

On the length of an interval that contains distinct multiples of the first n positive integers

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Abstract

Confirming a conjecture by Erdős and Pomerance, we prove that there exist intervals of length $\frac{cn \log n}{\log \log n}$ that do not contain distinct multiples of $1, 2, \dots, n$.

1 Introduction

Define $f(n, m)$ to be the least integer so that the interval $(m, m + f(n, m)]$ contains n distinct integers a_1, a_2, \dots, a_n such that i divides a_i for all i . Erdős and Pomerance conjectured in [3] that $\max_m f(n, m) - f(n, n)$ goes to infinity with n . In [2] Erdős even offered 1000 rupees for a solution, and it is now listed as (part of) problem #711 at Bloom's website [1]. In this short note we will settle their conjecture in the affirmative by proving the following theorem.

Theorem 1. *We have the lower bound*

$$\max_m f(n, m) - f(n, n) > \frac{0.36n \log n}{\log \log n}$$

for all large enough $n \in \mathbb{N}$. In particular, if n is sufficiently large, then an interval of length $\frac{0.36n \log n}{\log \log n}$ exists that does not contain distinct multiples of $1, 2, \dots, n$.

We note that the second sentence of Theorem 1 immediately follows from the first, as we trivially have $f(n, n) \geq 0$.

2 Proof of Theorem 1

The proof of Theorem 1 is based on the following fairly straight-forward, but surprisingly powerful, lemma.

Lemma 2. *For all positive integers k and n we have*

$$kn + f(kn, kn) \leq k^2n + f(n, k^2n). \quad (1)$$

Proof. Replacing both n and m in the definition of $f(n, m)$ by kn , we need to show that for every $1 \leq i \leq kn$ there is a multiple a_i of i with $a_i \in (kn, k^2n + f(n, k^2n)]$, where all a_i are distinct. Now, for every $i \in (n, kn]$ we simply choose $a_i = ki \in (kn, k^2n]$, which is certainly divisible by i , while all a_i are distinct as k is non-zero. On the other hand, by the definition of $f(n, m)$ with $m = k^2n$, it is for all $i \in [1, n]$ possible to choose distinct multiples $a_i \in (k^2n, k^2n + f(n, k^2n)]$. By combining the disjoint intervals we conclude that all a_i are indeed contained in $(kn, k^2n + f(n, k^2n)]$. \square

To apply Lemma 2, we will need lower and upper bounds on $f(n, n)$.

Lemma 3. *For all sufficiently large $n \in \mathbb{N}$ we have the following inequalities:*

$$\left(\frac{2}{\sqrt{e}} + o(1) \right) n \sqrt{\frac{\log n}{\log \log n}} < f(n, n) < (2 + o(1)) n \sqrt{\log n}.$$

Both the lower and the upper bound were already proven by Erdős and Pomerance in [3]. With these bounds we are ready to prove Theorem 1.

Proof of Theorem 1. With n a sufficiently large integer, define $k := \left\lceil 0.6 \sqrt{\frac{\log n}{\log \log n}} \right\rceil$ and choose $\epsilon := \frac{1}{100}$. Using the inequality $\frac{2}{\sqrt{e}} > 1.21$ and the fact that n is sufficiently large, the bounds from Lemma 3 then imply, in particular, that

$$f(kn, kn) > (2 + \epsilon)k^2 n \quad (2)$$

and

$$\epsilon k^2 n > f(n, n). \quad (3)$$

Combining Equations (1), (2), and (3) now finishes the proof. Indeed,

$$\begin{aligned} \max_m f(n, m) &\geq f(n, k^2 n) \\ &\geq kn + f(kn, kn) - k^2 n \\ &> (2 + \epsilon)k^2 n - k^2 n \\ &= \epsilon k^2 n + k^2 n \\ &> f(n, n) + \frac{0.36n \log n}{\log \log n}. \end{aligned} \quad \square$$

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References

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