Improved bounds for the Mayer-Erdős phenomenon on similarly ordered Farey fractions

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Abstract

Let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots$ be the Farey fractions of order n. We then prove that the inequality $(a_l - a_k)(b_l - b_k) \geq 0$ holds for all k and $l > k$ with $l - k \leq \left(\frac{1}{12} - o(1)\right) n$, sharpening an old result by Erdős. On the other hand, we will show that for all $n \geq 4$ there are k, l with $k < l < k + \frac{n}{4} + 5$ for which the product $(a_l - a_k)(b_l - b_k)$ is negative.

1 Introduction

If two fractions $\frac{a}{b}$ and $\frac{a'}{b'}$ are such that the product $(a' - a)(b' - b)$ is non-negative, then we say that $\frac{a}{b}$ and $\frac{a'}{b'}$ are similarly ordered. For example, $\frac{2}{5}$ and $\frac{3}{7}$ are similarly ordered, while $\frac{2}{5}$ and $\frac{3}{4}$ are not. With this definition in mind, let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots$ be the Farey sequence of order $n \geq 4$ and let $f(n)$ be the largest integer such that $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered for all k and l with $|l - k| \leq f(n)$. The condition $n \geq 4$ here ensures that the Farey sequence of order n actually contains fractions (e.g. $\frac{1}{4}$ and $\frac{2}{3}$) which are not similarly ordered, so that $f(n)$ is unambiguously defined.

In [1] Mayer proved $f(n) \geq 3$ for all $n \geq 5$, which he subsequently improved in [2] to $f(n) \to \infty$ if $n \to \infty$. This was further improved by Erdős in [3], where he showed $f(n) > cn$ for some suitable constant c. Moreover, his proof showed that one can take $c = \frac{1}{400}$. A generalization to arbitrary linear forms was then obtained by Zaharescu in [4] (with a constant $c = \frac{1}{480}$), after which Meng and Zaharescu generalized it even further in [5], to arbitrary linear forms in multiple variables.

Concerning the original problem, Erdős remarked in [3] that he was not able to find the optimal value of c. And as far as we are aware, in the better part of a century since, no improvements have occurred in the literature. In this paper we take another look at Erdős's proof, try to optimize its arguments, and find a better lower bound.

We start off by looking at upper bounds, however. We will prove that $f(n) \leq \frac{n}{4} + O(1)$ holds for all $n \geq 4$, and conjecture that this is optimal.

2 Upper bounds

Recall that $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots$ is the Farey sequence of order n, and, in order to upper bound $f(n)$, we aim to find k and $l > k$ with $(a_l - a_k)(b_l - b_k) < 0$ and $l - k$ as small as possible. We claim that such k and l exist with $l - k < \frac{n}{4} + 5$.

Theorem 1. *For all $n \geq 4$ we have $f(n) \leq \lfloor \frac{n}{4} \rfloor + d$, with $d = 1, 2, 2, 4$, depending on whether $n \equiv 0, 1, 2, 3 \pmod 4$.*

To prove this theorem, we will use the following well-known property of consecutive Farey fractions.

Lemma 1. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two reduced fractions with $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$. Then they are consecutive fractions in the Farey sequence of order n if, and only if, $bc - ad = 1$ and $\max(b, d) \leq n < b + d$.*

Proof of Theorem 1. If $n = 4m$ for some $m \in \mathbb{N}$, consider the fraction $\frac{a_k}{b_k} = \frac{2m-1}{4m}$. One can then check by Lemma 1 that the Farey sequence continues as follows:

$$\frac{m}{2m+1}, \frac{m+1}{2m+3}, \ldots, \frac{2m-1}{4m-1}, \frac{1}{2}, \frac{2m}{4m-1}.$$

With $\frac{a_l}{b_l}$ equal to this final fraction, we notice that $\frac{a_k}{b_k} = \frac{2m-1}{4m}$ and $\frac{a_l}{b_l} = \frac{2m}{4m-1}$ are not similarly ordered. Since $l = k + m + 2$, this shows $f(n) \leq m + 1$.

