Completeness of exponentially increasing sequences

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Abstract

For fixed positive reals α and t, consider the sequence $S_t(\alpha) = (s_1, s_2, \dots,)$ with $s_n = \lfloor t\alpha^n \rfloor$. In 1964, Graham managed to characterize those pairs (α, t) with $1 < \alpha < 2$ and $0 < t < 1$ for which every large enough integer can be written as the sum of distinct elements of $S_t(\alpha)$. In this paper we show that his methods can be applied to deal with many other pairs of (α, t) as well.

1 Introduction

For a sequence or multiset S of positive integers we define $P(S)$ as the set of all integers that can be written as the sum of distinct elements of S. We say that S is complete if $\mathbb{N} \setminus P(S)$ is finite, and we say that S is entirely complete if $P(S) = \mathbb{N}$. For positive real numbers α and t, in this paper we are interested in the completeness of the sequence $S_t(\alpha) = (s_1, s_2, \dots,)$, where $s_n = \lfloor t\alpha^n \rfloor$.

Research into this question started with [1], where Graham first showed that for $1 < \alpha \leq 5^{1/3}$, the sequence $S_t(\alpha)$ is entirely complete if $t < 1$. He then more generally determined the full set of pairs (α, t) with $1 < \alpha < 2$ and $t < 1$ for which $S_t(\alpha)$ is (entirely) complete. Interestingly, for the α and t in this square, $S_t(\alpha)$ is complete if, and only if, it is entirely complete.

With the aforementioned square having been dealt with, Erdős and Graham [2] ask in full generality for which values of $\alpha > 0$ and $t > 0$ the sequence $S_t(\alpha)$ is complete, which is now listed as problem #349 at [3]. And the goal of this paper is to revisit [1] and see how much further its ideas can be pushed.

We will generalize the result from Graham's paper that we just mentioned, and show that if $1 < \alpha \leq 5^{1/3}$, then $S_t(\alpha)$ is entirely complete if, and only if, $t < \min(\frac{2}{\alpha}, \frac{3}{\alpha^2}, \frac{5}{\alpha^3})$. On the other hand, if $\alpha \geq \frac{1+\sqrt{5}}{2}$ and $t \geq \max(\min(\frac{3}{\alpha^2}, \frac{5}{\alpha^3}), 1)$, then $S_t(\alpha)$ is not complete. Combined with Graham's results, this lets us finish off the case $\alpha \geq \frac{1+\sqrt{5}}{2}$.

As for $1 < \alpha < \frac{1+\sqrt{5}}{2}$, it is plausible that $S_t(\alpha)$ is complete for all $t > 0$, but this remains open in general. On the other hand, for any $\epsilon > 0$ and any T, in our final section we show how one can in principle prove this conjecture for all $\alpha < \frac{1+\sqrt{5}}{2} - \epsilon$ and all $t < T$ at once, by a finite computation. We then give an example of what such a proof might look like and show that $S_t(\alpha)$ is complete for all pairs (α, t) with $\alpha < \frac{5}{4}$ and $t < \frac{4}{\alpha}$. Finally, one idea used in this proof is then applied to furthermore show that $S_t(\alpha)$ is complete for all $\alpha < 1 + \frac{1}{2\lceil t \rceil}$.

2 Preliminary lemmas

Before we fully dive in, in this section we mention a few small general lemmas that we will repeatedly make use of. Most of them will be very elementary and well-known, but we will record them anyway. We start off with a lemma that Graham attributes in [1] to Folkman, although we have not been able to find a reference. For completeness' sake (no pun intended) we will also provide a proof.

Lemma 1. *Assume that a positive integer $m \notin P(S_t(\alpha))$ and a non-negative integer r exist with $s_1 + \ldots + s_r < m < s_{r+2}$ and $s_n + s_{n+1} \leq s_{n+2}$ for all $n > r$. Then $m + s_{r+3} + s_{r+5} + \ldots + s_{r+2k+1} \notin P(S_t(\alpha))$ for all $k \geq 1$.*

Proof. For an integer $k \geq 1$ define

$$m_k = m + \sum_{i=1}^{k-1} s_{r+2i+1},$$

and assume by induction $m_k \notin P(A)$ while

$$\sum_{i=1}^{r+2(k-1)} s_i < m_k < s_{r+2k}.$$

This is certainly true for $k = 1$ by assumption. Adding s_{r+2k+1} to these inequalities and using $s_n + s_{n+1} \leq s_{n+2}$ on both the left- and the right-hand sides, we deduce

$$\sum_{i=1}^{r+2k} s_i < m_{k+1} < s_{r+2k+2}.$$

We therefore see that, if $m_{k+1} \in P(S_t(\alpha))$, then you would need to use s_{r+2k+1} in any representation of m_{k+1}. But this would imply that $m_k = m_{k+1} - s_{r+2k+1}$ has a representation as well, which contradicts the induction hypothesis. □

Lemma 1 comes in handy whenever $\alpha \geq \frac{1+\sqrt{5}}{2}$, as we then have the following inequalities, the first part of which is Lemma 3 in [1].

