

第4章 不定积分习题课

本章主要内容

1. 原函数的概念.
2. 不定积分的概念与性质
3. 求不定积分的方法.
4. 特殊函数的不定积分求法

一. 原函数概念

1. 定义 设 $f(x)$ 在区间 I 有定义, 如果区间上可微函数 $F(x)$ 满足 $F'(x) = f(x)$ ($x \in I$), 则称 $F(x)$ 是 $f(x)$ 在 I 上的一个原函数.

注意定义中: $F(x)$ 是 $f(x)$: (1)在 I 上的(2)一个原函数.

2. 原函数的性质

- (i) 如果 $F(x)$ 是 $f(x)$ 的原函数, 则对任意常数 c , $F(x) + c$ 也是 $f(x)$ 的原函数.
- (ii) $f(x)$ 的任意两个原函数之差为常数, 所以

$$\{F(x) + c \mid c \in R\}$$

就是 $f(x)$ 的全体原函数组成的集合.

(iii) $f(x)$ 在区间 I 上的原函数 $F(x)$ 在 I 上一定是连续函数.

3. 原函数存在定理 设 $f(x)$ 在区间 I 上连续, 则在 I 上一定有原函数.

注: (1) 区间 I 上的连续函数一定有原函数, 但原函数未必能用初等函数表示. 例如

$$\frac{\sin x}{x}, \quad \frac{1}{\ln x}, \quad \frac{e^x}{x}, \quad e^{-x^2}, \quad \sqrt{1 - k^2 \sin^2 x} \quad (0 < k < 1)$$

(2) 函数若在区间 I 上有第一类间断点, 则一定在 I 上没有原函数.

三. 不定积分

1. 定义 设 $F(x)$ 是 $f(x)$ 的一个原函数, 则称集合 $\{F(x) + c \mid c \in R\}$ 为 $f(x)$ 的不定积分, 记成 $\int f(x)dx = F(x) + c$.

2. 性质

$$\begin{aligned} \frac{d}{dx} \int f(x)dx &= f(x), & d\left(\int f(x)dx\right) &= f(x)dx. \\ \int F'(x)dx &= F(x) + c, & \int dF(x) &= F(x) + c. \end{aligned}$$

3. 不定积分的几何意义

$\{F(x) + c \mid c \in R\}$ 是一族积分曲线, 它们是 $y = F(x)$ 沿 Oy 轴上下平移而得的. 在这些曲线上, 对应着 x 那一点的切线都有相同的斜率 $F'(x) = f(x)$.

4. 不定积分的运算法则

如果 $F(x)$ 和 $G(x)$ 分别是 $f(x)$ 和 $g(x)$ 的原函数, 则 $aF(x) + bG(x)$ 是 $af(x) + bg(x)$ 的原函数 (其中 a, b 是常数). $\int [af(x) + bg(x)]dx = a \int f(x)dx + b \int g(x)dx$.

5. 不定积分的公式

$$\begin{aligned} \int 0 dx &= c; & \int \sec x \tan x dx &= \sec x + c; \\ \int x^\mu dx &= \frac{x^{\mu+1}}{\mu+1} + c, \mu \neq -1; & \int \csc x \cot x dx &= -\csc x + c; \\ \int \frac{dx}{x} &= \ln|x| + c; & \int \frac{dx}{\sqrt{1-x^2}} &= \arcsin x + c; \\ \int a^x dx &= \frac{a^x}{\ln a} + c; & \int \frac{dx}{1+x^2} &= \arctan x + c; \\ \int \cos x dx &= \sin x + c; & \int \operatorname{ch} x dx &= \operatorname{sh} x + c; \\ \int \sin x dx &= -\cos x + c; & \int \operatorname{sh} x dx &= \operatorname{ch} x + c; \\ \int \csc^2 x dx &= -\cot x + c; & \int \sec^2 x dx &= \tan x + c; \\ \int \frac{1}{\sqrt{x^2-a^2}} dx &= \ln|x + \sqrt{x^2-a^2}| + c. \\ \int \frac{1}{\sqrt{x^2+a^2}} dx &= \ln(x + \sqrt{x^2+a^2}) + c; . \end{aligned}$$

