Partitions with prescribed sum of reciprocals: computational results

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Abstract

For a positive rational α, call a set of distinct positive integers $\{a_1, a_2, \ldots, a_r\}$ an α-partition of n, if the sum of the a_i is equal to n and the sum of the reciprocals of the a_i is equal to α. Define n_α to be the smallest positive integer such that for all $n \geq n_\alpha$ an α-partition of n exists and, for a positive integer $M \geq 2$, define N_M to be the smallest positive integer such that for all $n \geq N_M$ a 1-partition of n exists where M does not divide any of the a_i. In this paper we determine N_M for all M, and find the set of all α such that $n_\alpha \leq 100$.

1 Introduction and overview of results

Let us call a set of positive integers $\{a_1, \ldots, a_r\}$ an α-partition of n, if all a_i are distinct, $a_1 + \ldots + a_r = n$, and $\frac{1}{a_1} + \ldots + \frac{1}{a_r} = \alpha$. For a positive integer $M \geq 2$, define an α-partition to be M-free if none of the a_i are divisible by M. On the other hand, for a set of primes P, define an α-partition $\{a_1, \ldots, a_r\}$ to be P-full, if all prime factors of the a_i are contained in P. To give an example, $\{2, 3, 6\}$ is a 1-partition of 11, as $2 + 3 + 6 = 11$ and $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$. It is furthermore $\{2, 3\}$-full, as $2, 3$ and 6 only have 2 and 3 as prime factors. One can note that, whenever M is divisible by a prime $p \notin P$, any P-full partition is M-free.

Interest in M-free or P-full partitions emerges quite naturally from the study of α-partitions in the first place. For example, when Graham in [1] obtained the surprising result that 1-partitions of n exist for all $n \geq 78$, he actually proved that P-full 1-partitions of $n \geq 78$ exist, with P equal to $\{2, 3, 5, 7, 11, 13\}$. And in the same paper he then immediately uses this stronger result to prove the existence of 1-partitions of large n with arbitrarily large smallest element.

Where M-free and P-full partitions really shine however, is in the study of n_α; the smallest positive integer such that there exists an α-partition of n, for all $n \geq n_\alpha$. For example, it turns out that M-free α-partitions for just a single α (which will often be $\alpha = 1$ in our case), can help pin down the value of $n_{\alpha'}$, for infinitely many other values of α'. This paper is therefore dedicated to proving the existence of such M-free 1-partitions, and in a follow-up paper we will then apply these results to bound quantities like n_α and $|\{\alpha : n_\alpha \leq n\}|$.

And on the topic of M-free 1-partitions, in [3] Köhler and Spilker improve on Graham's original result, by showing that $\{2, 3, 5\}$-full 1-partitions of n exist for all $n \geq 113$. In particular this implies that M-free 1-partitions exist for all $n \geq 113$ and all M divisible by a prime larger than 5. They then asked whether this can be further improved to the existence of $\{2, 3\}$-full 1-partitions for all

large enough n. In Section 5 we will answer this question in the affirmative, by showing that $\{2,3\}$-full 1-partitions exist for all $n \geq 531$.

This leaves open the question whether M-free 1-partitions exist for large enough n, when M can be written as $M = 2^x 3^y$. In Section 6 we will then deal with the case $x + y \geq 2$, and show that M-free 1-partitions exist for all $n \geq 183$, if M is divisible by $4, 6$ or 9. So the only two remaining cases are $M = 2$ and $M = 3$.

For these latter two cases we come to a negative result, as we will show that a 2-free α-partition of n implies $n \equiv \alpha \pmod 8$, while a 3-free α-partition of n implies $n \equiv \alpha \pmod 3$. On the other hand, for $n \equiv 1 \pmod 3$ we prove in Section 7 that $\{2,5\}$-full 1-partitions exist if $n \geq 3634$. And in Section 8 we show for $n \equiv 1 \pmod 8$ that $\{3,5,7\}$-full 1-partitions exist if $n \geq 3609$, answering a question raised by Köhler and Spilker in [2]. Since all $M \geq 2$ are then dealt with, this concludes our results on M-free 1-partitions.

Let $A(n)$ be defined as the set of all positive rationals α with $n_\alpha \leq n$. After showing $1 \in A(78)$ in [1], Graham proved more generally that the infinite union $\bigcup_{n \geq 1} A(n)$ is equal to the set of all positive rationals. To date, this is (to the best of our knowledge) still all that is known about these sets $A(n)$. In Section 9 we will ameliorate this state of affairs by fully determining $A(n)$ for all $n \leq 100$. In particular, we will see that $A(65)$ is the empty set, while $A(100)$ turns out to contain 4314 rationals, with the set difference $A(n) \setminus A(n-1)$ non-empty for all n with $66 \leq n \leq 100$.

Perhaps the most interesting result we obtain along the way is a metatheorem we prove in Section 4. For a given set S of positive rationals and any property Q that an α-partition might have, this metatheorem provides a general framework for how a proof can be structured, if one wants to show that for all large enough n and all $\alpha \in S$, an α-partition of n with property Q exists.