If $n = 4m + 1$ or $n = 4m + 2$, consider $\frac{a_k}{b_k} = \frac{2m}{4m+1}$ instead. These are then the next Farey fractions:

$$\frac{1}{2}, \frac{2m+1}{4m+1}, \frac{2m}{4m-1}, \ldots, \frac{m+1}{2m+1}, \frac{2m+1}{4m}.$$

With $\frac{a_l}{b_l} = \frac{2m+1}{4m}$ we have $l = k + m + 3$ and $(a_l - a_k)(b_l - b_k) < 0$, so that $f(n) \leq m + 2$.

Finally, for $n = 4m + 3$ we also take $\frac{a_k}{b_k} = \frac{2m}{4m+1}$ and $\frac{a_l}{b_l} = \frac{2m+1}{4m}$. In this case however, the two fractions $\frac{2m+1}{4m+3}$ and $\frac{2m+2}{4m+3}$ are contained in the sequence we just mentioned as well (right before and right after $\frac{1}{2}$ respectively). We therefore have $l = k + m + 5$, implying $f(n) \leq m + 4$. □

Based on computer calculations we tentatively believe Theorem 1 to be optimal for large enough n.

Conjecture. *For all $n \geq 4$ we have $f(n) > \frac{n}{4}$. More precisely, for all $n \geq 92$ we have the equality $f(n) = \lfloor \frac{n}{4} \rfloor + d$, with d as in Theorem 1.*

We have checked this conjecture for all $n \leq 5000$ and have not been able to find any counterexamples. In fact, the only positive integers n with $4 \leq n < 92$ for which $f(n)$ is strictly smaller than the upper bound from Theorem 1 are $n = 7, 9, 11, 15, 19, 23, 25, 27, 31, 35, 39, 49, 51, 63, 91$.

It is possible to strengthen the above conjecture in the following way: given any integer d, it seems plausible that for large enough n one can actually classify all pairs of Farey fractions $(\frac{a_k}{b_k}, \frac{a_l}{b_l})$ with $l - k = \lfloor \frac{n}{4} \rfloor + d$ that are not similarly ordered. In particular, for every d there should be an e such that for all n there are at most e such pairs of fractions, with $e = 0$ for $d \leq 0$ in particular. We leave the exact formulation (and proof) of such a stronger conjecture to the interested reader.

3 Lower bounds

To improve upon the lower bound $f(n) > \frac{n}{400}$ that was proven in [3], we will first show that, given any fraction with small denominator, there is a small interval around it that only contains similarly ordered Farey fractions. To give an idea of what such an interval looks like, let us consider the fraction $\frac{4}{5}$. These are then the Farey fractions of order 40 around this fraction:

$$\frac{15}{19}, \frac{19}{24}, \frac{23}{29}, \frac{27}{34}, \frac{31}{39}, \frac{4}{5}, \frac{29}{36}, \frac{25}{31}, \frac{21}{26}, \frac{17}{21}.$$

One can notice that, to the left of $\frac{4}{5}$, both the numerators and the denominators form an increasing arithmetic progression (with common difference 4 and 5 respectively), whereas to the right of $\frac{4}{5}$ the numerators and denominators form decreasing arithmetic progressions. Such a result turns out to be true in general, which we will apply in the proof of our next lemma.

Lemma 2. *Let $\frac{a_k}{b_k}$, $\frac{a}{b}$ and $\frac{a_l}{b_l}$ be fractions in the Farey sequence of order n with $\frac{a_k}{b_k} \leq \frac{a}{b} \leq \frac{a_l}{b_l}$. Then $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered if $l - k \leq \frac{n+b+1}{2b}$.*

