1. $\begin{cases} [(1-x^2)y']' + \lambda y = 0 & 0 < x < 1 \\ y'(0) = 0, |y(1)|\text{有界} \end{cases}$,

(1)求上述固有值问题的解;

(2)将$f(x) = 2x + 1$用固有函数系展开.

1. 解：

（1） 分析过程：此方程为 Legendre 方程，如果没有$y'(0) = 0$及$0 < x < 1$这两个条件，则方程的固有值为$\lambda = n(n+1)$，固有函数为 Legendre 多项式$P_n(x)$。加上限制条件之后，注意到方程的固有函数为 Legendre 多项式，其中含有$P_{2n+1}(x)$和$P_{2n}(x), n = 0,1,2,\ldots$

而$y'(0) = 0$，说明$P_n'(0) = 0$。而$P_{2n}'(0) = 0, {P_{2n+1}}'(0) \neq 0$。

故固有值为$\lambda = 2n(2n+1)$，固有函数为 Legendre 多项式$P_{2n}(x), n = 0,1,2,\ldots$

（2） $f(x) = 2x + 1, 0 < x < 1$

设

$$f(x) = \sum_{n=0}^{+\infty} C_n P_{2n}(x)$$

$$C_0 = 1$$

$n \neq 0$时候

$$C_n = \frac{\langle f(x), P_{2n}(x) \rangle}{\langle P_{2n}(x), P_{2n}(x) \rangle} = \frac{\int_0^1 f(x) P_{2n}(x) dx}{\int_0^1 {P_{2n}}^2(x) dx} = \frac{\int_0^1 (2x+1) P_{2n}(x) dx}{\int_0^1 {P_{2n}}^2(x) dx}$$

其中由

$$\int_{-1}^{1} {P_n}^2(x) dx = \frac{2}{2n+1}$$

$$\int_0^1 {P_n}^2(x) dx = \frac{1}{2n+1}$$

$$\int_0^1 {P_{2n}}^2(x) dx = \frac{1}{4n+1}$$

由递推公式得,

$$x P_{2n}(x) = \frac{2n+1}{4n+1} P_{2n+1}(x) + \frac{2n}{4n+1} P_{2n-1}(x)$$

$$P_{2n+1}(x) = \frac{P'_{2n+2}(x) - P'_{2n}(x)}{4n+3}$$

$$P_{2n-1}(x) = \frac{P'_{2n}(x) - P'_{2n-2}(x)}{4n+1}$$

$$\int_0^1 P_{2n+1}(x) dx = \frac{P_{2n+2}(1) - P_{2n+2}(0) - [P_{2n}(1) - P_{2n}(0)]}{4n+3}$$

$$P_{2n+2}(1) = P_{2n}(1) = 0$$

$$P_{2n}(0)=\frac{(-1)^n(2n-1)!!}{(2n)!!}$$

$$P_{2n+2}(0)=\frac{(-1)^{n+1}(2n+1)!!}{(2n+2)!!}$$

$$P_{2n}(0)-P_{2n+2}(0)=(-1)^n\frac{(4n+3)(2n-1)!!}{(2n+2)!!}$$

$$\int_0^1 P_{2n+1}(x)dx=(-1)^n\frac{(2n-1)!!}{(2n+2)!!}$$

$$\int_0^1 P_{2n-1}(x)dx=(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}$$

$$\int_0^1 xP_{2n}(x)dx=(-1)^n\left[\frac{(2n+1)!!}{(4n+1)(2n+2)!!}-\frac{2n}{4n+1}\frac{(2n-3)!!}{(2n)!!}\right]$$

$$=\frac{(-1)^{n-1}}{4n+1}\frac{(2n-3)!!}{(2n+2)!!}$$

$$\int_0^1 P_{2n}(x)dx=0$$

$$C_n=\frac{2\frac{(-1)^{n-1}}{4n+1}\frac{(2n-3)!!}{(2n+2)!!}}{\frac{1}{4n+1}}=2(-1)^{n-1}\frac{(2n-3)!!}{(2n+2)!!}$$

综上

$$f(x)=1+2\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n+2)!!}P_{2n}(x)$$

2. 求解初值问题

$$\begin{cases}u_{tt}=u_{xx}+f(x,t) & -\infty<x<+\infty,t>0\\ u|_{t=0}=3x^2,u_t|_{t=0}=0\end{cases}$$

(1)求$f(x,t)=0$时问题的解；（2）求$f(x,t)=\cos 2x+x^2t^2$时问题的解.

