7:
$$\begin{aligned}F(t) &= \iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2)dxdydz \\ &= \iiint_{r^2\leqslant t^2} f(r^2)r^2\sin\theta drd\theta d\varphi \\ &= 4\pi\int_0^{|t|} f(r^2)r^2dr\end{aligned}$$
故$F'(t)=4\pi t^2 f(t^2)Sgn(t),\forall t\neq 0$,其中Sgn为符号函数

9: 这显然是一个用球极坐标换元的题目

使用球极坐标换元，$0 \leq r \leq t, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$

原式=$\int_0^{2\pi}\int_0^{\pi}\int_0^{t} f(r^2)r^2\sin\phi dr d\phi d\theta$

易见只要知道$G(t) = \int_0^t f(r^2)r^2 dr$的导数

一元变上限积分的导数利用莱布尼兹公式即得

10: $\int_D \rho dxdy = \frac{\pi}{2}ab$

11: 任取圆环上一圆周，圆周到圆心的距离为x

该圆周的面积ds=2πxdx,密度ρ=$\frac{1}{x}$

面质量dm=ρds=2πdx

质量m=$\int_L dm$=$\int_r^R 2\pi\, dx$=2π(R-r)

15: 不妨设$a, b, c > 0$

考虑坐标变换$\begin{cases} x & = ar\cos\theta \\ y & = br\sin\theta \\ z & = z \end{cases}$

$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = abr \neq 0$

则$z_G = \frac{\iiint_V zdxdydx}{\iiint_V dxdydx} = \frac{\int_0^c dz \int_0^{2\pi} d\theta \int_0^{\frac{z}{c}} abrzdz}{\int_0^c dz \int_0^{2\pi} d\theta \int_0^{\frac{z}{c}} abrdz} = \frac{\frac{1}{4}\pi abc^2}{\frac{1}{3}\pi abc} = \frac{3}{4}\pi$

由对称性可知$x_G = y_G = 0$

故重心为$(0, 0, \frac{3}{4}\pi)$

17: 本题直接套用本章正文的质心计算公式不难算出

注意到对称性x，y不用去算，只要算z

两个计算区域可以分别用柱坐标换元和球坐标换元计算，然后再相加

18: (1)(a)$I=\int_0^{2\pi}\int_0^R \rho r^3 dr d\theta=\frac{1}{2}mR^2$

(b)$I=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\rho x^2 dxdy=\frac{1}{4}mR^2$

(2)$\frac{3}{5}mR^2$

19: 取球心为坐标原点，z轴过顶点

则顶点坐标为(0,0,l=$\frac{R}{tan\alpha}$)

由积分对称性可得$F_x=F_y=0$

$F_z=km\rho\iiint_V \frac{z-l}{r^3}\,dx\,dy\,dz$

$r=\sqrt{x^2+y^2+(z-l)^2}$,$V=V_1+V_2$

V_1为半球体 $x^2+y^2+z^2\le R^2$,($z\le 0$)

V_2为锥体 $x^2+y^2=((l-z)\tan\alpha)^2\le R^2$,($0\le z\le l$)

将物体带入积分区域，仿照书本10.3节例10.3.13积分可得最终结果

10.4 n重积分

1: n重积分的计算转化为累次积分的计算

(1)

$$\int\cdots\int_{[0,1]^n} x_1^2+\cdots+x_n^2\mathrm{dx_1}\cdots\mathrm{dx_n}=\int_0^1\mathrm{dx_n}\cdots\int_0^1\mathrm{dx_2}\int_0^1 x_1^2+\cdots+x_n^2\mathrm{dx_1} \tag{10.1}$$

$$=\int_0^1 x_n^2\mathrm{dx_n}+\cdots+\int_0^1 x_1^2\mathrm{dx_1} \tag{10.2}$$

$$=\frac{n}{3} \tag{10.3}$$

(2) 引理：对任意$0 \le i \le n$有

$$\int_0^1 (x_i + \cdots + x_n)^2 \mathrm{dx_i} = \frac{1}{12} + \left(\frac{1}{2} + x_{i+1} + \cdots + x_n\right)^2$$

