Divide and Conquer

- Mergesort
- Closest Pair
- Integer Multiplication
- Matrix Multiplication
- Maximum Sum Subarray

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}$n.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: n log n.

Maximum Sum Subarray

Maximum Sum Subarray

Problem: Given a one dimensional array A[1..n] of numbers. Find a contiguous subarray with largest sum within A.

Assume an empty subarray has sum 0.

Example:

4	-7	12	5	-2	3	-5	1	5	-8	2	5

4	-7	12	5	-2	3	-5	1	5	-8	2	5

Algorithm (brute-force)

Observation: Let S[i] = A[1]+…+A[i]. We have A[i]+…+A[j]=S[j]-S[i-1]

```
Pre-Processing
S[0] = 0
for i = 1 to n do
    S[i] = S[i-1]+A[i]
```

Running time of pre-processing: $T(n) = O(n)$

```
sol = 0
for i = 1 to n do
   for j = i to n do
        if S[j]-S[i-1] > sol then
           sol = S[j]-S[i-1]
return sol
```

Running time: $T(n) = O(n^2)$

Algorithm (divide and conquer)

The general strategy: Divide into 2 equal-size subarrays

Case 1: optimal solution is in one subarray

Case 2: optimal solution crosses the splitting line

4	-7	12	5	-2	3	-5	1	5	-8	2	5

```
MCS(A[1..n])
if n = 1 then return max(0, a[1])
sol = max(MCS(A[1…n/2]), MCS(A[n/2+1…n])
Lsol = 0
for i = n/2 downto 1 do
    if S[n/2]-S[i-1] > Lsol then
        Lsol = S[n/2]-S[i-1]
Rsol = 0
for i = n/2+1 to n do
    if S[i]-S[n/2-1] > solR then
        Rsol = S[i]-S[n/2-1]
return max(sol, Lsol+Rsol)
```

Running time: $T(n) = 2T\left(\frac{n}{2}\right) + O(n) \rightarrow T(n) = O(n \log n)$

Mergesort

Sorting

Sorting. Given n elements, rearrange in ascending order.

Applications.

- Sort a list of names.
- Organize an MP3 library.
- Find the median.
- Find the closest pair.
- Binary search in a database.
- Find duplicates in a mailing list.
- . . .

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

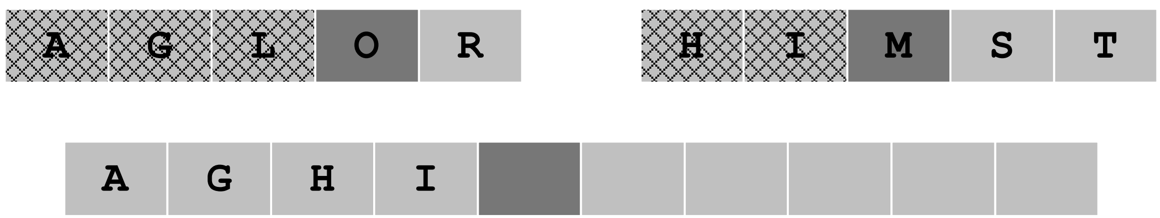
A	L	G	O	R	I	T	H	M	S		
A	L	G	O	R	I	T	H	M	S	divide	$O(1)$
A	G	L	O	R	H	I	M	S	T	sort	$2T(n/2)$
A	G	H	I	L	M	O	R	S	T	merge	$O(n)$

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

- Linear number of comparisons.
- Use temporary array.



Challenging version. In-place merge

using only a constant amount of extra storage

A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

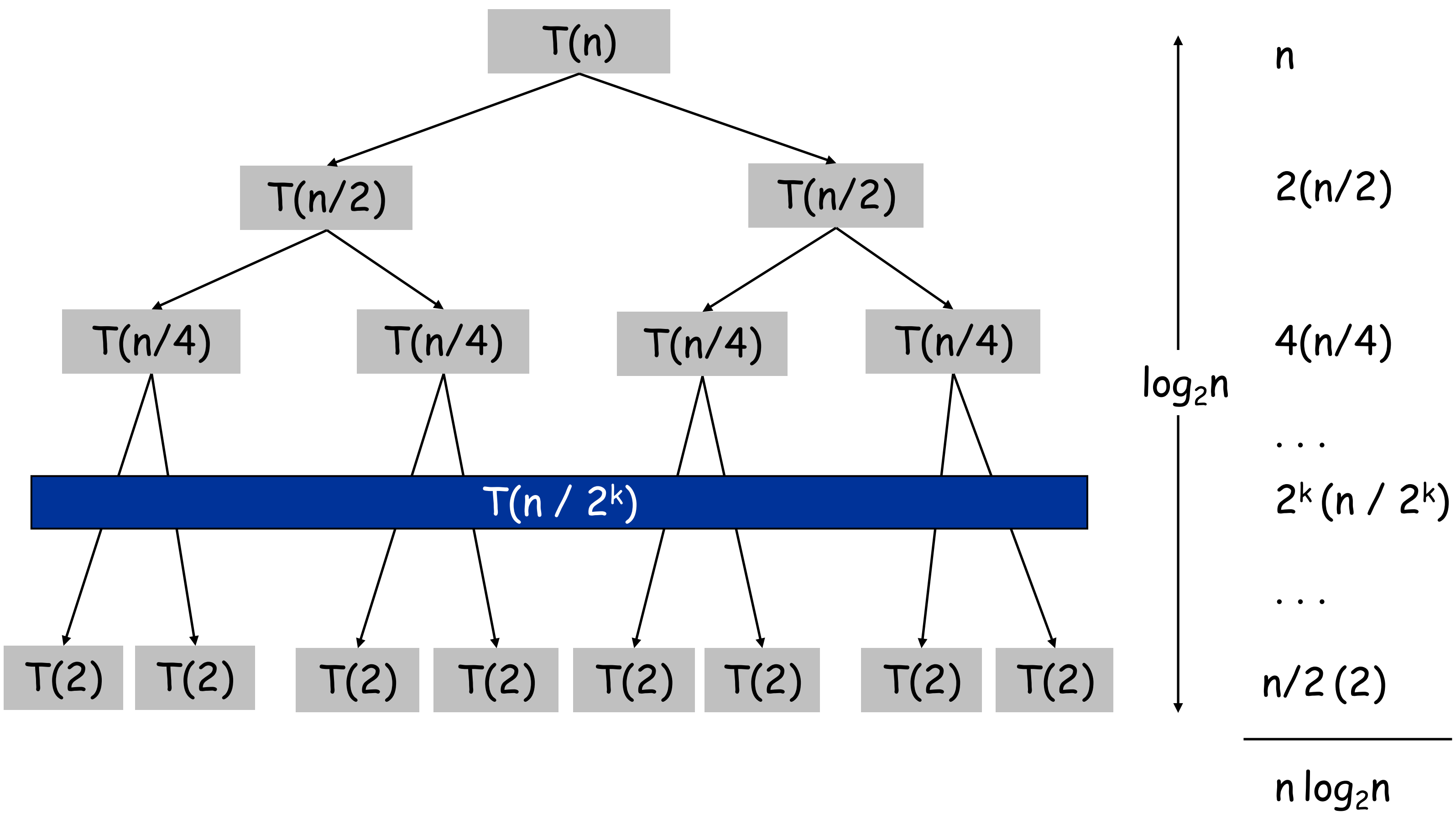
Merge sort recurrence.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

