Completeness of exponentially increasing sequences II

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Abstract

For two positive reals α and $\beta \notin \mathbb{Q}$ we prove that every sufficiently large integer can be written as a sum of distinct elements of the sequence

$$(\lceil \alpha \rceil, \lceil \beta \rceil, \lceil 2\alpha \rceil, \lceil 2\beta \rceil, \lceil 4\alpha \rceil, \lceil 4\beta \rceil, \ldots).$$

Hereby we do not answer a question of Graham.

1 Notation and definitions

For a sequence $S = (s_1, s_2, \ldots)$ of positive integers, we denote by $P(S)$ the set of all integers that can be written as a sum of distinct elements of S. We then say that S is complete if $P(S)$ contains all large enough integers, while S is precomplete if, for every positive integer k, $P(S)$ contains k consecutive integers. We remark that this is the terminology used in [1]; in [2] this latter property is called nearly complete. We call a sequence strongly (pre)complete if it remains (pre)complete after removing any finite subsequence, and we say that S is a Σ-sequence if $s_{n+1} - \sum_{i=1}^{n} s_i = O(1)$.

The greek letters α and β will always denote positive reals, and we define the sequences $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$ by $a_n = \lceil 2^{n-1}\alpha \rceil$ and $b_n = \lceil 2^{n-1}\beta \rceil$ respectively, with the a_i and b_i implicitly dependent on α and β. In this paper we are interested in whether the sequence $S(\alpha, \beta) := (a_1, b_1, a_2, b_2, \ldots)$ is (strongly) complete. It is of course not hard to find α and β for which $S(\alpha, \beta)$ is not complete (for example, take $\alpha = \beta = 2$), and in this regard it turns out to be useful to let D be the set of positive dyadic rationals. That is, $d \in D$ if, and only if, integers $m \geq 0$ and $n \geq 1$ exist with $d = \frac{n}{2^m}$. We finally define $i_1 < i_2 < \ldots$ to be the unique sequence of positive integers such that $\beta = N - \sum_{j=1}^{\infty} 2^{-i_j}$ with $N \in \mathbb{N}$, and we note that this sequence is infinite precisely when $\beta \notin D$.

Now, if we replace all ceiling functions by floor functions in the definition of A and B, then the question whether the corresponding sequence is complete (for all positive reals α and β with $\frac{\alpha}{\beta}$ irrational) was asked by Graham in [4]. This question was then repeated in [3] and is now listed as Problem #354 at [5]. Somewhat surprisingly, as it currently stands, the methods we employ in this paper (which are ideas that are mainly taken from [1] and [2]) do not immediately resolve this question that was actually asked. For now, we will therefore focus on the version with ceilings instead, although it seems entirely plausible that some slight tweaking does answer the original question.

2 Preliminaries

In order to work towards our main goal of deciding the completeness of $S(\alpha, \beta)$, we need some preliminary lemmas. For a start, the following two results are (modulo the addition of 'strongly') taken from [2]; there they are Lemma 1 and Lemma 2 as well.

Lemma 1. *If* $(s_1, s_2, \ldots)$ *is a* Σ*-sequence and* $(t_1, t_2, \ldots,)$ *is strongly precomplete, then* $(s_1, t_1, s_2, t_2, \ldots)$ *is strongly complete.*

Lemma 2. *If* $s_{n+1} \leq 2s_n$ *for all large enough* n*, then* $(s_1, s_2, \ldots)$ *is a* Σ*-sequence.*

For the specific sequence we are interested in, we have the following straightforward result.

Lemma 3. *For all* $n \geq 1$ *we have* $a_{n+1} \in \{2a_n - 1, 2a_n\}$*.*

Proof. Let $k \in \mathbb{N}$ and $\epsilon \in (0, 1]$ be such that $2^{n-1}\alpha = k + \epsilon$, so that $a_n = k + 1$. We then see

$$\begin{aligned} a_{n+1} &= \lceil 2^n \alpha \rceil \\ &= \lceil 2k + 2\epsilon \rceil. \end{aligned}$$

And on the one hand we have $\lceil 2k + 2\epsilon \rceil \geq 2k + 1 = 2a_n - 1$, while on the other hand we have $\lceil 2k + 2\epsilon \rceil \leq 2k + 2 = 2a_n$. $\square$

By combining Lemma 2 and Lemma 3 we immediately deduce the following.