With Theorem 6 in [6], Alekseyev proved a special case of such a metatheorem, in order to show that for all positive integers $n > 15707$ a set $\{a_1, \ldots, a_r\}$ of positive integers exists with $\frac{1}{a_1} + \ldots + \frac{1}{a_r} = 1$, $a_1^2 + \ldots + a_r^2 = n$, and $a_i \geq 6$ for all i.

Finally, a few words on the notation we use. If $A = \{a_1, \ldots, a_r\}$ is a set of positive integers and $m \in \mathbb{N}$, then $\sum A$ denotes $a_1 + \ldots + a_r$, $\sum A^{-1}$ is defined as the sum $\frac{1}{a_1} + \ldots + \frac{1}{a_r}$, and we write mA for the set $\{ma_1, \ldots, ma_r\}$.

2 Values of N_M for all $M \geq 2$

Define N_M to be the smallest positive integer such that for all integers $n \geq N_M$ an M-free 1-partition of n exists. As it turns out, for $M = 2$ and $M = 3$ such an N_M does not exist, due to the following lemma.

Lemma 1. *Let A be a set of integers. If all elements of A are odd, then $\sum A \equiv \sum A^{-1} \pmod 8$. Similarly, if none of the integers in A are divisible by 3, then $\sum A \equiv \sum A^{-1} \pmod 3$.*

Proof. With M' equal to either 8 or 3, this follows from the fact that every integer coprime to M' is its own inverse modulo M'. That is, $a \equiv \frac{1}{a} \pmod{M'}$ for all integers a coprime to M', and hence $\sum A \equiv \sum A^{-1} \pmod{M'}$. $\square$

In particular, if $\sum A^{-1} = 1$, then $\sum A \equiv 1 \pmod{M'}$. For $M = 2$ and $M = 3$ we will therefore redefine N_M to the smallest positive integer such that an M-free 1-partition of n exists for $n = N_M$, and for all integers $n \geq N_M$ with $n \equiv 1 \pmod 8$ or $n \equiv 1 \pmod 3$ respectively. We are then able to state the full theorem on the values of N_M, that we will then prove in the coming sections.

Theorem 1. *The values of N_M can be found in the following table. For all positive integers $M \geq 2$ not present in this table we have $N_M = 78$.*

M	2	3	4	5	6	7	8	9	10	11	12
N_M	≤ 737	154	155	126	183	97	101	91	108	92	98

M	14	15	16	18	20	21	22	24	28	30	33
N_M	81	108	78	91	106	81	92	80	81	108	92

We will have to split up the proof of Theorem 1 into a few different cases, depending on the existence of certain divisors of M. For example, in [3] Köhler and Spilker prove that $\{2, 3, 5\}$-full 1-partitions exist for all $n \geq 113$. This in principle already takes care of all M with a prime divisor larger than 5, and in Section 3 we will do just that. In the sections afterwards we will then deal with the case where the largest prime factor of M is at most 5. As for $M = 2$, it is possible to combine a brute-force search with arguments similar to ones used in [5] (which, for example, imply that every M-free 1-partition of $n \leq 729$ can only use integers whose largest prime factor is either 47 or at most 31) to prove that N_M is exactly equal to 737, but this becomes hard to verify. We therefore only claim the upper bound $N_M \leq 737$ (as opposed to equality) for $M = 2$.

3   When M has a prime divisor larger than 5

In [3] Köhler and Spilker prove that 91 is the only positive integer $n \geq 78$ for which a $\{2, 3, 5, 7\}$-full 1-partition does not exist. This is due to the fact that $\{3, 4, 6, 11, 12, 22, 33\}$ is the unique 1-partition of 91. This immediately gives the following corollary.

Corollary 1. *If $M \in \{11, 22, 33\}$, then $N_M = 92$. And if the largest prime divisor of M is at least 11 with $M \notin \{11, 22, 33\}$, then $N_M = 78$.*

When the largest prime divisor of M is (exactly) equal to 7, we apply the following lemma.

Lemma 2. *If the largest prime divisor of M is equal to 7, then either $M \leq 28$, or M is divisible by $35, 42, 49, 56$ or 63.*

Proof. If $\frac{M}{7}$ is divisible by 5 or 7, then M is divisible by 35 or 49. On the other hand, if $\frac{M}{7}$ is not divisible by 5 or 7, then the largest prime divisor of $\frac{M}{7}$ is at most 3. Now there are three different cases to consider: either $\frac{M}{7}$ is not divisible by 2, or $\frac{M}{7}$ is not divisible by 3, or $\frac{M}{7}$ is divisible by both 2 and 3. In the first case, either $M \leq 21$ or 63 divides M. In the second case, either $M \leq 28$ or 56 divides M. And in the third case, M is divisible by 42. $\qquad \square$

Corollary 2. *If $M = 7$, then $N_M = 97$. If $M \in \{14, 21, 28\}$, then $N_M = 81$. And if the largest prime divisor of M is equal to 7 with $M > 28$, then $N_M = 78$.*

Proof. On the author's GitHub page[1] one can find all 1-partions of n for all $n \leq 200$. In particular, if M is divisible by $35, 42, 49, 56$ or 63, then M-free 1-partitions of n exist if $78 \leq n \leq 112$. As for $M \in \{14, 21, 28\}$, we note that $\{2, 4, 10, 15, 21, 28\}$ is the unique 1-partition of 80, while M-free 1-partitions of n exist if $81 \leq n \leq 112$. And for $M = 7$, there are a total of four 1-partitions of 96, but all of them contain an integer divisible by M, while M-free 1-partitions of n exist if $97 \leq n \leq 112$. Combining all this with the proof by Köhler and Spilker [3] that $\{2, 3, 5\}$-full 1-partitions exist for all $n \geq 113$, finishes the proof. $\quad \square$

4 A metatheorem

Since most of the proofs in the rest of this paper have essentially the same structure, we will first prove a kind of metatheorem that shows how specific proofs fit into a more general framework.