2. 解：

（1） $f(t,x)=0$时，由达朗贝尔公式

$$u(t,x)=\frac{3(x-t)^2+3(x+t)^2}{2}=3(x^2+t^2)$$

（2） 设$u(t,x)=v(t,x)+w(t,x)$

其中$v(t,x),w(t,x)$满足

$$\begin{cases}v_{tt}=v_{xx},-\infty<x<+\infty,t>0\\ v|_{t=0}=3x^2,v_t|_{t=0}=0\end{cases}$$

$$\begin{cases}w_{tt}=w_{xx}+cos2x+x^2t^2,-\infty<x<+\infty,t>0\\ v|_{t=0}=0,v_t|_{t=0}=0\end{cases}$$

则由（1）

$$v(t,x)=\frac{3(x-t)^2+3(x+t)^2}{2}=3(x^2+t^2)$$

由齐次化原理$W(t,\tau,x)$满足

$$\begin{cases} W_{tt} = W_{xx}, -\infty < x < +\infty, t > \tau \\ W|_{t=\tau} = 0, W_t|_{t=\tau} = cos2x + x^2\tau^2 \end{cases}$$

$$W(t,\tau,x) = \frac{1}{2}\int_{x-(t-\tau)}^{x+t-\tau} cos2\xi + \xi^2\tau^2\, d\xi$$

$$= \frac{sin2(t-\tau)cos2x}{2} + \frac{\tau^2}{3}x[4x^2 + x(t-\tau) - (t-\tau)^2]$$

$$w(t,x) = \int_0^t W(t,\tau,x)d\tau$$

$$= \int_0^t \frac{sin2(t-\tau)cos2x}{2} + \frac{\tau^2}{3}x[4x^2 + x(t-\tau) - (t-\tau)^2]d\tau$$

$$w(t,x) = cos2x\frac{1-cos2t}{4} + \frac{4}{9}x^3t^3 + \frac{x^2t^4}{36} - \frac{xt^5}{90}$$

综上

$$u(t,x) = 3(x^2+t^2) + cos2x\frac{1-cos2t}{4} + \frac{4}{9}x^3t^3 + \frac{x^2t^4}{36} - \frac{xt^5}{90}$$

3. 求解混合问题 $\begin{cases} u_{tt} = u_{xx} + e^{-x}, \quad 0 < x < 4, t > 0 \\ u|_{x=0} = u|_{x=4} = 0 \\ u|_{t=0} = \sin \pi x, u_t|_{t=0} = 0 \end{cases}$

3. 解：

设$u(t,x) = v(x) + w(t,x)$

其中$v(x), w(t,x)$满足

$$\begin{cases} 0 = v'' + e^{-x}, 0 < x < 4 \\ v(0) = v(4) = 0 \end{cases}$$

解得

$$v(x) = -e^{-x} + \frac{e^{-4}-1}{4}x + 1$$

$$\begin{cases} w_{tt} = w_{xx}, 0 < x < 4, t > 0 \\ w|_{x=0} = w|_{x=4} = 0 \\ w|_{t=0} = sin\pi x - \left(-e^{-x} + \frac{e^{-4}-1}{4}x + 1\right), w_t|_{t=0} = 0 \end{cases}$$