Proof. If $b = 1$ the result is trivial as it forces either $\frac{a_k}{b_k} = \frac{0}{1}$ or $\frac{a_l}{b_l} = \frac{1}{1}$ in which case $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are certainly similarly ordered, so without loss of generality we may assume $b \geq 2$. Moreover, if $\frac{n+b+1}{2b} < 3$, then the lemma follows from Mayer's result in [1], so we may further assume $n \geq 5b - 1$. Now, in the Farey sequence of order b, let $\frac{p}{q}$ and $\frac{r}{s}$ be the two fractions immediately to the left and right of $\frac{a}{b}$ respectively, and note that both q and s are smaller than b. Then, analogously to what we saw earlier in the case $\frac{a}{b} = \frac{4}{5}$, it follows from Lemma 1 that the segment of the Farey sequence of order n around $\frac{a}{b}$ is as follows:

$$\frac{p+ca}{q+cb}, \frac{p+(c+1)a}{q+(c+1)b}, \ldots, \frac{p+da}{q+db}, \frac{a}{b}, \frac{r+d'a}{s+d'b}, \frac{r+(d'-1)a}{s+(d'-1)b}, \ldots, \frac{r+c'a}{s+c'b}.$$

Here, $c = \left\lfloor \frac{n-2q-b}{2b} \right\rfloor + 1$, $c' = \left\lfloor \frac{n-2s-b}{2b} \right\rfloor + 1$, $d = \left\lfloor \frac{n-q}{b} \right\rfloor$, and $d' = \left\lfloor \frac{n-s}{b} \right\rfloor$. The values of c and c' ensure that any sum of two consecutive denominators is larger than n (which is required by Lemma 1), while d and d' are the largest values for which all denominators are smaller than or equal to n.

In order to prove Lemma 2, we now have three different cases to consider: either $\frac{a_k}{b_k} = \frac{a}{b}$, or $\frac{a_l}{b_l} = \frac{a}{b}$, or $\frac{a_k}{b_k} < \frac{a}{b} < \frac{a_l}{b_l}$. As for the first case, it is clear that $\frac{a_k}{b_k} = \frac{a}{b}$ and $\frac{a_l}{b_l}$ are similarly ordered if $\frac{a_l}{b_l}$ is one of the elements in the segment, as both $a_l > a$ and $b_l > b$. Moreover, if $\frac{a_l}{b_l}$ is the smallest Farey fraction larger than

$\frac{r+c'a}{s+c'b}$, then we claim $b_l > 2b$. Indeed, applying Lemma 1 and $n \geq 5b-1$,

$$\begin{aligned} b_l &\geq n+1-(s+c'b) \\ &\geq n+1-\left(s+\frac{n-2s-b}{2}+b\right) \\ &= \frac{n-b+2}{2} \\ &> 2b. \end{aligned}$$

By the inequalities $s+c'b < (c'+1)b \leq 2c'b$ and the fact that $\frac{r+c'a}{s+c'b}$ and $\frac{a_l}{b_l}$ are consecutive Farey fractions, we (once again by Lemma 1) then get

$$\begin{aligned} a_l &= \frac{1+b_l(r+c'a)}{s+c'b} \\ &> \frac{2bc'a}{2c'b} \\ &= a. \end{aligned}$$

Since both $a_l > a$ and $b_l > b$, we deduce that, even when $\frac{a_l}{b_l} > \frac{a}{b}$ is the smallest Farey fraction outside of the segment, $\frac{a}{b}$ and $\frac{a_l}{b_l}$ are still similarly ordered. We therefore conclude that $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered in this case if $l-k \leq d'-c'+2$ holds, so in particular whenever $l-k \leq \min(d-c, d'-c')+2$.

Analogously, if $\frac{a_l}{b_l} = \frac{a}{b}$, then $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered as well, as long as $l-k \leq \min(d-c, d'-c')+2$.