Let S be a set of positive rationals such that for all $\alpha \in S$ we want to prove that for all large enough n, α-partitions of n exist with a certain property Q. To prove this, we choose for every $\alpha \in S$ a positive integer l, positive integers $m_1, \ldots, m_l$, rationals $\beta_1, \ldots, \beta_l$ and sets of positive integers $A_1, \ldots, A_l$ (with A_i non-empty if $m_i = 1$), with the following five properties:

1. For all i, $\beta_i \in S$.

2. For every positive integer n, there exists an i with $n \equiv \sum A_i \pmod{m_i}$.

3. For all i, $\alpha = \sum A_i^{-1} + \frac{\beta_i}{m_i}$.

4. For all i and every β_i-partition B with property Q, we have $A_i \cap m_i B = \emptyset$.

5. For all i and every β_i-partition B with property Q, the union $A_i \cup m_i B$ also has property Q.

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[1] See https://github.com/Woett/Partition-data.

Theorem 2. *Assume that for all $\alpha \in S$ such m_i, β_i, A_i exist, satisfying the above five properties. Further assume that a positive integer X exists such that for all $\alpha \in S$ and all n with $X \le n \le \max_{i,\alpha}\left(\sum A_i + m_i(X-1)\right)$ an α-partition of n exists with property Q. Then α-partitions of n with property Q exist for all $\alpha \in S$ and all $n \ge X$.*

Proof. By induction we may assume that $n > \max_{i,\alpha}\left(\sum A_i + m_i(X-1)\right)$ is a positive integer such that for all $\alpha \in S$ and all n' with $X \le n' < n$ an α-partition of n' with property Q exists. The goal is to prove that for all $\alpha \in S$ an α-partition of n with property Q exists as well. So let $\alpha \in S$ be given, and let i be such that $n \equiv \sum A_i \pmod{m_i}$. By writing $n = \sum A_i + m_i n'$, we claim $n' \ge X$. This latter inequality follows, because if $n' \le X - 1$, then $n \le \sum A_i + m_i(X-1)$, contrary to the assumption $n > \max_{i,\alpha}\left(\sum A_i + m_i(X-1)\right)$. We may therefore apply the induction hypothesis to n', so let B be a β_i-partition of n' with property Q. Then we claim that the union $A := A_i \cup m_i B$ is an α-partition of n with property Q.

Indeed, A_i and $m_i B$ are disjoint by the fourth property above, while A has property Q by the fifth property. We get that the sum $\sum A^{-1}$ of reciprocals is equal to $\sum A_i^{-1} + \sum (m_i B)^{-1} = \sum A_i^{-1} + \frac{\beta_i}{m_i} = \alpha$ by the third property. And finally, the sum $\sum A$ is equal to $\sum A_i + \sum m_i B = \sum A_i + m_i n' = n$. $\square$

To give an example, let us say $S = \{1\}$ and let Q be the property of not containing 1 or 39. We can then define $l = 2$, $m_1 = m_2 = 2$, $\beta_1 = 1$, $\beta_2 = 1$, $A_1 = \{3, 7, 78, 91\}$ and $A_2 = \{2\}$. The reader can then verify that the five properties listed above all hold, where for the fourth property one should realize that $2B$ only contains even integers, but -due to property Q- does not contain 2 or 78.

By Theorem 2 it is then sufficient to find a positive integer X and 1-partitions of n with property Q for all n with $X \le n \le \max_i\left(\sum A_i + m_i(X-1)\right) = 2X + 177$, and it turns out that $X = 78$ works. This is how one can recreate Graham's original proof [1] that every positive integer $n \ge 78$ has a 1-partition with property Q, as a specific instance of our metatheorem.

With $m = m_1 = m_2$, one could have also written the above proof more succinctly as the following table, which in principle contains all necessary data, besides the verification for all n with $X \le n \le 2X + 177$.

α	m	β_1	β_2	A_1	A_2
1	2	1	1	$\{3, 7, 78, 91\}$	$\{2\}$

For most of our proofs we will actually have $l = m_1 = \ldots = m_l$, in which case we will denote the integer that is equal to all of these by m. Moreover, the sets A_i will then often be such that m does not divide any of the elements in A_i,

which makes verifying the fourth property trivial.