证明：只需要关于x_i展开，积分后重新配方即可，易证

$$\int \cdots \int_{[0,1]^n} (x_1 + \cdots + x_n)^2 \mathrm{dx_1} \cdots \mathrm{dx_n} = \frac{1}{12} + \int \cdots \int_{[0,1]^{n-1}} (\frac{1}{2} + x_2 + \cdots + x_n)^2 \mathrm{dx_2} \cdots \mathrm{dx_n} \tag{10.4}$$

$$= \cdots = \frac{n}{12} + (\frac{n}{2})^2 \tag{10.5}$$

$$\tag{10.6}$$

(3)

$$\int_0^1 \mathrm{dx_1} \int_0^{x_1} \mathrm{dx_2} \cdots \int_0^{x_{n-1}} x_1 \cdots x_n \mathrm{dx_n} = \int_0^1 \mathrm{dx_1} \int_0^{x_1} \mathrm{dx_2} \cdots \int_0^{x_{n-2}} \frac{1}{2} x_1 \cdots x_{n-1}^3 \mathrm{dx_{n-1}} \tag{10.7}$$

$$= \cdots = \int_0^1 \frac{1}{2^{n-1}(n-1)!} x_1^{2n-1} \mathrm{dx_1} = \frac{1}{2^n n!} \tag{10.8}$$

$$\tag{10.9}$$

2: $\frac{\prod_{i=1}^n a_i}{n!}$

3: 证明：采用数学归纳法，当$n=1$时，结论显然成立。假设$n=k$时，结论成立，即

$$\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{k-1}}f(x_k)dx_k=\frac{1}{(k-1)!}\int_0^a f(t)(a-t)^{k-1}dt$$

那么，当$n=k+1$时，有

$$\begin{aligned}\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{k-1}}dx_k\int_0^{x_k}f(x_{k+1})dx_{k1}&=\int_0^a dx_1\frac{1}{(k-1)!}\int_0^{x_1}f(t)(x_1-t)^{k-1}dt\\&=\int_0^a dt\frac{1}{(k-1)!}\int_t^a f(t)(x_1-t)^{k-1}dx_1=\int_0^a\frac{1}{k!}f(t)(a-t)^k dt\end{aligned}$$

10.5 综合习题

2: I可以化简为$\int_{-1}^1\int_{-1}^1\frac{ArcTan(\frac{1}{\sqrt{1+x^2+y^2}})}{\sqrt{1+x^2+y^2}}dydx$
结果的数值近似是0.308425

3: (1)

$$I_1=\int_0^1 sin(ln\frac{1}{x})\cdot\frac{x^b-x^a}{lnx}dx=\int_0^1 sin(ln\frac{1}{x})\int_a^b x^y dydx=\int_a^b dy\int_0^1 sin(ln\frac{1}{x})x^y dx$$

计算

$$\begin{aligned}J_1&=\int_0^1 sin(ln\frac{1}{x})x^y dx=\int_0^1\frac{sin(ln(\frac{1}{x}))}{y+1}dx^{y+1}=\frac{sin(ln(\frac{1}{x}))x^{y+1}}{y+1}\Big|_0^1+\int_0^1\frac{x^{y+1}}{y+1}\frac{cos(ln\frac{1}{x})}{x}dx\\&=\int_0^1\frac{cos(ln\frac{1}{x})}{(y+1)^2}dx^{y+1}=\frac{cos(ln\frac{1}{x})x^{y+1}}{(y+1)^2}\Big|_0^1-\int_0^1\frac{x^y}{(y+1)^2}sin(ln\frac{1}{x})dx=\frac{1}{(y+1)^2}-\frac{1}{(y+1)^2}J_1\end{aligned}$$

解得

$$J_1=\frac{1}{(y+1)^2+1}$$

那么

$$I_1=\int_a^b J_1dy=\int_a^b\frac{1}{(y+1)^2+1}dy=\int_{a+1}^{b+1}\frac{1}{t^2+1}dt$$
$$=arctan(t)|_{a+1}^{b+1}=arctan(b+1)-arctan(a+1)=arctan\frac{b-a}{1+(a+1)(b+1)}$$

