

Any multiset which is complete after removing any two  
elements remains complete after removing a specific  
infinite set

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### Abstract

For an infinite multiset  $A = \{a_1, a_2, \dots\}$  of positive integers, define

$$P(A) = \left\{ \sum_{n \in B} B \subset A \text{ finite} \right\},$$

and assume that the equality  $P(A \setminus \{a_i, a_j\}) = \mathbb{N}$  holds for all indices  $i$  and  $j$ . We then prove that a set  $S \subset A$  of infinite cardinality exists for which  $P(A \setminus S)$  still contains all positive integers.

## 1 Introduction

The set  $A$  of powers of two has the property  $P(A) = \mathbb{N}$ , while  $P(B)$  misses infinitely many positive integers for any proper subset  $B$  of  $A$ . Similarly, the set  $A$  of Fibonacci numbers has the property that for all  $a, b \in A$ ,  $P(A \setminus \{a\}) = \mathbb{N}$ , while  $|\mathbb{N} \setminus P(A \setminus \{a, b\})| = \infty$ .

In [1] Erdős and Graham more generally ask for which non-negative integers  $m < n$  a multiset  $A$  exists for which  $\mathbb{N} \setminus P(A \setminus S)$  is finite for any set  $S$  with  $|S| = m$ , whereas  $\mathbb{N} \setminus P(A \setminus S)$  is infinite for any set  $S$  with  $|S| = n$ . This is listed as problem #348 at [2] and remains unsolved as of writing this. However, if we replace 'finite' and 'infinite' by 'empty' and 'non-empty' instead, then the aim of this short note is to show that for all  $m \geq 2$  such an  $A$  does not exist for any  $n$ . In fact, we will even see that it is possible to remove an infinite subset  $S$  from  $A$  while still having  $P(A \setminus S) = \mathbb{N}$ .

## 2 Main result

Let  $A = \{a_1, a_2, \dots\}$  with  $1 \leq a_1 \leq a_2 \leq \dots$  be a multiset of positive integers and let  $m \geq 2$  be an integer. We can then state our main result right away.

**Theorem.** *If  $P(A \setminus S) = \mathbb{N}$  for all subsets  $S \subset A$  with  $|A| = m$ , then an infinite subset  $S \subset A$  exists such that  $P(A \setminus S)$  still contains all positive integers.*

*Proof.* To prove this, we use the following well-known lemma.

**Lemma 1.** *If  $B = \{b_1, b_2, \dots\} \subset A$  with  $1 \leq b_1 \leq b_2 \leq \dots$ , then  $P(B) = \mathbb{N}$  if, and only if, the following inequality holds for all  $k \geq 0$ :*

$$b_{k+1} \leq 1 + \sum_{i=1}^k b_i. \tag{1}$$

A quick corollary to Lemma 1 is the following.

**Lemma 2.** *The multiset  $A$  has the property that  $P(A \setminus S)$  contains all positive integers for all  $S \subset A$  with  $|S| = m$  if, and only if, the following inequality holds for all  $k \geq -m$ :*

$$a_{k+m+1} \leq 1 + \sum_{i=1}^k a_i. \quad (2)$$

*Proof.* If inequality (2) holds for all  $k \geq -m$ , then inequality (1) also holds for  $B = A \setminus S$  with  $|S| = m$ . Conversely, if (2) does not hold for some  $k$ , then the integer  $a_{k+m+1} - 1$  is not contained in  $P(A \setminus \{a_{k+1}, a_{k+2}, \dots, a_{k+m}\})$ .  $\square$

Assume from now on that  $A$  indeed has the property that  $P(A \setminus S)$  contains all positive integers, for all  $S \subset A$  with  $|S| = m$ . We may furthermore assume without loss of generality that  $a_i$  goes to infinity with  $i$ . The goal is to remove an infinite subset from  $A$  to get a multiset  $B$  for which  $P(B)$  still contains all positive integers.

Choose  $s_1 = a_1$  and, for any  $j \geq 2$ ,  $s_j = a_i$  where  $i$  is the smallest index such that  $a_{i-1} > s_1 + \dots + s_{j-1}$ . Note that this latter inequality implies in particular that for all  $i \geq 2$ , out of  $a_{i-1}$  and  $a_i$  at most one is in  $S$ . With  $B = \{b_1, b_2, \dots\} = A \setminus S$ , we then need to show inequality (1) for all  $k \geq 0$ .

If  $k = 0$ , we deduce  $b_{k+1} = a_2 = 1$  by Lemma 2. On the other hand, if  $k \geq 1$ , let  $j$  be the largest index such that with  $s_j = a_{i_1}$  and  $b_k = a_{i_2}$  we have  $i_1 < i_2$ . We then get  $b_k = a_{k+j}$ , and there are two possibilities. Either  $b_{k+1} = a_{k+j+1}$  or  $a_{k+j+1} = s_{j+1}$  and  $b_{k+1} = a_{k+j+2}$ . In the first case we get

$$\begin{aligned} 1 + \sum_{i=1}^k b_i &= 1 + \sum_{i=1}^{k+j} a_i - \sum_{i=1}^j s_j \\ &> 1 + \sum_{i=1}^{k+j-2} a_i \\ &\geq a_{k+j+m-1} \\ &\geq a_{k+j+1} \\ &= b_{k+1}. \end{aligned}$$

In the second case we get

$$\begin{aligned}
1 + \sum_{i=1}^k b_i &= 1 + \sum_{i=1}^{k+j} a_i - \sum_{i=1}^j s_j \\
&> 1 + \sum_{i=1}^{k+j-1} a_i \\
&\geq a_{k+j+m} \\
&\geq a_{k+j+2} \\
&\geq b_{k+1}.
\end{aligned}$$

□

## References

- [1] P. Erdős, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), Volume 28, Enseignement Math., Geneva, 1980. Also available here.
- [2] T. F. Bloom, <https://www.erdosproblems.com>.