Another thing we would like to note in regards to Theorem 2 is that the easiest type of α to deal with are generally those with $\alpha < 2$ and $\alpha - 1 \in S$. For those α, one can (unless adding 1 to a β_1-partition ruins property Q) simply choose $m = 1$, $\beta_1 = \alpha - 1$ and $A_1 = \{1\}$. And as long as $X > 1$, this works for $\alpha = 2$ as well. In fact, adding $\alpha + 1$ to S for some $\alpha \leq 1$ is one example of many where it can be helpful to increase the size of S. Even though increasing S means we need to find more m_i, β_i, A_i, it also means we have more β_i to work with, in order to satisfy the third property. Going back to Graham's proof, we could have also used $S = \{1, \frac{4}{3}, 2\}$, together with the following proof table.

α	m	β_1	β_2	A_1	A_2
1	2	4/3	2	$\{3\}$	$\emptyset$
4/3	2	2	4/3	$\{3\}$	$\{3, 5, 9, 45\}$
2	1	1	-	$\{1\}$	-

With this data it can be checked that $X = 79$ works. And not only do we now prove something about $\alpha = \frac{4}{3}$ and $\alpha = 2$ as well, $\max_{i,\alpha} \sum A_i$ has decreased from 179 to 62 in the process. The latter means that we have to verify fewer partitions to get the induction proof started.

And on the topic of verifying partitions, usually with an induction proof you start out with one or more base cases and then do the induction step. However, for a proof in the style of Theorem 2 it often make sense to do the induction step at the start, as for the base cases you have to check all n with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m_i(X - 1) \right)$. But for this latter quantity to make sense, you first have to figure out what the values of m_i, β_i and A_i are going to be.

In any case, for our research the computer has been an invaluable tool, both for finding m_i, β_i, A_i for every $\alpha \in S$, and especially for verifying that appropriate α-partitions exist for all n with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m_i(X - 1) \right)$, for some X. Hence the title of this paper.

5 When M is divisible by 5

In [3] Köhler and Spilker ask whether every large enough integer n has a 1-partition A such that the largest prime factor of a is at most 3 for all $a \in A$. This turns out to be true.

Proposition 1. *If M is divisible by 5, then M-free 1-partitions exist for all $n \geq 126$. Moreover, $\{2, 3\}$-full 1-partitions exist for all $n \geq 531$.*

Proof. For an odd prime p, define the sets $S_p := \{\frac{4}{p^2}, \frac{6}{p^2}, \ldots, \frac{2p^2 - 2p}{p^2}, 1, 2\}$ and $P_p := \{2, p\}$. Our proof will then fit in the framework of Theorem 2, with $p = 3$,

$S = S_p$ and the property Q the property of being P_p-full. For the rest of this proof we will write p instead of 3, in order to come back to this proof later on and find out what needs to be changed in order to make it work for $p > 3$ as well.

In any case, as we mentioned in Section 4, for $\alpha = 2$, we can choose $m = 1$, $\beta_1 = 1$ and $A_1 = \{1\}$, while for all $\alpha \neq 2$, we choose $m = 2$. With $\alpha = \frac{s}{p^2}$ (where $4 \leq s \leq 2p^2 - 2p$), the rationals β_1 and β_2, and the sets A_1 and A_2, will then depend on the value of s.

For example, for $s \leq p^2 - p$ we choose $\beta_1 = \frac{2s-2}{p^2}$ and $\beta_2 = \frac{2s}{p^2}$, which we claim work with $A_1 = \{p^2\}$ and $A_2 = \emptyset$. Let us check the five listed properties to verify this. First of all, $\beta_1, \beta_2 \in S_p$, since $6 \leq 2s - 2 < 2s \leq 2p^2 - 2p$. As for the second property, $\sum A_1 = p^2 \equiv 1 \pmod 2$, while $\sum A_2 = 0 \equiv 0 \pmod 2$. Thirdly, $\alpha = \frac{s}{p^2}$ is indeed equal to $\sum A_1^{-1} + \frac{\beta_1}{2} = \frac{1}{p^2} + \frac{2s-2}{2p^2} = \frac{2s}{2p^2} = \sum A_2^{-1} + \frac{\beta_2}{2}$. And finally, the fourth and fifth property follow because all sets A_i that we create in this proof will only contain powers of p.

As for $\alpha = \frac{s}{p^2}$ with $p^2 - p < s \leq 2p^2 - 2p$, there are three different cases we need to consider. For all these three types of α we once again need to choose β_1, β_2, A_1 and A_2 in such a way that the five properties hold. And analogously to what we saw in Section 4, we can succinctly write this data down in a table. The motivated reader can then verify that in all three cases the five properties do indeed hold.

$\alpha = sp^{-2}$	$\beta_1 p^2$	$\beta_2 p^2$	A_1	A_2
$p^2 - p < s \leq p^2$	$2s - 2p$	$2s - 2p - 2$	$\{p\}$	$\{p, p^2\}$
$s = p^2 + 1$	$2p^{-2}$	$2p^2 - 2p$	$\{p^2\}$	$\{p, p^2\}$
$p^2 + 3 \leq s \leq 2p^2 - 2p$	$2s - 2p^2$	$2s - 2p^2 - 2$	$\{1\}$	$\{1, p^2\}$

With $X = 814$, on the author's GitHub page one can find 5-free 1-partitions of n for all n with $126 \leq n \leq 530$, P_3-full 1-partitions of n for all n with $531 \leq n < X$, and P_3-full α-partitions of n for all n with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m(X-1) \right) = 2X + 10 = 1638$ and all $\alpha \in S_3 \setminus \{2\}$. By Theorem 2, this finishes the proof of Proposition 1. $\qquad\square$

Proposition 1 takes care of all M which have 5 as their largest prime factor, via the following lemma and subsequent corollary.

