# Partitions with prescribed sum of reciprocals: explicit exponential bounds

Wouter van Doorn

**Abstract**

In [1] Graham showed that for all positive rationals $\alpha$ and all positive integers $m$, there exists an $n_{\alpha,m}$ such that for every positive integer $n \geq n_{\alpha,m}$ distinct positive integers $a_1, a_2, \ldots, a_r$ exist, all larger than or equal to $m$, for which $a_1 + a_2 + \ldots + a_r$ is equal to $n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_r}$ is equal to $\alpha$. The first (asymptotic) upper bounds on $n_{\alpha,1}$ and $n_{\alpha,m}$ were recently established in [2]. To complement these asymptotic bounds, in this paper we will provide a significantly weaker though fully explicit bound on $n_{\alpha,m}$ which is exponential in $M$ and doubly exponential in $\alpha$.

## 1 A quick recap of Graham's first proof

In [1] it is proved that every positive integer $n \geq 78$ can be written as $\displaystyle\sum_{i=1}^{r} a_i = n$ with $\displaystyle\sum_{i=1}^{r} \frac{1}{a_i} = 1$ and all $a_i$ distinct. The proof depends on a table[1] with explicit decompositions of $n$ for certain $n \leq 333$, and then an induction step based on the identities $\frac{1}{a_1} + \ldots + \frac{1}{a_r} = \frac{1}{2} + \frac{1}{2a_1} + \ldots + \frac{1}{2a_r}$ and $\frac{1}{a_1} + \ldots + \frac{1}{a_r} = \frac{1}{3} + \frac{1}{7} + \frac{1}{78} + \frac{1}{91} + \frac{1}{2a_1} + \ldots + \frac{1}{2a_r}$. The first identity shows that a decomposition of $n$ induces a decomposition of $2n+2$, and the second identity shows that a decomposition of $n$ (that does not contain 39 as one of the $a_i$) induces a decomposition of $2n + 179$.

By checking the table in [1], one can see that the largest odd divisor of any integer occurring as an $a_i$ is equal to 99. Moreover, the identities used in the induction step do not add any denominators whose largest odd divisor is larger than 99. Graham's result can therefore be phrased as follows.

**Lemma 1.** *For every positive integer $n \geq 78$ there exist distinct positive integers $a_1, a_2, \ldots, a_r$, all of which have largest odd divisor at most 99, and such that $n = \displaystyle\sum_{i=1}^{r} a_i$ and $\displaystyle\sum_{i=1}^{r} \frac{1}{a_i} = 1$.*

Similar to [1], in the next section we will generalize Lemma 1 to ensure that, at the cost of potentially increasing the (finite) number of counterexamples, for all $m \in \mathbb{N}$ and all positive rationals $\alpha$, a partition of $n$ exists with all parts larger than or equal to $m$ and such that the sum of reciprocals is equal to $\alpha$.

[1] Bill Stephenson noted that there are 3 minor mistakes in this table; the decomposition of 183 omits 33, the decomposition of 231 contains 6 instead of 4, and the decomposition of 237 contains 72 instead of 84.

## 2 Decompositions with large denominators

Let $\alpha = \frac{p}{q}$ be a positive rational with $p$ and $q$ coprime. Let $m$ be a positive integer larger than or equal to $\max\left(10^4, 2q\right)$. In this section we will prove the following generalization of Lemma 1, which gives a specific bound for Theorem 3 in [1], and which is a more explicit (though weaker) version of Theorem 1 and Theorem 3 in [2].

**Theorem 1.** *We have $n_{\alpha,m} \leq 3^{me^{2\alpha}}$ uniformly in $\alpha$ and $m$. That is, every positive integer $n \geq 3^{me^{2\alpha}}$ can be written as a sum of distinct positive integers $a_1, a_2, \dots, a_r$, all larger than or equal to $m$, for which $\displaystyle\sum_{i=1}^{r} \frac{1}{a_i} = \alpha$.*

What we will do to prove Theorem 1 is build up a sum of unit fractions (which will eventually be equal to $\alpha$) step-by-step. In step $i$, we will denote this sum by a simplified fraction $\alpha_i$. In every step we will check two properties:

- First property: either $\alpha_i < \alpha$ and $\alpha - \alpha_i$ has odd denominator, or $\alpha_i = \alpha$.
- Second property: all denominators that occur in the definition of $\alpha_i$ are distinct.

