

# The cardinality of a set containing the pairwise sums of four integers

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## Abstract

Choi, Erdős and Szemerédi proved the existence of a constant  $c$  such that for any set  $A \subseteq \{1, 2, \dots, 2n\}$  with  $|A| \geq n + c$  there exist four integers  $b_1, b_2, b_3, b_4$  with  $b_i + b_j \in A$  for  $1 \leq i < j \leq 4$ . We show that one can take  $c = 2338$ .

## 1 Introduction

For an integer  $k \geq 3$ , let  $g_k(n)$  be the smallest integer such that for all sets  $A \subseteq \{1, 2, \dots, 2n\}$  with  $|A| \geq n + g_k(n)$ , one can find distinct integers  $b_1, b_2, \dots, b_k$  (not necessarily contained in  $A$ ) with  $b_i + b_j \in A$  for  $1 \leq i < j \leq k$ . Choi, Erdős and Szemerédi introduced this function in [1], and proved many bounds on it. For example, they proved  $g_3(n) = 2$ ,  $g_4(n) \leq c$  for some constant  $c$ ,  $c_1 \log(n) < g_5(n) < c_2 \log(n)$  for some constants  $c_1, c_2$  and  $c_3 \sqrt{n} < g_6(n) < c_4 \sqrt{n}$  for some constants  $c_3, c_4$ . They did not necessarily attempt to optimize all these constants however, and estimating the value of  $g_k(n)$  is listed as problem #866 at [2]. In this short note we will (without claiming much originality ourselves) revisit their proof that  $g_4(n)$  is bounded, and find an explicit value for it.

## 2 An explicit upper bound

The goal is to find an explicit value of  $c$  in the theorem of Choi, Erdős and Szemerédi.

**Theorem 1.** *Let  $c$  be equal to 2338. If  $A \subseteq \{1, 2, \dots, 2n\}$  is any set with  $|A| \geq n + c$  elements, then there exist distinct integers  $b_1, b_2, b_3, b_4$  with  $b_i + b_j \in A$  for  $1 \leq i < j \leq 4$ .*

In order to be able to prove this, we need a few lemmas, the first of which will essentially be the base case of Lemma A in [1].

**Lemma 1.** *Let  $y_1 < y_2 < \dots < y_t$  be a sequence of integers and set  $y := y_t - y_1$ . If  $t \geq y^{1/2} + y^{1/4} + 2$ , then there exist integers  $x_1, x_2, x_3$  with  $x_2 \neq x_3$  such that  $x_1, x_1 + x_2, x_1 + x_3, x_1 + x_2 + x_3$  are all elements of the sequence. That is, the sequence contains all subset sums containing  $x_1$ .*

*Proof.* If a positive integer  $m$  and indices  $i_1 < i_2, j_1 < j_2$  exist with  $y_{j_1} - y_{i_1} = y_{j_2} - y_{i_2} = m$ , then one can choose  $x_1 = y_{i_2}, x_2 = y_{i_1} - y_{i_2}, x_3 = m$ . However, if such an  $m$  and indices do not exist, then the sequence forms a so-called Golomb ruler; a sequence whose pairwise differences are all distinct. For such a sequence we must have  $t^2 - 2t\sqrt{t} < y$ , by Theorem 4.9 in [3, p. 32], but one can check that this inequality contradicts the assumption  $t \geq y^{1/2} + y^{1/4} + 2$ .  $\square$

We will use the previous lemma to prove what is essentially the induction step of Lemma A in [1].

**Lemma 2.** *Let  $y_1 < y_2 < \dots < y_t$  be a sequence of integers and set  $y := y_t - y_1$ . If  $t \geq 2y^{3/4} + 2y^{1/2} + 1$ , then there exist integers  $x_1, x_2, x_3, x_4$  with  $x_2, x_3, x_4$  distinct such that the sequence contains all subset sums containing  $x_1$ .*