6. 求不定积分的方法

- (i) 直接利用公式及运算法则
- (ii) 第一类换元法(凑微分法)
- (iii) 第二类换元法
- (iv) 分部积分法

7. 常见凑微分

$$\begin{aligned}
dx &= \frac{1}{a}d(ax+b); & x^n dx &= \frac{1}{n+1}d(x^{n+1}); \\
\frac{1}{x}dx &= d(\ln|x|); & e^x dx &= d(e^x); \\
\cos x dx &= d(\sin x); & \sin x &= -d(\cos x); \\
\sec^2 x dx &= d(\tan x); & \csc^2 x dx &= -d(\cot x); \\
\frac{1}{1+x^2}dx &= d(\arctan x); & \operatorname{ch}x dx &= d\operatorname{sh}x; \\
\frac{1}{1+x^2}dx &= -d(\operatorname{arccot}x); & \operatorname{sh}x dx &= d\operatorname{ch}x; \\
\frac{1}{2\sqrt{x}}dx &= d(\sqrt{x}); & \frac{1}{\sqrt{1-x^2}}dx &= d(\arcsin x).
\end{aligned}$$

8. 常用换元法

(1.)三角代换 (2.) 双曲代换 (3.)倒代换

一. 求不定积分

要求熟练掌握凑微分法、第二换元法、分部积分法、有理函数及三角有理函数的积分法

1. (20) (6分) 求不定积分 $\int \frac{\sin^3 x}{\cos x} dx$.
2. (19) (6分) 求不定积分 $\int \frac{\sin x}{(\sin x + \cos x)^3} dx$.
3. (19) (6分) 求不定积分 $\int \sin x \cdot \min\{\frac{1}{2}, x\} dx$.
4. (18) (6分) 求不定积分 $\int |x^2 - 1| dx$.
5. (18) (6分) 求不定积分 $\int \sin x \sin 2x dx$.
6. (17) (6分) 求不定积分 $\int \max\{1, x^2\} dx$.
7. (17) (6分) 求不定积分 $I = \int \sqrt{a^2 + x^2} dx$.
8. (17) (6分) 求不定积分 $I = \int \frac{dx}{1+x^3}$.
9. (16) (6分) 求不定积分 $\int \min\{x^2, x^5\} dx \quad (-\infty < x < +\infty)$.

10. (15,16) (5分,6分)求不定积分 $\int x^2 \arctan x dx$.
11. (15) (5分)求不定积分 $\int \frac{1}{x(1+x^4)} dx$.
12. (14) (10分)已知 $f''(x)$ 连续, $f'(x) \neq 0$, 求 $\int \left[\frac{f(x)}{f'(x)} - \frac{f^2(x)f''(x)}{(f'(x))^3} \right] dx$.
13. (14) (6分) $\int |\ln x| dx$.
14. (14) (6分) $\int \frac{x^3 - x}{1 + x^4} dx$.
15. (13) (6分) $\int \frac{1}{1 - x^4} dx$.
16. (13) (6分) $\int \max\{x^2, x^4\} dx$.
17. (13) (10分) 设 $f(x)$ 可微, 且 $\int x^3 f'(x) dx = x^2 \cos x - 4x \sin x - 6 \cos x + C$, 求 $f(x)$.
18. (13) (8分) 设 $f(x)$ 是 $(-\infty, +\infty)$ 上的可微函数且有反函数, 已知 $F(x)$ 是 $f(x)$ 的一个原函数, 求 $\int f^{-1}(x) dx$.
19. (12) (5分) $\int x(x-1)^n dx \quad (n > 0)$
20. (12) (5分) $\int \sin(2x) \cos^2 x dx$
21. (12) (5分) $\int \sin \sqrt{x} dx$
22. (12) (5分) $\int \ln(x + \sqrt{x^2 + 1}) dx$
23. (11)(8分) $\int \frac{xe^x}{\sqrt{e^x - 1}} dx$
24. (10) (5分) $\int x^2 e^x dx$
25. (10) (5分) $\int \frac{1}{\sqrt{e^x + 1}} dx$
26. (09) (8分) $\int x^2 \ln^2 x dx$

