

# Completeness of exponentially increasing sequences II

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## Abstract

For two positive reals  $\alpha$  and  $\beta \notin \mathbb{Q}$  we prove that every sufficiently large integer can be written as a sum of distinct elements of the sequence

$$(\lceil \alpha \rceil, \lceil \beta \rceil, \lceil 2\alpha \rceil, \lceil 2\beta \rceil, \lceil 4\alpha \rceil, \lceil 4\beta \rceil, \dots).$$

Hereby we do not answer a question of Graham.

## 1 Notation and definitions

For a sequence  $S = (s_1, s_2, \dots)$  of positive integers, we denote by  $P(S)$  the set of all integers that can be written as a sum of distinct elements of  $S$ . We then say that  $S$  is complete if  $P(S)$  contains all large enough integers, while  $S$  is precomplete if, for every positive integer  $k$ ,  $P(S)$  contains  $k$  consecutive integers. We remark that this is the terminology used in [1]; in [2] this latter property is called nearly complete. We call a sequence strongly (pre)complete if it remains (pre)complete after removing any finite subsequence. Finally, if  $s_{n+1} - \sum_{i=1}^n s_i = O(1)$ , then we say that  $S$  is a  $\Sigma$ -sequence.

The greek letters  $\alpha$  and  $\beta$  will always denote positive reals, and we define the sequences  $A = (a_1, a_2, \dots)$  and  $B = (b_1, b_2, \dots)$  by  $a_n = \lceil 2^{n-1} \alpha \rceil$  and  $b_n = \lceil 2^{n-1} \beta \rceil$  respectively, with the  $a_i$  and  $b_i$  implicitly dependent on  $\alpha$  and  $\beta$ . In this paper we are interested in whether the sequence  $S(\alpha, \beta) := (a_1, b_1, a_2, b_2, \dots)$  is (strongly) complete. It is of course not hard to find  $\alpha$  and  $\beta$  for which  $S(\alpha, \beta)$  is not complete (for example, take  $\alpha = \beta = 2$ ), and in this regards it turns out to be useful to let  $D$  be the set of positive dyadic rationals. That is,  $d \in D$  if, and only if, integers  $m \geq 0$  and  $n \geq 1$  exist with  $d = \frac{n}{2^m}$ . We finally define  $i_1 < i_2 < \dots$  to be the unique sequence of positive integers such that  $\beta = N - \sum_{j=1}^{\infty} 2^{-i_j}$  with  $N \in \mathbb{N}$ , and we note that this sequence is infinite precisely when  $\beta \notin D$ .

Now, if we replace all ceiling functions by floor functions in the definition of  $A$  and  $B$ , then the question whether the corresponding sequence is complete (for all positive reals  $\alpha$  and  $\beta$  with  $\frac{\alpha}{\beta}$  irrational) was asked by Graham in [4]. This question was then repeated in [3] and is now listed as Problem #354 at [5]. Somewhat surprisingly, as it currently stands, the methods we employ in this paper (which are ideas that are mainly taken from [1] and [2]) do not immediately resolve this question that was actually asked. For now, we will therefore focus on the version with ceilings instead.

## 2 Preliminaries

In order to work towards our main goal of deciding the completeness of  $S(\alpha, \beta)$ , we need some preliminary lemmas. For a start, the following two results are

taken from [2]; there they are Lemma 1 and Lemma 2 as well.

**Lemma 1.** *If  $S = (s_1, s_2, \dots)$  is a  $\Sigma$ -sequence and  $T = (t_1, t_2, \dots)$  is (strongly) precomplete, then  $(s_1, t_1, s_2, t_2, \dots)$  is (strongly) complete.*

**Lemma 2.** *If  $s_{n+1} \leq 2s_n$  for all large enough  $n$ , then  $(s_1, s_2, \dots)$  is a  $\Sigma$ -sequence.*

For the specific sequence we are interested in, we have the following straightforward result.