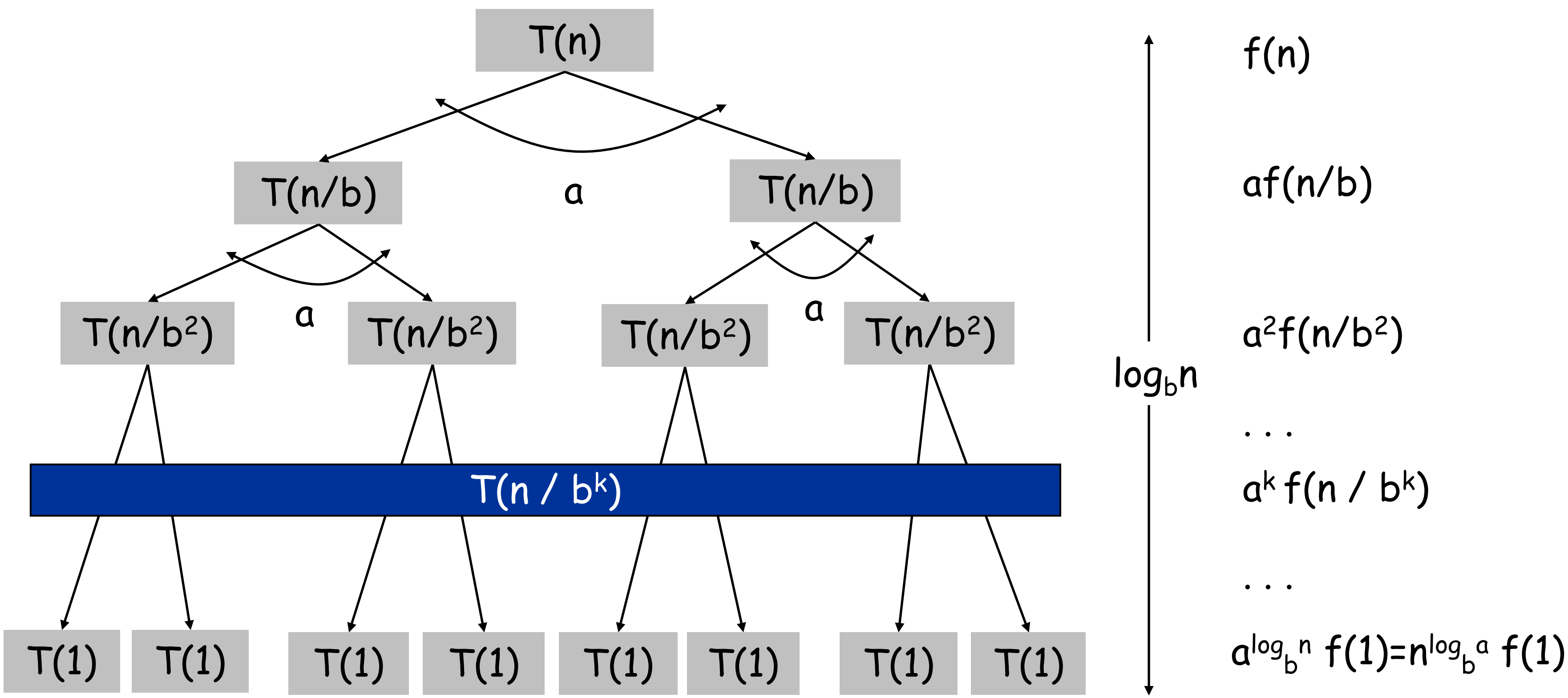
Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{aT(n/b)}_{\text{sub-problems}} + \underbrace{f(n)}_{\text{merging}} & \text{otherwise} \end{cases}$$



$\Sigma a^k f(n / b^k)$

#leaves = $n^{\log_b a}$

Mergesort

What happen if we divide the array into more sub-problems?.

- We have to find the minimum among a numbers in the merging step.
- So,

$$T(n)=aT(n/a)+an$$

- It is easy to see T(n) is minimum when a=2

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

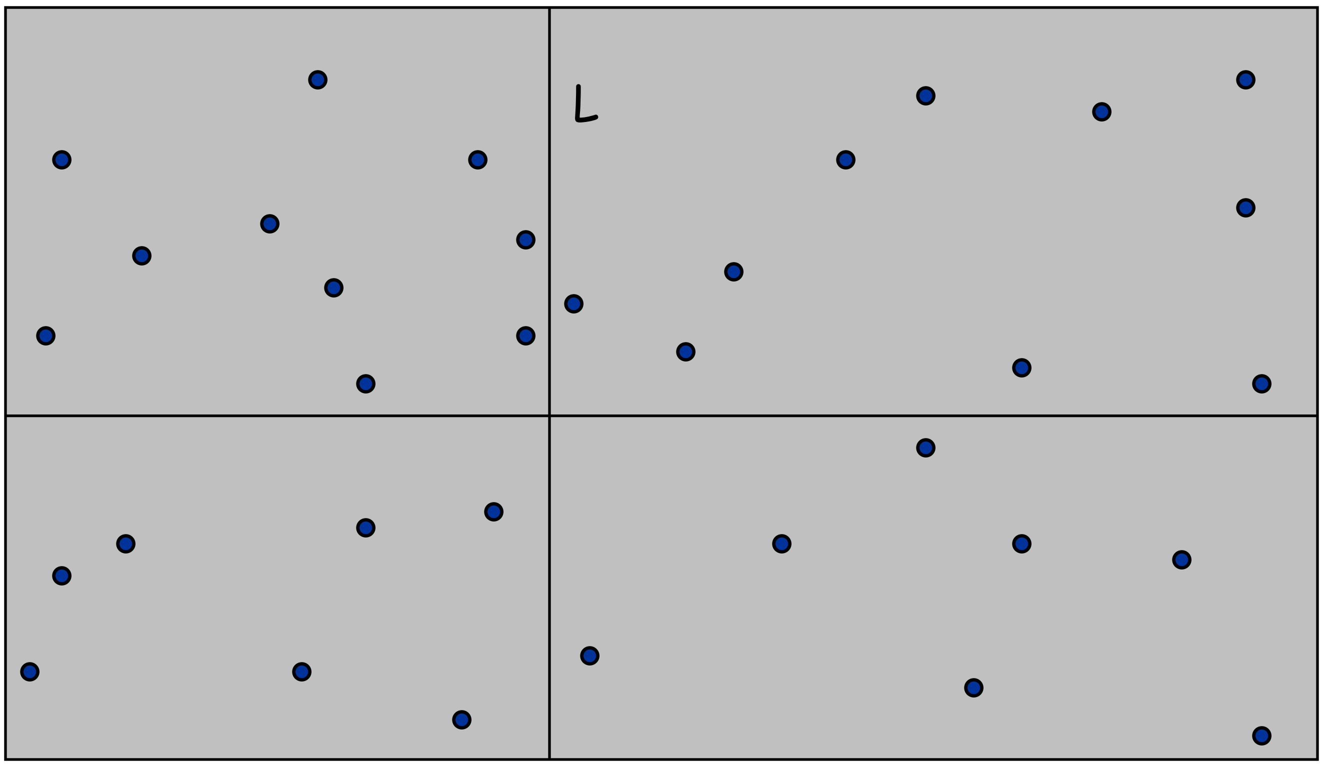
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