Lemma 3. *If the largest prime divisor of M is equal to 5, then either $M \in \{5, 10, 15, 20, 30\}$, or M is divisible by $25, 40, 45$ or 60.*

Proof. One can prove this in analogy with the proof of Lemma 2. $\qquad\square$

Corollary 3. *If $M = 5$, then $N_M = 126$. If $M \in \{10, 15, 30\}$, then $N_M = 108$, while $N_M = 106$ if $M = 20$. And if the largest prime divisor of M is equal to 5 with $M \notin \{5, 10, 15, 20, 30\}$, then $N_M = 78$.*

Proof. We leave this one for the interested reader, and it can of course be done analogously to the proof of Corollary 2. □

In a follow-up paper we will need a result on 5-free partitions for some other rationals as well, so let us quickly state and prove it here.

Lemma 4. *For every integer $k \geq 2$ there exists an $N(k) \leq 106 \cdot 4^k - 98 \cdot 3^k$ such that for every $\alpha \in \{\frac{2}{3^{k-1}}, \frac{4}{3^k}\}$ and every $n \geq N(k)$ a 5-free α-partition of n exists.*

Proof. For $k = 2$ this follows from the data and the proof of Proposition 1, so assume by induction $k \geq 3$ and $N(k-1) \leq 106 \cdot 4^{k-1} - 98 \cdot 3^{k-1}$. For $\alpha = \frac{2}{3^{k-1}}$, define $l = 3$, $m_1 = m_3 = 4$, $m_2 = 2$, $\beta_1 = \beta_2 = \beta_3 = \frac{4}{3^{k-1}}$, and define the sets A_i as follows:

$$A_1 := \{3^{k-1}\}$$
$$A_2 := \emptyset$$
$$A_3 := \{2 \cdot 3^{k-1}, 3^k, 2 \cdot 3^k\}$$

We then get $\sum A_1 \equiv \sum A_3 \equiv 1 \pmod 2$, while $\sum A_1 \not\equiv \sum A_3 \pmod 4$, so that for all $n \in \mathbb{N}$ there exists an i with $n \equiv \sum A_i \pmod{m_i}$.

Let $n \geq 4N(k-1) + 11 \cdot 3^{k-1}$ be a positive integer, let i be such that $n \equiv \sum A_i \pmod{m_i}$, and write $n = \sum A_i + m_i n'$ with $n' \geq N(k-1)$. With B a 5-free β_i-partition of n', one can then check that $A_i \cup m_i B$ is a 5-free α-partition of n.

As for $\alpha = \frac{4}{3^k}$, define $m = 4$, $\beta_1 = \beta_3 = \beta_4 = \frac{4}{3^{k-1}}$, $\beta_2 = \frac{2}{3^{k-1}}$, and define the sets A_i as follows:

$$A_1 := \{3^k\}$$
$$A_2 := \{2 \cdot 3^{k-1}, 2 \cdot 3^k, 3^{k+1}, 3^{k+2}, 2 \cdot 3^{k+2}\}$$
$$A_3 := \{2 \cdot 3^k, 3^{k+1}, 2 \cdot 3^{k+1}\}$$
$$A_4 := \{2 \cdot 3^k, 3^{k+1}, 3^{k+2}, 2 \cdot 3^{k+2}\}$$

Here we have $\sum A_1 \equiv \sum A_3 \equiv 1 \pmod 2$ and $\sum A_2 \equiv \sum A_4 \equiv 0 \pmod 2$, while on the other hand, $\sum A_1 \not\equiv \sum A_3 \pmod 4$, and $\sum A_2 \not\equiv \sum A_4 \pmod 4$.

The rest of the proof for $n \geq 4N(k-1) + 98 \cdot 3^{k-1}$ follows analogously to the proof for $\alpha = \frac{2}{3^{k-1}}$, and we conclude that $N(k)$ exists with $N(k) \leq 4N(k-1) + 98 \cdot 3^{k-1} \leq 106 \cdot 4^k - 98 \cdot 3^k$. □

6   When M is divisible by 4, 6 or 9

The only M which are not covered by the results in [3] or Proposition 1, are those which can be written as $M = 2^x 3^y$ for non-negative integers x, y. In this section we will assume $x + y \geq 2$, in which case M-free 1-partitions exist for all $n \geq 183$.

