

The set  $\{x^2 + \frac{1}{x}\}$  is strongly complete

Wouter van Doorn

### Abstract

By combining proofs of Graham and Alekseyev we show that, with  $A = \{x^2 + \frac{1}{x}\}$ , the set  $\{\sum_{x \in X} X \subset A \setminus B \text{ finite}\}$  contains all sufficiently large integers for any finite set  $B$ .

### Main result

Define  $A = \{x^2 + \frac{1}{x} | x \in \mathbb{N}\}$ , let  $B = \{b_1^2 + \frac{1}{b_1}, \dots, b_t^2 + \frac{1}{b_t}\}$  be any finite subset of  $A$  and let  $m$  be larger than  $\max_i b_i$ . It is then sufficient to show that for all large enough integers  $n$  there exist distinct positive integers  $a_1, a_2, \dots, a_r$ , all larger than or equal to  $m$ , for which the sum of their reciprocals is equal to 1 and their sum of squares is equal to  $n$ . As we then have  $n+1 = (a_1^2 + \frac{1}{a_1}) + \dots + (a_r^2 + \frac{1}{a_r})$ , while  $a_i^2 + \frac{1}{a_i} \in A \setminus B$  for all  $i$ .

We will essentially copy-paste Graham's proof of his Theorem 2 in [1], with making as few changes as possible in order to adapt it to our situation. We will make use of the same lemma that Graham uses, that he himself proved in [2]:

**Lemma.** *Let  $\frac{p}{q}$  be a positive rational and let  $t$  be an integer relatively prime to  $q$ . Then for all  $s$  there exist positive integers  $k, c_1, \dots, c_k$  such that*

$$s < c_1 < c_2 < \dots < c_k$$

and

$$\frac{p}{q} = \sum_{i=1}^k \frac{1}{tc_i - 1}.$$

Then let us begin the proof of our theorem. By Dirichlet's theorem on primes in an arithmetic progression there is an  $h$  such that  $mh - 1$  is a prime greater than  $10^6$ . Consider the quantity

$$\frac{m-1}{m} - \frac{1}{m(mh-1)} = \frac{(m-1)h-1}{mh-1}.$$

By the lemma (and the fact that there are infinitely many primes) there exist primes  $q_1, q_2, \dots, q_{m^2}$  and positive integers  $k, c_1, c_2, \dots, c_k$  such that:

1.  $mh - 1 < q_1$ .
2.  $mq_i(mq_i - 1) < q_{i+1}$  for  $1 \leq i < m^2$ .
3.  $\frac{(m-1)h-1}{mh-1} > \frac{1}{mq_1-1} + \dots + \frac{1}{mq_{m^2}-1}$ .
4.  $mq_{m^2}(mq_{m^2} - 1) < mc_1 - 1$ .

5.  $c_i < c_{i+1}$  for  $1 \leq i \leq k-1$ .

$$6. \frac{(m-1)h-1}{mh-1} - \left( \frac{1}{mq_1-1} + \dots + \frac{1}{mq_{m^2}-1} \right) = \frac{1}{mc_1-1} + \dots + \frac{1}{mc_k-1}.$$

The preceding lemma implies that 4., 5. and 6. can be satisfied. Thus we have

$$1 = \frac{1}{m} + \frac{1}{m(mh-1)} + \frac{1}{mq_1-1} + \dots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \dots + \frac{1}{mc_k-1}.$$

Notice that if  $(mq_i-1)^{-1}$  is replaced by

$$\frac{1}{mq_i} + \frac{1}{mq_i(mq_i-1)}$$

then all denominators are still distinct and the sum of their squares modulo  $m^2$  has been decreased by 1. Consider the  $m^2$  representations of 1 given by:

$$1 = \frac{1}{m} + \frac{1}{m(mh-1)} + \left( \frac{1}{mq_1} + \frac{1}{mq_1(mq_1-1)} \right) + \dots + \left( \frac{1}{mq_j} + \frac{1}{mq_j(mq_j-1)} \right) \\ + \frac{1}{mq_{j+1}-1} + \dots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \dots + \frac{1}{mc_k-1}$$

for  $1 \leq j \leq m^2$ . Let  $U_j$  denote the sum of the squares of the denominators of the  $j$ -th representation of 1. Then for  $1 \leq j \leq m^2$  the  $U_j$  run through a complete residue system modulo  $m^2$ . By the proof of Theorem 1 of [4] any  $n$  exceeding 8542 is the sum of the squares of denominators of some representation of 1, where the denominators are distinct and are only divisible by primes smaller than  $10^6$ . If

$$1 = \frac{1}{d_1} + \frac{1}{d_2} + \dots + \frac{1}{d_w}$$

for some  $w$  where the  $d_i$  are distinct and  $U = d_1^2 + \dots + d_w^2$ , then

$$1 = \frac{1}{md_1} + \dots + \frac{1}{md_w} + \frac{1}{m(mh-1)} + \left( \frac{1}{mq_1} + \frac{1}{mq_1(mq_1-1)} \right) + \dots \\ + \left( \frac{1}{mq_j} + \frac{1}{mq_j(mq_j-1)} \right) + \frac{1}{mq_{j+1}-1} + \dots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \dots + \frac{1}{mc_k-1}$$

where the sum of the squares of the denominators in the new representation of 1 is

$$m^2U + U_j - m^2.$$

Thus, by using the representations of all numbers greater than 8542 given by Theorem 1 of [4] and applying to each of these the  $m^2$  transformations arising from the  $m^2$  representations of 1 given in the previous paragraph, it follows at once that every integer exceeding  $8542m^2 + U_{m^2} - m^2$  occurs as the sum of the

squares of the denominators of at least one of the new representations of 1. But in each one of these representations all the denominators used are greater than or equal to  $m$ . It remains only to check that any one of these representations contains distinct denominators. The only way this could fail to happen is for some  $md_i$ , to be equal to one of the integers  $mq_j, mq_j(mq_j - 1)$  or  $m(mh - 1)$ . However, since  $mh - 1$  is a prime greater than  $10^6$  as are the  $q_j$ , and all  $d_i$  are only divisible by primes smaller than  $10^6$ , then we can never have equality. Thus, the denominators in each new representation are distinct. This proves the theorem.

## References

- [1] R.L. Graham, *A theorem on partitions*. Journal of the Australian Mathematical Society, vol. 3, 435–441, 1963. Also available here.
- [2] R.L. Graham, *On finite sums of unit fractions*. Proc. London Math. Soc., vol. 14, 193–207, 1964. Also available here.
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- [4] Alekseyev, M. A., *On partitions into squares of distinct integers whose reciprocals sum to 1*. 2019. Available here.
- [5] T. F. Bloom, Erdős Problem #351, <https://www.erdosproblems.com>, accessed 2025-09-15.