The smallest set such that every positive integer is the sum of a square and an element from this set

Wouter van Doorn

Abstract

We construct a set A such that every positive integer can be written as the sum of a square and an element from A, with $\limsup \frac{|A\cap\{1,2,\ldots,N\}|}{\sqrt{N}} = 2\varphi^{5/2}$.

Construction of a small valid set

Let A be a set of positive integers with the property that for every positive integer k there exists an integer $n \geq 0$ and an $a \in A$ with $k = n^2 + a$. For ease of reference, we will call such sets *valid*. Furthermore, denote by $A(x) = |A \cap \{1, 2, \ldots, \lfloor x \rfloor\}|$ the number of elements in A that do not exceed x. In [1] Erdős observed that the limit superior $\limsup \frac{A(x)}{\sqrt{x}}$ must be larger than 1 for any valid set A. He furthermore stated that there do exist valid sets for which this limit is finite, so it is natural to wonder what the smallest possible value is, among all valid sets. Determining this minimal value is listed as Problem #33 at [2]. In this short note we will optimize the construction that Erdős gave, and provide the first known improvement on the upper bound.

Theorem. *There exists a valid set A with $\limsup \frac{A(x)}{\sqrt{x}} = 2\varphi^{5/2}$. Here, the constant $\varphi \approx 1.618$ is the golden ratio, and $2\varphi^{5/2} \approx 6.66$.*

Proof. As stated, let φ be the golden ratio. We then define A to be the set of all positive integers a such that $\varphi^{2j} \leq a < \varphi^{2j} + 2\varphi^{j+1/2} + 1$ for some $j \geq 0$. We then have to prove that A is valid and find the value of $\limsup \frac{A(x)}{\sqrt{x}}$. Let us start with its validity.

Let k be any positive integer and let j be such that $\varphi^{2j} \leq k < \varphi^{2j+2}$. With n defined as $\left\lfloor \sqrt{k - \varphi^{2j}} \right\rfloor$, we claim that $k - n^2$ is an element of A.

On the one hand we have $\varphi^{2j} \leq k - n^2$ by definition of n, while on the other hand we have $(n+1)^2 > k - \varphi^{2j}$ and

$$\begin{aligned} n &= \left\lfloor \sqrt{k - \varphi^{2j}} \right\rfloor \\ &\leq \sqrt{k - \varphi^{2j}} \\ &< \sqrt{\varphi^{2j+2} - \varphi^{2j}} \\ &= \varphi^j \sqrt{\varphi^2 - 1} \\ &= \varphi^{j+1/2}, \end{aligned}$$

by the fact that φ is the golden ratio. Combining the lower bound on $(n+1)^2$ and the upper bound on n gives us

$$\begin{aligned} k - n^2 &= k - (n+1)^2 + 2n + 1 \\ &< \varphi^{2j} + 2n + 1 \\ &< \varphi^{2j} + 2\varphi^{j+1/2} + 1. \end{aligned}$$

We thusly conclude $\varphi^{2j} \leq k - n^2 < \varphi^{2j} + 2\varphi^{j+1/2} + 1$, so that $k - n^2 \in A$ and A is indeed a valid set.

Secondly, if $\varphi^{2j} \leq N < \varphi^{2j+2}$, then for an upper bound on the limit superior, note that for every i there are at most $2 + 2\varphi^{i+1/2}$ different values of a with $\varphi^{2i} \leq a < \varphi^{2i} + 2\varphi^{i+1/2} + 1$. We therefore have

$$\begin{aligned} A(N) &\leq \sum_{i=0}^{j} \left(2 + 2\varphi^{i+1/2}\right) \\ &= 2j + 2 + \frac{2\varphi^{j+3/2} - 2\sqrt{\varphi}}{\varphi - 1} \\ &\leq \left(2\varphi^{5/2} + o(1)\right)\sqrt{N}. \end{aligned}$$

On the other hand, with N equal to $\varphi^{2j} + 2\varphi^{j+1/2} + 1$, we obtain $A(N) \geq \left(2\varphi^{5/2} - o(1)\right)\sqrt{N}$, so that the limit superior is indeed equal to $2\varphi^{5/2}$. $\square$

References

[1] P. Erdős, *Problems and results in additive number theory.* Colloque sur la Théorie des Nombres, 127–137, 1956. Also available here.

[2] T. F. Bloom, Erdős Problem #33, https://www.erdosproblems.com, accessed.