Completeness of exponentially increasing sequences

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Abstract

For fixed positive reals α and t, consider the sequence $S_t(\alpha) = (s_1, s_2, \ldots,)$ with $s_n = \lfloor t\alpha^n \rfloor$. In 1964, Graham managed to characterize those pairs (α, t) with $1 < \alpha < 2$ and $0 < t < 1$ for which every large enough integer can be written as the sum of distinct elements of $S_t(\alpha)$. In this paper we show that his methods can be applied to deal with many other pairs of (α, t) as well.

1 Introduction

For a sequence or multiset S of positive integers we define $P(S)$ as the set of all integers that can be written as the sum of distinct elements of S. We say that S is complete if $\mathbb{N} \setminus P(S)$ is finite, and we say that S is entirely complete if $P(S) = \mathbb{N}$. For real numbers $\alpha > 0, t > 0$, in this paper we are interested in the completeness of the sequence $S_t(\alpha) = (s_1, s_2, \ldots,)$, where $s_n = \lfloor t\alpha^n \rfloor$.

In [1] Graham showed that $S_t(\alpha)$ is entirely complete if $0 < t < 1$ and $1 < \alpha \leq 5^{1/3}$. More generally, he fully determined the set of pairs (α, t) with $1 < \alpha < 2$ and $0 < t < 1$ for which $S_t(\alpha)$ is (entirely) complete. Interestingly, for the α and t in this square, $S_t(\alpha)$ is complete if, and only if, it is entirely complete. As we will see, this property holds for all $\alpha \geq \frac{1+\sqrt{5}}{2}$, regardless of the value of t.

With the square $1 < \alpha < 2, 0 < t < 1$ having been dealt with, [2] Erdős and Graham ask in full generality for which values of $\alpha > 0$ and $t > 0$ the sequence $S_t(\alpha)$ is complete, which is now listed as problem #349 at [3]. The goal of this paper is to revisit [1] and see how much further its ideas can be pushed. We will see in particular that they can be extended to deal with all pairs (α, t) with $\alpha \geq \frac{1+\sqrt{5}}{2}$ and all $t > 0$, which in most cases leads to sequences that are not complete. On the other hand, for $1 < \alpha < \frac{1+\sqrt{5}}{2}$, we will find various intervals of α and t where the resulting sequence is complete.

2 Preliminary lemmas

Before we fully dive in, in this section we mention a few small general lemmas that we will repeatedly make use of. Most of them will be very elementary and well-known, but we will record them anyway. We start off with a lemma that Graham attributes in [1] to Folkman, although we have not been able to find a reference. For completeness' sake (no pun intended) we will also provide a proof.

Lemma 1. *Assume that a positive integer $m \notin P(S_t(\alpha))$ and a non-negative integer r exist with $s_1 + \ldots + s_r < m < s_{r+2}$ and $s_n + s_{n+1} \leq s_{n+2}$ for all $n > r$. Then $m + s_{r+3} + s_{r+5} + \ldots + s_{r+2k+1} \notin P(S_t(\alpha))$ for all $k \geq 1$.*

Proof. For an integer $k \geq 1$ define

$$m_k = m + \sum_{i=1}^{k-1} s_{r+2i+1},$$

and assume by induction $m_k \notin P(A)$ while

$$\sum_{i=1}^{r+2(k-1)} s_i < m_k < s_{r+2k}.$$

This is certainly true for $k = 1$ by assumption. Adding s_{r+2k+1} to these inequalities and using $s_n + s_{n+1} \leq s_{n+2}$ on both the left- and the right-hand sides, we deduce

$$\sum_{i=1}^{r+2k} s_i < m_{k+1} < s_{r+2k+2}.$$

We therefore see that, if $m_{k+1} \in P(S_t(\alpha))$, then you would need to use s_{r+2k+1} in any representation of m_{k+1}. But this would imply that $m_k = m_{k+1} - s_{r+2k+1}$ has a representation as well, which contradicts the induction hypothesis. $\square$

Lemma 1 comes in handy whenever $\alpha \geq \frac{1+\sqrt{5}}{2}$, as we then have the following inequalities, the first part of which is Lemma 3 in [1].