to make presentation cleaner

Closest Pair of Points: First Attempt

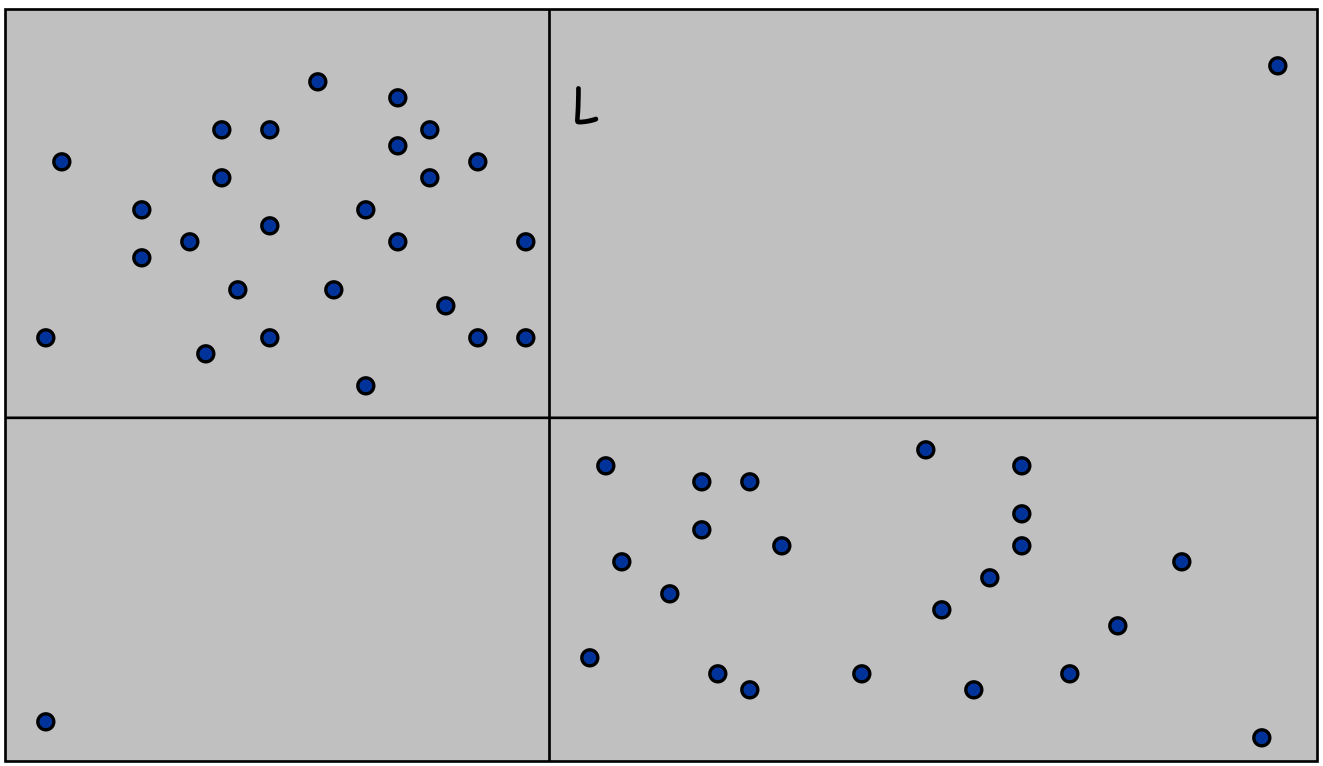
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