As for the third and final case, assume that $\frac{a_k}{b_k} = \frac{p+ea}{q+eb}$ and $\frac{a_l}{b_l} = \frac{r+e'a}{s+e'b}$ are two fractions contained in the segment, with $\frac{a_k}{b_k} < \frac{a}{b} < \frac{a_l}{b_l}$, $c \leq e \leq d$ and $c' \leq e' \leq d'$. We then aim to prove that they are similarly ordered too. Define $X := a_l - a_k = r+e'a-p-ea$ and $Y := b_l - b_k = s+e'b-q-eb$. We then get

$$\begin{aligned} bX - aY &= (br-as)+(aq-bp) \\ &= 1+1. \end{aligned}$$

Here, the second equality follows from the fact that $\frac{p}{q}$, $\frac{a}{b}$ and $\frac{r}{s}$ were consecutive fractions in the Farey sequence of order b. Since $bX - aY = 2$ with $a \geq 1$ and $b \geq 2$, this implies that X and Y cannot have opposite signs, which is what we wanted to show. So in this third case we conclude that $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered whenever $l-k \leq \min(d-c, d'-c')+2$ as well.

It therefore remains to calculate this latter quantity. By applying the aforementioned values of c, c', d, d' we obtain

$$\begin{aligned}\min(d-c, d'-c') &= \min\left(\left\lfloor\frac{n-q}{b}\right\rfloor - \left\lfloor\frac{n-2q-b}{2b}\right\rfloor, \left\lfloor\frac{n-s}{b}\right\rfloor - \left\lfloor\frac{n-2s-b}{2b}\right\rfloor\right) - 1 \\ &\geq \min\left(\frac{n-q}{b} - \frac{n-2q-b-1}{2b}, \frac{n-s}{b} - \frac{n-2s-b-1}{2b}\right) - 2 \\ &= \frac{n+b+1}{2b} - 2.\end{aligned}$$

We conclude that if $l-k \leq \frac{n+b+1}{2b}$, then $l-k \leq \min(d-c, d'-c')+2$, which in all three cases was sufficient to deduce that $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered. □

Note that, in light of the proof of Theorem 1, Lemma 2 is essentially optimal for $b=2$. Now, before we continue with the statement and proof of our main lower bound, we need two more preliminary lemmas, where we define N to be the number of Farey fractions of order n.

Lemma 3. *For all positive integers n we have $N > \frac{n^2}{4}$.*

Proof (sketch). With a computer one can check the inequality for all $n < 56$, so assume $n \geq 56$. With $\varphi(n)$ Euler's totient function, we have $N = 1 + \sum_{i \leq n} \varphi(i)$. By applying Möbius inversion to the identity $n = \sum_{d|n} \varphi(d)$ and rewriting the sum $\sum_{i\leq n} \varphi(i)$, we obtain $N = 1 + \frac{1}{2}\sum_{i\leq n} \mu(i) \left\lfloor \frac{n}{i} \right\rfloor \left(\left\lfloor \frac{n}{i}\right\rfloor + 1\right)$. Since $\left\lfloor \frac{n}{i} \right\rfloor \left(\left\lfloor \frac{n}{i}\right\rfloor + 1\right) > \frac{n^2}{i^2} - \frac{n}{i}$, $\sum_{i\geq 1} \frac{\mu(i)}{i^2} = \frac{6}{\pi^2}$ and $\sum_{i \leq n} \frac{1}{i} < \log(n) + 1$, with some algebra one can deduce $N > \frac{3n^2}{\pi^2} - \frac{n}{2}(\log(n)+2)$ for all $n \geq 1$. Since the latter is larger than $\frac{n^2}{4}$ for $n \geq 56$, this finishes the proof. □

We will furthermore make use of the following tight result that was obtained by Dress in [6].

Lemma 4. *For $\alpha \in [0,1]$, let $A_n(\alpha)$ be the number of Farey fractions of order n in the interval $(0, \alpha)$. For all $\alpha \in [0,1]$ and all $n \in \mathbb{N}$ we then have the bounds*

$$N\left(\alpha - \frac{1}{n}\right) \leq A_n(\alpha) \leq N\left(\alpha + \frac{1}{n}\right).$$

We are now ready to prove our main lower bound.