**Lemma 3.** *For all  $n \geq 1$  we have  $a_{n+1} \in \{2a_n - 1, 2a_n\}$ .*

*Proof.* Let  $k \in \mathbb{N}$  and  $\epsilon \in (0, 1]$  be such that  $2^{n-1}\alpha = k + \epsilon$ , so that  $a_n = k + 1$ . We then see

$$\begin{aligned} a_{n+1} &= \lceil 2^n \alpha \rceil \\ &= \lceil 2k + 2\epsilon \rceil. \end{aligned}$$

And on the one hand we have  $\lceil 2k + 2\epsilon \rceil \geq 2k + 1 = 2a_n - 1$ , while on the other hand we have  $\lceil 2k + 2\epsilon \rceil \leq 2k + 2 = 2a_n$ .  $\square$

By combining Lemma 2 and Lemma 3 we immediately deduce the following.

**Lemma 4.** *The sequence  $A$  is a  $\Sigma$ -sequence for all  $\alpha \in \mathbb{R}^+$ .*

In order to apply some of the above lemmas, let us mention two more sequence-specific results on (pre)completeness, where we denote by  $B_\beta$  the corresponding sequence  $(\lceil \beta \rceil, \lceil 2\beta \rceil, \lceil 4\beta \rceil, \dots)$ .

**Lemma 5.** *If  $B_{2^k \beta}$  is precomplete for some  $k \in \mathbb{N}$ , then so is  $B_\beta$ . Moreover, if  $B_\beta$  is precomplete for all  $\beta \notin D$ , then  $B_\beta$  is strongly precomplete for all  $\beta \notin D$ .*

*Proof.* Both claims immediately follow from the fact that  $B_{2^k \beta}$  is a subsequence of  $B_\beta$ . More precisely, if  $B_\beta = (b_1, b_2, \dots)$ , then  $B_{2^k \beta} = (b_{k+1}, b_{k+2}, \dots)$ .  $\square$

**Lemma 6.** *If  $P(B)$  contains two consecutive integers for all  $\beta \notin D$ , then  $B$  is strongly precomplete for all  $\beta \notin D$ .*

*Proof.* By Lemma 5 we may assume  $\beta > 1$  and it is sufficient to prove that  $B$  is precomplete. Now, let  $k, k+1, \dots, k+l$  be any consecutive elements of  $P(B_\beta)$  (with  $l \geq 1$ ), while  $k'$  and  $k'+1$  are both elements of  $P(B_{2^{k+l}\beta})$ . Then  $k'$  and  $k'+1$  are contained in  $P(B_\beta)$  as well, while neither of their representations has any overlap with the representations of  $k, k+1, \dots, k+l$ . In particular, by adding the appropriate representations, we see that  $k+k', k+k'+1, \dots, k+k'+l+1$  are all contained in  $P(B_\beta)$ , and we are done by induction.  $\square$

We remark that Lemma 6 is essentially an application of Lemma 3.3 in [1]. With all of the above in mind, our main result is not far off.

### 3 Main result

**Theorem 1.** *Let  $\alpha$  and  $\beta$  be positive reals with  $\beta \notin D$ . Then the sequence  $S(\alpha, \beta)$  is strongly complete.*

Combining Lemma 1, Lemma 4 and Lemma 6, we see that, in order to prove Theorem 1, it is sufficient to show that  $P(B)$  contains two consecutive integers for all positive reals  $\beta \notin D$ . This however immediately follows from the following Lemma.

**Lemma 7.** *We have the equality*

$$\sum_{i=1}^{i_{N-1}} b_i = b_{i_N} - 1.$$

*Proof.* Still to be written. □

### References

- [1] S.A. Burr, P. Erdős, *Completeness properties of perturbed sequences*. J. Number Theory Volume 31, no. 4, 446–455, 1981. Also available here.
- [2] R.L. Graham, *Complete sequences of polynomial values*. Duke Math. Jour., Volume 31, 275–286, 1964. Also available here.
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- [5] T. F. Bloom, Erdős Problem #354, <https://www.erdosproblems.com>, accessed 2025-10-09.