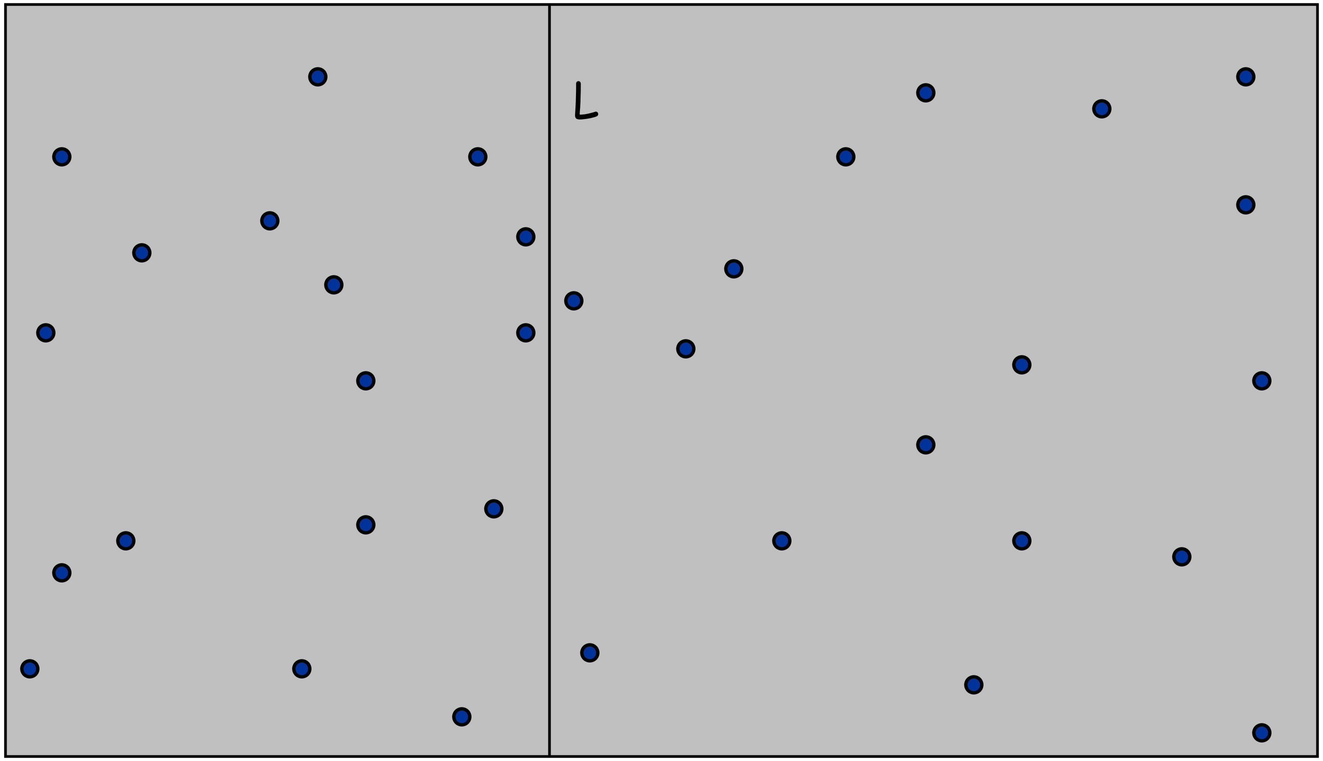
Divide. Sub-divide region into 4 quadrants.

Obstacle. Impossible to ensure n/4 points in each piece.



Algorithm.

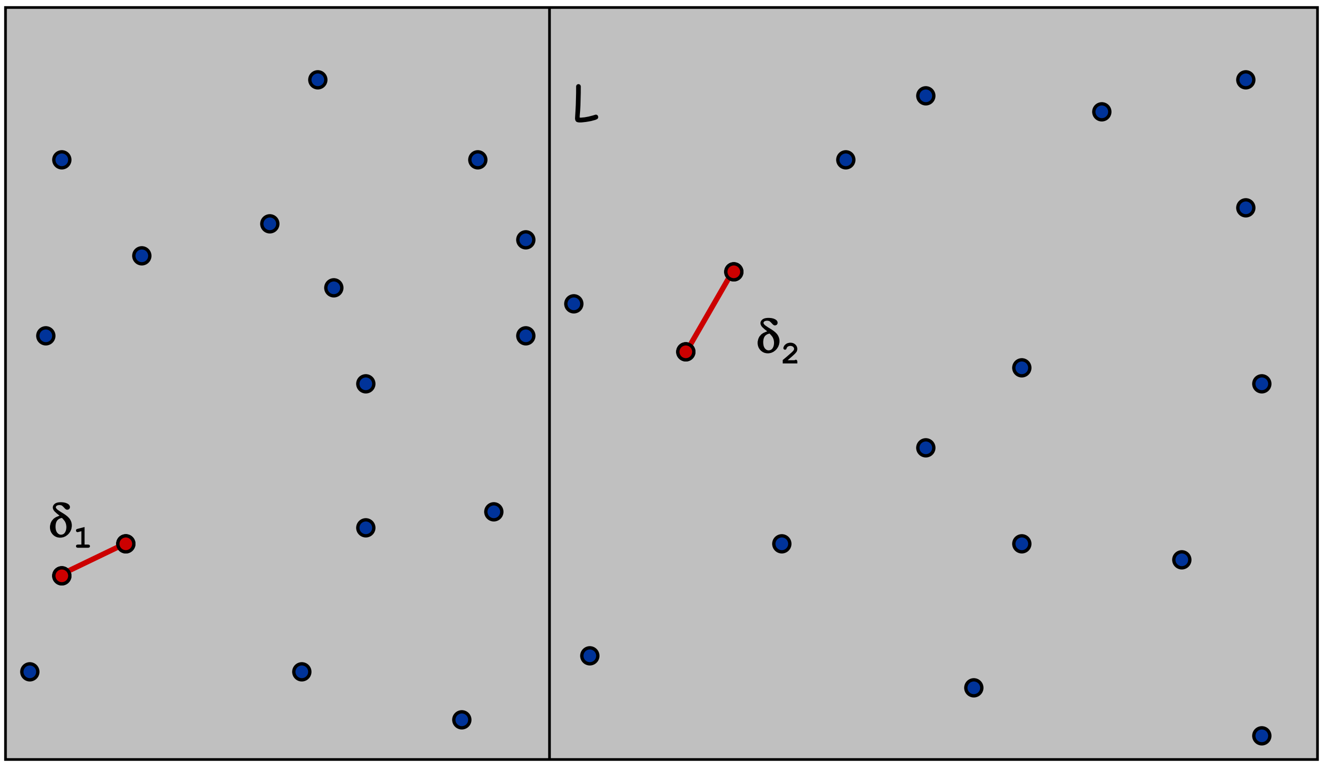
- Divide: draw vertical line L so that roughly ½n points on each side.



