Any multiset which is entirely complete after removing any two elements remains entirely complete after removing a specific infinite set

Wouter van Doorn

Abstract

For an infinite multiset $A = \{a_1, a_2, \dots, \}$ of positive integers, define

$$P(A) = \left\{ \sum_{n \in B} B \subset A \text{ finite} \right\},$$

and assume that the equality $P(A \setminus \{a_i, a_j\}) = \mathbb{N}$ holds for all indices i and j. We then prove that a set $S \subset A$ of infinite cardinality exists for which $P(A \setminus S)$ still contains all positive integers.

1 Introduction

The set A of powers of two has the property $P(A) = \mathbb{N}$, while $P(B)$ misses infinitely many positive integers for any proper subset B of A. Similarly, the set A of Fibonacci numbers has the property that for all $a, b \in A$, $P(A \setminus \{a\}) = \mathbb{N}$, while $|\mathbb{N} \setminus P(A \setminus \{a, b\})| = \infty$.

In [1] Erdős and Graham more generally ask for which non-negative integers $m < n$ a multiset A exists for which $\mathbb{N} \setminus P(A \setminus S)$ is finite for any set S with $|S| = m$, whereas $\mathbb{N} \setminus P(A \setminus S)$ is infinite for any set S with $|S| = n$. This is listed as problem #348 at [2] and remains unsolved as of writing this. However, if we replace 'finite' and 'infinite' by 'empty' and 'non-empty' instead, then the aim of this short note is to show that for all $m \geq 2$ such an A does not exist for any $n > m$. In fact, we will even see that it is possible to remove an infinite subset S from A while still having $P(A \setminus S) = \mathbb{N}$.

2 Main result

Let $A = \{a_1, a_2, \dots\}$ with $1 \leq a_1 \leq a_2 \leq \dots$ be a multiset of positive integers and let $m \geq 2$ be an integer. We can then state our main result right away.

Theorem. *If $P(A \setminus S) = \mathbb{N}$ for all subsets $S \subset A$ with $|A| = m$, then an infinite subset $S \subset A$ exists such that $P(A \setminus S)$ still contains all positive integers.*

Proof. To prove this, we use the following well-known lemma.

Lemma 1. *If $B = \{b_1, b_2 \dots\} \subset A$ with $1 \leq b_1 \leq b_2 \leq \dots$, then $P(B) = \mathbb{N}$ if, and only if, the following inequality holds for all $k \geq 0$:*

$$b_{k+1} \leq 1 + \sum_{i=1}^{k} b_i. \tag{1}$$

A quick corollary to Lemma 1 is the following.

Lemma 2. *The multiset A has the property that $P(A \setminus S)$ contains all positive integers for all $S \subset A$ with $|S| = m$ if, and only if, the following inequality holds for all $k \geq -m$:*

$$a_{k+m+1} \leq 1 + \sum_{i=1}^{k} a_i. \tag{2}$$

Proof. If inequality (2) holds for all $k \geq -m$, then inequality (1) also holds for $B = A \setminus S$ with $|S| = m$. Conversely, if (2) does not hold for some k, then the integer $a_{k+m+1} - 1$ is not contained in $P(A \setminus \{a_{k+1}, a_{k+2}, \ldots, a_{k+m}\})$. □

Assume from now on that A indeed has the property that $P(A \setminus S)$ contains all positive integers, for all $S \subset A$ with $|S| = m$. We may furthermore assume without loss of generality that a_i goes to infinity with i. The goal is to remove an infinite subset from A to get a multiset B for which $P(B)$ still contains all positive integers.

Choose $s_1 = a_1$ and, for any $j \geq 2$, $s_j = a_i$ where i is the smallest index such that $a_{i-1} > s_1 + \ldots + s_{j-1}$. Note that this latter inequality implies in particular that for all $i \geq 2$, out of a_{i-1} and a_i at most one is in S. With $B = \{b_1, b_2, \ldots\} = A \setminus S$, we then need to show inequality (1) for all $k \geq 0$.

If $k = 0$, we deduce $b_{k+1} = a_2 = 1$ by Lemma 2. On the other hand, if $k \geq 1$, let j be the largest index such that with $s_j = a_{i_1}$ and $b_k = a_{i_2}$ we have $i_1 < i_2$. We then get $b_k = a_{k+j}$, and there are two possibilities. Either $b_{k+1} = a_{k+j+1}$ or $a_{k+j+1} = s_{j+1}$ and $b_{k+1} = a_{k+j+2}$. In the first case we get

$$\begin{aligned} 1 + \sum_{i=1}^{k} b_i &= 1 + \sum_{i=1}^{k+j} a_i - \sum_{i=1}^{j} s_j \\ &> 1 + \sum_{i=1}^{k+j-2} a_i \\ &\geq a_{k+j+m-1} \\ &\geq a_{k+j+1} \\ &= b_{k+1}. \end{aligned}$$

In the second case we get

$$\begin{aligned}
1+\sum_{i=1}^{k} b_i &= 1+\sum_{i=1}^{k+j} a_i - \sum_{i=1}^{j} s_j \\
&> 1+\sum_{i=1}^{k+j-1} a_i \\
&\geq a_{k+j+m} \\
&\geq a_{k+j+2} \\
&= b_{k+1}. \qquad \square
\end{aligned}$$

References

[1] P. Erdős, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. Enseign. Math. (2), Volume 28, Enseignement Math., Geneva, 1980. Also available here.

[2] T. F. Bloom, https://www.erdosproblems.com.