From step 2 onwards, to prove the second property it is sufficient to check that all denominators that are newly added in that step, did not already occur in the previous step.

In order to get started, we need a couple of definitions. For a prime $p$, let $v_p(x)$ denote the $p$-adic valuation of $x$. That is, $v_p(x)$ is such that $p^{v_p(x)}$ is the largest power of $p$ that divides $x$. Let $k \in [m, m+111]$ be such that $v_7(k) \geq 1$ and $3 \leq v_2(k) \neq v_2(q)$. Since out of two consecutive multiples of 56 at least one has the property that its 2-adic valuation is different from $v_2(q)$, such a $k$ indeed exists in the interval $[m, m+111]$. With $e = \max\left(v_2(k), v_2(q)\right)$, write $\alpha - \frac{1}{k} = \frac{p_0}{q_0 2^e}$ with $p_0$ and $q_0$ odd and $e \geq 3$. And with $d$ the smallest (odd) integer larger than $13k$ for which $d \equiv q_0 p_0^{-1} \pmod{2^e}$, define $w = d2^e$. For an odd integer $z$, further define $S_z$ to be the set of all odd integers $i$ with $k < i \leq z$. Finally, define the set $T = \{1, k-1\} \cup \{2^l + 1 | 1 \leq l \leq \frac{k-2}{\log(2)}\}$ and the constant $\beta = k\left\lceil e^{2\alpha}\right\rceil - 1$. With these last two definitions, note that, since $k$ is divisible by both 2 and 7, $\beta$ is not. Similarly, $v_2(i) = v_7(i) = 0$ for all $i \in T$, where the 7-adic valuation follows from the fact that $2^l$ is never congruent to $6 \pmod 7$.

**Step 1.**
Let $\alpha_1$ be the simplified fraction for which the following holds:

$$\alpha_1 := \frac{1}{k} + \frac{1}{w} + \sum_{i \in T} \left(\frac{1}{21 i\beta} + \frac{1}{28 i\beta}\right) + \sum_{i \in T} \frac{1}{196 i\beta}$$

*Proof of first property:* The difference $\alpha - \frac{1}{k} - \frac{1}{w}$ is equal to $\frac{p_0}{q_0 2^e} - \frac{1}{d2^e} = \frac{dp_0 - q_0}{dq_0 2^e}$. This is equal to a fraction with odd denominator, because $dp_0 \equiv q_0 \pmod{2^e}$. On the other hand, $\frac{1}{21i\beta} + \frac{1}{28i\beta} + \frac{1}{196i\beta} = \frac{13}{147i\beta}$. This latter fraction has odd denominator since $\beta$ is odd, while $i$ is odd for all $i \in T$ as well. The fraction $\alpha - \alpha_1$ is therefore equal to a difference of fractions with odd denominator, which makes its denominator odd as well. To see that $\alpha_1$ is smaller than $\alpha$, note that $\sum_{i \in T} \frac{1}{i} < 2$. As $w = d2^e > d > 13k$, we get $\alpha_1 < \frac{1}{k} + \frac{1}{13k} + \frac{26}{147k} < \frac{2}{k} < \alpha$.

*Proof of second property:* Since $w > 13k$, we see that $k \neq w$. Moreover, $k$ and $w$ are not equal to any of the other denominators either, as $v_2(w) \geq v_2(k) \geq 3$, while $v_2(28i\beta) = v_2(196i\beta) = 2$ and $v_2(21i\beta) = 0$. These latter equalities imply that the only remaining possibility is $28i_1\beta = 196i_2\beta$ for some $i_1, i_2 \in T$. This is also impossible however, as $v_7(28i_1\beta) = 1$, while $v_7(196i_2\beta) = 2$.