(2) 同理

$$I_2=\int_0^1 cos(ln\frac{1}{x})\cdot\frac{x^b-x^a}{lnx}dx=\int_0^1 cos(ln\frac{1}{x})\int_a^b x^ydydx=\int_a^b dy\int_0^1 cos(ln\frac{1}{x})x^ydx$$

计算

$$J_2=\int_0^1 cos(ln\frac{1}{x})x^ydx=\int_0^1\frac{cos(ln(\frac{1}{x}))}{y+1}dx^{y+1}=\frac{cos(ln(\frac{1}{x}))x^{y+1}}{y+1}\Big|_0^1-\int_0^1\frac{x^{y+1}}{y+1}\frac{sin(ln\frac{1}{x})}{x}dx$$
$$=\frac{1}{y+1}-\int_0^1\frac{sin(ln\frac{1}{x})}{(y+1)^2}dx^{y+1}=\frac{1}{y+1}-\frac{sin(ln\frac{1}{x})x^{y+1}}{(y+1)^2}\Big|_0^1-\int_0^1\frac{sin(ln\frac{1}{x})x^y}{(y+1)^2}dx$$
$$=\frac{1}{y+1}-\frac{1}{(y+1)^2}J_2$$

解得

$$J_2=\frac{y+1}{(y+1)^2+1}$$

那么

$$I_2=\int_a^b J_2dy=\int_a^b\frac{y+1}{(y+1)^2+1}dy\overset{t=y+1}{=}\int_{a+1}^{b+1}\frac{t}{t^2+1}dt\overset{u=t^2}{=}\int_{(a+1)^2}^{(b+1)^2}\frac{1}{2(u+1)}du$$
$$=\frac{1}{2}ln(u+1)\Big|_{(a+1)^2}^{(b+1)^2}=\frac{1}{2}ln\frac{(b+1)^2+1}{(a+1)^2+1}$$

6: 解：考虑到$y\geqslant 0$和曲面关于x,z对称，因此对于此体积分可以化归为第一象限内的体积分，也即：

$$\iiint_V dxdydz=4\iiint_{V'}dxdydz$$

其中V'是V在第一象限的部分。根据重积分的定理，可知：

$$\iiint_{V'} dxdydz = \int_0^{z_0} dz \iint_D dxdy$$

其中z_0可以根据柱坐标系代换得知：

$$z^4 = r\sin\theta - r^4 \geqslant r - r^4 \geqslant \frac{3}{4}\cdot 2^{-\frac{2}{3}}$$

因此：

$$z_0 = (\frac{3}{4}\cdot 2^{-\frac{2}{3}})^{\frac{1}{4}}$$

而对于内部的二重积分，则有r,θ满足：

$$z^4 = r\sin\theta - r^4$$

其中r,θ具体关系，涉及到四次方程的判别式因此该问题难以求得解析解。若使用数学软件Mathematica，亦无法获得解析解，但可以求得数值解，其解为：0.3702402451

7: 考虑坐标变换$\begin{cases} t &= xy \\ m &= x \end{cases}$

则$\begin{cases} x &= m \\ y &= \frac{m}{t} \end{cases}$

$\frac{\partial(x,y)}{\partial(m,t)} = \frac{1}{m} \neq 0$

记$D = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

则$D' = \{(m,t)|0 \leqslant m \leqslant 1, 0 \leqslant \frac{t}{m} \leqslant 1\} = \{(m,t)|0 \leqslant t \leqslant 1, t \leqslant m \leqslant 1\}$

而

$$\lim_{t\to 0^+} t^t = \lim_{t\to 0^+} e^{t\log t} = 1$$

则 $\iint_D (xy)^{xy}dxdy = \iint_{D'} t^t \frac{1}{m} dmdt$

$$
\begin{aligned}
&= \int_0^1 -t^t \log t dt \\
&= -t^t|_{t\to 0^+}^1 + \int_0^1 t^t dt \\
&= \int_0^1 t^t dt
\end{aligned}
$$