Lemma 2. *If $\alpha \geq \frac{1+\sqrt{5}}{2}$, then $s_n + s_{n+1} \leq s_{n+2}$ for all $n \in \mathbb{N}$. And if $\alpha > 2$, then $1 + s_1 + \ldots + s_n < s_{n+1}$ if n is large enough.*

Proof. If $\alpha > 2$, let n be large enough so that $t\alpha^{n+1}(\alpha - 2) > 2\alpha - 2$. Then $\frac{t\alpha^{n+1}}{\alpha-1} < t\alpha^{n+1} - 2$, as

$$(t\alpha^{n+1} - 2)(\alpha - 1) - t\alpha^{n+1} = t\alpha^{n+1}(\alpha - 2) - 2\alpha + 2 > 0.$$

We therefore get

$$\begin{aligned}1+\sum_{i=1}^{n} s_i &\le 1+\sum_{i=1}^{n} t\alpha^i\\ &= 1+\frac{t(\alpha^{n+1}-\alpha)}{\alpha-1}\\ &< 1+\frac{t\alpha^{n+1}}{\alpha-1}\\ &< t\alpha^{n+1}-1\\ &< s_{n+1}.\end{aligned}$$ □

Combining Lemma 1 and the first part of Lemma 2 provides the following corollary.

Corollary 1. *If $\alpha \ge \frac{1+\sqrt{5}}{2}$ and $s_1+\ldots+s_r < m < s_{r+1}$ for some $m \ge 1$, $r \ge 0$, then $S_t(\alpha)$ is not complete.*

In the next section we will apply Corollary 1 to show that certain pairs of (α, t) lead to sequences which are not complete. But first we will quickly mention one more general lemma and two lemmas specific to our sequence, all geared towards showing that some sequences are complete.

Lemma 3. *Assume an non-negative integer r exists such that the inequality $s_{n+1} \le 1+s_1+\ldots+s_n$ holds for all $n \le r$, while $s_{n+1} \le 2s_n$ holds for all $n > r$. Then $S_t(\alpha)$ is entirely complete.*

Proof. This is Lemma 2 in [1]. □

Lemma 4. *Assume $t \ge 1$. If $1 < \alpha < \frac{3}{2}$, then the inequality $s_{n+1} \le 2s_n$ holds for all $n \ge 1$. If $\frac{3}{2} \le \alpha < \frac{1+\sqrt{5}}{2}$, then $s_{n+1} \le 2s_n$ holds for all $n \ge 2$. And if $\frac{1+\sqrt{5}}{2} \le \alpha < 5^{1/3}$, then $s_{n+1} \le 2s_n$ holds holds for all $n \ge 3$.*

Proof. One can check that these claims follow from the first part of Lemma 4 in [1]. □

Lemma 5. *Let $t \ge 1$ and $1 < \alpha < \frac{1+\sqrt{5}}{2}$, and assume that positive integers r and X exist such that $m \in P(\{s_1, \ldots, s_r\})$ for all m with $X \le m < X + s_{r+1}$. Then $S_t(\alpha)$ is complete.*

Proof. The hypothesis implies in particular that $m \in P(\{s_1, \ldots, s_r, s_{r+1}\})$ for all m with $X + s_{r+1} \le m < X + 2s_{r+1}$. Since $s_{r+2} \le 2s_{r+1}$ by Lemma 4, we deduce $m \in P(\{s_1, \ldots, s_r, s_{r+1}\})$ for all m with $X \le m < X + s_{r+2}$, so we are done by induction, and see that all $m \ge X$ are representable. □

3 Main results

We are now ready to deal with the completeness of $S_t(\alpha)$. And to provide the full picture, we will first swiftly deal with the near-trivial case where α does not belong to the interval $(1, 2)$, before moving on to the more interesting cases.