**Step 2.**
Let $z$ be the largest odd integer for which the following sum is smaller than or equal to $\alpha$:

$$\begin{aligned}\alpha_2 &:= \frac{1}{k} + \frac{1}{w} + \sum_{i \in S_z} \frac{1}{i} + \sum_{i \in T} \left( \frac{1}{21i\beta} + \frac{1}{28i\beta} \right) + \sum_{i \in T} \frac{1}{196i\beta} \\ &= \alpha_1 + \sum_{i \in S_z} \frac{1}{i}.\end{aligned}$$

Since $S_{k-1} = \emptyset$ and $\alpha_1 < \alpha$, we see that $z$ exists and $z \geq k-1$. In fact, we can say a bit more about $z$.

**Lemma 2.** *We have the inequalities* $(k-6)e^{2\alpha} - 1 < z < ke^{2\alpha} - 1$.

*Proof.* For the upper bound, define $z'$ to be the largest odd integer smaller than $ke^{2\alpha} + 1$, and note that $z' \geq ke^{2\alpha} - 1$. It then suffices to show $\frac{1}{k} + \sum_i \frac{1}{i} > \alpha$, where the sum is taken over all $i \in S_{z'}$, as that would imply $z \leq z' - 2 < ke^{2\alpha} - 1$. We will do this by applying the inequality $\frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{b} > \log(b) - \log(a)$ with $a = \frac{k}{2}$ and $b = \frac{z'+1}{2} \geq \frac{ke^{2\alpha}}{2}$.

$$\begin{aligned}\frac{1}{k} + \sum_{i \in S_{z'}} \frac{1}{i} &> \sum_{i=k,\ i \text{ even}}^{z'+1} \frac{1}{i} \\ &= \frac{1}{2} \sum_{i=\frac{k}{2}}^{\frac{z'+1}{2}} \frac{1}{i} \\ &> \frac{1}{2} \left( \log\left( \frac{ke^{2\alpha}}{2} \right) - \log\left( \frac{k}{2} \right) \right) \\ &= \alpha.\end{aligned}$$

For the lower bound, define $z'$ to be the largest odd integer smaller than or equal to $(k-6)e^{2\alpha} + 1$. Since $\alpha_1 < \frac{2}{k}$, in this case it suffices to show $\frac{2}{k} + \sum_i \frac{1}{i} < \alpha$, where the sum is again taken over all $i \in S_{z'}$, as that would imply $z \geq z' > (k-6)e^{2\alpha} - 1$. We will do this by applying the inequality $\frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{b} < \log(b) - \log(a-1)$ with $a = \frac{k-4}{2}$ and $b = \frac{z'-1}{2} \leq \frac{(k-6)e^{2\alpha}}{2}$.

$$\begin{aligned} \frac{2}{k} + \sum_{i \in S_{z'}} \frac{1}{i} &< \sum_{i=k-4,\ i \text{ even}}^{z'-1} \frac{1}{i} \\ &= \frac{1}{2} \sum_{i=\frac{k-4}{2}}^{\frac{z'-1}{2}} \frac{1}{i} \\ &< \frac{1}{2}\left(\log\left(\frac{(k-6)e^{2\alpha}}{2}\right) - \log\left(\frac{k-6}{2}\right)\right) \\ &= \alpha. \end{aligned}$$

□

*Proof of first property:* The inequality $\alpha_2 \leq \alpha$ follows by definition of $S_z$. And since $S_z$ is also by definition a set of odd integers, we see that the difference $\alpha_2 - \alpha_1$ is equal to a sum of fractions with odd denominator. Since $\alpha - \alpha_1$ has odd denominator, we conclude that $\alpha - \alpha_2 = (\alpha - \alpha_1) - (\alpha_2 - \alpha_1)$ does as well.

*Proof of second property:* Again, $S_z$ only contains odd integers, so we only have to check $21i\beta \notin S_z$, as all other denominators are even. But by Lemma 2 we see $z < \beta$, so we are done immediately.