10:

$$
\begin{aligned}
\int_a^b dx \int_{\phi(x)}^{\psi(x)} f^2(x,y)dy &= \int_a^b dx \int_{\phi(x)}^{\psi(x)} (\int_{\phi(x)}^{y} f'_y(x,t)dt)^2 dy && (10.10)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy \int_{\phi(x)}^{y} (f'y(x,t)^2)dt \int_{\phi(x)}^{y} dt && (10.11)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy \int_{\phi(x)}^{\psi(x)} (f'y(x,t))^2 dt \int_{\phi(x)}^{\psi(x)} dt \quad (using\ Cauchy\ inequality) && (10.12)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy (f'(x,y))^2 \quad (define\ M := \max_{a\le x\le b} |\psi(x)-\phi(x)|) && (10.13)
\end{aligned}
$$

11: 做变元代换

$$x_i = at_i, i = 1, \cdots, n, \quad \frac{\partial(x_1,\cdots,x_n)}{\partial(t_1,\cdots,t_n)} = a^n$$

因此积分可以写成

$$I_n(a) = \int\cdots\int_{\Omega_n(a)} x_1x_2\cdots x_n dx_1x_2\cdots x_n = a^{2n}\int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n dt_1t_2\cdots t_n = a^{2n}I_n(1)$$

区域也可写成

$$\Omega_n(1) : 0 \le x_n \le 1, 0 \le \sum_{i=1}^{n-1} x_i \le 1 - x_n, x_1, x_2, \cdots x_{n-1} \ge 0$$

因此积分可以写成

$$
\begin{aligned}
I_n(1) &= \int_0^1 tdt \int \cdots \int_{\Omega_{n-1}(1-t)} t_1 t_2 \cdots t_{n-1} dt_1 t_2 \cdots t_{n-1} = \int_0^1 t I_{n-1}(1-t)dt \\
&= \int_0^1 t(1-t)^{2n-2} I_{n-1}(1)dt = I_{n-1}(1)\int_0^1 t(1-t)^{2n-2} = \int_0^1 (1-t)^{2n-2}dt - \int_0^1 (1-t)^{2n-1} \\
&= \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)}
\end{aligned}
$$

又显然的$I_1(1) = \int_0^1 xdx = \frac{1}{2}$，所以

$$I_n(1) = \frac{1}{(2n)!}, I_n(a) = \frac{a^{2n}}{(2n)!}$$

Chapter 11

曲线积分和曲面积分

11.1 数量场在曲线上的积分

1: (1)

根据弧长公式

$$\int_0^{2\pi} \sqrt{[(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + e^{2t}]}dt = \sqrt{3}\int_0^{2\pi} e^t dt = \sqrt{3}(e^{2\pi} - 1)$$

(2)

根据弧长公式有

$$\begin{aligned}
&\int_0^1 \sqrt{9 + (6t)^2 + (6t^2)^2} \\
&= \int_0^1 \sqrt{9 + 36t^2 + 36t^4} \\
&= \int_0^1 6t^2 + 3dt = 5
\end{aligned}$$

(3)

由弧长公式得到

$$\begin{aligned}
&\sqrt{(a\sin t)^2+(a\cos t)^2+\left[\frac{a\sin t}{\cos t}\right]^2} \\
&=\int_0^{\frac{\pi}{4}}\frac{a}{\cos t}dt \\
&=\int_0^{\frac{\pi}{4}}\frac{a\cot t}{\cos^2 t}dt=\int_0^{\frac{\sqrt{2}}{2}}\frac{a}{1-x^2}dx=\frac{a}{2}\ln(3+2\sqrt{2})
\end{aligned}$$

(4)

可以解出曲线的参数方程为

$$\begin{cases} x &= \frac{z^2}{2a} \\ y &= \sqrt{\frac{8z^3}{9a}} \\ z &= z \end{cases}$$

