The smallest set such that every positive integer is the sum of a square and an element from this set

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Abstract

We construct a set A such that every positive integer can be written as the sum of a square and an element from A, with $|A \cap \{1, 2, \ldots, \lfloor x \rfloor\}| < 2\varphi^{5/2}\sqrt{x}$.

Construction of a small valid set

Let A be a set of positive integers with the property that for every positive integer k there exists an integer $n \geq 0$ and an $a \in A$ with $k = n^2 + a$. For ease of reference, we will call such sets *valid*. Furthermore, denote by $A(x) = |A \cap \{1, 2, \ldots, \lfloor x \rfloor\}|$ the number of elements in A that do not exceed x. In [1] Erdős observed that the upper limit $\limsup \frac{A(x)}{\sqrt{x}}$ must be larger than 1 for any valid set A. He furthermore stated that there do exist valid sets for which this limit is finite, so it is natural to wonder what the smallest possible value is, among all valid sets. Determining this minimal value is listed as Problem #33 at [2]. In this short note we will optimize the construction that Erdős gave, and provide the first known improvement on the upper bound.

Theorem. *There exists a valid set A such that $A(x) < 2\varphi^{5/2}\sqrt{x}$ for all $x > 0$. Here, the constant $\varphi \approx 1.618$ is the golden ratio, and $2\varphi^{5/2} \approx 6.66$.*

Proof. As stated, let φ be the golden ratio. We then define A to be the set of all positive integers a such that $\varphi^{2j} \leq a < \varphi^{2j} + 2\varphi^{j+1/2} - 1$ for some $j \geq 0$. We then have to prove that A is valid and find an upper bound on $\frac{A(x)}{\sqrt{x}}$. Let us start with its validity.

Let k be any positive integer and let j be the non-negative integer for which $\varphi^{2j} \leq k < \varphi^{2j+2}$. With $n \in \mathbb{N}_0$ and $\epsilon \in [0, 1)$ defined by $n + \epsilon = \sqrt{k - \varphi^{2j}}$, we claim that $k - n^2$ is an element of A.

On the one hand we have $n \leq \sqrt{k - \varphi^{2j}}$ implying $\varphi^{2j} \leq k - n^2$, while on the other hand we have

$$\begin{aligned}
k - n^2 &= k - (n + \epsilon)^2 + 2\epsilon n + \epsilon^2 \\
&= \varphi^{2j} + 2\epsilon\left(\sqrt{k - \varphi^{2j}} - \epsilon\right) + \epsilon^2 \\
&= \varphi^{2j} + 2\epsilon\sqrt{k - \varphi^{2j}} - \epsilon^2 \\
&< \varphi^{2j} + 2\epsilon\sqrt{\varphi^{2j+2} - \varphi^{2j}} - \epsilon^2 \\
&= \varphi^{2j} + 2\epsilon\varphi^{j+1/2} - \epsilon^2 \\
&< \varphi^{2j} + 2\varphi^{j+1/2} - 1.
\end{aligned}$$

Here, the final inequality follows from the fact that $2\epsilon\varphi^{j+1/2} - \epsilon^2$ is an increasing function of ϵ on the interval $[0, 1]$. We thusly conclude $\varphi^{2j} \leq k - n^2 < \varphi^{2j} + 2\varphi^{j+1/2} - 1$, so that $k - n^2 \in A$ and A is indeed a valid set.

As for bounding $\frac{A(x)}{\sqrt{x}}$, if $\varphi^{2j} \leq x < \varphi^{2j+2}$, then note that for every i with $0 \leq i \leq j$ there are at most $2\varphi^{i+1/2}$ different values of a with $\varphi^{2i} \leq a < \varphi^{2i} + 2\varphi^{i+1/2} - 1$. We therefore have

$$\begin{aligned} A(x) &\leq \sum_{i=0}^{j} 2\varphi^{i+1/2} \\ &= \frac{2\varphi^{j+3/2} - 2\varphi^{1/2}}{\varphi - 1} \\ &= 2\varphi^{j+5/2} - 2\varphi^{3/2} \\ &< 2\varphi^{5/2}\sqrt{x}. \end{aligned}$$ □

One can check that, for the above set A, $\limsup \frac{A(x)}{\sqrt{x}}$ is actually equal to $2\varphi^{5/2}$, by letting x be equal to $\varphi^{2j} + 2\varphi^{j+1/2} - 1$ for a large j.

References

[1] P. Erdős, *Problems and results in additive number theory.* Colloque sur la Théorie des Nombres, 127–137, 1956. Also available here.

[2] T. F. Bloom, Erdős Problem #33, https://www.erdosproblems.com, accessed 03-10-2025.