设$w(t,x) = T(t)X(x)$, 完成分离变量手续，有

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(4) = 0 \end{cases}$$

解得固有值$\lambda = \left(\frac{n\pi}{4}\right)^2$, 固有函数$X_n(x) = sin\frac{n\pi}{4}x, n = 1,2,3 \ldots$

$$\begin{cases} T''(t) + \lambda T(t) = 0 \\ T'(0) = 0 \end{cases}$$

$$T_n(t) = cos\frac{n\pi}{4}t, n = 1,2,3 \ldots$$

故

$$w(t,x)=\sum_{n=1}^{+\infty}A_n cos\frac{n\pi}{4}tsin\frac{n\pi}{4}x$$

$$w(0,x)=\sum_{n=1}^{+\infty}A_n sin\frac{n\pi}{4}x=sin\pi x-\left(-e^{-x}+\frac{e^{-4}-1}{4}x+1\right)$$

则有$A_4=1$

$n\neq 4$时

$$A_n=\frac{1}{2}\int_0^4\left[\left(e^{-x}-\frac{e^{-4}-1}{4}x-1\right)\right]sin\frac{n\pi}{4}xdx$$

$$\int_0^4 x\,sin\frac{n\pi}{4}xdx=-\frac{4}{n\pi}\int_0^4 xdcos\frac{n\pi}{4}x=(-1)^{n-1}\frac{16}{n\pi}$$

$$\int_0^4 sin\frac{n\pi}{4}xdx=\frac{4}{n\pi}[1-(-1)^n]$$

$$\int_0^4 e^{-x}\,sin\frac{n\pi}{4}xdx=-\frac{4}{n\pi}\int_0^4 e^{-x}dcos\frac{n\pi}{4}x=-\frac{4}{n\pi}\left[(-1)^n e^{-4}-1+\int_0^4 e^{-x}cos\frac{n\pi}{4}xdx\right]$$

$$\int_0^4 e^{-x}cos\frac{n\pi}{4}xdx=\frac{4}{n\pi}\int_0^4 e^{-x}dsin\frac{n\pi}{4}x=\frac{4}{n\pi}\int_0^4 e^{-x}\,sin\frac{n\pi}{4}xdx$$

$$\int_0^4 e^{-x}\,sin\frac{n\pi}{4}xdx=\frac{4}{n\pi}[1-(-1)^n e^{-4}]-\frac{16}{n^2\pi^2}\int_0^4 e^{-x}\,sin\frac{n\pi}{4}xdx$$

$$\int_0^4 e^{-x}\,sin\frac{n\pi}{4}xdx=\frac{4n\pi}{n^2\pi^2+16}[1-(-1)^n e^{-4}]$$

$$A_n=-32\frac{[1+(-1)^{n-1}e^{-4}]}{n\pi(n^2\pi^2+16)}$$

$$w(t,x)=cos\pi tsin\pi x-32\sum_{n=1,n\neq 4}^{+\infty}\frac{[1+(-1)^{n-1}e^{-4}]}{n\pi(n^2\pi^2+16)}cos\frac{n\pi}{4}tsin\frac{n\pi}{4}x$$

$$u(t,x)=cos\pi tsin\pi x-32\sum_{n=1,n\neq 4}^{+\infty}\frac{[1+(-1)^{n-1}e^{-4}]}{n\pi(n^2\pi^2+16)}cos\frac{n\pi}{4}tsin\frac{n\pi}{4}x-e^{-x}+\frac{e^{-4}-1}{4}x+1$$

4. 求解圆内热传导问题 $\begin{cases}u_t=u_{xx}+u_{yy}, \quad 0<r=x^2+y^2<1,t>0\\ u|_{r=1}=0,u|_{r=0}\text{有界}\\ u|_{t=0}=\varphi(r)=J_0(ar)+3J_0(br),0<a<b,J_0(a)=J_0(b)=0\end{cases}$

4. 解：

依题，利用极坐标系，有

$$\begin{cases}u_t=u_{rr}+\frac{1}{r}u_r\\ u|_{r=1}=0,u|_{r=0}\text{有界}\\ u|_{t=0}=\varphi(r)=J_0(ar)+3J_0(br),0<a<b\\ J_0(a)=J_0(b)=0\end{cases}$$

