Divide and Conquer

- Polynomial Multiplication
- Integer Multiplication

Polynomial Multiplication

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add: O(n) arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate: O(n) using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

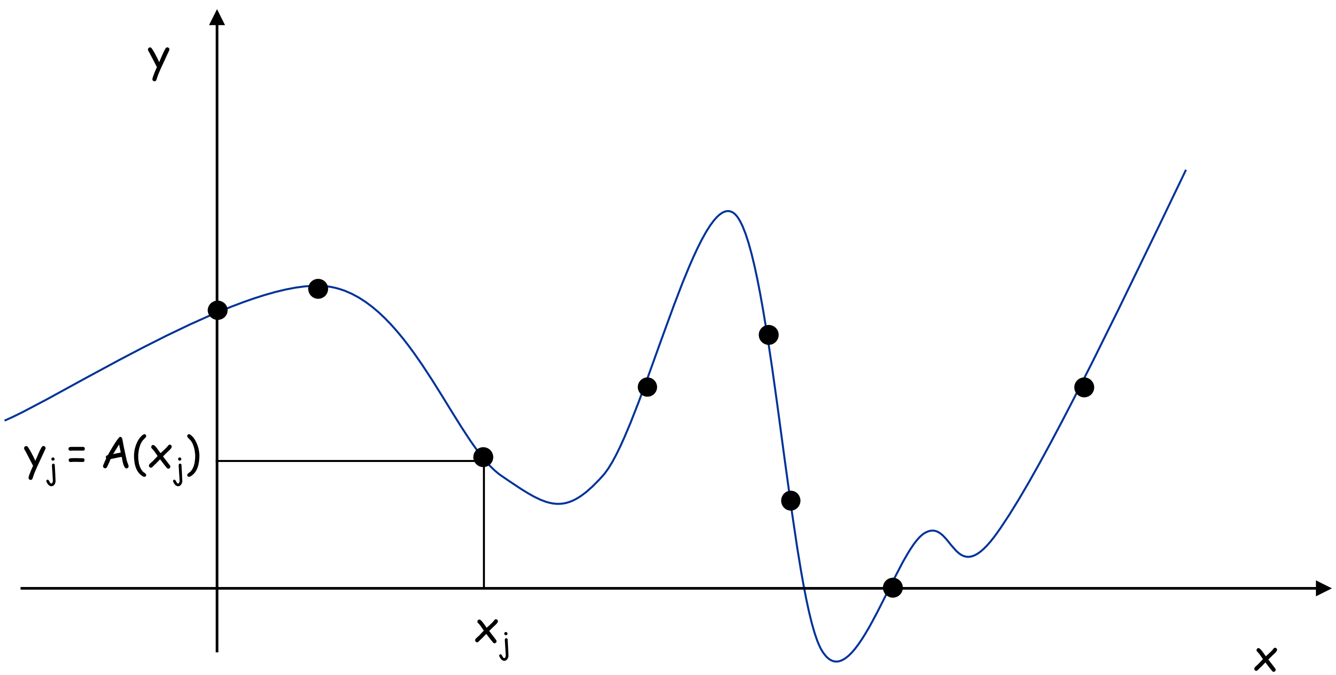
Multiply (convolve): $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

Polynomials: Point-Value Representation

Fundamental theorem of algebra. [Gauss, PhD thesis] A degree n polynomial with complex coefficients has n complex roots.

Corollary. A degree n-1 polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): \ (\mathrm{x}_0, y_0), \ \ldots, (\mathrm{x}_{\mathrm{n-1}}, y_{n-1})$$
$$B(x): \ (\mathrm{x}_0, \ z_0), \ \ldots, (\mathrm{x}_{\mathrm{n-1}}, z_{n-1})$$

Add: O(n) arithmetic operations.

$$A(x) + B(x): \ (\mathrm{x}_0, y_0 + z_0), \ldots, (\mathrm{x}_{\mathrm{n-1}}, y_{n-1} + z_{n-1})$$

Multiply: O(n), but need 2n-1 points.

