On the length of an interval that contains distinct multiples of the first n positive integers

Wouter van Doorn

Abstract

Confirming a conjecture by Erdős and Pomerance, we prove that there exist intervals of length $\frac{cn \log n}{\log \log n}$ that do not contain distinct multiples of $1, 2, \ldots, n$.

A new lower bound

Define $f(n, m)$ to be the least integer so that the interval $(m, m + f(n, m)]$ contains n distinct integers $a_1, a_2, \ldots, a_n$ such that i divides a_i for all i. Erdős and Pomerance conjectured in [1] that $\max_m f(n, m) - f(n, n)$ goes to infinity with n, and in [2] Erdős even offered 1000 rupies for a solution. In this very short note we will settle their conjecture in the affirmative.

Theorem. *We have the lower bound* $\max_m f(n, m) - f(n, n) > \frac{0.36n \log n}{\log \log n}$ *for all large enough* $n \in \mathbb{N}$.

Proof. With n a large enough integer, define $x := \left\lceil 0.6\sqrt{\frac{\log n}{\log \log n}} \right\rceil$ and $\epsilon := \frac{1}{100}$. From the inequalities $1.21n\sqrt{\frac{\log n}{\log \log n}} < f(n, n) < 1.74n\sqrt{\log n}$ (proven in [1]), we in particular obtain $(2 + \epsilon)nx < f(n, n) < \epsilon nx^2$ for sufficiently large n. We will then prove the lower bound $f(n, nx^2) - f(n, n) > nx^2$, which suffices.

With $N := nx$ we can upper bound $f(N, N)$ as follows: for every $i \in (n, N]$ choose $a_i = xi \in (N, Nx]$, while for $i \in [1, n]$ it is by definition of $f(n, m)$ with $m = Nx$ possible to choose $a_i \in (Nx, Nx + f(n, Nx)]$. Now assume by contradiction $f(n, Nx) - f(n, n) \leq nx^2$. We then get the following contradictory string of inequalities:

$$\begin{aligned} f(N, N) &\leq Nx + f(n, Nx) \\ &\leq Nx + f(n, n) + nx^2 \\ &< Nx + \epsilon nx^2 + nx^2 \\ &= (2 + \epsilon)Nx \\ &< f(N, N). \end{aligned}$$

□

References

[1] P. Erdős, C. Pomerance, *Matching the natural numbers up to n with distinct multiples of another interval.* Indagationes Mathematicae, Volume 83, Issue 2, 147–161, 1980. Also available here.

[2] P. Erdős, *Some of my forgotten problems in number theory.* Hardy-Ramanujan Journal, Volume 15, 34–50, 1992. Also available here.