No sequence that grows at least as fast as the Fibonacci sequence is strongly complete

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Abstract

For a sequence $S = (s_1, s_2, \dots)$ of positive integers we say that it is complete if every sufficiently large integer can be written as the sum of distinct elements of S, and we say that S is strongly complete if $S \setminus T$ is complete for any finite subsequence T of S. In [1] the following questions are attributed to Erdős: given $\epsilon > 0$, is there a strongly complete sequence for which $\frac{s_{n+1}}{s_n}$ is larger than $2 - \epsilon$ for all sufficiently large n? Can $\frac{s_{n+1}}{s_n}$ converge to 2 as n goes to infinity? In this short note we will show that the answer is no to both questions. In fact, we will see that any strongly complete sequence must have $\frac{s_{n+1}}{s_n} < \frac{1+\sqrt{5}}{2}$ infinitely often. By a theorem of Graham stating that the sequence $s_n = F_n - (-1)^n$ (with F_n the n-th Fibonacci number) is strongly complete, this is seen to best possible.

1 Introduction

A sequence S of positive integers is called complete if every large enough integer can be written as the sum of distinct elements of S, and it is called strongly complete if $S \setminus T$ remains complete, for any finite subsequence T of S. To give an example, one can show that the sequence defined by $s_n = n^2$ is strongly complete. More generally, in [4] Graham characterized all real-valued polynomials $f(x)$ for which the sequence $(f(1), f(2), \dots)$ is strongly complete, and a further review of the literature (at least up to 1971) on complete and strongly complete sequences can be found in [2]. In this very same paper, Graham mentions the following two questions by Erdős:

Is there, for every $\epsilon > 0$, a strongly complete sequence $(s_1, s_2, \dots)$ for which $\frac{s_{n+1}}{s_n} > 2 - \epsilon$ holds for all large enough n? Is $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 2$ possible?

In this short note we will answer both of these questions in the negative.

2 Main result

Let us state our main result.

Theorem 1. *If $S = (s_1, s_2, \dots)$ is strongly complete, then there are infinitely many n with $\frac{s_{n+1}}{s_n} < \frac{1+\sqrt{5}}{2}$.*

In fact, we will show the following slightly stronger statement:

Theorem 2. *If $S = (s_1, s_2, \dots)$ is a complete sequence for which $\frac{s_{n+1}}{s_n} \geq \frac{1+\sqrt{5}}{2}$ holds for all $n \geq 1$, then $s_1 = 1$.*

To see why Theorem 2 implies Theorem 1, let us take the contrapositive and assume that the inequality $\frac{s_{n+1}}{s_n} < \frac{1+\sqrt{5}}{2}$ holds for only finitely many n, say $n_1, n_2, \ldots, n_k$. Then $S' = (s_{n_k+2}, s_{n_k+3}, \ldots) = (s'_1, s'_2, \ldots)$ is a sequence for which $\frac{s'_{n+1}}{s'_n} \geq \frac{1+\sqrt{5}}{2}$ holds for all n, while $s'_1 \neq 1$. By Theorem 2 S' is not complete, which in particular implies that S is not strongly complete.

Now, before we start the (short) proof of Theorem 2, let us be clear and state that we do not claim much originality in any of this. Indeed, everything is already implicit in [2], in particular in Graham's proof of Theorem 3 of that paper. However, it still seems worth it to us to explicitly write it down, not in the least because Graham himself did not seem to appreciate its relevancy! Even though [2] was referenced just one page before the question by Erdős on the existence of strongly complete sequences with $\frac{s_{n+1}}{s_n} > 2 - \epsilon$ was stated, it seems to have not been realized that Graham's own ideas were already sufficient to settle this matter. In fact, our theorem follows immediately from the theorem by Folkman that was referenced as Theorem 1 in [2], but for the sake of being self-contained, let us prove it ourselves.

Proof of Theorem 2. From the inequality $\frac{s_{n+1}}{s_n} \geq \frac{1+\sqrt{5}}{2}$ it is quickly seen that $s_n + s_{n+1} \leq s_{n+2}$ holds for all $n \geq 1$. Indeed,

$$\begin{aligned} s_n + s_{n+1} &\leq \left(\left(\frac{2}{1+\sqrt{5}} \right)^2 + \frac{2}{1+\sqrt{5}} \right) s_{n+2} \\ &= s_{n+2}. \end{aligned}$$

Now, let us prove the contrapositive and assume $s_1 > 1$. In order to show that S is not complete, define $P(S)$ as the set of all integers that can be written as the sum of distinct elements of S. We then claim that no integer of the form $1 + s_2 + s_4 + \ldots + s_{2k}$ is an element of $P(S)$, which would indeed show that S is not complete. To see this, define

$$m_k = 1 + \sum_{i=1}^{k-1} s_{2i},$$

for any positive integer k (with $m_1 = 1$), and assume by induction $m_k \notin P(A)$ while

$$\sum_{i=1}^{2k-3} s_i < m_k < s_{2k-1}.$$

This is certainly true for $k = 1$ by assumption, as the sum on the left-hand side is empty while $s_{2k-1} = s_1 > 1$. Now, adding s_{2k} to these inequalities and using $s_n + s_{n+1} \leq s_{n+2}$ on both the left- and the right-hand sides, and we deduce

$$\sum_{i=1}^{2k-1} s_i < m_{k+1} < s_{2k+1}.$$

We therefore see that, if $m_{k+1} \in P(S)$, then you would need to use s_{2k} in any representation of m_{k+1}. But this would imply that $m_k = m_{k+1} - s_{2k}$ has a representation as well, which contradicts the induction hypothesis and finishes the proof. $\square$

3 Aftermath

In [3] Graham himself showed that the sequence $S = (s_1, s_2, \ldots)$ defined by $s_n = F_n - (-1)^n$ (with F_n the n-th Fibonacci number) is strongly complete, which shows that Theorem 1 is best possible. Indeed, this gives an example of a strongly complete sequence with $s_{n+1} > \frac{1+\sqrt{5}}{2}s_n - O(1)$ for all $n \geq 1$, barely leaving room to satisfy the inequality from Theorem 1. Moreover, this sequence of s_n has the remarkable property that, even though it is strongly complete, it is no longer complete if any infinite set is removed from it. Graham [3] therefore asked whether any strongly complete sequence $(s_1, s_2, \ldots)$ with the property that it is no longer complete by removing any infinite set, must have $\lim \frac{s_{n_1}}{s_n} = \frac{1+\sqrt{5}}{2}$. This question was repeated in [1] and [5], and can be found as Problem #346 at [6]. Theorem 1 above shows half of this; $\liminf \frac{s_{n_1}}{s_n} \leq \frac{1+\sqrt{5}}{2}$.

A weaker version of Erdős' question also appeared in [5] and can now be found as Problem #347 at [7]; is it possible for a sequence $S = (s_1, s_2, \ldots)$ with $\lim \frac{s_{n_1}}{s_n} = 2$ to have the property that $P(S \setminus T)$ has density 1 for every finite subsequence T? It does not seem that the above ideas shed a lot of light on this question, and it remains open for now.

A couple of days after having written the above, the author found out that the above theorems were already proven by Burr and Erdős in [8], back in 1981; see pages 452 and 453 of their paper.

References

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