$$A(x) \times B(x): \ (\mathrm{x}_0, y_0 \times z_0), \ldots, (\mathrm{x}_{\mathrm{2n-1}}, y_{2n-1} \times z_{2n-1})$$

Evaluate: O(n^2) using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

Representation	Multiply	Evaluate
Coefficient	$O(n^2)$	$O(n)$
Point-value	$O(n)$	$O(n^2)$

Goal. Make all ops fast by efficiently converting between two representations.

$$\boxed{a_0, a_1, \ldots, a_{n-1}} \rightleftarrows \boxed{(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})}$$

coefficient representation

point-value representation

Converting Between Two Polynomial Representations: Brute Force

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$O(n^2)$ for matrix-vector multiply

$O(n^3)$ for Gaussian elimination

Vandermonde matrix is invertible iff x_i distinct

Point-value to coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) \;\;= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(\ x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose two points to be ± 1.

- $A(\ 1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$

Can evaluate polynomial of degree $\leq$ n at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}$n at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A(x) \;=\; a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7.$
- $A_{even}(x) \;=\; a_0 + a_2x + a_4x^2 + a_6x^3.$
- $A_{odd}(x) \;=\; a_1 + a_3x + a_5x^2 + a_7x^3.$
- $A(\ x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose four points to be $\pm 1, \pm i$.

- $A(\ 1) = A_{even}(\ 1) + 1\, A_{odd}(\ 1).$
- $A(-1) = A_{even}(\ 1) - 1\, A_{odd}(\ 1).$
- $A(\ i) = A_{even}(-1) + i\, A_{odd}(-1).$
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1).$

> Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 points.

Discrete Fourier Transform

Coefficient to point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea: choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{\text{Discrete Fourier transform}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}}_{\text{Fourier matrix } F_n} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Roots of Unity

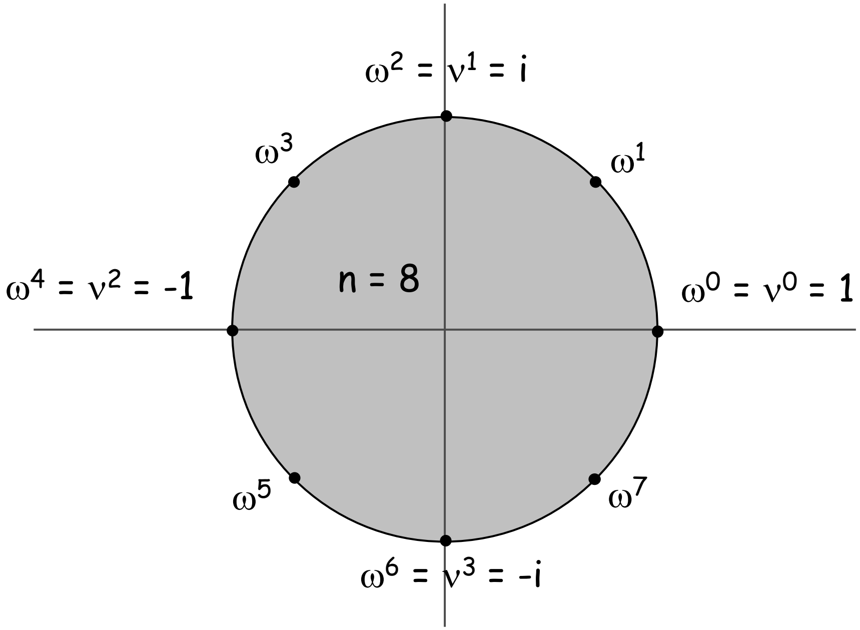
Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = e^{4\pi i / n}$.