Proposition 1. *If $\alpha \notin [1, 2]$, then $S_t(\alpha)$ is not complete for any $t > 0$. And if $\alpha = 1$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t \in [1, 2)$.*

Proof. If $\alpha < 1$, then $s_n = 0$ for all large n, so that $P(S_t(\alpha))$ is actually finite. If $\alpha = 1$, then $s_n = \lfloor t \rfloor$ for all n, so that $P(S_t(\alpha)) = \lfloor t \rfloor \mathbb{N}$. Finally, if $\alpha > 2$, then $s_n - 1 \notin P(S_t(\alpha))$ for all large enough n, by the second part of Lemma 2. $\square$

Proposition 2. *If $\alpha = 2$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t = \frac{1}{2^k}$ for some $k \geq 1$.*

Proof. For $t = \frac{1}{2^k}$ the completeness of $S_t(\alpha)$ follows from the completeness of the sequence of powers of two, while for $t \geq 1$ we can apply Corollary 1 with $r = 0, m = 1$ to deduce that $S_t(\alpha)$ is not complete. Finally, if $t < 1$ and $t \neq \frac{1}{2^k}$, let n be the smallest index with $s_n \geq 1$. We then see $s_n = 1$, so write $t2^n = 1 + \epsilon$ for some $\epsilon \in (0, 1)$ and define $j = \left\lceil \frac{-\log \epsilon}{\log 2} \right\rceil$. We then have $s_{n+i} = 2^i$ for all i with $0 \leq i < j$ and $s_j = 2^j + 1$, so that we can apply Corollary 1 with $r = j - 1, m = 2^j$. $\square$

With these trivial cases out of the way, we may from now on assume $1 < \alpha < 2$. We will split up this interval into four different regions; $1 < \alpha < \frac{3}{2}$, $\frac{3}{2} \leq \alpha < \frac{1+\sqrt{5}}{2}$, $\frac{1+\sqrt{5}}{2} \leq \alpha < 5^{1/3}$ and $5^{1/3} \leq \alpha < 2$. We will deal with them in order from large to small.

Proposition 3. *If $5^{1/3} \leq \alpha < 2$, there is no value of $t \geq 1$ for which $S_t(\alpha)$ is complete.*

Proof. If $t \geq \frac{2}{\alpha}$ we see $s_1 \geq 2$, which means we can apply Corollary 1 with $m = 1$, $r = 0$ to deduce that $S_t(\alpha)$ is not complete. So we may assume that t is smaller than $\frac{2}{\alpha}$, implying $s_1 \leq 1$. Now, if $t \geq \frac{3}{\alpha^2}$, then $s_2 \geq 3$, in which case we apply Corollary 1 with $m = 2$, $r = 1$. So we may further assume $t < \frac{3}{\alpha^2}$, which gives $s_2 \leq 2$. But we then have $t \geq 1 \geq \frac{5}{\alpha^3}$, implying $s_3 \geq 5$. And in this case we apply Corollary 1 with $m = 4$, $r = 2$. $\square$

Proposition 4. *If $\frac{1+\sqrt{5}}{2} \leq \alpha < 5^{1/3}$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t < \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^3})$.*

Proof. For $t \geq \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^3})$ we can apply the same proof that we just used for Proposition 3 to conclude that $S_t(\alpha)$ is not complete. So it remains to show the converse; prove the completeness of $S_t(\alpha)$ if $t < \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^3})$, and by Theorem 2 in [1] we may assume $t \geq 1$.

By the assumed bounds on α and t we claim $s_1 = 1, s_2 = 2, s_3 = 4$. Indeed,

$$\begin{aligned} 1 < \alpha \leq t\alpha < \frac{3}{\alpha} < 2, \\ 2 < \alpha^2 \leq t\alpha^2 < 3, \\ 4 < \alpha^3 \leq t\alpha^3 < 5. \end{aligned}$$

Since $s_{n+1} \leq 2s_n$ for all $n \geq 3$ by Lemma 4, Lemma 3 implies that $S_t(\alpha)$ is entirely complete. $\square$

We remark that the case $\alpha \geq \frac{1+\sqrt{5}}{2}$ is now fully dealt with, as is $t < 1$, by [1]. We therefore only need to consider those pairs (α, t) with $1 < \alpha < \frac{1+\sqrt{5}}{2}$ and $t \geq 1$, and it is thought (see e.g. [2, p. 57]) that $S_t(\alpha)$ is complete for all these values. We will not be able to fully settle this problem, but we will at least determine all those pairs for which $S_t(\alpha)$ is entirely complete. For example, for all α with $1 < \alpha < \frac{1+\sqrt{5}}{2}$, we will see that the resulting sequence is entirely complete for all $t \leq \frac{9-3\sqrt{5}}{2} \approx 1.15$.