**Step 3.**
If $\alpha - \alpha_2 < \frac{1}{21\beta k+1}$, define $U = \emptyset$. Otherwise, note that, by definition of $z$, $\alpha - \alpha_2 < \frac{1}{z+2} \leq \frac{3}{\beta+12}$, where the last inequality follows from Lemma 2 and the definition of $\beta$. In this case define $u_1$ with $\frac{\beta+12}{3} < u_1 \leq 21\beta k + 1$ to be the smallest odd integer with $u_1 \not\equiv 0 \pmod 7$ for which $\frac{1}{u_1} \leq \alpha - \alpha_2$. Then $\alpha - \alpha_2 - \frac{1}{u_1} < \frac{1}{u_1-4} - \frac{1}{u_1} = \frac{4}{u_1(u_1-4)} < \frac{36}{\beta^2+12\beta}$. If $\alpha - \alpha_2 - \frac{1}{u_1} < \frac{1}{21\beta k+1}$, define $U = \{u_1\}$. Otherwise, define $U = \{u_1, u_2\}$ where $u_2$ with $\frac{\beta^2+12\beta}{36} < u_2 \leq 21\beta k + 1$ is the smallest odd integer with $u_2 \not\equiv 0 \pmod 7$ for which $\frac{1}{u_2} \leq \alpha - \alpha_2 - \frac{1}{u_1}$. Then $\alpha - \alpha_2 - \frac{1}{u_1} - \frac{1}{u_2} < \frac{1}{u_2-4} - \frac{1}{u_2} = \frac{4}{u_2(u_2-4)} < \frac{5184}{\beta^4} < \frac{1}{21\beta k+1}$, where the final inequality uses $\beta > k \geq 10^4$. This leads us to the following sum:

$$\begin{aligned} \alpha_3 &:= \frac{1}{k} + \frac{1}{w} + \sum_{i \in S_z} \frac{1}{i} + \sum_{i \in T}\left(\frac{1}{21i\beta} + \frac{1}{28i\beta}\right) + \sum_{i \in T} \frac{1}{196i\beta} + \sum_{i \in U} \frac{1}{i} \\ &= \alpha_2 + \sum_{i \in U} \frac{1}{i}. \end{aligned}$$

*Proof of first property:* By the definitions of $u_1$ and $u_2$ we deduce $\alpha_3 \leq \alpha$. Moreover, since $i$ is odd for all $i \in U$, we also immediately get that $\alpha_3 - \alpha_2$ can be written as a fraction with odd denominator.

*Proof of second property:* None of $k, w, 21i\beta, 28i\beta, 196i\beta$ is coprime to 14, so we only have to check that $S_z \cap U = \emptyset$. Since $i > z$ for all $i \in U$, this is immediate.

**Step 4.**
Let $L_z$ be the least common multiple of all odd positive integers at most $z$ and define $L$ to be equal to $3d\beta L_z \prod_{i\in U} i$. Then $\alpha - \alpha_3$ can be written as a fraction $\frac{p_1}{L}$ since both $\alpha - \frac{1}{k} - \frac{1}{w} = \frac{dp_0 - q_0}{dq_0 2^e}$ and $\frac{1}{21i\beta} + \frac{1}{28i\beta} + \frac{1}{196i\beta} = \frac{13}{147i\beta}$ for $i \in T$ have denominator dividing $L$, while all other fractions in its sum have a denominator dividing $L$ as well. If $p_1$ is non-zero, then $3p_1$ can be written as a sum of distinct (odd) divisors of $3L$, say $3p_1 = f_1 + f_2 + \ldots + f_l$, by the proofs and discussion in Section 5 of [3]. Now define the set $V = \left\{\frac{3L}{f_1}, \frac{3L}{f_2}, \ldots, \frac{3L}{f_l}\right\}$, or define $V = \emptyset$ if $p_1 = 0$. We then get:

$$\begin{aligned}\alpha_3 &= \alpha - \frac{3p_1}{3L}\\ &= 1 - \frac{\sum_{i=1}^{l} f_i}{3L}\\ &= 1 - \sum_{i\in V}\frac{1}{i}.\end{aligned}$$