Proposition 2. *If* 4 *divides* M*, then* M*-free* 1*-partitions exist for all* $n \geq 155$*.* *If* 6 *divides* M*, then* M*-free* 1*-partitions exist for all* $n \geq 183$*. And if* 9 *divides* M*, then* M*-free* 1*-partitions exist for all* $n \geq 91$*.*

Proof. Let M be divisible by 4, 6 or 9. In the framework of Theorem 2 with property Q the property of being M-free, we choose $S = \{\frac{5}{6}, 1\}$ with $m = 5$ and $\beta_i = \frac{5}{6}$ for all $\alpha \in S$ and all i. For $\alpha = 1$ the sets $A_1, \ldots, A_5$ are defined as follows:

$$A_1 := \{2, 11, 13, 21, 22, 26, 33, 273\}$$
$$A_2 := \{2, 7, 11, 21, 22, 154\}$$
$$A_3 := \{2, 7, 13, 14, 39, 91, 182\}$$
$$A_4 := \{2, 7, 14, 21, 23, 46, 161\}$$
$$A_5 := \{2, 3\}$$

For $\alpha = \frac{5}{6}$ the sets $A_1, \ldots, A_5$ are defined as follows:

$$A_1 := \{3, 7, 13, 14, 26, 273\}$$
$$A_2 := \{2, 14, 22, 33, 77, 154\}$$
$$A_3 := \{2, 11, 22, 33\}$$
$$A_4 := \{3, 7, 13, 14, 39, 91, 182\}$$
$$A_5 := \{2, 11, 26, 33, 143\}$$

For all α and i we now have $\sum A_i \equiv i \pmod{5}$ and $\sum A_i^{-1} = \alpha - \frac{1}{6}$, thereby satisfying the second and third property. Since none of the A_i contain any multiples of 4, 5, 6 or 9, one sees that the fourth and fifth properties are also satisfied.

With $X = 211$ and $Y \in \{155, 183, 91\}$ depending on whether M is divisible by $4, 6$ or 9, on the author's GitHub page one can find M-free 1-partitions of n for all n with $Y \leq n \leq \max_{i, \alpha} \left(\sum A_i + m(X - 1) \right) = 5X + 396 = 1451$ and M-free $\frac{5}{6}$-partitions of n for all n with $X \leq n \leq 5X + 396$, finishing the proof. $\qquad\square$

The N_M values in Theorem 1 for all $M > 3$ that are not divisible by a prime larger than 3 now follow from the following lemma, to be proven in analogy with the proof of Lemma 2.

Lemma 5. *If the largest prime divisor of $M > 3$ is at most 3, then either $M \in \{4, 6, 8, 9, 12, 18, 24\}$, or M is divisible by $16, 27$ or 36.*

This settles the N_M values for all $M > 3$.

7   When M equals 3

Even though the proof of the induction step in Proposition 1 does still work when p is larger than 3, the base case of Theorem 2 cannot be true, due to Lemma 1;

if M equals 2 or 3, then not all n with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m_i(X-1) \right)$ can have an M-free α-partition. With M' equal to 8 or 3 (depending on whether $M = 2$ or $M = 3$ respectively), at best only those n with $n \equiv \alpha \pmod{M'}$ can have an M-free α-partition. Fortunately, we can modify Theorem 2 as follows.

Lemma 6 (Modification of Theorem 2). *Assume that for all $\alpha \in S$, m_i, β_i, A_i exist which satisfy the five properties mentioned at the start of Section 4, and such that $\gcd(m_i, M') = \gcd(a, M') = 1$, for all α, i and $a \in A_i$. Further assume that a positive integer X exists such that for all $\alpha \in S$ and all $n \equiv \alpha$ $\pmod{M'}$ with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m_i(X-1) \right)$ an M-free α-partition of n exists with property Q. Then M-free α-partitions of $n \equiv \alpha \pmod{M'}$ with property Q exist for all $\alpha \in S$ and all $n \geq X$.*

Proof. Recall that, for a given $\alpha \in S$ and i such that $n \equiv \sum A_i \pmod{m_i}$, in the proof of Theorem 2 we write $n = \sum A_i + m_i n'$. We then apply the induction hypothesis to n' and assume a β_i-partition of n'. But with the amended base case, this assumption would only be valid if $n' \equiv \beta_i \pmod{M'}$. Fortunately, this turns out to be true due to the second property, if $n \equiv \alpha \pmod{M'}$.

$$\begin{aligned} n' &= m_i^{-1} \left(n - \sum A_i \right) \\ &\equiv m_i \left(\alpha - \sum A_i^{-1} \right) \pmod{M'} \\ &= \beta_i \end{aligned}$$

The rest of the proof of Theorem 2 still holds verbatim. $\qquad\square$

We conclude that the framework of Theorem 2 does still suit our needs, even for M-free partitions with $M \in \{2, 3\}$, if we use the amended base case and as long as all m_i (or all m if the m_i are the same for all i) and $a \in A_i$ are coprime to M' for all α and i. Now, the proof of Proposition 1 uses sets A_i that only contain powers of p, while $m \in \{1, 2\}$ for all $\alpha \in S_p$. So we indeed get $\gcd(m, M') = \gcd(a, M') = 1$, and it is therefore sufficient to verify the base case.

