The set $\{n^2 + \frac{1}{n}\}$ is strongly complete

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Abstract

By combining proofs of Graham and Alekseyev we show that, with $A = \{x^2 + \frac{1}{x}\}$, the set $\{\sum_{x \in X} X \subset A \setminus B \text{ finite}\}$ contains all sufficiently large integers for any finite set B.

1 Introduction

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2 Main result

It is sufficient to show that for all positive integers m and all large enough integers n there exist distinct positive integers $a_1, a_2, \ldots, a_r$, all larger than or equal to m, for which their sum of reciprocals is equal to 1 and their sum of squares is equal to n. Because in that case, if the largest element of B is smaller than m, then $\{\sum_{x \in X} X \subset A \setminus B \text{ finite}\}$ contains all integers larger than n. We will essentially copy-paste Graham's proof of his Theorem 2 in [1], with making as few changes as possible in order to adapt it to our situation. We will make use of the same lemma that Graham uses, that he himself proved in [2]:

Lemma. *Let $\frac{p}{q}$ be a positive rational and let t be an integer relatively prime to q. Then for all s there exist positive integers $k, c_1, \ldots, c_k$ such that*

$$s < c_1 < c_2 < \ldots < c_k$$

and

$$\frac{p}{q} = \sum_{i=1}^{k} \frac{1}{tc_i - 1}.$$

By Dirichlet's theorem on primes in an arithmetic progression there is an h such that $mh - 1$ is a prime greater than 10^6. Consider the quantity

$$\frac{m-1}{m} - \frac{1}{m(mh-1)} = \frac{(m-1)h - 1}{mh - 1}.$$

By the lemma (and the fact that there are infinitely many primes) there exist primes $q_1, q_2, \ldots, q_{m^2}$ and positive integers $k, c_1, c_2, \ldots, c_k$ such that:

1. $mh - 1 < q_1$.
2. $mq_i(mq_i - 1) < q_{i+1}$ for $1 \leq i < m^2$.
3. $\dfrac{(m-1)h - 1}{mh - 1} > \dfrac{1}{mq_1 - 1} + \ldots + \dfrac{1}{mq_{m^2} - 1}$.

4. $mq_{m^2}(mq_{m^2}-1) < mc_1 - 1$.

5. $c_i < c_{i+1}$ for $1 \le i \le k-1$.

6. $\dfrac{(m-1)h-1}{mh-1} - \left(\dfrac{1}{mq_1-1} + \ldots + \dfrac{1}{mq_{m^2}-1}\right) = \dfrac{1}{mc_1-1} + \ldots + \dfrac{1}{mc_k-1}$.

The preceding lemma implies that 4., 5. and 6. can be satisfied. Thus we have

$$1 = \frac{1}{m} + \frac{1}{m(mh-1)} + \frac{1}{mq_1-1} + \ldots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \ldots + \frac{1}{mc_k-1}.$$

Notice that if $(mq_i-1)^{-1}$ is replaced by

$$\frac{1}{mq_i} + \frac{1}{mq_i(mq_i-1)}$$

then all denominators are still distinct and the sum of their squares modulo m^2 has been decreased by 1. Consider the m^2 representations of 1 given by:

$$1 = \frac{1}{m} + \frac{1}{m(mh-1)} + \left(\frac{1}{mq_1} + \frac{1}{mq_1(mq_1-1)}\right) + \ldots + \left(\frac{1}{mq_j} + \frac{1}{mq_j(mq_j-1)}\right)$$
$$+ \frac{1}{mq_{j+1}-1} + \ldots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \ldots + \frac{1}{mc_k-1}$$

for $1 \le j \le m^2$. Let U_j denote the sum of the squares of the denominators of the j-th representation of 1. Then for $1 \le j \le m^2$ the U_j run through a complete residue system modulo m^2. By the proof of Theorem 1 of [4] any n exceeding 8542 is the sum of the squares of denominators of some representation of 1, where the denominators are distinct and are only divisible by primes smaller than 10^6. If

$$1 = \frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_w}$$

for some w where the d_i are distinct and $U = d_1^2 + \ldots + d_w^2$, then

$$1 = \frac{1}{md_1} + \ldots + \frac{1}{md_w} + \frac{1}{m(mh-1)} + \left(\frac{1}{mq_1} + \frac{1}{mq_1(mq_1-1)}\right) + \ldots$$
$$+ \left(\frac{1}{mq_j} + \frac{1}{mq_j(mq_j-1)}\right) + \frac{1}{mq_{j+1}-1} + \ldots + \frac{1}{mq_{m^2}-1} + \frac{1}{mc_1-1} + \ldots + \frac{1}{mc_k-1}$$

where the sum of the squares of the denominators in the new representation of 1 is

$$m^2U + U_j - m^2.$$

Thus, by using the representations of all numbers greater than 8542 given by Theorem 1 of [4] and applying to each of these the m^2 transformations arising from the m^2 representations of 1 given in the previous paragraph, it follows at once that every integer exceeding $8542m^2 + U_{m^2} - m^2$ occurs as the sum of the squares of the denominators of at least one of the new representations of 1. But in each one of these representations all the denominators used are greater than or equal to m. It remains only to check that any one of these representations contains distinct denominators. The only way this could fail to happen is for some md_i, to be equal to one of the integers $mq_j, mq_j(mq_j - 1)$ or $m(mh - 1)$. However, since $mh - 1$ is a prime greater than 10^6 as are the q_j, and all d_i are only divisible by primes smaller than 10^6, then we can never have equality. Thus, the denominators in each new representation are distinct. This proves the theorem.

References

[1] R.L. Graham, *A theorem on partitions*. Journal of the Australian Mathematical Society, vol. 3, 435–441, 1963. Also available here.

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[3] P. Erdős, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), Volume 28, Enseignement Math., Geneva, 1980. Also available here.

[4] Alekseyev, M. A., *On partitions into squares of distinct integers whose reciprocals sum to 1*. 2019. Available here.

[5] T. F. Bloom, Erdős Problem #351, https://www.erdosproblems.com, accessed 2025-09-15.