27. (08) (5分) $\int \frac{\arctan x}{x^3} dx$

28. (06) (9分) $\int \frac{\cos x}{\sin x(\sin^2 x + 1)} dx$

29. (05) (9分) $\int \frac{\ln x}{\sqrt{x-1}} dx$

30. (04) (6分) $\int \frac{\sin 2x}{\sqrt{1 + \cos^2 x}} dx$

31. (03) (5分) $\int x \tan x \sec^2 x dx$

32. (02) (8分) $\int e^{\sqrt{x}} dx,$

33. (02) (8分) $\int \frac{dx}{\sin^2 x - 3 \sin x \cos x + 2 \cos^2 x}$

例题:

1. $\int x \tan x \sec^4 x dx.$

解:

$$\begin{aligned} \int x \tan x \sec^4 x dx &= \int x \sec^3 x d \sec x = \frac{1}{4} \int x d(\sec^4 x) \\ &= \frac{1}{4} (x \sec^4 x - \int \sec^4 x dx) = \frac{1}{4} x \sec^4 x - \frac{1}{4} \int (1 + \tan^2 x) d(\tan x) \\ &= \frac{1}{4} x \sec^4 x - \frac{1}{4} \tan x - \frac{1}{12} \tan^3 x + c \end{aligned}$$

2. $\int \frac{x e^x}{\sqrt{e^x - 2}} dx.$

解:

$$\begin{aligned} \int \frac{x e^x}{\sqrt{e^x - 2}} dx &= \int x \frac{d(e^x - 2)}{\sqrt{e^x - 2}} = 2 \int x d\sqrt{e^x - 2} = 2x\sqrt{e^x - 2} - 2 \int \sqrt{e^x - 2} dx \\ &= \text{(以下自己完成)} \end{aligned}$$

3. $\int \frac{x(\arcsin x)^2}{\sqrt{1-x^2}} dx.$

解:

$$\begin{aligned}
& \int \frac{x(\arcsin x)^2}{\sqrt{1-x^2}} dx = - \int (\arcsin x)^2 d\sqrt{1-x^2} \\
&= -(\arcsin x)^2 \sqrt{1-x^2} + \int \frac{\sqrt{1-x^2} 2 \arcsin x}{\sqrt{1-x^2}} dx \\
&= -(\arcsin x)^2 \sqrt{1-x^2} + 2x \arcsin x - 2 \int \frac{x}{\sqrt{1-x^2}} dx \\
&= -(\arcsin x)^2 \sqrt{1-x^2} + 2x \arcsin x + 2\sqrt{1-x^2} + c
\end{aligned}$$

$$4. \int \frac{\sin 2x}{\sqrt{3-\cos^4 x}} dx.$$

解:

$$\int \frac{\sin 2x}{\sqrt{3-\cos^4 x}} dx = - \int \frac{d(\cos^2 x)}{\sqrt{3-\cos^4 x}} = - \arcsin \frac{\cos^2 x}{\sqrt{3}} + c.$$

$$5. \int \frac{x \ln x}{\sqrt{(x^2-1)^3}} dx.$$

解:

$$\begin{aligned}
& \int \frac{x \ln x}{\sqrt{(x^2-1)^3}} dx = - \int \ln x d\left(\frac{1}{\sqrt{x^2-1}}\right) = -\frac{\ln x}{\sqrt{x^2-1}} + \int \frac{1}{x\sqrt{x^2-1}} dx \\
&= -\frac{\ln x}{\sqrt{x^2-1}} + \int \frac{dx}{x^2 \sqrt{1-\frac{1}{x^2}}} = -\frac{\ln x}{\sqrt{x^2-1}} - \int \frac{d(\frac{1}{x})}{\sqrt{1-(\frac{1}{x})^2}} \\
&= -\frac{\ln x}{\sqrt{x^2-1}} - \arcsin \frac{1}{x} + c.
\end{aligned}$$

$$6. \int \frac{x^2}{1+x^2} \arctan x dx.$$

解:

$$\begin{aligned}
& \int \frac{x^2}{1+x^2} \arctan x dx = \int \frac{x^2-1+1}{1+x^2} \arctan x dx = \int \arctan x dx - \int \frac{\arctan x}{1+x^2} dx \\
&= x \arctan x - \int \frac{x}{1+x^2} dx - \frac{1}{2} \int d(\arctan x)^2 \\
&= x \arctan x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\arctan x)^2 + c.
\end{aligned}$$

$$7. \int \frac{x e^x}{(1+x)^2} dx.$$

解:

$$\begin{aligned}
& \int \frac{x e^x}{(1+x)^2} dx = \int \frac{(1+x-1)e^x}{(1+x)^2} dx = \int \frac{e^x}{1+x} dx - \int \frac{e^x}{(1+x)^2} dx \\
&= \int \frac{d(e^x)}{1+x} - \int \frac{e^x}{(1+x)^2} dx = \frac{e^x}{1+x} + \int \frac{e^x}{(1+x)^2} dx - \int \frac{e^x}{(1+x)^2} dx = \frac{e^x}{1+x} + c.
\end{aligned}$$

8. $\int \frac{\ln x - 1}{(\ln x)^2} dx.$

解法一：

$$\begin{aligned} \int \frac{\ln x - 1}{(\ln x)^2} dx &= \int \left(\frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) dx = \int \frac{dx}{\ln x} - \int \frac{1}{\ln^2 x} dx \\ &= \frac{x}{\ln x} + \int \frac{x}{\ln^2 x} \cdot \frac{1}{x} dx - \int \frac{1}{\ln^2 x} dx = \frac{x}{\ln x} + c. \end{aligned}$$

解法二：

$$\int \frac{\ln x - 1}{(\ln x)^2} dx \stackrel{\ln x = t}{=} \int \frac{t - 1}{t^2} \cdot e^t dt.$$

9. $\int \frac{e^x(1 + \sin x)}{1 + \cos x} dx.$

解:

$$\begin{aligned} & \int \frac{e^x(1 + \sin x)}{1 + \cos x} dx = \int \frac{e^x}{1 + \cos x} dx + \int \frac{e^x \sin x}{1 + \cos x} dx \\ = & \int \frac{e^x}{2 \cos^2 \frac{x}{2}} dx + \int e^x \tan \frac{x}{2} dx = \int e^x d(\tan \frac{x}{2}) + \int e^x \tan \frac{x}{2} dx = e^x \tan \frac{x}{2} + c. \end{aligned}$$

10. $\int (1 + x - \frac{1}{x}) e^{x + \frac{1}{x}} dx.$

解:

$$\begin{aligned} & \int (1 + x - \frac{1}{x}) e^{x + \frac{1}{x}} dx = \int e^{x + \frac{1}{x}} dx + \int (x - \frac{1}{x}) e^{x + \frac{1}{x}} dx \\ = & x e^{x + \frac{1}{x}} - \int x e^{x + \frac{1}{x}} (1 - \frac{1}{x^2}) dx + \int (x - \frac{1}{x}) e^{x + \frac{1}{x}} dx = x e^{x + \frac{1}{x}} + c. \end{aligned}$$

11. $\int \frac{x}{x^8 - 1} dx.$

解:

$$\int \frac{x}{x^8 - 1} dx \stackrel{x^2=t}{=} \frac{1}{2} \int \frac{dt}{t^4 - 1} = \frac{1}{4} \int \left(\frac{-1}{2(t+1)} + \frac{1}{2(t-1)} - \frac{1}{t^2+1} \right) dt.$$

12. $\int \frac{dx}{x^8(1+x^2)}.$

解:

$$\begin{aligned} & \int \frac{dx}{x^8(1+x^2)} = \int \frac{1 - x^8 + x^8}{x^8(1+x^2)} dx = \int \frac{(1+x^4)(1+x^2)(1-x^2)}{x^8(1+x^2)} dx + \int \frac{dx}{1+x^2} dx \\ = & \int \left(\frac{1}{x^8} - \frac{1}{x^6} + \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1+x^2} \right) dx = -\frac{1}{7x^7} + \frac{1}{5x^5} - \frac{1}{3x^3} + \frac{1}{x} + \arctan x + c. \end{aligned}$$