Lemma 2. *If $\alpha \geq \frac{1+\sqrt{5}}{2}$, then $s_n + s_{n+1} \leq s_{n+2}$ for all $n \in \mathbb{N}$. And if $\alpha > 2$, then $1 + s_1 + \ldots + s_n < s_{n+1}$ if n is large enough.*

Proof. If $\alpha > 2$, let n be large enough so that $t\alpha^{n+1}(\alpha - 2) > 2\alpha - 2$. Then $\frac{t\alpha^{n+1}}{\alpha-1} < t\alpha^{n+1} - 2$, as

$$(t\alpha^{n+1} - 2)(\alpha - 1) - t\alpha^{n+1} = t\alpha^{n+1}(\alpha - 2) - 2\alpha + 2 > 0.$$

We therefore get

$$\begin{aligned} 1 + \sum_{i=1}^{n} s_i &\leq 1 + \sum_{i=1}^{n} t\alpha^i \\ &= 1 + \frac{t(\alpha^{n+1} - \alpha)}{\alpha - 1} \\ &< 1 + \frac{t\alpha^{n+1}}{\alpha - 1} \\ &< t\alpha^{n+1} - 1 \\ &< s_{n+1}. \end{aligned}$$

$\square$

Combining Lemma 1 and the first part of Lemma 2 provides the following corollary.

Corollary 1. *If $\alpha \geq \frac{1+\sqrt{5}}{2}$ and $s_1 + \ldots + s_r < m < s_{r+1}$ for some $m \geq 1$, $r \geq 0$, then $S_t(\alpha)$ is not complete.*

In the next section we will apply Corollary 1 to show that certain pairs of (α, t) lead to sequences which are not complete. But first we will quickly mention three more lemmas.

Lemma 3. *Assume $t \geq 1$. If $\alpha < \frac{3}{2}$, then the inequality $s_{n+1} \leq 2s_n$ holds for all $n \geq 1$. If $\alpha < \frac{1+\sqrt{5}}{2}$, then $s_{n+1} \leq 2s_n$ holds for all $n \geq 2$. And if $\alpha < 5^{1/3}$, then $s_{n+1} \leq 2s_n$ holds whenever $s_n \geq 3$.*

Proof. One can check that these claims follow from the first part of Lemma 4 in [1]. $\square$

Lemma 4. *Assume an non-negative integer r exists such that the inequality $s_{n+1} \leq 1 + s_1 + \ldots + s_n$ holds for all $n \leq r$, while $s_{n+1} \leq 2s_n$ holds for all $n > r$. Then $S_t(\alpha)$ is entirely complete.*

Proof. This is Lemma 2 in [1]. $\square$

Lemma 5. *Let $t \geq 1$, $\alpha < \frac{1+\sqrt{5}}{2}$, and assume that positive integers r and X exist such that $m \in P(\{s_1, \ldots, s_r\})$ for all m with $X \leq m < X + s_{r+1}$. Then $S_t(\alpha)$ is complete.*

Proof. The hypothesis implies in particular that $m \in P(\{s_1, \ldots, s_r, s_{r+1}\})$ for all m with $X + s_{r+1} \leq m < X + 2s_{r+1}$. Since $s_{r+2} \leq 2s_{r+1}$ by Lemma 3, we deduce $m \in P(\{s_1, \ldots, s_r, s_{r+1}\})$ for all m with $X \leq m < X + s_{r+2}$, so we are done by induction, and see that all $m \geq X$ are representable. $\square$

3 Main results

We are now ready to deal with the completeness of $S_t(\alpha)$. And to provide the full picture, we will first swiftly deal with the near-trivial case where α does not belong to the interval $(1, 2)$, before moving on to the more interesting cases.