Theorem 2. *If $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l} > \frac{a_k}{b_k}$ are two fractions in the Farey sequence of order n with $l - k \leq \frac{n}{12}\left(1 - \frac{4}{n^{1/3}}\right)$, then $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are similarly ordered.*

Proof. Taking the contrapositive, let us assume that $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are not similarly ordered. We then see $\frac{a_l}{b_l} \geq \frac{a_k+1}{b_k-1} > \frac{a_k+1}{b_k} \geq \frac{a_k}{b_k} + \frac{1}{n}$, so write $\frac{a_l}{b_l} - \frac{a_k}{b_k} = \frac{x}{n}$ for some $x > 1$. We now aim to show $l-k > \frac{n}{12}\left(1 - \frac{4}{n^{1/3}}\right)$, and by Lemma 2 we may assume $b_i > 6$ for all i with $k \leq i \leq l$. We may further assume $n \geq 4^3 = 64$, as otherwise our upper bound is negative and the statement is trivially true.

Let S_1 be the set of indices i with $k \leq i \leq l-1$ and $\min(b_1, b_{i+1}) \leq \frac{n}{6}$, and let S_2 be those i with $\min(b_1, b_{i+1}) > \frac{n}{6}$. Furthermore, let $i_1, i_2, \ldots, i_t$ be the actual indices for which $b_{i_j} \leq \frac{n}{6}$. With these definitions in mind, we can show that we may assume that at least one of b_{i_1}, b_{i_t} is larger than $n^{1/3}$.

Lemma 5. *If $n \geq 64$, $t \geq 2$, and $\max(b_{i_1}, b_{i_t}) \leq n^{1/3}$, then $l - k > \frac{n}{2}$.*

Proof. If $\max(b_{i_1}, b_{i_t}) \leq n^{1/3}$, then $\frac{a_l}{b_l} - \frac{a_k}{b_k} \geq \frac{a_{i_t}}{b_{i_t}} - \frac{a_{i_1}}{b_{i_1}} \geq \frac{1}{b_{i_1} b_{i_t}} \geq \frac{1}{n^{2/3}}$. Applying Lemma 4 with $\alpha = \frac{a_k}{b_k}$ and $\alpha = \frac{a_k}{b_k} + \frac{1}{n^{2/3}}$, and we obtain that there are at least $N\left(\frac{1}{n^{2/3}} - \frac{2}{n}\right) = \frac{N(n^{1/3}-2)}{n}$ Farey fractions in between $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$. Since $\frac{N(n^{1/3}-2)}{n} > \frac{n(n^{1/3}-2)}{4}$ by Lemma 3 and the latter is at least $\frac{n}{2}$ for $n \geq 64$, the proof is finished. □

With the help of Lemma 5 we can bound the sum of the reciprocals of the b_{i_j}.

Lemma 6. *We have the upper bound*

$$\sum_{j=1}^{t} \frac{1}{b_{i_j}} < \frac{x}{6} + \frac{1}{n^{1/3}}.$$

Proof. If $t = 1$, then we are done by the assumption $b_{i_1} > 6$. If $t > 1$, then

$$\begin{aligned}
\frac{x}{n} + \frac{6}{n^{4/3}} &\geq \frac{6}{n^{4/3}} + \frac{a_{i_t}}{b_{i_t}} - \frac{a_{i_1}}{b_{i_1}} \\
&= \frac{6}{n^{4/3}} + \sum_{j=1}^{t-1} \left(\frac{a_{i_{j+1}}}{b_{i_{j+1}}} - \frac{a_{i_j}}{b_{i_j}} \right) \\
&\geq \frac{6}{n^{4/3}} + \sum_{j=1}^{t-1} \frac{1}{b_{i_j} b_{i_{j+1}}} \\
&\geq \frac{6}{n} \left(\max\left(\sum_{j=1}^{t-1} \frac{1}{b_{i_j}}, \sum_{j=2}^{t} \frac{1}{b_{i_j}} \right) + \frac{1}{n^{1/3}} \right) \\
&> \frac{6}{n} \sum_{j=1}^{t} \frac{1}{b_{i_j}},
\end{aligned}$$

where the final inequality uses Lemma 5. Multiplying both sides by $\frac{n}{6}$ gives the desired result. □