Proposition 5. *If $\frac{3}{2} \leq \alpha < \frac{1+\sqrt{5}}{2}$, then $S_t(\alpha)$ is entirely complete if, and only if, $t < \frac{3}{\alpha^2}$. In particular, if $t \leq \frac{9-3\sqrt{5}}{2}$, then $S_t(\alpha)$ is entirely complete for all these values of α.*

Proof. We will again assume $t \geq 1$. Analogously to the second part of the proof of Proposition 4, by the given bounds on α and t we then, once again, deduce $s_1 = 1, s_2 = 2, s_3 = 4$. And by the same reasoning, $S_t(\alpha)$ is entirely complete. Conversely, if $t \geq \frac{3}{\alpha^2}$, then $s_2 \geq 3$, so that $\{1, 2\} \not\subset P(S_t(\alpha))$. $\square$

Proposition 6. *If $1 < \alpha < \frac{3}{2}$, then $S_t(\alpha)$ is entirely complete if, and only if, $t < \frac{2}{\alpha}$. In particular, if $t \leq \frac{4}{3}$, then $S_t(\alpha)$ is entirely complete for all these values of α.*

Proof. If $t \geq \frac{2}{\alpha}$, then $1 \notin P(S_t(\alpha))$, so that $S_t(\alpha)$ is not entirely complete. On the other hand, if $1 \leq t < \frac{2}{\alpha}$, then we deduce $s_1 = 1$, while $s_{n+1} \leq 2s_n$ for all $n \geq 1$ by Lemma 4. Once more, we are done by Lemma 3. $\square$

The only remaining case left is when $1 < \alpha \leq \frac{1+\sqrt{5}}{2}$ and $t \geq \min\left(\frac{2}{\alpha}, \frac{3}{\alpha^2}\right)$. This is furthermore the only case where $S_t(\alpha)$ can be complete without being entirely complete. And this brings us to a new chapter.

4 Computational possibilities

In this final section we will give a very quick taste of where this research could be taken next. We know by the results in the previous section that we may assume $1 < \alpha \leq \frac{1+\sqrt{5}}{2}$ and $t \geq \min\left(\frac{2}{\alpha}, \frac{3}{\alpha^2}\right)$, in which case $S_t(\alpha)$ is necessarily not entirely complete, as either $s_1 \geq 2$, or $s_1 = 1$ and $s_2 \geq 3$. However, for any given pair of (α, t) it is sufficient to provide r and X for which one can apply

Lemma 5 to still conclude that $S_t(\alpha)$ is complete. Moreover, if such r and X are found, then one can note that the first $r+1$ elemens of $S_t(\alpha)$ actually coincide with the first $r+1$ elements of $S_{t+\epsilon}(\alpha+\epsilon)$, as long as ϵ is sufficiently small. So for every $(\alpha, t) \in \mathbb{R}^2$ for which we prove the completeness of $S_t(\alpha)$ this way, we actually get a set $U \subset \mathbb{R}^2$ for which $S_t(\alpha)$ is complete for all $(\alpha, t) \in U$.

Working in the other direction and starting with any bounded set $U \subset \mathbb{R}^2$, there are only finitely many distinct possibilities for the values of, say, $s_1, s_2, \ldots s_{10}$. All of these possible values can in principle be enumerated, and if it turns out that every single possibility leads to the existence of X and r with which to apply Lemma 5, then $S_t(\alpha)$ is complete for all $(\alpha, t) \in U$.

To show the flavor of such computational considerations, we will provide one construction, which partially extends Proposition 6.

Proposition 7. *If $1 < \alpha \leq \frac{5}{4}$ and $t < \frac{4}{\alpha}$, then $S_t(\alpha)$ is complete. In particular, if $t < \frac{16}{5}$, then $S_t(\alpha)$ is complete for all these values of α.*

Before we start the proof, let us quickly mention and prove two small, but valuable lemmas.