Proposition 1. *If $\alpha \notin [1, 2]$, then $S_t(\alpha)$ is not complete for any $t > 0$. And if $\alpha = 1$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t \in [1, 2)$.*

Proof. If $\alpha < 1$, then $s_n = 0$ for all large n, so that $P(S_t(\alpha))$ is actually finite. If $\alpha = 1$, then $s_n = \lfloor t \rfloor$ for all n, so that $P(S_t(\alpha)) = \lfloor t \rfloor \mathbb{N}$. Finally, if $\alpha > 2$, then $s_n - 1 \notin P(S_t(\alpha))$ for all large enough n, by the second part of Lemma 2. $\square$

Proposition 2. *If $\alpha = 2$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t = \frac{1}{2^k}$ for some $k \geq 1$.*

Proof. For $t = \frac{1}{2^k}$ the completeness of $S_t(\alpha)$ follows from the completeness of the sequence of powers of two, while for $t \geq 1$ we can apply Corollary 1 with $r = 0, m = 1$ to deduce that $S_t(\alpha)$ is not complete. Finally, if $t < 1$ and $t \neq \frac{1}{2^k}$, let n be the smallest index with $s_n \geq 1$. We then see $s_n = 1$, so write $t2^n = 1 + \epsilon$ for some $\epsilon \in (0, 1)$ and define $j = \left\lceil \frac{-\log \epsilon}{\log 2} \right\rceil$. We then have $s_{n+i} = 2^i$ for all i with $0 \leq i < j$ and $s_j = 2^j + 1$, so that we can apply Corollary 1 with $r = j - 1, m = 2^j$. □

With these trivial cases out of the way, we may from now on assume $1 < \alpha < 2$. We will split up this interval into four different regions; $1 < \alpha < \frac{3}{2}$, $\frac{3}{2} \leq \alpha < \frac{1+\sqrt{5}}{2}$, $\frac{1+\sqrt{5}}{2} \leq \alpha < 5^{1/3}$ and $5^{1/3} \leq \alpha < 2$. We will initially attempt to deal with them in order from large to small.

Proposition 3. *If $5^{1/3} \leq \alpha < 2$, there is no value of $t \geq 1$ for which $S_t(\alpha)$ is complete.*

Proof. If $t \geq \frac{2}{\alpha}$ we see $s_1 \geq 2$, which means we can apply Corollary 1 with $m = 1$, $r = 0$ to deduce that $S_t(\alpha)$ is not complete. So we may assume that t is smaller than $\frac{2}{\alpha}$, implying $s_1 \leq 1$. Now, if $t \geq \frac{3}{\alpha^2}$, then $s_2 \geq 3$, in which case we apply Corollary 1 with $m = 2$, $r = 1$. So we may further assume $t < \frac{3}{\alpha^2}$, which gives $s_2 \leq 2$. But we then have $t \geq 1 \geq \frac{5}{\alpha^3}$, implying $s_3 \geq 5$. And in this case we apply Corollary 1 with $m = 4$, $r = 2$. □

Proposition 4. *If $\frac{1+\sqrt{5}}{2} \leq \alpha < 5^{1/3}$, then $S_t(\alpha)$ is (entirely) complete if, and only if, $t < \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^{1/3}})$.*

Proof. For $t \geq \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^{1/3}})$ we can apply the same proof that we just used for Proposition 3 to conclude that $S_t(\alpha)$ is not complete. So it remains to show the converse; prove the completeness of $S_t(\alpha)$ if $t < \min(\frac{3}{\alpha^2}, \frac{5}{\alpha^{1/3}})$, and by Theorem 2 in [1] we may assume $t \geq 1$.

By the assumed bounds on α and t we claim $s_1 = 1, s_2 = 2, s_3 = 4$. Indeed,

$$\begin{aligned} &1 < \alpha \leq t\alpha < \frac{3}{\alpha} < 2, \\ &2 < \alpha^2 \leq t\alpha^2 < 3, \\ &4 < \alpha^3 \leq t\alpha^3 < 5. \end{aligned}$$

Since $s_{n+1} \leq 2s_n$ for all $n \geq 3$ by Lemma 3, Lemma 4 implies that $S_t(\alpha)$ is entirely complete. □

We remark that the case $\alpha \geq \frac{1+\sqrt{5}}{2}$ is now fully dealt with, as is $t < 1$, by [1]. We therefore only need to consider those pairs (α, t) with $1 < \alpha < \frac{1+\sqrt{5}}{2}$ and $t \geq 1$, and it is thought (see e.g. [2, p. 57]) that $S_t(\alpha)$ is complete for all these values. We will not be able to fully settle this problem, but we will at least determine all those pairs for which $S_t(\alpha)$ is entirely complete. For example, for all α with $1 < \alpha < \frac{1+\sqrt{5}}{2}$, we will see that the resulting sequence is entirely complete for all $t \leq \frac{9-3\sqrt{5}}{2} \approx 1.15$.