In the spirit of Erdős [3], we will now write $\frac{x}{n}$ as the sum of two sums.

$$
\begin{aligned}
\frac{x}{n} &= \frac{a_l}{b_l} - \frac{a_k}{b_k} \\
&= \sum_{i=k}^{l-1} \left(\frac{a_{i+1}}{b_{i+1}} - \frac{a_i}{b_i} \right) \\
&= \sum_{i=k}^{l-1} \frac{1}{b_i b_{i+1}} \\
&= \sum_{i \in S_1} \frac{1}{b_i b_{i+1}} + \sum_{i \in S_2} \frac{1}{b_i b_{i+1}}
\end{aligned}
$$

Applying $b_i + b_{i+1} > n$ for all i, we see that for the second sum (where $\min(b_i, b_{i+1}) > \frac{n}{6}$) we have $b_i b_{i+1} > \frac{n}{6}\frac{5n}{6} = \frac{5n^2}{36}$. This gives

$$
\sum_{i \in S_2} \frac{1}{b_i b_{i+1}} < \frac{36(l-k)}{5n^2},
$$

or

$$
l - k > \frac{5n^2}{36} \sum_{i \in S_2} \frac{1}{b_i b_{i+1}}.
$$

As for the first sum we have $b_i b_{i+1} > \min(b_i, b_{i+1})\frac{5n}{6}$, while every element in S_1 occurs at most twice as an i with $\min(b_i, b_{i+1}) \leq \frac{n}{6}$. By furthermore applying Lemma 6 we then get

$$
\begin{aligned}
\sum_{i \in S_1} \frac{1}{b_i b_{i+1}} &< \frac{6}{5n} \sum_{i \in S_1} \frac{1}{\min(b_i, b_{i+1})} \\
&\leq \frac{12}{5n} \sum_{j=1}^{t} \frac{1}{b_{i_j}} \\
&< \frac{12}{5n} \left(\frac{x}{6} + \frac{1}{n^{1/3}} \right) \\
&= \frac{2x}{5n} - \frac{12}{5n^{4/3}}.
\end{aligned}
$$

We can now finish our proof as follows:

$$\begin{aligned}
l-k &> \frac{5n^2}{36}\sum_{i\in S_2}\frac{1}{b_ib_{i+1}}\\
&= \frac{5n^2}{36}\left(\frac{x}{n}-\sum_{i\in S_1}\frac{1}{b_ib_{i+1}}\right)\\
&> \frac{5n^2}{36}\left(\frac{x}{n}-\frac{2x}{5n}-\frac{12}{5n^{4/3}}\right)\\
&= \frac{nx}{12}-\frac{n^{2/3}}{3}\\
&> \frac{n}{12}\left(1-\frac{4}{n^{1/3}}\right). \qquad \square
\end{aligned}$$

4 A few final remarks

The proof of Theorem 2 more generally shows the following result on the local density of Farey fractions.

Theorem 3. *Let $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ be two Farey fractions of order n with $\frac{a_l}{b_l}-\frac{a_k}{b_k}=\frac{x}{n}$ for some $x>0$. Then either there exists a Farey fraction $\frac{a}{b}$ with $b<\frac{6}{x}$ and $\frac{a_k}{b_k}\le\frac{a}{b}\le\frac{a_l}{b_l}$, or $l-k>nx\left(\frac{1}{12}-o(1)\right)$.*

However, one can check that a direct application of Lemma 4 already improves upon this more general theorem for $x>2.76$, so its value seems to stem mostly from small values of x.

And on that note, for $\frac{a_k}{b_k}\ge\frac{1}{2}-o(1)$ we have $x\ge\frac{3}{2}-o(1)$ if $\frac{a_k}{b_k}$ and $\frac{a_l}{b_l}$ are not similarly ordered. In this case we get the improved lower bound $l-k>n\left(\frac{1}{8}-o(1)\right)$ which in turn is at most a factor 2 off from optimal, by the proof of Theorem 1.

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