Fact. $\omega^2 = \nu$ and $(\omega^2)^k = \nu^k$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1} x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break polynomial up into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n/2-2} x^{(n-1)/2}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n/2-1} x^{(n-1)/2}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate degree $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+n}) = A_{even}(\nu^k) - \omega^k A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\omega^{k+\frac{1}{2}n} = -\omega^k$

$\nu^k = (\omega^k)^2 = (\omega^{k+\frac{1}{2}n})^2$

FFT Algorithm

```
fft(n, a₀,a₁,…,aₙ₋₁) {
   if (n == 1) return a₀

   (e₀,e₁,…,eₙ/₂₋₁) ← FFT(n/2, a₀,a₂,a₄,…,aₙ₋₂)
   (d₀,d₁,…,dₙ/₂₋₁) ← FFT(n/2, a₁,a₃,a₅,…,aₙ₋₁)

   for k = 0 to n/2 - 1 {
      ωᵏ ← e^(2πik/n)
      yₖ     ← eₖ + ωᵏ dₖ
      yₖ₊ₙ/₂ ← eₖ - ωᵏ dₖ
   }

   return (y₀,y₁,…,yₙ₋₁)
}
```

FFT Summary

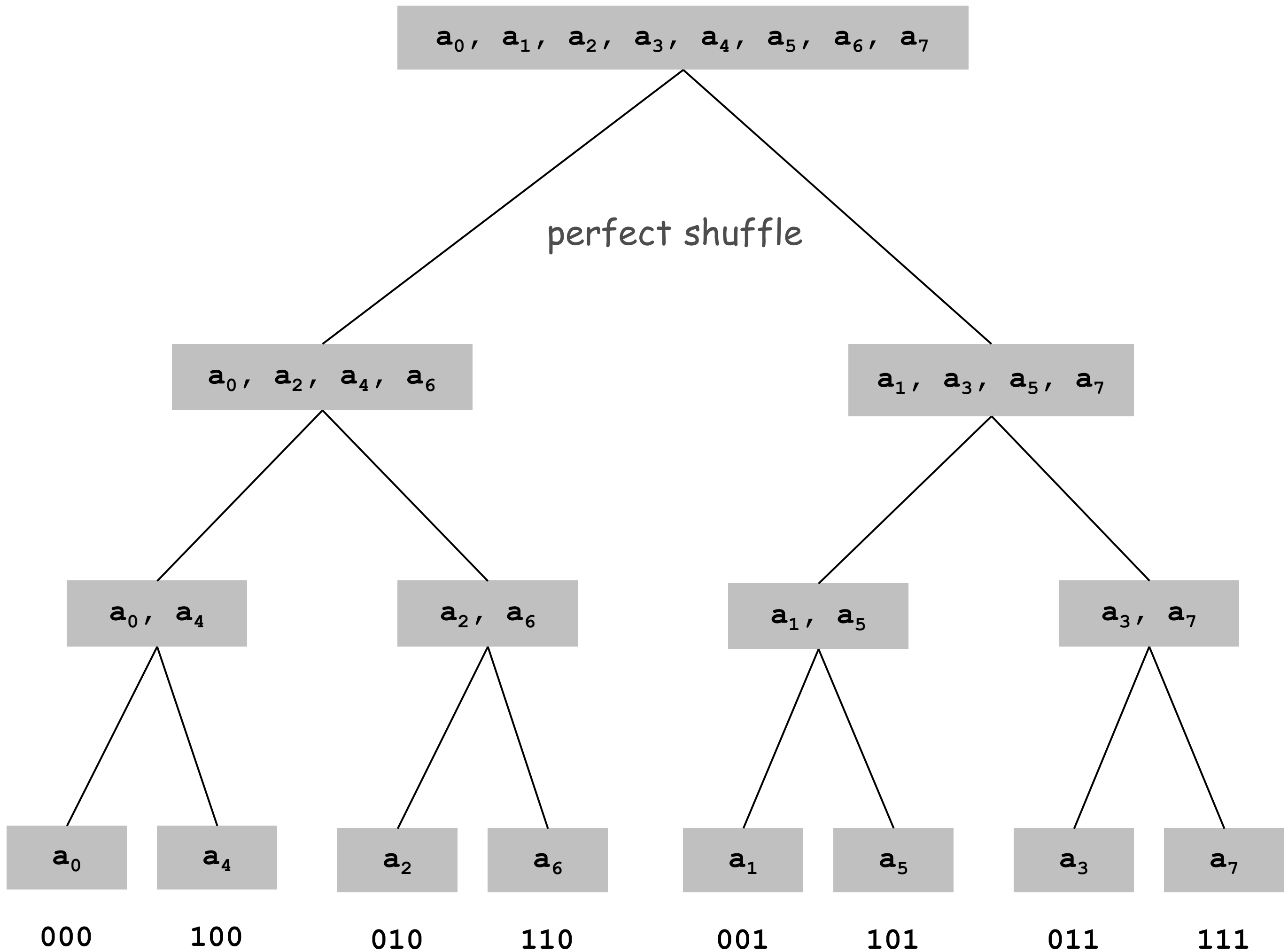
Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