Proposition 5. *If $\frac{3}{2} \leq \alpha < \frac{1+\sqrt{5}}{2}$, then $S_t(\alpha)$ is entirely complete if, and only if, $t < \frac{3}{\alpha^2}$. In particular, if $t \leq \frac{9-3\sqrt{5}}{2}$, then $S_t(\alpha)$ is entirely complete for all these values of α.*

Proof. We may once again assume $t \geq 1$. Analogously to the second part of the proof of Proposition 4, by the given bounds on α and t we once again deduce $s_1 = 1, s_2 = 2, s_3 = 4$. And by the same reasoning, $S_t(\alpha)$ is entirely complete. Conversely, if $t \geq \frac{3}{\alpha^2}$, then $s_2 \geq 3$, so that $\{1, 2\} \not\subset P(S_t(\alpha))$. $\square$

Proposition 6. *If $1 < \alpha < \frac{3}{2}$, then $S_t(\alpha)$ is entirely complete if, and only if, $t < \frac{2}{\alpha}$. In particular, if $t \leq \frac{4}{3}$, then $S_t(\alpha)$ is entirely complete for all these values of α.*

Proof. If $t \geq \frac{2}{\alpha}$, then $1 \notin P(S_t(\alpha))$, so that $S_t(\alpha)$ is not entirely complete. On the other hand, if $1 \leq t < \frac{2}{\alpha}$, then we deduce $s_1 = 1$, while $s_{n+1} \leq 2s_n$ for all $n \geq 1$ by Lemma 3. We are once again done by Lemma 4. $\square$

The only remaining case left is when $1 < \alpha \leq \frac{1+\sqrt{5}}{2}$ and $t \geq \min\left(\frac{2}{\alpha}, \frac{3}{\alpha^2}\right)$. This is furthermore the only case where $S_t(\alpha)$ can be complete without being entirely complete. And this brings us to a new chapter.

4 Computational possibilities

In this final section we will give a very quick taste of where this research could be taken next. We know by the results in the previous section that we may assume $1 < \alpha \leq \frac{1+\sqrt{5}}{2}$ and $t \geq \min\left(\frac{2}{\alpha}, \frac{3}{\alpha^2}\right)$, in which case $S_t(\alpha)$ is necessarily not entirely complete, as either $s_1 \geq 2$, or $s_1 = 1$ and $s_2 \geq 3$. However, for any given pair of (α, t) it is sufficient to provide r and X for which one can apply Lemma 5 to still conclude that $S_t(\alpha)$ is complete. Moreover, if such r and X are found, then one can note that the first $r + 1$ elemens of $S_t(\alpha)$ actually coincide with the first $r + 1$ elements of $S_{t+\epsilon}(\alpha + \epsilon)$, as long as ϵ is sufficiently small. So for every $(\alpha, t) \in \mathbb{R}^2$ for which we prove the completeness of $S_t(\alpha)$ this way, we actually get a set $U \subset \mathbb{R}^2$ for which $S_t(\alpha)$ is complete for all $(\alpha, t) \in U$.

Working in the other direction and starting with any bounded set $U \subset \mathbb{R}^2$, there are only finitely many distinct possibilities for the values of, say, $s_1, s_2, \ldots s_{10}$. All of these possible values can in principle be enumerated, and if it turns out that every single possibility leads to the existence of X and r with which to apply Lemma 5, then $S_t(\alpha)$ is complete for all $(\alpha, t) \in U$.

To show the flavor of such computational considerations, we will provide one construction, which extends Proposition 6. The reader is certainly invited to further extend these results (potentially with the help of a computer), as there is lots of low-hanging fruit left.