*Proof.* There are a total of  $\frac{1}{2}t(t-1)$  differences  $y_j - y_i$  in the interval  $[1, y-1]$ . Since one can check the inequality  $\frac{1}{2}t(t-1) > 2y(y^{1/2} + y^{1/4} + 2)$ , this implies there is an integer  $m$  that can be written in more than  $t' := y^{1/2} + y^{1/4} + 2$  fully disjoint ways as a difference  $y_{j_1} - y_{i_1} = y_{j_2} - y_{i_2} = \dots = y_{j_{t'}} - y_{i_{t'}} = m$ . Now we apply Lemma 1 to this sequence  $y_{i_1}, y_{i_2}, \dots, y_{i_{t'}}$  to obtain  $x_1, x_2, x_3$ , and set  $x_4 = m$ . And  $x_4$  is distinct from  $x_2$  and  $x_3$ , because if, say,  $x_2 = m$ , then  $x_1 + x_2 + x_4$  is both a  $y_{i_1}$  and a  $y_{j_{t'}}$ , contradicting the fact that these were disjoint.  $\square$

As in [1], the previous lemma can be used to deduce the existence of integers whose pairwise sums are contained in  $A$ , if  $A$  contains sufficiently many even integers.

**Lemma 3.** *Let  $y_1 < y_2 < \dots < y_t$  be a sequence of even integers and set  $y := y_t - y_1$ . If  $t \geq 2y^{3/4} + 2y^{1/2} + 1$ , then there exist distinct integers  $b_1, b_2, b_3, b_4$  such that the sequence contains all pairwise sums  $b_i + b_j$  with  $1 \leq i < j \leq 4$ .*

*Proof.* Apply Lemma 2 with  $b_1 = \frac{1}{2}x_1$  and  $b_i = \frac{1}{2}x_1 + x_i$  for  $2 \leq i \leq 4$ .  $\square$

We are now ready to prove our main result.

*Proof of Theorem 1.* With  $t$  the number of even elements of  $A$ , write  $t = c + d$  for some  $d \geq 0$ . Let us first assume that there are no even elements of  $A$  contained in the interval  $[4d + 4, 2n - 4d - 4]$ .

In that case, there are either at least  $t' := \lceil \frac{1}{2}t \rceil \geq \frac{1}{2}c + \frac{1}{2}d$  even elements of  $A$  in  $[2, 4d + 2]$ , or at least  $t'$  even elements of  $A$  in  $[2n - 4d - 2, 2n]$ . In either situation, with  $y = 4d + 2$  we have an interval of length  $y$  with at least  $t'$  even elements. If we can prove  $t' \geq 2y^{3/4} + 2y^{1/2} + 1$ , then we are done with this case by Lemma 3. But this follows from the inequality  $t' \geq \frac{1}{2}c + \frac{1}{8}(y - 2)$  and the fact that  $\frac{1}{2}c + \frac{1}{8}(y - 2) \geq 2y^{3/4} + 2y^{1/2} + 1$  holds for all  $y \geq 1$ .

We may therefore freely assume the existence of an even element  $2m \in A$  with  $2d + 2 \leq m \leq n - 2d - 2$ . Now let  $(x, y)$  be a pair of odd integers in the interval  $I := [m - 4d - 2, m + 4d + 2]$  with  $x < y$  and  $x + y = 2m$ . With  $b_1, b_2$  even integers such that  $b_1 + b_2 = 2m$  and  $m - 2 \leq b_1 < b_2 \leq m + 2$ , we have  $x + b_1 \geq m - 4d - 2 + m - 2 = 0$ , which implies  $x + b_1 \geq 1$  as this sum is odd. On the other hand,  $y + b_2 \leq m + 4d + 2 + m + 2 = 2m + 4d + 4 \leq 2n$ . Since we have  $2d + 1$  such distinct pairs  $(x, y)$  in  $I$ , while  $A$  misses  $t - c = d$  odd integers in the interval  $[1, 2n]$ , there must be a pair  $(x, y)$  in  $I$  such that  $b_1 + x, b_2 + x, b_1 + y, b_2 + y$  are all in  $A$ . We can then take  $b_3 = x, b_4 = y$ .  $\square$

## References

- [1] S.L.G. Choi, P. Erdős, E. Szemerédi, *Some additive and multiplicative problems in number theory*. Acta Arithmetica, vol. 27, 37–50, 1975. Also available here.
- [2] T. F. Bloom, <https://www.erdosproblems.com>.
- [3] A. Dimitromanolakis, *Analysis of the Golomb ruler and the Sidon set problems and determination of large, near-optimal Golomb rulers..* Diploma thesis, Department of Electronic and Computer Engineering, Technical University of Crete, 2002. Also available here.