所以由曲线的弧长公式

$$\int_0^{2a}\sqrt{(\frac{z}{a})^2+\frac{2z}{a}+1}dz=4|a|$$

(5)

令$y+z=0$， 那么可以解出参数方程为

$$\begin{cases} x &= \frac{t^2}{4a} \\ y &= \frac{1}{2}\left(t-\frac{t^3}{12a^2}\right) \\ z &= \frac{1}{2}\left(t+\frac{t^3}{12a^2}\right) \end{cases}$$

所以根据弧长公式

$$\begin{aligned}
&\int\sqrt{\left(\frac{t}{2a}\right)^2+\frac{1}{4}(1-\frac{t^2}{4a^2})^2+\frac{1}{4}(1+\frac{t^2}{4a^2})^2} \\
&=\int\frac{t^2+4a^2}{4\sqrt{2}a^2}=\int_0^2\sqrt{2}zdz=\frac{\sqrt{2}}{2}z^2
\end{aligned}$$

11.2 数量场在曲面上的积分

11.3 向量场在曲线上的积分

1: 第二型曲线积分基本计算方法是利用积分曲线的参数方程转化为第一型曲线积分

(1)两段分别为：$L_1 : y = x, L_2 : x + y = 2$，可以都选取$x$为参数

$$\int_{L_1} (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \int_0^1 2x^2\mathrm{dx} = \frac{2}{3}$$

$$\int_{L_2} (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \int_1^2 2(2 - x)^2\mathrm{dx} = \frac{2}{3}$$

故，原积分为$\int_L (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ 注：也可以用Green公式

(2)积分曲线为：$|x| + |y| = 1$，故在此曲线上，被积表达式为$\int_L \mathrm{dx} + \mathrm{dy}$

因为在直线AB, CD上，$\mathrm{dx} = -\mathrm{dy}$，积分为0

在直线BC, DA上，$\mathrm{dx} = \mathrm{dy}$，但是由于方向相反，在用参数方程表达时上下限相反，故求和为0

综上，此积分在正方形$ABCD$上的值为0

(3)思路同2，此积分表达式限制在积分曲线上可以表示为$\frac{1}{a^2} \int_L -x\mathrm{dx} + y\mathrm{dy}$

利用圆的参数方程$x(t) = a\cos t, y(t) = a\sin t$以及相应地$\mathrm{dx} = -a\sin t\mathrm{dt}, \mathrm{dy} = a\cos t\mathrm{dt}$,

带入表达式有$\int_0^{2\pi} \sin 2t\mathrm{dt} = 0$

(4)$OA: 0 \le x \le 1, y = 0, z = 0$，此时对应表达式为0，相应的积分值为0

$AB: x = 1, 0 \le y \le 1, z = 0$，此时对应表达式为$\int_0^1 y\mathrm{dy} = \frac{1}{2}$ $BC: x = 1, y = 1, 0 \le z \le 1$，此时对应表达式为$\int_0^1 z\mathrm{dz} = \frac{1}{2}$

故积分值为1

(5)A对应$\phi = 0$，B对应$\phi = \frac{\pi}{2}$，故原积分表达式可化为

$$\int_0^{\frac{\pi}{2}} e^{\cos\phi + \sin\phi + \frac{\phi}{\pi}} \left(\cos\phi - \sin\phi + \frac{1}{\pi}\right) \mathrm{d}\phi = e^{\cos\phi + \sin\phi + \frac{\phi}{\pi}}|_0^{\frac{\pi}{2}} = e^{\frac{3}{2}} - e$$

(6)L的参数方程为$x = 1 - \cos t, y = 1 + \cos t, z = \sqrt{2}\sin t$，注意到原点看是顺时针，$0 \le t \le 2\pi$，带入被积表达式有$\int_0^{2\pi}(1 + \cos t)\sin t\mathrm{dt} = 0$

11.4 向量场在曲面上的积分

11.5 Gauss定理和Stokes定理

11.6 其它形式的曲线曲面积分*

1: 参照课本推导，结果为

$$\frac{1}{r}\frac{\partial rF_r}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$