Lemma 4. *The sequence* A *is a* Σ*-sequence for all* $\alpha \in \mathbb{R}^+$*.*

In order to apply some of the above lemmas, let us mention two more sequence-specific results on (pre)completeness, where we denote by B_β the corresponding sequence $(\lceil \beta \rceil, \lceil 2\beta \rceil, \lceil 4\beta \rceil, \ldots)$.

Lemma 5. *If* $B_{2^k\beta}$ *is precomplete for some* $k \in \mathbb{N}$*, then so is* B_β*. Moreover, if* B_β *is precomplete for all* $\beta \notin D$*, then* B_β *is strongly precomplete for all* $\beta \notin D$*.*

Proof. Both claims immediately follow from the fact that $B_{2^k\beta}$ is a subsequence of B_β. More precisely, if $B_\beta = (b_1, b_2, \ldots)$, then $B_{2^k\beta} = (b_{k+1}, b_{k+2}, \ldots)$. $\square$

Lemma 6. *If* $P(B)$ *contains two consecutive integers for all* $\beta \notin D$*, then* B *is strongly precomplete for all* $\beta \notin D$*.*

Proof. By Lemma 5 we may assume $\beta > 1$ and it is sufficient to prove that B is precomplete. Now, let $k, k+1, \ldots, k+l$ be any consecutive elements of $P(B_\beta)$ (with $l \geq 1$), while k' and $k'+1$ are both elements of $P(B_{2^{k+l}\beta})$. Then k' and $k'+1$ are contained in $P(B_\beta)$ as well, while neither of their representations has any overlap with the representations of $k, k+1, \ldots, k+l$. In particular, by adding the appropriate representations, we see that $k+k', k+k'+1, \ldots, k+k'+l+1$ are all contained in $P(B_\beta)$, and we are done by induction. $\square$

We remark that Lemma 6 is essentially an application of Lemma 3.3 in [1]. With all of the above in mind, our main result is not far off.

3 Main result

Theorem 1. *Let α and β be positive reals with $\beta \notin D$. Then the sequence $S(\alpha, \beta)$ is strongly complete.*

Combining Lemma 1, Lemma 4 and Lemma 6, we see that, in order to prove Theorem 1, it is sufficient to show that $P(B)$ contains two consecutive integers for all $\beta \notin D$. This however immediately follows from the following Lemma.

Lemma 7. *We have the equality*

$$\sum_{i=1}^{i_{N-1}} b_i = b_{i_N} - 1.$$

Proof. If $i_j \leq k < i_{j+1}$, we more generally claim

$$\sum_{i=1}^{k-1} b_i = b_k - N + j.$$

To see this, we note that, analogously to Lemma 3, we have $b_{n+1} \in \{2b_n - 1, 2b_n\}$. More precisely, the equality $b_{n+1} = 2b_n - 1$ holds precisely when $n + 1$ occurs as one of the i_j. Now, the equality we want to prove definitely holds for $k = 1$, as the left-hand side is 0, while the right-hand side is equal to $b_1 - N = 0$. By induction we now see (assuming $i_j \leq k < i_{j+1}$)

$$\begin{aligned} \sum_{i=1}^{k} b_i &= b_k + \sum_{i=1}^{k-1} b_i \\ &= 2b_k - N + j, \end{aligned}$$

which is exactly equal to either $b_{k+1} - N + j$ or $b_{k+1} - N + (j + 1)$, depending on whether $b_{k+1} < i_{j+1}$ or $b_{k+1} = i_{j+1}$. $\square$

References

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