设$u(t,r)=T(t)R(r)$

完成分离变量手续，得

$$\begin{cases} r^2R''(r)+rR'(r)+\lambda r^2R(r)=0 \\ R(1)=0, R(0)\text{有界} \end{cases}$$

其固有值为$\lambda=\omega_n^2$，固有函数为$J_0(\omega_n r)$，其中ω_n为方程$J_0(r)=0$的正根。

$$T'(t)+\lambda T(t)=0$$

$$T_n(t)=e^{-\omega_n^2 t}$$

$$u(t,r)=\sum_{n=1}^{+\infty}A_n e^{-\omega_n^2 t}J_0(\omega_n r)$$

$$u(0,r)=\sum_{n=1}^{+\infty}A_n J_0(\omega_n r)=J_0(ar)+J_0(br)$$

依题，设a为$J_0(r)=0$的第i个正根，设b为$J_0(r)=0$的第j个正根，$1\le i<j$

则$A_i=1, A_j=3, A_n=0, (n\neq i,j)$

$$u(t,r)=e^{-a^2t}J_0(ar)+3e^{-b^2t}J_0(br)$$

5. (1)求解无界问题$\begin{cases} u_t=u_{xx}+2u_x+5u & -\infty<x<+\infty, t>0 \\ u|_{t=0}=\varphi(x) \end{cases}$

(2)求出当$\varphi(x)=4x$时方程的解。

5. 解：

1. 依题，用傅里叶变换，记其符号为F

$$\tilde{u}=\int_{-\infty}^{+\infty}u(t,x)e^{i\lambda x}dx$$

则有

$$\begin{cases} \dfrac{d\tilde{u}}{dt}=(-2i\lambda-\lambda^2+5)\tilde{u} \\ \tilde{u}|_{t=0}=F[\varphi(x)] \end{cases}$$

$$\tilde{u}=F[\varphi(x)]e^{(-2i\lambda-\lambda^2+5)t}$$

$$F^{-1}\left[e^{-(2i\lambda-\lambda^2+5)t}\right]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{(-2i\lambda-\lambda^2+5)t}e^{-i\lambda x}d\lambda=\frac{e^{5t}}{2\pi}\int_{-\infty}^{+\infty}e^{-t\lambda^2-i\lambda(2t+x)}d\lambda$$

$$=\frac{e^{5t}}{2\pi}\int_{-\infty}^{+\infty}exp\left\{-\left(\sqrt{t}\lambda+\frac{i(x+2t)}{2\sqrt{t}}\right)^2-\frac{(x+2t)^2}{4t}\right\}d\lambda=\frac{e^{5t-\frac{(x+2t)^2}{4t}}}{2}\sqrt{\frac{\pi}{t}}$$

$$u(t,x)=\varphi(x)*\frac{e^{-\frac{(x+2t)^2}{4t}+5t}}{2}\sqrt{\frac{\pi}{t}}=\frac{e^{5t}}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi+2t)^2}{4t}}d\xi$$

2. $\varphi(x)=4x$

$$u(t,x)=\frac{2e^{5t}}{\sqrt{\pi t}}\int_{-\infty}^{+\infty}\xi e^{-\frac{(x-\xi+2t)^2}{4t}}d\xi$$

令$\frac{x-\xi+2t}{2\sqrt{t}}=\eta$

$$\int_{-\infty}^{+\infty}\xi e^{\frac{(x-\xi+2t)^2}{4t}}d\xi=2\sqrt{t}\int_{-\infty}^{+\infty}\left(x+2t-2\sqrt{t}\eta\right)e^{-\eta^2}d\eta=2(x+2t)\sqrt{\pi t}$$

$$u(t,x)=4e^{5t}(x+2t)$$

6. $\begin{cases}\Delta_3 u=0, & r^2=x^2+y^2+z^2>9\\ u|_{r=3}=\cos 2\theta-4, & \lim\limits_{r\to\infty}u=2020\end{cases}$