Closest Pair of Points

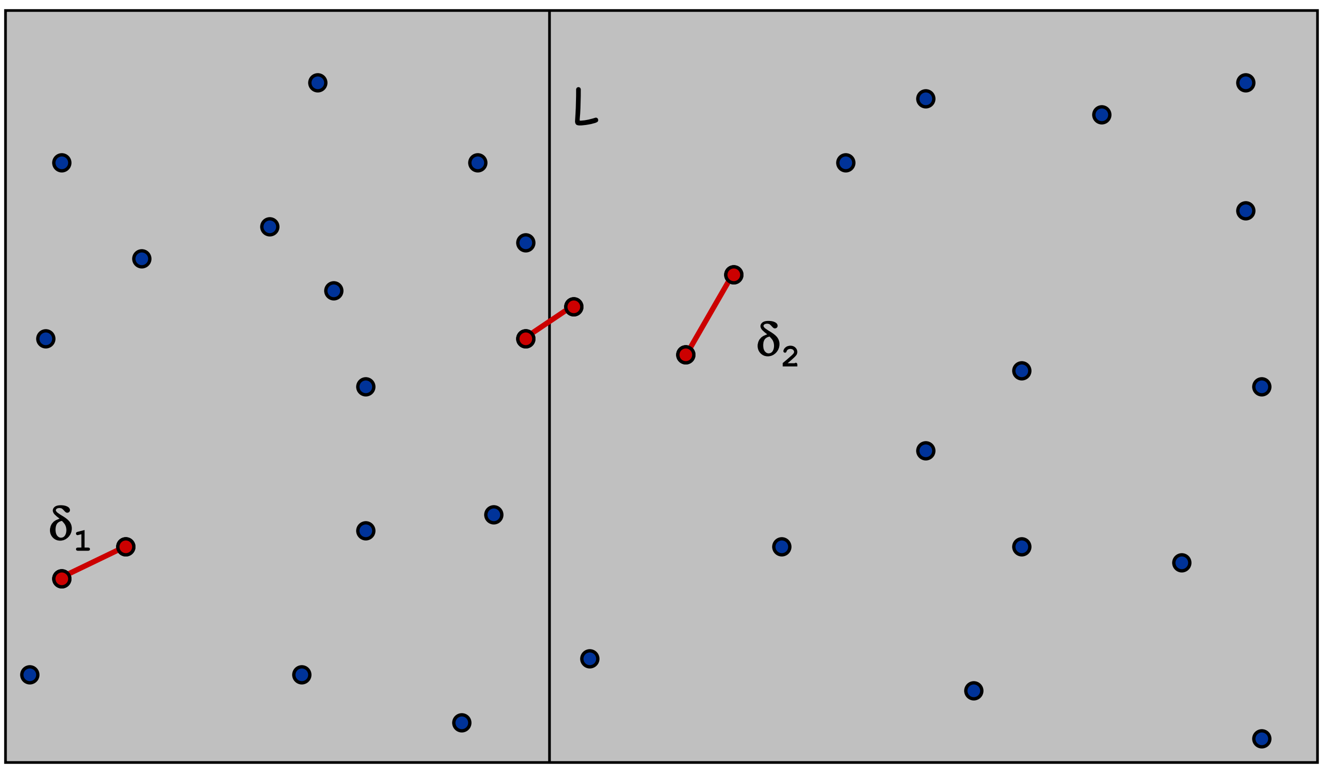
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



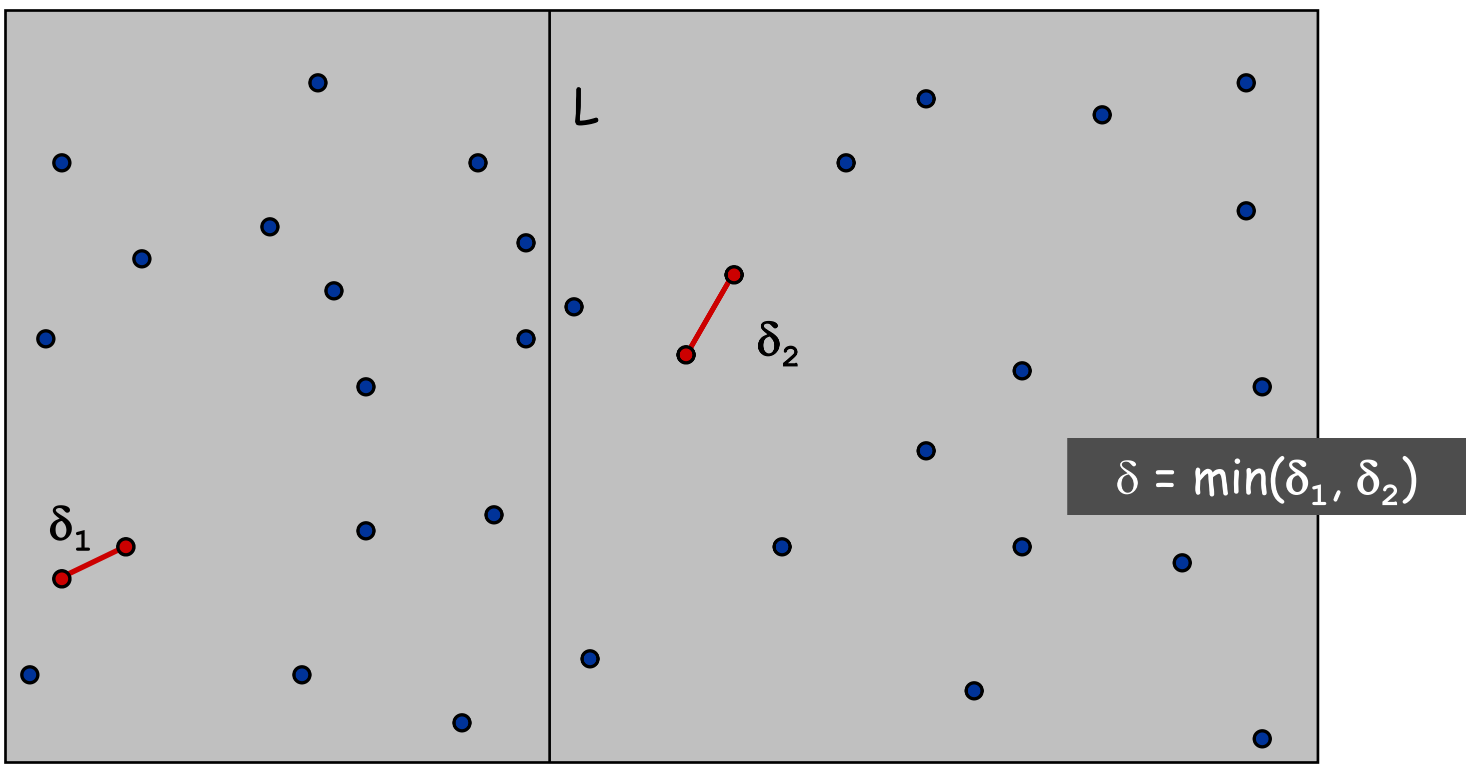
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