We then consider the following sum:

$$\begin{aligned}\alpha_4 := \frac{1}{k} + \frac{1}{w} + \sum_{i\in S_z}\frac{1}{i} + \sum_{i\in T}\left(\frac{1}{21i\beta} + \frac{1}{28i\beta}\right) + \sum_{i\in T}\frac{1}{196i\beta} + \sum_{i\in U}\frac{1}{i} + \sum_{i\in V}\frac{1}{i}\\ = \alpha_3 + \sum_{i\in V}\frac{1}{i}.\end{aligned}$$

*Proof of first property:* The fraction $\alpha_4$ is equal to $\alpha$ by definition of $V$.

*Proof of second property:* For all $i \in V$ we have that $i$ is odd and $\frac{1}{i} \le \alpha - \alpha_3 < \frac{1}{21\beta k+1}$ by definition of $U$. This implies $i > 21\beta k + 1$, whereas all the other denominators are either even or smaller than or equal to $21\beta k + 1$.

Let $Y$ be the sum of all denominators in the definition of $\alpha_4$, and define $X = Y + \sum_{i\in T} i\beta + 76k$.

**Step 5.**
Let $n$ be any positive integer larger than $X$, let $T_2 \subset T$ be such that $j := \sum_{i\in T_2} i \equiv Y - n \pmod{k}$, and define $T_1 = T \setminus T_2$. Note that such a set $T_2$ exists by definition of $T$. Further define $n' = \frac{n-Y+k-j\beta}{k}$ and note that $n'$ is an integer, since $j\beta \equiv -j \equiv n - Y \pmod{k}$. Now let $a'_1, a'_2, \ldots, a'_{r'}$ be distinct positive integers with $\sum_{i=1}^{r'} a'_i = n'$ and $\sum_{i=1}^{r'} \frac{1}{a'_i} = 1$. Such a decomposition exists,

as we claim that $n'$ is larger than 77. Indeed:

$$\begin{aligned} n' &= \frac{1}{k}\left(n - Y + k - j\beta\right) \\ &> \frac{1}{k}\left(X - Y + k - j\beta\right) \\ &= \frac{1}{k}\left(77k + \sum_{i\in T} i\beta - \sum_{i\in T_2} i\beta\right) \\ &\geq 77. \end{aligned}$$

We now consider the following sum:

$$\begin{aligned} \alpha_5 := \sum_{i=1}^{r'} \frac{1}{ka'_i} + \frac{1}{w} + \sum_{i\in S_z}\frac{1}{i} + \sum_{i\in T_1}\left(\frac{1}{21i\beta} + \frac{1}{28i\beta}\right) + \sum_{i\in T_2}\left(\frac{1}{20i\beta} + \frac{1}{30i\beta}\right) \\ + \sum_{i\in T}\frac{1}{196i\beta} + \sum_{i\in U}\frac{1}{i} + \sum_{i\in V}\frac{1}{i} \\ = \alpha_4 + \left(\sum_{i=1}^{r'}\frac{1}{ka'_i} - \frac{1}{k}\right) + \left(\sum_{i\in T_2}\left(\frac{1}{20i\beta} + \frac{1}{30i\beta}\right) - \sum_{i\in T_2}\left(\frac{1}{21i\beta} + \frac{1}{28i\beta}\right)\right). \end{aligned}$$

By noticing the differences between $\alpha_4$ and $\alpha_5$, we see that the sum of the denominators is equal to

$$\begin{aligned} Y + n'k - k + \sum_{i\in T_2}(20i\beta + 30i\beta) - \sum_{i\in T_2}(21i\beta + 28i\beta) &= Y + n'k - k + \sum_{i\in T_2} i\beta \\ &= Y + n'k - k + j\beta \\ &= n. \end{aligned}$$

*Proof of first property:* The two differences between $\alpha_4$ and $\alpha_5$ are that $\frac{1}{k}$ is replaced by $\sum \frac{1}{ka'_i}$ and that $\frac{1}{21i\beta} + \frac{1}{28i\beta}$ for $i \in T_2$ is replaced by $\frac{1}{20i\beta} + \frac{1}{30i\beta}$. Since both equalities are quickly checked, we still get $\alpha_5 = \alpha_4 = \alpha$.