6. 解：

依题，$u=u(r,\theta)$

则

$$\begin{cases}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial u}{\partial\theta}\right)=0\\ u|_{r=3}=cos2\theta-4,\ \lim\limits_{r\to+\infty}u=2020\end{cases}$$

设$u=R(r)\Theta(\theta)$

完成分离变量手续，有

$$\begin{cases}\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right)+\lambda=0\\ \Theta(0),\Theta(\pi)\text{有界}\end{cases}$$

其固有值为$\lambda=n(n+1)$, 固有函数为$P_n(cos\theta), n=0,1,2,\ldots$

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=0$$

$$R_n(r)=A_n r^{-(n+1)}+B_n r^n, n=0,1,2,\ldots$$

$$u(r,\theta)=\sum_{n=0}^{+\infty}\left(A_n r^{-(n+1)}+B_n r^n\right)P_n(cos\theta)$$

$$\lim_{r\to+\infty}u=2020\Rightarrow B_1=2020, B_n=0, n=1,2,3\ldots$$

$$u(r,\theta)=2020+\sum_{n=0}^{+\infty}A_n r^{-(n+1)}P_n(cos\theta)$$

$$u|_{r=3}=cos2\theta-4=\frac{4}{3}P_2(cos\theta)-\frac{13}{3}P_0(cos\theta)$$

$$2020+\sum_{n=0}^{+\infty}A_n 3^{-(n+1)}P_n(cos\theta)=\frac{4}{3}P_2(cos\theta)-\frac{13}{3}P_0(cos\theta)$$

所以

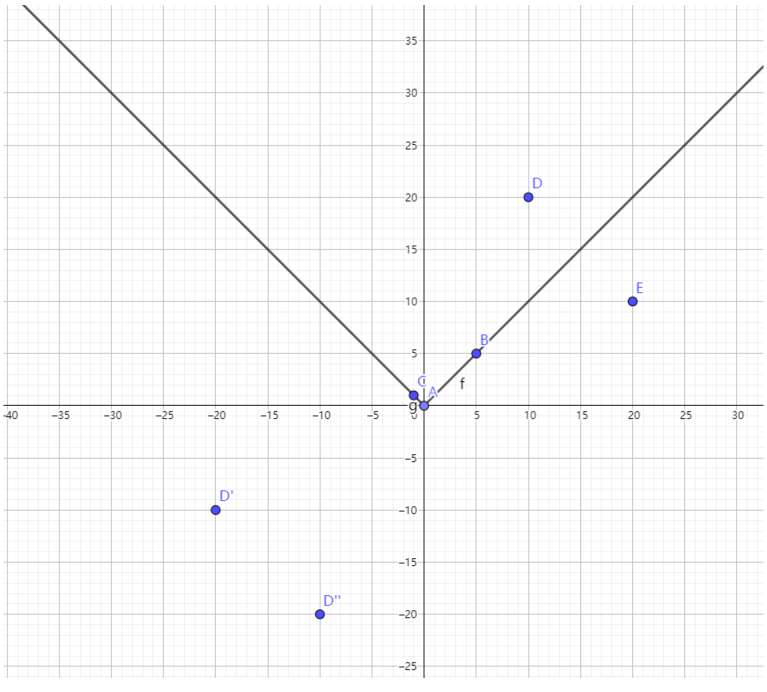
$$\begin{cases} \dfrac{A_2}{27} = \dfrac{4}{3} \\ 2020 + \dfrac{A_0}{3} = -\dfrac{13}{3} \\ A_n = 0\text{，其他} \end{cases}$$

$$\begin{cases} A_2 = 36 \\ A_0 = -6073 \\ A_n = 0\text{，其他} \end{cases}$$

$$u(r,\theta) = 2020 - \frac{6073}{r} + \frac{18}{r^3}(3cos^2\theta - 1)$$

7. 平面区域$D = \{(x,y)|y > |x| > 0\}$, 记$L$为$D$的边界，求$D$内Poisson方程第一边值问题的Green函数，并利用Green函数求解$\begin{cases} \Delta_2 u = 0 & (x,y) \in D \\ u|_L = \begin{cases} g(x), & x \geq 0 \\ 0, & x < 0 \end{cases} \end{cases}$

7. 解：如图



在D点(ξ,η)处D点位于所给平面区域内。放置一电荷量大小为ε_0的电荷，在E点(η,ξ)处放置一电荷量大小为$-\varepsilon_0$的电荷，保证了在$y = x, x > 0$的边界上电势为 0。此时$y = -x, x < 0$的边界上电势不为 0.