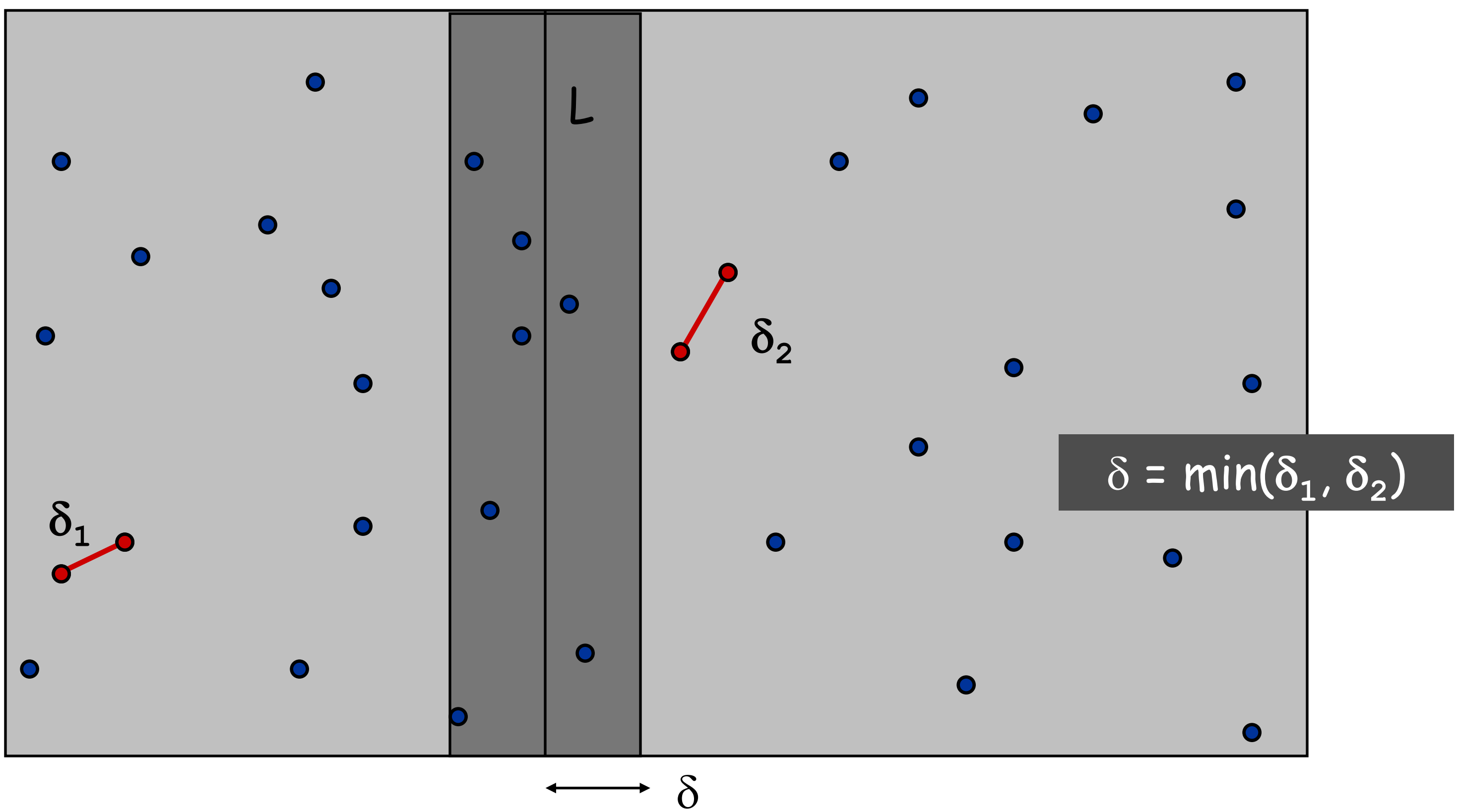
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

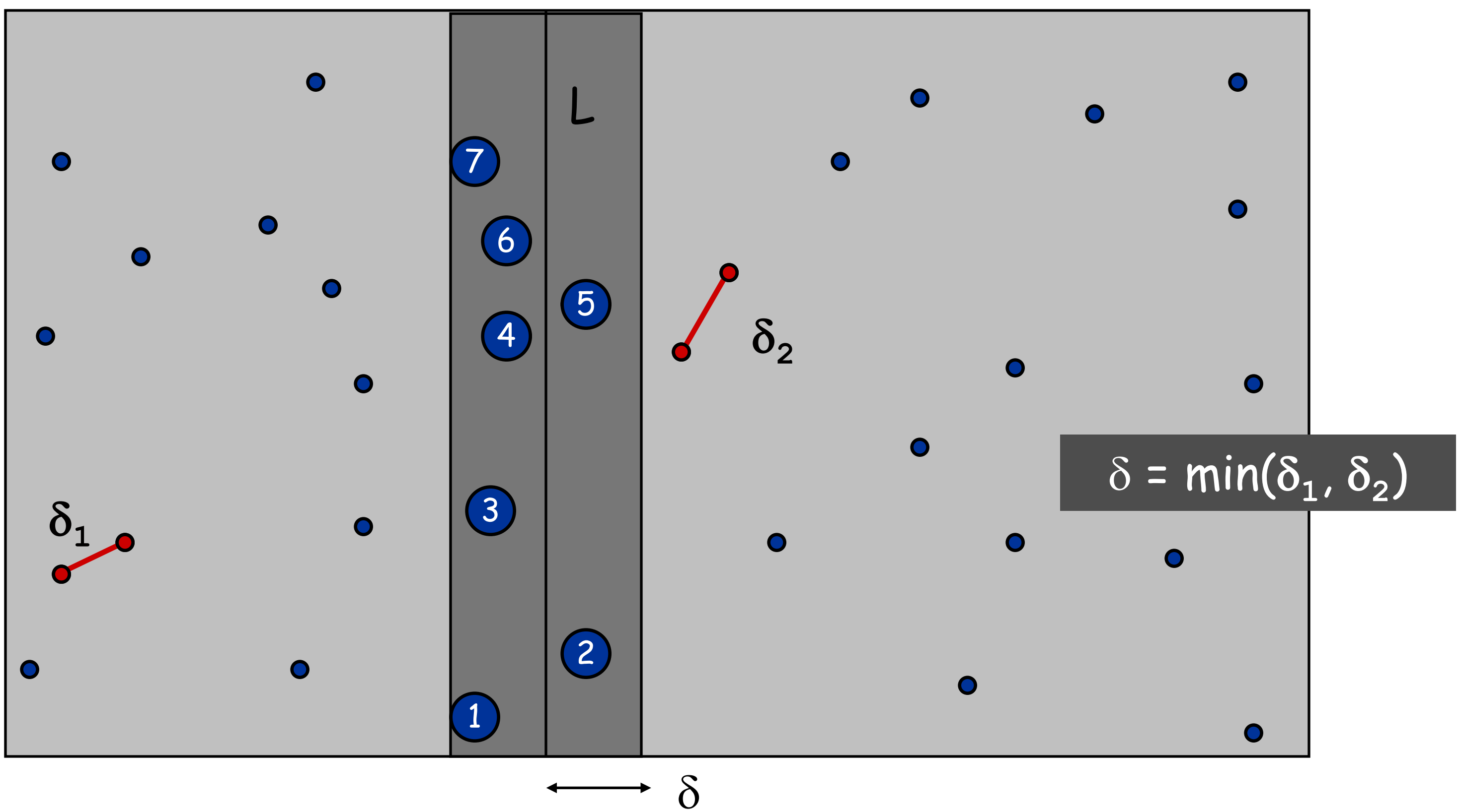
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