13. $\int \frac{x^9 - 8}{x^{10} + 8x} dx.$

解:

$$\begin{aligned} & \int \frac{x^9 - 8}{x^{10} + 8x} dx = \int \frac{x^8(x^9 - 8)}{x^9(x^8 + 8)} dx = \frac{1}{9} \int \frac{x^9 - 8}{x^9(x^9 + 8)} d(x^9) \stackrel{x^9=t}{=} \frac{1}{9} \int \frac{t - 8}{t(t+8)} dt \\ = & \frac{1}{9} \int \left(\frac{2}{t+8} - \frac{1}{t} \right) dt = \frac{1}{9} \ln \frac{(t+8)^2}{t} + c = \frac{1}{9} \ln \frac{(x^9+8)^2}{x^9} + c. \end{aligned}$$

14. $\int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx.$

解:

$$\begin{aligned} & \int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx = \int \frac{4 \sin x + 2 \cos x + 2 \cos x - \sin x}{2 \sin x + \cos x} dx = 2 \int dx + \int \frac{d(2 \sin x + \cos x)}{2 \sin x + \cos x} \\ = & 2x + \ln |2 \sin x + \cos x| + c. \end{aligned}$$

15. $\int \frac{\ln(1+x^2)}{x^3} dx.$

解:

$$\begin{aligned} \int \frac{\ln(1+x^2)}{x^3} dx &= -\frac{1}{2} \int \ln(1+x^2) d\left(\frac{1}{x^2}\right) = -\frac{1}{2} \left[\frac{\ln(1+x^2)}{x^2} - \int \frac{1}{x^2} \cdot \frac{2x}{1+x^2} dx \right] \\ &= -\frac{\ln(1+x^2)}{2x^2} + \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx = -\frac{\ln(1+x^2)}{2x^2} + \ln|x| - \frac{1}{2} \ln(1+x^2) + c. \end{aligned}$$

16. $\int \frac{1}{\sin^3 x \cos^5 x} dx.$

解:

$$\begin{aligned} \int \frac{1}{\sin^3 x \cos^5 x} dx &= \int \frac{1}{\tan^3 x \cos^8 x} dx = \int \frac{(1+\tan^2 x)^3}{\tan^3 x} d(\tan x) \\ &= \frac{-1}{2 \tan x^2} + \frac{3}{2} \tan^2 x + 3 \ln|\tan x| + \frac{1}{4} \tan^4 x + c. \end{aligned}$$

17. $\int \frac{x^5}{\sqrt[4]{x^3+1}} dx.$

解:

$$\begin{aligned} \int \frac{x^5}{\sqrt[4]{x^3+1}} dx &= \frac{1}{3} \int \frac{x^3}{\sqrt[4]{x^3+1}} d(x^3) \stackrel{t=x^3}{=} \frac{1}{3} \int \frac{t dt}{\sqrt[4]{t+1}} = \frac{1}{3} \int \left[(t+1)^{\frac{3}{4}} - (t+1)^{\frac{-1}{4}} \right] dt \\ &= \frac{4}{21} (1+t)^{\frac{7}{4}} - \frac{4}{9} (1+t)^{\frac{3}{4}} + c = \frac{4}{21} (1+x^3)^{\frac{7}{4}} - \frac{4}{9} (1+x^3)^{\frac{3}{4}} + c \end{aligned}$$

18. $\int e^x \left(3^x + \frac{e^{-x}}{x \ln x} \right) dx.$

解:

$$\int e^x \left(3^x + \frac{e^{-x}}{x \ln x} \right) dx = \int (3e)^x dx + \int \frac{1}{\ln x} d(\ln x) = \frac{(3e)^x}{\ln(3e)} + \ln|\ln x| + c.$$