则需要在D'点$(-\eta,-\xi)$(D点关于$y = -x$对称点)处放置一电荷量大小为$-\varepsilon_0$的电荷，则需要在D''点$(-\xi,-\eta)$(E点关于$y = -x$对称点)处放置一电荷量大小为ε_0的电荷，

记$M(x,y)$则所求格林函数为

$$G = -\frac{1}{2\pi} ln \frac{r(M,D)r(M,D'')}{r(M,E)r(M,D')}$$

$$G=-\frac{1}{4\pi}\{ln[(x-\xi)^2+(y-\eta)^2]+ln[(x+\xi)^2+(y+\eta)^2]-ln[(x-\eta)^2+(y-\xi)^2]$$
$$-ln[(x+\eta)^2+(y+\xi)^2]\}$$
$$\frac{\partial G}{\partial x}=-\frac{1}{2\pi}\left[\frac{x-\xi}{(x-\xi)^2+(y-\eta)^2}+\frac{x+\xi}{(x+\xi)^2+(y+\eta)^2}-\frac{x-\eta}{(x-\eta)^2+(y-\xi)^2}\right.$$
$$\left.-\frac{x+\eta}{(x+\eta)^2+(y+\xi)^2}\right]$$
$$\frac{\partial G}{\partial y}=-\frac{1}{2\pi}\left[\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}+\frac{y+\eta}{(x+\xi)^2+(y+\eta)^2}-\frac{y-\xi}{(x-\eta)^2+(y-\xi)^2}\right.$$
$$\left.-\frac{y+\xi}{(x+\eta)^2+(y+\xi)^2}\right]$$

当边界为$y=x$时，单位外法向向量为$\frac{1}{\sqrt{2}}(1,-1)$

$$\frac{\partial G}{\partial n}=\left(\frac{\partial G}{\partial x},\frac{\partial G}{\partial y}\right)\cdot\frac{1}{\sqrt{2}}(1,-1)=\frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)$$
$$u(\xi,\eta)=-\int_{y=x}g(x)\frac{1}{\sqrt{2}}\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)|_{y=x}dl$$

依题，

$$dl=\sqrt{1+y'(x)^2}dx=\sqrt{2}dx$$
$$x:0\longrightarrow+\infty$$
$$\frac{\partial G}{\partial x}|_{y=x}=-\frac{\xi-\eta}{2\pi}\left[-\frac{1}{(x-\xi)^2+(x-\eta)^2}+\frac{1}{(x+\xi)^2+(x+\eta)^2}\right]$$
$$\frac{\partial G}{\partial y}|_{y=x}=-\frac{\xi-\eta}{2\pi}\left[\frac{1}{(x-\xi)^2+(x-\eta)^2}-\frac{1}{(x+\xi)^2+(x+\eta)^2}\right]$$
$$\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)|_{y=x}=-\frac{\xi-\eta}{\pi}\left[-\frac{1}{(x-\xi)^2+(x-\eta)^2}+\frac{1}{(x+\xi)^2+(x+\eta)^2}\right]$$
$$u(\xi,\eta)=-\int_0^{+\infty}g(x)\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)|_{y=x}dx$$
$$=\frac{\xi-\eta}{\pi}\int_0^{+\infty}g(x)\left[-\frac{1}{(x-\xi)^2+(x-\eta)^2}+\frac{1}{(x+\xi)^2+(x+\eta)^2}\right]dx$$