Lemma 6. *For all $n \geq 2$ we have $s_n < \alpha(s_{n-1}+1)$.*

Proof. An easy calculation;

$$\begin{aligned} s_n &= \lfloor t\alpha^n \rfloor \\ &\leq \alpha(t\alpha^{n-1}) \\ &< \alpha(s_{n-1}+1). \end{aligned}$$

□

Lemma 7. *If $1 < \alpha \leq 1 + \frac{1}{x}$ for some $x > t$, then $m \in S_t(\alpha)$ for all m with $s_1 \leq m \leq x$.*

Proof. Without loss of generality assume $s_1 < m \leq x$, and let $n \geq 2$ be the largest integer with $s_{n-1} < m$. By Lemma 6 we then have

$$\begin{aligned} m &\leq s_n \\ &< \alpha(s_{n-1}+1) \\ &\leq \alpha m \\ &\leq m + \frac{m}{x} \\ &\leq m+1. \end{aligned}$$

Since $m \leq s_n < m+1$ we indeed get $s_n = m$. □

Proof of Proposition 7. As a first remark, by applying Lemma 7 with $x = 4$, we deduce $\{3, 4\} \subset S_t(\alpha)$.

Now, if $\alpha \leq \frac{7}{6}$, then we even have $\{3, 4, 5, 6\} \subset S_t(\alpha)$, by applying Lemma 7 with $x = 6$ instead. And by employing Lemma 6 with n the largest index with $s_{n-1} < 7$, we moreover get $s_n \in \{7, 8\}$. As $P(\{3, 4, 5, 6\})$ contains $\{3, 4, \ldots, 11\}$, we can finish this case by applying Lemma 5 with $X = 3, r = n - 1$.

We will therefore assume $\frac{7}{6} < \alpha \leq \frac{5}{4}$ from now on, and we may further assume $\{5, 6\} \not\subset S_t(\alpha)$, as we would otherwise be done by the previous argument.

By Lemma 6 and the assumption $\alpha \leq \frac{5}{4}$, we note that $s_n \in \{5, 6\}$ if s_{n-1} is the largest element smaller than 5. Depending on the value of s_n we now have to distinguish between two cases, and let us first assume $s_n = 6$. For s_{n+1} we then, on the one hand, get

$$\begin{aligned} t\alpha^{n+1} &> \frac{7}{6} t\alpha^n \\ &\geq \frac{7}{6} s_n \\ &= 7. \end{aligned}$$

While on the other hand,

$$\begin{aligned} t\alpha^{n+1} &\leq \left(\frac{5}{4}\right)^2 t\alpha^{n-1} \\ &< \frac{125}{16} \\ &< 8. \end{aligned}$$

And we conclude $s_{n+1} = 7$. For s_{n+2} we analogously have $8 \leq s_{n+2} \leq 9$, while $s_{n+3} \leq 12$.

Regardless of whether $\{3, 4, 6, 7, 8\}$ or $\{3, 4, 6, 7, 9\}$ is contained in $S_t(\alpha)$, in both cases one can check that we have $\{9, 10, \ldots, 20\} \subset P(\{s_1, \ldots, s_{n+2}\})$. This in turn implies that we are done by applying Lemma 5 with $X = 9, r = n + 2$.

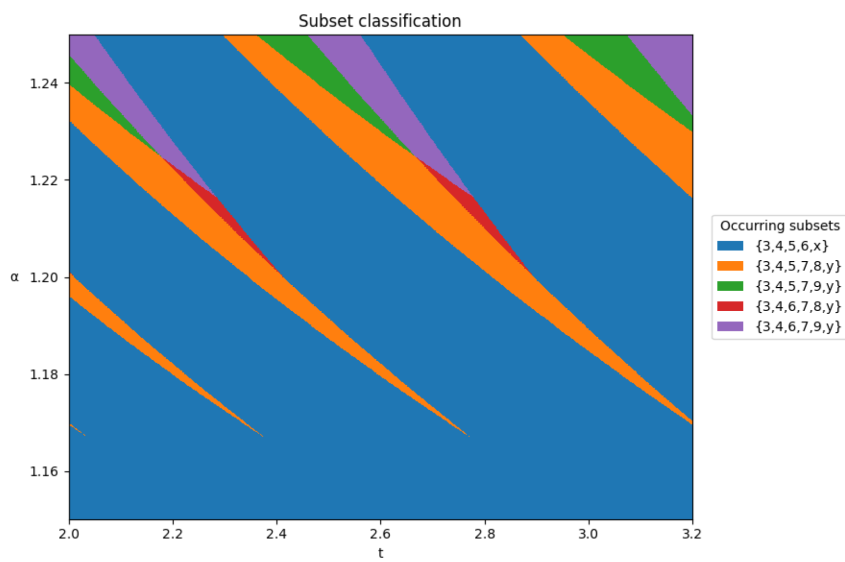
The second case we have to deal with is $s_n = 5$ with $6 \notin S_t(\alpha)$. As it turns out however, this case works analogously, and we get that either $S_1 = \{3, 4, 5, 7, 8\}$ or $S_2 = \{3, 4, 5, 7, 9\}$ is contained in $S_t(\alpha)$, while $S_t(\alpha)$ also contains either 10, 11 or 12. With both S_1 and S_2 we are again lucky enough to get $\{9, 10, \ldots, 20\} \subset P(S_i)$, so one final application of Lemma 5 finishes the proof. $\square$