19. $\int \frac{e^{-\frac{1}{x}}}{x^4} dx.$

解:

$$\begin{aligned} \int \frac{e^{-\frac{1}{x}}}{x^4} dx &\stackrel{\frac{1}{x}=t}{=} \int \frac{t^4}{e^t} \cdot \left(-\frac{1}{t^2}\right) dt = \int t^2 d(e^{-t}) = t^2 e^{-t} + 2t e^{-t} + 2e^{-t} + c \\ &= \left(\frac{1}{x}\right)^2 e^{-\frac{1}{x}} + 2t e^{-\frac{1}{x}} + 2e^{-\frac{1}{x}} + c \end{aligned}$$

20. $\int \frac{x e^{\arctan x}}{(1+x^2)^2} dx.$

解:

$$\begin{aligned}
& \int \frac{x e^{\arctan x}}{(1+x^2)^2} dx \stackrel{\arctan x=t}{=} \int \frac{\tan t e^t \sec^2 t}{(1+\tan^2 t)^2} dt = \frac{1}{2} \int e^t \sin 2t dt \\
& \int e^t \sin 2t dt = \int \sin 2t e^t = e^t \sin 2t - 2 \int e^t \cos 2t dt = e^t \sin 2t - 2 \int \cos 2t e^t \\
& = e^t \sin 2t - 2e^t \cos 2t - 4 \int e^t \sin 2t dt \\
& \int e^t \sin 2t dt = \frac{1}{5} (e^t \sin 2t - 2e^t \cos 2t). \\
& \int \frac{x e^{\arctan x}}{(1+x^2)^2} dx = \frac{1}{10} (e^t \sin 2t - 2e^t \cos 2t) = \frac{1}{10} (e^{\arctan x} \sin 2 \arctan x - 2e^{\arctan x} \cos 2 \arctan x) + c.
\end{aligned}$$

21. 设

$$f'(\ln x) = \begin{cases} 1, & 0 < x \leq 1 \\ x, & 1 < x < +\infty \end{cases}, \text{ 且 } f(0) = 0, \text{ 求 } f(x).$$

解: 令 $\ln x = t$, $x = e^t$

$$f'(t) = \begin{cases} 1, & 0 < e^t \leq 1 \iff -\infty < t \leq 0 \\ e^t, & 1 < e^t < +\infty \iff 0 < t < +\infty \end{cases}$$

$$f(t) = \begin{cases} t + c_1, & -\infty < t \leq 0 \\ e^t + c_2, & 0 < t < +\infty \end{cases}$$

$$f(0-0) = f(0+) = f(0), \text{ 得 } c_1 = 1 + c_2$$

$$f(x) = \begin{cases} x + 1 + c, & -\infty < t \leq 0 \\ e^x + c, & 0 < t < +\infty \end{cases}$$

因为 $f(0) = 0$, 可得 $c = -1$ 所以

$$f(x) = \begin{cases} x, & -\infty < t \leq 0 \\ e^x - 1, & 0 < t < +\infty \end{cases}$$

$$22. \text{ 已知 } f(x) = \frac{e^x}{x}, \text{ 求 } \int x f''(x) dx.$$

解:

$$\begin{aligned}
\int x f''(x) dx &= \int x df'(x) = x f'(x) - \int f'(x) dx \\
&= x f'(x) - \int df(x) = x f'(x) - f(x) + c = \left(\frac{e^x}{x}\right)' x - \frac{e^x}{x} + c.
\end{aligned}$$

练习题

1. $\int \frac{\ln x}{(1-x)^2} dx$
2. $\int \frac{dx}{\sin 2x + 2 \sin x}$
3. $\int \frac{\arctan x}{x^2(1+x^2)} dx$
4. $\int e^{2x}(\tan x + 1)^2 dx$
5. $\int \frac{dx}{x\sqrt{1+x^2}}$
6. $\int \frac{dx}{x + \sqrt{x^2 - 1}}$
7. $\int x^2 \sin^2 x dx$
8. $\int \frac{dx}{a \sin^2 x + b \cos^2 x}$
9. 设 $f(x^2 - 1) = \ln \frac{x^2}{x^2 - 2}$, 且 $f(\varphi(x)) = \ln x$, 求 $\int \varphi(x) dx$.
10. 设 $f(\ln x) = \frac{\ln(1+x)}{x}$, 计算 $\int f(x) dx$.