

Completeness of exponentially increasing sequences II

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Abstract

For two positive reals α and $\beta \notin \mathbb{Q}$ we prove that every sufficiently large integer can be written as a sum of distinct elements of the sequence

$$(\lceil \alpha \rceil, \lceil \beta \rceil, \lceil 2\alpha \rceil, \lceil 2\beta \rceil, \lceil 4\alpha \rceil, \lceil 4\beta \rceil, \dots).$$

Hereby we do not answer a question of Graham.

1 Notation and definitions

For a sequence $S = (s_1, s_2, \dots)$ of positive integers, we denote by $P(S)$ the set of all integers that can be written as a sum of distinct elements of S . We then say that S is complete if $P(S)$ contains all large enough integers, while S is precomplete if, for every positive integer k , $P(S)$ contains k consecutive integers. We remark that this is the terminology used in [1]; in [2] this latter property is called nearly complete. We call a sequence strongly (pre)complete if it remains (pre)complete after removing any finite subsequence. Finally, if $s_{n+1} - \sum_{i=1}^n s_i = O(1)$, then we say that S is a Σ -sequence.

The greek letters α and β will always denote positive reals, and we define the sequences $A = (a_1, a_2, \dots)$ and $B = (b_1, b_2, \dots)$ by $a_n = \lceil 2^{n-1}\alpha \rceil$ and $b_n = \lceil 2^{n-1}\beta \rceil$ respectively, with the a_i and b_i implicitly dependent on α and β . In this paper we are interested in whether the sequence $S(\alpha, \beta) := (a_1, b_1, a_2, b_2, \dots)$ is (strongly) complete. It is of course not hard to find α and β for which $S(\alpha, \beta)$ is not complete (for example, take $\alpha = \beta = 2$), and in this regards it turns out to be useful to let D be the set of positive dyadic rationals. That is, $d \in D$ if, and only if, integers $m \geq 0$ and $n \geq 1$ exist with $d = \frac{n}{2^m}$. We finally define $i_1 < i_2 < \dots$ to be the unique sequence of positive integers such that $\beta = N - \sum_{j=1}^{\infty} 2^{-i_j}$ with $N \in \mathbb{N}$, and we note that this sequence is infinite precisely when $\beta \notin D$.

Now, if we replace all ceiling functions by floor functions in the definition of A and B , then the question whether the corresponding sequence is complete (for all positive reals α and β with $\frac{\alpha}{\beta}$ irrational) was asked by Graham in [4]. This question was then repeated in [3] and is now listed as Problem #354 at [5]. Somewhat surprisingly, as it currently stands, the methods we employ in this paper (which are ideas that are mainly taken from [1] and [2]) do not immediately resolve this question that was actually asked. For now, we will therefore focus on the version with ceilings instead.

2 Preliminaries

In order to work towards our main goal of deciding the completeness of $S(\alpha, \beta)$, we need some preliminary lemmas. For a start, the following two results are

taken from [2]; there they are Lemma 1 and Lemma 2 as well.

Lemma 1. *If $S = (s_1, s_2, \dots)$ is a Σ -sequence and $T = (t_1, t_2, \dots)$ is (strongly) precomplete, then $(s_1, t_1, s_2, t_2, \dots)$ is (strongly) complete.*

Lemma 2. *If $s_{n+1} \leq 2s_n$ for all large enough n , then (s_1, s_2, \dots) is a Σ -sequence.*

For the specific sequence we are interested in, we have the following straightforward result.

Lemma 3. *For all $n \geq 1$ we have $a_{n+1} \in \{2a_n - 1, 2a_n\}$.*

Proof. Let $k \in \mathbb{N}$ and $\epsilon \in (0, 1]$ be such that $2^{n-1}\alpha = k + \epsilon$, so that $a_n = k + 1$. We then see

$$\begin{aligned} a_{n+1} &= \lceil 2^n \alpha \rceil \\ &= \lceil 2k + 2\epsilon \rceil. \end{aligned}$$

And on the one hand we have $\lceil 2k + 2\epsilon \rceil \geq 2k + 1 = 2a_n - 1$, while on the other hand we have $\lceil 2k + 2\epsilon \rceil \leq 2k + 2 = 2a_n$. \square

By combining Lemma 2 and Lemma 3 we immediately deduce the following.

Lemma 4. *The sequence A is a Σ -sequence for all $\alpha \in \mathbb{R}^+$.*

In order to apply some of the above lemmas, let us mention two more sequence-specific results on (pre)completeness, where we denote by B_β the corresponding sequence $(\lceil \beta \rceil, \lceil 2\beta \rceil, \lceil 4\beta \rceil, \dots)$.

Lemma 5. *If $B_{2^k \beta}$ is precomplete for some $k \in \mathbb{N}$, then so is B_β . Moreover, if B_β is precomplete for all $\beta \notin D$, then B_β is strongly precomplete for all $\beta \notin D$.*

Proof. Both claims immediately follow from the fact that $B_{2^k \beta}$ is a subsequence of B_β . More precisely, if $B_\beta = (b_1, b_2, \dots)$, then $B_{2^k \beta} = (b_{k+1}, b_{k+2}, \dots)$. \square

Lemma 6. *If $P(B)$ contains two consecutive integers for all $\beta \notin D$, then B is strongly precomplete for all $\beta \notin D$.*

Proof. By Lemma 5 we may assume $\beta > 1$ and it is sufficient to prove that B is precomplete. Now, let $k, k+1, \dots, k+l$ be any consecutive elements of $P(B_\beta)$ (with $l \geq 1$), while k' and $k'+1$ are both elements of $P(B_{2^{k+l} \beta})$. Then k' and $k'+1$ are contained in $P(B_\beta)$ as well, while neither of their representations has any overlap with the representations of $k, k+1, \dots, k+l$. In particular, by adding the appropriate representations, we see that $k+k', k+k'+1, \dots, k+k'+l+1$ are all contained in $P(B_\beta)$, and we are done by induction. \square

We remark that Lemma 6 is essentially an application of Lemma 3.3 in [1]. With all of the above in mind, our main result is not far off.

3 Main result

Theorem 1. *Let α and β be positive reals with $\beta \notin D$. Then the sequence $S(\alpha, \beta)$ is strongly complete.*

Combining Lemma 1, Lemma 4 and Lemma 6, we see that, in order to prove Theorem 1, it is sufficient to show that $P(B)$ contains two consecutive integers for all positive reals $\beta \notin D$. This however immediately follows from the following Lemma.

Lemma 7. *We have the equality*

$$\sum_{i=1}^{i_N-1} b_i = b_{i_N} - 1.$$

Proof. Still to be written. □

References

- [1] S.A. Burr, P. Erdős, *Completeness properties of perturbed sequences*. J. Number Theory Volume 31, no. 4, 446–455, 1981. Also available here.
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