Even though it was not the most thrilling proof of all time, let us quickly recap it anyway, to see how one could perhaps automate it and extend Proposition 7 to larger intervals of α and t. By repeated applications of Lemma 6 we deduced that one of the following sets must be contained in $S_t(\alpha)$ (where $7 \leq x \leq 8$ and

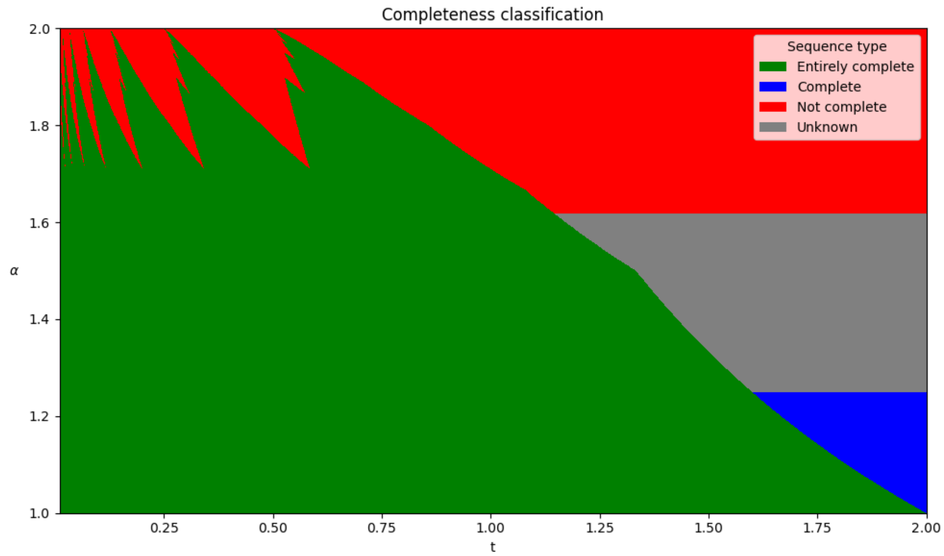
$9 \leq y \leq 12$):

$$\{3, 4, 5, 6, x\}$$
$$\{3, 4, 5, 7, 8, y\}$$
$$\{3, 4, 5, 7, 9, y\}$$
$$\{3, 4, 6, 7, 8, y\}$$
$$\{3, 4, 6, 7, 9, y\}$$

And in every case, the elements excluding the last one generate a large enough interval to apply Lemma 5 with. To give a visual representation of what this looks like, here is a plot of which values of α and t lead to which of the above subsets.



Now, if we were to extend the range of α or t, the potential number of subsets one has to check increases quite rapidly of course. On the other hand, a computer seems very amenable to deal with such issues, so this is where we heartily invite the interested reader to further improve upon these results and see if this search can be automated. There is certainly lots of low-hanging fruit left. What we have so far -combining our results with those from [1]- provides the following picture.



For a final note, we would like to mention one nice application of Lemma 7. If one plots (α, t) in the plane, then up till now, the region for which we have managed to show that $S_t(\alpha)$ is complete, is bounded. So to finish this paper, let us prove it for a region with infinite area instead.

Proposition 8. *The sequence $S_t(\alpha)$ is complete for all α with $1 < \alpha \leq 1 + \frac{1}{2\lceil t \rceil}$.*

Proof. We may assume $t > 1$ by Proposition 6. With r such that $s_r = 2\lceil t \rceil$ (which exists by Lemma 7), we have $s_{r+1} \leq 2\lceil t \rceil + 2$ by Lemma 6. By applying Lemma 7 again and taking sums of one or two elements, we deduce that $m \in P(\{s_1, \ldots, s_r\})$ for all m with $\lceil t \rceil \leq m < 4\lceil t \rceil$. We may therefore apply Lemma 5 with $X = \lceil t \rceil$, as $4\lceil t \rceil \geq \lceil t \rceil + (2\lceil t \rceil + 2) \geq X + s_{r+1}$ by $\lceil t \rceil \geq 2$. $\square$

References

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