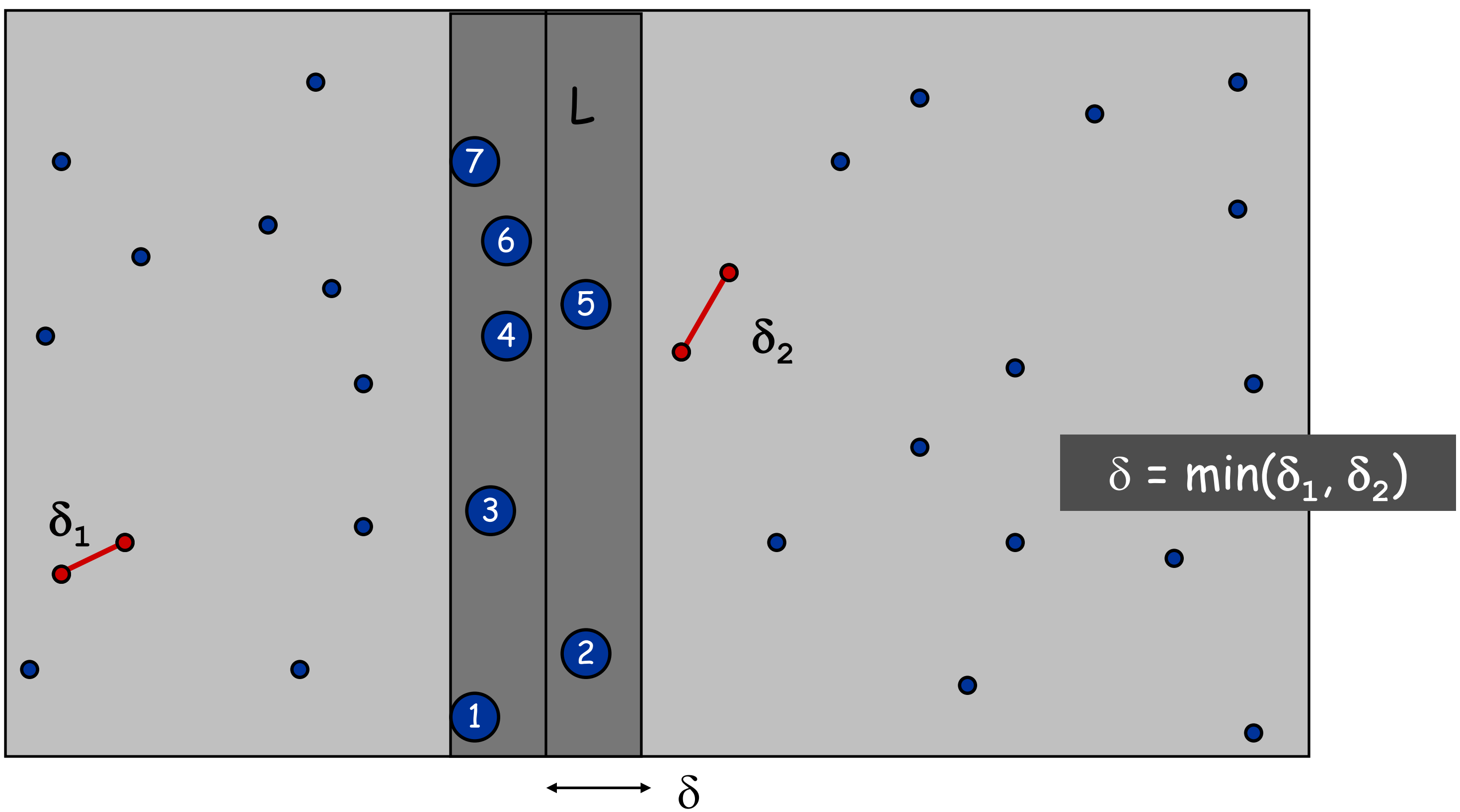
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

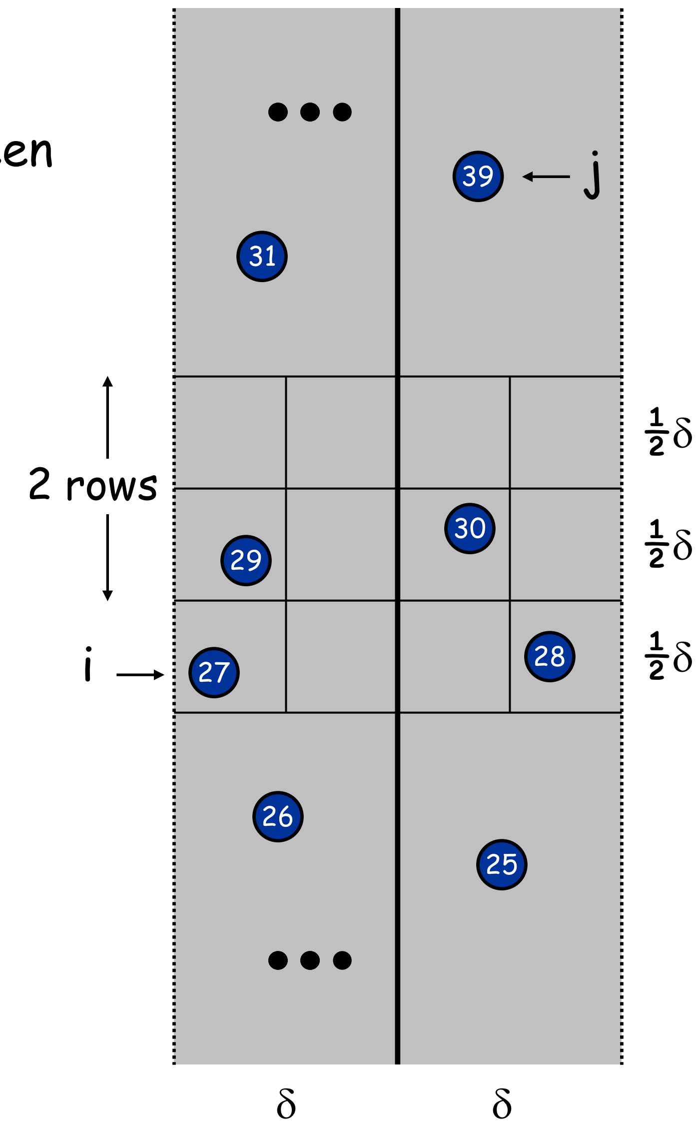
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

$O(n)$

$2T(n / 2)$

$O(n)$

$O(n \log n)$

$O(n)$

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \;\Rightarrow\; T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \;\Rightarrow\; T(n) = O(n \log n)$$

Master Theorem

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{aT(n/b)}_{\text{sub-problems}} + \underbrace{f(n)}_{\text{merging}} & \text{otherwise} \end{cases}$$

You can imagine above as a recursive function which calls itself: a times, each with an input of size n/b, and merge their outputs in f(n) time.