Proposition 3. *Assume $n \equiv 1 \pmod 3$. Then 3-free 1-partitions exist if $n \geq$ 154, while $\{2,5\}$-full 1-partitions exist if $n \geq 3634$.*

Proof. With S_p and P_p as in the proof of Proposition 1 and $X = 6482$, on the author's GitHub page one can find 3-free 1-partitions of n for all $n \equiv 1 \pmod 3$ with $154 \leq n < 3634$, P_5-full 1-partitions of n for all $n \equiv 1 \pmod 3$ with $3634 \leq n < X$, and P_5-full α-partitions of n for all $n \equiv \alpha \pmod 3$ with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m(X-1) \right) = 2X + 28 = 12992$ and all $\alpha \in S_5 \setminus \{2\}$. $\square$

On the other hand, all 34 1-partitions of 151 contain an integer divisible by 3, and we get that N_M is exactly equal to 154 for $M = 3$.

8 When M equals 2

In analogy with Proposition 3, we can prove that 2-free 1-partitions do exist for all large enough integers n with $n \equiv 1 \pmod 8$.

Proposition 4. *Assume $n \equiv 1 \pmod 8$. Then 2-free 1-partitions exist if $n \geq 737$, while $\{3,5,7\}$-full 1-partitions exist if $n \geq 3609$.*

Proof. With $\alpha = 1$, we choose $S = \{\alpha\}$, $m = 15$ (which is coprime to 8), $\beta_i = \alpha$ for all i, and we define the sets $A_1, \ldots, A_{15}$ as follows:

$$A_1 := \{3, 5, 7, 9, 21, 27, 35, 81, 147, 189, 245, 441, 567, 3969\}$$
$$A_2 := \{3, 5, 7, 9, 15, 25, 63, 81, 189, 441, 567, 1225, 1323, 3969\}$$
$$A_3 := \{3, 5, 7, 9, 15, 25, 49, 125, 147, 441, 1225, 1715, 3087, 6125\}$$
$$A_4 := \{3, 5, 7, 9, 21, 27, 35, 63, 147, 189, 245, 1323\}$$
$$A_5 := \{3, 5, 7, 9, 21, 25, 27, 125, 189, 245, 441, 1225, 1323, 6125\}$$
$$A_6 := \{3, 5, 7, 9, 21, 25, 35, 63, 147, 245, 441, 1225\}$$
$$A_7 := \{3, 5, 7, 9, 15, 27, 49, 81, 189, 441, 567, 3969\}$$
$$A_8 := \{3, 5, 7, 9, 21, 25, 35, 63, 175, 245, 441, 1029, 1225, 8575\}$$
$$A_9 := \{3, 5, 7, 9, 15, 25, 35, 147, 441, 1225, 1715, 3087\}$$
$$A_{10} := \{3, 5, 7, 9, 15, 27, 49, 63, 189, 1323\}$$
$$A_{11} := \{3, 5, 7, 9, 15, 27, 49, 63, 343, 441, 1323, 9261\}$$
$$A_{12} := \{3, 5, 7, 9, 15, 25, 49, 63, 441, 1225\}$$
$$A_{13} := \{3, 5, 7, 9, 15, 21, 35, 441, 1715, 3087\}$$
$$A_{14} := \{3, 5, 7, 9, 21, 27, 35, 49, 189, 245, 441, 1323\}$$
$$A_{15} := \{3, 5, 7, 9, 25, 27, 35, 63, 81, 189, 245, 567, 1225, 3969\}$$

One can now check $\sum A_i \equiv i \pmod{15}$ and $\sum A_i^{-1} = \frac{14}{15}$ for all i, taking care of the second and third property. With property Q defined as the property of being $\{3,5,7\}$-full and not containing 1, the fourth and fifth property are also clear, since all sets A_i consist of integers only divisible by $3, 5$ or 7, and do not contain any multiples of 15 larger than 15 itself.

With $X = 3609$, on the author's GitHub page one can find 2-free 1-partitions of n for all $n \equiv 1 \pmod 8$ with $737 \leq n < X$, and 1-partitions of n with property Q for all $n \equiv 1 \pmod 8$ with $X \leq n \leq \max_{i,\alpha} \left(\sum A_i + m(X-1) \right) = 15X + 12963 = 67098$, which finishes the proof. $\square$

For future reference we note that all 2-free 1-partitions A of $n \geq 737$ we create have $1 \notin A$ and $\{3,5\} \subset A$. Considering the set $C = A \setminus \{3,5\}$ then provides the following corollary.

Corollary 4. *If $n \geq 729$ and $n \equiv 1 \pmod 8$, then a 2-free $\frac{7}{15}$-partition C of n exists, with $C \cap \{1,3,5\} = \emptyset$.*

9   The set of all α with $n_\alpha \leq 100$

Define $B(n)$ to be the set of all rationals α for which an α-partition of n exists, and let us define $B(n, N) := \bigcap_{i=n}^{N} B(i)$. Since $A(n)$ is equal to the infinite intersection, we get in particular $A(n) \subseteq B(n, N)$ for all $N \geq n$. The following pseudo-code shows how to create these sets $B(n)$ and $B(n, N)$, where in practice the function `forpart` from the computer program PARI/GP can be applied to loop over all partitions of a positive integer.