Running time. $T(2n) = 2T(n) + O(n) \Rightarrow T(n) = O(n \log n)$.

$$\underbrace{a_0, a_1, \ldots, a_{n-1}}_{\text{coefficient representation}} \quad \overset{O(n \log n)}{\rightleftarrows} \quad \underbrace{(\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})}_{\text{point-value representation}}$$

Recursion Tree



"bit-reversed" order

Point-Value to Coefficient Representation: Inverse DFT

Goal. Given the values $y_0, \ldots, y_{n-1}$ of a degree n-1 polynomial at the n points $\omega^0, \omega^1, \ldots, \omega^{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

Inverse FFT

Claim. Inverse of Fourier matrix is given by following formula.

$$G_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n\, G_n\right)_{k\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{k\,j}\, \omega^{-j\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{k\,j} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \Rightarrow$ sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \Rightarrow$ sums to 0. ▪

Inverse FFT: Algorithm

```
ifft(n, a₀,a₁,…,aₙ₋₁) {
   if (n == 1) return a₀

   (e₀,e₁,…,eₙ/₂₋₁) ← FFT(n/2, a₀,a₂,a₄,…,aₙ₋₂)
   (d₀,d₁,…,dₙ/₂₋₁) ← FFT(n/2, a₁,a₃,a₅,…,aₙ₋₁)

   for k = 0 to n/2 - 1 {
      ωᵏ ← e^(-2πik/n)
      yₖ     ← (eₖ + ωᵏ dₖ) / n
      yₖ₊ₙ/₂ ← (eₖ - ωᵏ dₖ) / n
   }

   return (y₀,y₁,…,yₙ₋₁)
}
```

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in O(n log n) steps.

assumes n is a power of 2

$$a_0, a_1, \ldots, a_{n-1} \underset{O(n \log n)}{\overset{O(n \log n)}{\rightleftarrows}} (\omega^0, y_0), \ldots, (\omega^{n-1}, y_{n-1})$$

coefficient representation

point-value representation

Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in O(n log n) steps.

coefficient representation

$a_0, a_1, \ldots, a_{n-1}$
$b_0, b_1, \ldots, b_{n-1}$

FFT O(n log n)

$A(x_0), \ldots, A(x_{2n-1})$
$B(x_0), \ldots, B(x_{2n-1})$

point-value multiplication

O(n)

$C(x_0), C(x_1), \ldots, C(x_{2n-1})$

inverse FFT O(n log n)

coefficient representation

$c_0, c_1, \ldots, c_{2n-2}$

Integer Multiplication

Integer Multiplication

Integer multiplication. Given two n bit integers $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$, compute their product $c = a \times b$.

Convolution algorithm.

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

- Form two polynomials.
- Note: $a = A(2)$, $b = B(2)$.
- Compute $C(x) = A(x) \times B(x)$.
- Evaluate $C(2) = a \times b$.
- Running time: $O(n \log n)$ complex arithmetic operations.

Theory. [Schönhage-Strassen 1971] $O(n \log n \log \log n)$ bit operations.
Theory. [Fürer 2007] $O(n \log n \, 2^{O(\log^* n)})$ bit operations.

References

References

- Section 5.6 of the text book “algorithm design” by Jon Kleinberg and Eva Tardos
- The original slides were prepared by Kevin Wayne. The slides are distributed by Pearson Addison-Wesley.