*Proof of second property:* No denominator $ka'_i$ can be equal to $w$, as $w$ is divisible by $d$, which is an odd integer larger than $13k$, while we may assume by Lemma 1 that $a'_i$ is not divisible by an odd integer larger than 100, so that the largest odd divisor of $ka'_i$ is at most $\frac{100k}{8} < 13k$. Furthermore, $ka'_i$ cannot equal any other denominator either, as $v_2(k) \geq 3$, while all other denominators (other than $w$) are at most divisible by 4. On a similar note, $v_2(30i\beta) = 1$ and these are the only denominators in the entire sum divisible by 2 but not by 4. And we finally have to check that $20i\beta$ with $i \in T_2$ does not occur among the other denominators. By realizing $v_2(20i\beta) = 2$ we conclude that the only possibilities that we cannot exclude right away are $20i_1\beta = 28i_2\beta$ and $20i_3\beta = 196i_4\beta$. But $v_7(20i\beta) = 0 < v_7(28i\beta) = 1 < v_7(196i\beta) = 2$.

Since the final step was possible for all $n > X$, to finish off the proof of Theorem 1 it suffices to prove the following Lemma.

**Lemma 3.** *We have the upper bound* $X < 3^{me^{2\alpha}}$.

*Proof.* By combining known bounds on the second Chebyshev function [8] with Lemma 2 and the fact that $z > 2^{13}$, we conclude $L_z < 2^{-13} \cdot 2.83^{ke^{2\alpha}}$. We furthermore use $d < 14k$ and $\beta < 2ke^{2\alpha}$ to calculate

$$\begin{aligned}12L &= 36d\beta L_z \prod_{i \in U} i \\ &< 36 \cdot 14k \cdot 2ke^{2\alpha} \cdot 2^{-13} \cdot 2.83^{ke^{2\alpha}} \cdot 42^2 \cdot e^{4\alpha} \cdot k^2 \\ &< 220k^4 e^{6\alpha} \cdot 2.83^{ke^{2\alpha}}.\end{aligned}$$

Since $Y - k - w > Y - 15k$ is at most the sum of divisors of $12L$, by known bounds on the sum-of-divisors function [9] we deduce

$$\begin{aligned}X &< Y + 3e^{2\alpha}k^3 \\ &< 2(12L)\log\log(12L) + 15k + 3e^{2\alpha}k^3 \\ &< 10^4 k^5 e^{8\alpha} \cdot 2.83^{ke^{2\alpha}} \\ &< (ke^{2\alpha})^6 \cdot 2.83^{ke^{2\alpha}} \\ &< 1.01^{ke^{2\alpha}} \cdot 2.83^{ke^{2\alpha}} \\ &< 2.86^{ke^{2\alpha}} \\ &< 3^{me^{2\alpha}},\end{aligned}$$

where we used the inequalities $10^4 \leq m \leq k \leq m + 111$. □

# 3 A quick way to see that all denominators are distinct

For a compact recap on why all denominators that occur in the definition of $\alpha_5$ are distinct, let us say that a denominator has small size if it is smaller than or equal to $z$, large size if it is larger than $21k\beta + 1$, and medium size if it is neither small nor large. The following table then shows that for every pair of denominators, they differ in at least one property.

| Denominator | $v_2$ | $v_7$ | Odd divisor $> 13m$? | Size |
|---|---|---|---|---|
| $ka'_i$ | $\geq 3$ | | No | |
| $w$ | $\geq 3$ | | Yes | |
| $i \in S_z$ | 0 | | | Small |
| $20i\beta$ | 2 | 0 | | |
| $21i\beta$ | 0 | 1 | | Medium |
| $28i\beta$ | 2 | 1 | | |
| $30i\beta$ | 1 | | | |
| $196i\beta$ | 2 | 2 | | |
| $i \in U$ | 0 | 0 | | Medium |
| $i \in V$ | 0 | | | Large |

# References


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