Fighting between #leaves and f(n)

- If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
- If $f(n)$ polynomially greater than $n^{\log_b a}$, then $T(n) = \Theta(f(n))$
- If $n^{\log_b a}$ polynomially greater than $f(n)$, then $T(n) = \Theta(n^{\log_b a})$

Note. The total input injecting to sub-problems is (a/b)n. Then if a/b is smaller, your running time is better.

A Note on Master Theorem

Consider the following recurrent equation:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$c^* = \log_b^a$$

- Case I: $f(n) = O(n^c)$ and $c < c^*$ then $T(n) = \Theta\left(n^{c^*}\right)$
- Case II: $f(n) = \Theta\left(n^{c^*} \log^k n\right)$ then $T(n) = \Theta\left(n^{c^*} \log^{k+1} n\right)$
- Case III: $f(n) = \Omega(n^c)$ and $c > c^*$ then $T(n) = \Theta\big(f(n)\big)$ if

$af\left(\frac{n}{b}\right) \leq kf(n)$

Integer Multiplication

Integer Addition

Addition. Given two n-bit integers a and b, compute $a + b$.
Grade-school. $\Theta(n)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Remark. Grade-school addition algorithm is optimal.

Integer Multiplication

Multiplication. Given two n-bit integers a and b, compute $a \times b$.

Grade-school. $\Theta(n^2)$ bit operations.

```
                1 1 0 1 0 1 0 1
              × 0 1 1 1 1 1 0 1
              -----------------
                1 1 0 1 0 1 0 1
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
-------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1
```

Q. Is grade-school multiplication algorithm optimal?

Divide-and-Conquer Multiplication: Warmup

To multiply two n-bit integers a and b:
- Multiply four $\frac{1}{2}n$-bit integers, recursively.
- Add and shift to obtain result.

$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= \left(2^{n/2} \cdot a_1 + a_0\right)\left(2^{n/2} \cdot b_1 + b_0\right) = 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0
\end{aligned}
$$

Ex. $a = \underbrace{1000}_{a_1}\underbrace{1101}_{a_0}$ $\quad b = \underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$

$$
T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)
$$

Karatsuba Multiplication

To multiply two n-bit integers a and b:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= 2^{n} \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^{n} \cdot \underbrace{a_1 b_1}_{1} + 2^{n/2} \cdot \left(\underbrace{(a_1 + a_0)(b_1 + b_0)}_{2} - \underbrace{a_1 b_1}_{1} - \underbrace{a_0 b_0}_{3}\right) + \underbrace{a_0 b_0}_{3}
\end{aligned}$$

Karatsuba Multiplication

To multiply two n-bit integers a and b:

- Add two $\frac{1}{2}n$ bit integers.
- Multiply three $\frac{1}{2}n$-bit integers, recursively.
- Add, subtract, and shift to obtain result.

$$
\begin{aligned}
a &= 2^{n/2} \cdot a_1 + a_0 \\
b &= 2^{n/2} \cdot b_1 + b_0 \\
ab &= 2^n \cdot a_1 b_1 + 2^{n/2} \cdot \left(a_1 b_0 + a_0 b_1\right) + a_0 b_0 \\
&= 2^n \cdot \underset{1}{a_1 b_1} + 2^{n/2} \cdot \left(\underset{2}{(a_1 + a_0)(b_1 + b_0)} - \underset{1}{a_1 b_1} - \underset{3}{a_0 b_0}\right) + \underset{3}{a_0 b_0}
\end{aligned}
$$

$$
T(n) \le \underbrace{T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + T\left(1 + \lceil n/2 \rceil\right)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \Rightarrow T(n) = O(n^{\lg 3}) = O(n^{1.585})
$$

Matrix Multiplication

Dot Product

Dot product. Given two length n vectors a and b, compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i \, b_i$$

$$a = \begin{bmatrix} .70 & .20 & .10 \end{bmatrix}$$
$$b = \begin{bmatrix} .30 & .40 & .30 \end{bmatrix}$$
$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Remark. Grade-school dot product algorithm is optimal.

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

$$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm optimal?

Block Matrix Multiplication

C_{11} A_{11} A_{12} B_{11}

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

B_{11}

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix Multiplication: Warmup

To multiply two n-by-n matrices A and B:

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Strassen's Algorithm

Key idea. multiply 2-by-2 blocks with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned}
P_1 &= A_{11} \times (B_{12} - B_{22}) \\
P_2 &= (A_{11} + A_{12}) \times B_{22} \\
P_3 &= (A_{21} + A_{22}) \times B_{11} \\
P_4 &= A_{22} \times (B_{21} - B_{11}) \\
P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\
P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12})
\end{aligned}$$

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}$$

- 7 multiplications.
- $18 = 8 + 10$ additions and subtractions.

Strassen's algorithm works for any ring, such as plus/multiply, but not all semirings, such as min-plus or boolean algebra, where the naive algorithm still works.

Strassen's Algorithm

To multiply two n-by-n matrices A and B

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Compute: 14 $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices via 10 matrix additions.
- Conquer: multiply 7 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- $T(n)$ = # arithmetic operations.

$$T(n) = \underbrace{7\,T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Fast Matrix Multiplication: Theory

Better results:

- $O(n^{2.376})$ [Coppersmith-Winograd, 1987]
- $O(n^{2.374})$ [Stothers, 2010]
- $O(n^{2.3728642})$ [Virginia Vassilevska Williams,2011]
- $O(n^{2.3728639})$ [François Le Gall, 2014]

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. Theoretical improvements to Strassen are progressively less practical.

References

References

- Sections 5.1, 5.2, 5.4, and 5.5 of the text book "algorithm design" by Jon Kleinberg and Eva Tardos
- Section 4.1 of the text book "introduction to algorithms" by CLRS, 3rd edition.
- The original slides were prepared by Kevin Wayne. The slides are distributed by Pearson Addison-Wesley.