Proposition 7. *If $1 < \alpha \leq \frac{5}{4}$ and $t < \frac{4}{\alpha}$, then $S_t(\alpha)$ is complete. In particular, if $t < \frac{16}{5}$, then $S_t(\alpha)$ is complete for all these values of α.*

Proof. As we may assume $t \geq \frac{2}{\alpha}$ by Proposition 6, we have $s_1 \in \{2, 3\}$. We furthermore claim that the integers 3 and 4 are both elements of $S_t(\alpha)$. To see this, let a and b be the largest indices with $s_a \leq 2$ (where $a = 0$ if $s_1 = 3$) and $s_b \leq 3$ respectively. Since $\alpha \leq \frac{5}{4}$, we have for all positive integers n,

$$\begin{aligned} s_{n+1} &= \left\lfloor t\alpha^{n+1} \right\rfloor \\ &\leq \frac{5}{4} t\alpha^n \\ &< \frac{5}{4}(s_n + 1). \end{aligned}$$

This shows $2 < s_{a+1} < \frac{5}{4}(2 + 1)$, which gives $s_{a+1} = 3$. Similarly, $s_{b+1} = 4$. From here we need to consider a few different cases separately.

First of all, if $\alpha \leq \frac{7}{6}$, then $s_{n+1} < \frac{7}{6}(s_n + 1)$ for all n, so by the same reasoning as before we have $\{3, 4, 5, 6\} \subset S_t(\alpha)$ and $\{7, 8\} \cap S_t(\alpha) \neq \emptyset$, say $s_c \in \{7, 8\}$. As $P(\{3, 4, 5, 6\})$ contains $\{3, 4, \ldots, 11\}$, we can finish this case by applying Lemma 5 with $X = 3, r = c - 1$.

We will therefore assume $\frac{7}{6} < \alpha \leq \frac{5}{4}$ from now on. If we still have $\{3, 4, 5, 6\} \subset S_t(\alpha)$, we are done by the argument above, which means we can further assume that either $S_t(\alpha) \cap \{5, 6\} = 5$ or $S_t(\alpha) \cap \{5, 6\} = 6$, as we note that $S_t(\alpha) \cap \{5, 6\} = \emptyset$ is not possible by the assumption $\alpha \leq \frac{5}{4}$. Let us start with the latter case $S_t(\alpha) \cap \{5, 6\} = 6$.

Let now c be such that $s_{c-1} < 5 < 6 \leq s_c$. We then have

$$\begin{aligned} 6 &\leq s_c \\ &< \frac{5}{4}(s_{c-1} + 1) \\ &< \frac{25}{4}, \end{aligned}$$

so that $s_c = 6$. As for s_{c+1}, on the one hand we get

$$\begin{aligned} t\alpha^{c+1} &> \frac{7}{6} t\alpha^c \\ &\geq \frac{7}{6} s_c \\ &= 7. \end{aligned}$$

While on the other hand,

$$
\begin{aligned}
t\alpha^{c+1} &\leq \left(\frac{5}{4}\right)^2 t\alpha^{c-1} \\
&< \frac{125}{16} \\
&< 8.
\end{aligned}
$$

And we conclude $s_{c+1} = 7$. Finally, for s_{c+2} we analogously have $8 \leq s_{c+2} \leq 9$, while $s_{c+3} \leq 12$. Regardless of whether $\{3,4,6,7,8\}$ or $\{3,4,6,7,9\}$ is contained in $S_t(\alpha)$, in both cases we have $\{9,10,\ldots,20\} \subset P(\{s_1,\ldots,s_{c+2}\})$ and we are done by applying Lemma 5 with $X = 9, r = c+2$. This finishes the case $S_t(\alpha) \cap \{5,6\} = 6$, so we are free to assume $\{3,4,5\} \subset S_t(\alpha)$ while $6 \notin S_t(\alpha)$.

As it turns out however, this final case works completely analogously, and with $s_c = 5$ we still have $s_{c+1} = 7$, $8 \leq s_{c+2} \leq 9$, and $s_{c+3} \leq 12$. And again we are lucky enough to get $\{9,10,\ldots,20\} \subset P(\{s_1,\ldots,s_{c+2}\})$, so one more application of Lemma 5 finishes the proof. $\square$

References

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