Algorithm: Create_B(n)

1. Initialize `B(n) = {}`.
2. For `partition p of n`:
    If `p only contains distinct integers`:
        a. α = `sum of reciprocals of p`.
        b. `B(n) = B(n)` $\cup \{\alpha\}$.
3. Return `B(n)`.

Algorithm: Create_B(n,N)

1. Initialize `B(n,n) = Create_B(n)`.
2. For `i going from n+1 to N`:
    Initialize `B(n,i) = {}`.
    For `partition p of i`:
        If `p only contains distinct integers`:
            a. α = `sum of reciprocals of p`.
            b. If $\alpha \in$ `B(n,i-1)`:
                `B(n,i) = B(n,i)` $\cup \{\alpha\}$.
3. Return `B(n,N)`.

Graham proved that for every positive rational α there exists an n_α such that $\alpha \in A(n_\alpha)$. On the other hand, a uniform lower bound exists.

Theorem 3. *For all positive rationals α we have $n_\alpha \geq 66$.*

Proof. Writing and running the code for *Create_B(n,N)* with $n = 65$ and $N = 78$ gives $B(n, N) = \emptyset$. In particular, $A(65) \subseteq B(65, 78) = \emptyset$. So $\alpha \notin A(m) \subseteq A(65)$ for all α and all $m \leq 65$, so that $n_\alpha \geq 66$. $\square$

Theorem 4. *There exist exactly 4314 positive rationals α with $n_\alpha \leq 100$.*

Proof. Writing and running the code for *Create_B(n,N)* with $n = 100$ and $N = 136$ gives $|B(n, N)| = 4314$, which provides an upper bound on the number of α with $n_\alpha \leq 100$. To prove that for all $\alpha \in B(n, N)$ we actually have $\alpha \in A(n)$, we once again apply the framework of Theorem 2.

For all $\alpha \in B(n, N)$ with $\alpha \leq 2$ and $\alpha - 1 \in B(n, N)$ (and there are 772 of these), we choose $m = 1$, $\beta_1 = \alpha - 1$ and $A_1 = \{1\}$. For all other α we choose $m = 2$. This does mean that, in total, we need to choose more than 7000 rationals β_i and more than 7000 sets A_i. As the margins of this paper are unfortunately too narrow to contain all of this, these $\beta_1, \beta_2, A_1, A_2$ can all be found on the author's GitHub page, along with α-partitions of n' for all $\alpha \in B(n, N)$ and all n' with $X = 100 \leq n' \leq \max_{i,\alpha} \left(\sum A_i + m(X - 1) \right) = 240$. $\qquad \square$

As a small compensation and to at least show something, in the following table one can find the values of m, β_i, A_i for all nine rationals α with $n_\alpha \leq 70$. We should point out however that this does not comprise a self-contained proof for those α with $n_\alpha \leq 70$, as it uses rationals $\beta_i \in A(100) \setminus A(70)$, violating the first property of Theorem 2.

α	m	β_1	β_2	A_1	A_2
4/5	2	14/15	8/5	$\{3\}$	$\emptyset$
7/12	2	1/2	7/6	$\{3\}$	$\emptyset$
9/5	1	4/5	-	$\{1\}$	-
11/12	2	7/6	11/6	$\{3\}$	$\emptyset$
13/12	2	3/2	13/6	$\{3\}$	$\emptyset$
19/12	1	7/12	-	$\{1\}$	-
23/12	1	11/12	-	$\{1\}$	-
25/12	2	13/6	3/2	$\{1\}$	$\{1,3\}$
97/60	1	37/60	-	$\{1\}$	-

With the data we have generated, we can also look at the growth rate of $|A(n)|$ as a function of n. Down below, we have tabulated the number of α for which $n_\alpha = n$, for all n with $65 \leq n \leq 100$. Equivalently, this is the cardinality of the set $A(n) \setminus A(n - 1)$.

n	65	66	67	68	69	70	71	72	73	74	75	76
$|\{\alpha : n_\alpha = n\}|$	0	2	2	2	1	2	4	5	5	7	7	5

n	77	78	79	80	81	82	83	84	85	86	87	88
$|\{\alpha : n_\alpha = n\}|$	12	18	22	32	38	41	48	57	76	82	74	97

n	89	90	91	92	93	94	95	96	97	98	99	100
$|\{\alpha : n_\alpha = n\}|$	117	155	170	194	228	277	306	332	430	473	483	510

The two elements in $A(66)$ are $\frac{4}{5}$ and $\frac{11}{12}$, which in turn implies that the two elements in $A(67) \setminus A(66)$ are $\frac{9}{5}$ and $\frac{23}{12}$. And of course, $\alpha = 1$ shows up as one of the eighteen elements with $n_\alpha = 78$.

As it turns out, it is possible to use the existence of M-free α-partitions in order to non-trivially relate the growth rates of $|A(n)|$ and $|B(n)|$ to one another, and show the following inequalities:

$$e^{\left(\frac{\pi}{\sqrt{3}}-o(1)\right)\sqrt{\frac{n}{\log(n)}}} < |A(n)| \leq |B(n)| < e^{\pi\sqrt{\frac{n}{3}}}$$

As the proofs of these statements are more theoretical in nature and less reliant on computational data, we shall leave this for a follow-up paper.

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