Any multiset which is complete after removing any two elements remains complete after removing a specific infinite set

Wouter van Doorn

Abstract

For an infinite multiset $A = \{a_1, a_2, \dots, \}$ of positive integers, define

$$P(A) = \left\{ \sum_{n \in B} B \subset A \text{ finite} \right\},$$

and assume that the equality $P(A \setminus \{a_i, a_j\}) = \mathbb{N}$ holds for all indices i and j. We then prove that a set $S \subset A$ of infinite cardinality exists for which $P(A \setminus S)$ still contains all positive integers.

1 Introduction

The set A of powers of two has the property $P(A) = \mathbb{N}$, while $P(B)$ misses infinitely many positive integers for any proper subset B of A. Similarly, the set A of Fibonacci numbers has the property that for all $a, b \in A$, $P(A \setminus \{a\}) = \mathbb{N}$, while $|\mathbb{N} \setminus P(A \setminus \{a, b\})| = \infty$.

In [1] Erdős and Graham more generally ask for which non-negative integers $m < n$ a multiset A exists for which $\mathbb{N} \setminus P(A \setminus S)$ is finite for any set S with $|S| = m$, whereas $\mathbb{N} \setminus P(A \setminus S)$ is infinite for any set S with $|S| = n$. This is listed as problem #348 at [2] and remains unsolved as of writing this. However, if we replace 'finite' and 'infinite' by 'empty' and 'non-empty' instead, then the aim of this short note is to show that for all $m \geq 2$ such an A does not exist for any n. In fact, we will even see that it is possible to remove an infinite subset S from A while still having $P(A \setminus S) = \mathbb{N}$.

2 Main result

Let $A = \{a_1, a_2, \dots\}$ with $1 \leq a_1 \leq a_2 \leq \dots$ be a multiset of positive integers and let $m \geq 2$ be an integer. We can then state our main result right away.

Theorem. *If $P(A \setminus S) = \mathbb{N}$ for all subsets $S \subset A$ with $|A| = m$, then an infinite subset $S \subset A$ exists such that $P(A \setminus S)$ still contains all positive integers.*

Proof. To prove this, we use the following well-known lemma.

Lemma 1. *If $B = \{b_1, b_2 \dots\} \subset A$ with $1 \leq b_1 \leq b_2 \leq \dots$, then $P(B) = \mathbb{N}$ if, and only if, the following inequality holds for all $k \geq 0$:*

$$b_{k+1} \leq 1 + \sum_{i=1}^{k} b_i. \tag{1}$$

A quick corollary to Lemma 1 is the following.

Lemma 2. *The multiset A has the property that $P(A \setminus S)$ contains all positive integers for all $S \subset A$ with $|S| = m$ if, and only if, the following inequality holds for all $k \geq -m$:*

$$a_{k+m+1} \leq 1 + \sum_{i=1}^{k} a_i. \tag{2}$$

Proof. If inequality (2) holds for all $k \geq -m$, then inequality (1) also holds for $B = A \setminus S$ with $|S| = m$. Conversely, if (2) does not hold for some k, then the integer $a_{k+m+1} - 1$ is not contained in $P(A \setminus \{a_{k+1}, a_{k+2}, \ldots, a_{k+m}\})$. □

Assume from now on that A indeed has the property that $P(A \setminus S)$ contains all positive integers, for all $S \subset A$ with $|S| = m$. We may furthermore assume without loss of generality that a_i goes to infinity with i. The goal is to remove an infinite subset from A to get a multiset B for which $P(B)$ still contains all positive integers.

From Lemma 2 we deduce in particular, $a_1 = a_2 = \ldots = a_{m+1} = 1$, so let $t \geq m+1 \geq 3$ be such that $a_t = 1$ and $a_{t+1} > 1$. Then Lemma 2 furthermore implies the inequality $a_{t+i} \leq t - m + i$ for all i with $1 \leq i \leq m+1$. Now define $S = \{s_1, s_2, \ldots,\}$ as follows: $s_1 = s_2 = 1, s_3 = a_{t+2}$ and, for any $j \geq 4$, $s_j = a_k$ with a_k such that $a_k > a_{k-1} \geq s_1 + \ldots + s_{j-1}$. Note that this latter inequality implies in particular that for all $k \geq 2$, out of a_{k-1} and a_k at most one is in S. With $B = \{b_1, b_2, \ldots\} = A \setminus S$, we then need to show inequality (1) for all k.

First of all, (1) holds for all $k \leq t-1$, as we have

$$\begin{aligned}
1 + \sum_{i=1}^{k} b_i &\geq 1 = b_1 = b_2 = \ldots = b_{t-2}, \\
1 + \sum_{i=1}^{t-2} b_i &= t - 1 \\
&\geq t - m + 1 \\
&\geq a_{t+1} \\
&= b_{t-1}, \\
1 + \sum_{i=1}^{t-1} b_i &= t - 1 + b_{t-1} \\
&\geq t + 1 \\
&\geq t - m + 3 \\
&\geq a_{t+3} \\
&= b_t.
\end{aligned}$$

Secondly, assume $k \geq t$ with $b_{k+1} < s_4$. Then $b_k = a_{k+3}, b_{k+1} = a_{k+4}$ and

$$\begin{aligned}
1 + \sum_{i=1}^{k} b_i &= 1 + \sum_{i=1}^{k+3} a_i - (s_1 + s_2 + s_3) \\
&= 1 + \sum_{i=1}^{k+1} a_i + (a_{k+2} + a_{k+3}) - (s_1 + s_2 + s_3) \\
&\geq 1 + \sum_{i=1}^{k+1} a_i + (2 + s_3) - (2 + s_3) \\
&\geq a_{k+m+2} \\
&\geq a_{k+4} \\
&= b_{k+1}.
\end{aligned}$$

Thirdly, assume $b_k < s_4 = a_{k+4} \leq b_{k+1} \leq a_{k+5}$. Then

$$\begin{aligned}
1 + \sum_{i=1}^{k} b_i &= 1 + \sum_{i=1}^{k+2} a_i + (a_{k+3}) - (s_1 + s_2 + s_3) \\
&\geq 1 + \sum_{i=1}^{k+2} a_i \\
&\geq a_{k+m+3} \\
&\geq a_{k+5} \\
&\geq b_{k+1}.
\end{aligned}$$

Finally, let $s_1, s_2, \ldots, s_j = a_{j'}$ be those elements of S with $j' < k$ for some $j \geq 4$. Then $b_k = a_{k+j}$, $b_{k+1} \leq a_{k+j+2}$ and

$$\begin{aligned}
1 + \sum_{i=1}^{k} b_i &= 1 + \sum_{i=1}^{k+j} a_i - \sum_{i=1}^{j} s_j \\
&\geq 1 + \sum_{i=1}^{k+j-1} a_i \\
&\geq a_{k+j+m} \\
&\geq a_{k+j+2} \\
&\geq b_{k+1}.
\end{aligned}$$

□

References

[1] P. Erdős, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), Volume 28, Enseignement Math., Geneva, 1980. Also available here.

[2] T. F. Bloom, https://www.erdosproblems.com.