1 Question: Calculate $\int_0^\infty \frac{sin(x)}{x} dx$

This integral is impossible to calculate using standard methods like substitution or integration by parts. Any trigonometry trick won't work neither, so therefore the undefined integral does not exists. However we have here the defined integral (from 0 to infinity) so it has to be possible to get the exact value. To solve this integral we need to use the Feynman's rule. We need to introduce another variable and put it into this integral in the way that does not change the value of the integral. The derivative of the new variable should help us calculate the original integral. But not every function of this new variable will work. We need to find the function that the derivative of it will produce an extra x at the numerator so it will cancel out with the x in the denominator, because this x at the bottom makes this integral hard. The result of the derivative of the new variable should also be possible to calculate so for example $sin(tx)/x$ does not work, because we will get $\int_0^\infty cos(tx)dx$ which clearly does not convert for $t \neq 0$. You can now try some other functions and check the result of the derivation but we are going to take the right function.

$$\int_0^\infty \frac{sin(x) \cdot e^{-tx}}{x} dx \tag{1}$$

Why e^{-tx} ? Because it will produce the x from the chain rule. What's more we are able to calculate the integral of sin times exponential, so it will work. Why not e^{tx}? Because x goes to infinity so the exponential function will be infinity as well and if we have a minus then the exponential function does not go to infinity but to zero.

We shouldn't forget that we can't change the value of the function so we need to find the value of **t** that fixes. As we know exponential of 0 always is equal to 1 and anything times one is one.

$$I(t) = \int_0^\infty \frac{sin(x) \cdot e^{-tx}}{x} dx \tag{2}$$

$$I(0) = \int_0^\infty \frac{sin(x) \cdot e^{-0 \cdot x}}{x} dx = \int_0^\infty \frac{sin(x) \cdot 1}{x} dx = \int_0^\infty \frac{sin(x)}{x} dx \tag{3}$$

Now let's calculate the partial derivative of /textbft and see why this function is so helpful for us.

$$\frac{d}{dt} I(t) = \frac{\partial}{\partial t} \int_0^\infty \frac{sin(x) \cdot e^{-tx}}{x} dx \tag{4}$$

$$= \int_0^\infty \frac{\partial}{\partial t} \frac{sin(x) \cdot e^{-tx}}{x} dx \tag{5}$$

$$= \int_0^\infty \frac{-x \cdot sin(x) \cdot e^{-tx}}{x} dx \tag{6}$$

$$= -\int_0^\infty sin(x) \cdot e^{-tx} dx \tag{7}$$

We can think of **t** as a constant and now this integral is possible to calculate using **standard** methods of integration. Let's calculate the undefined integral to make it as simple as possible. We will get the result doing an integration by parts twice and it doesn't really matter if we integrate sin part and derive exponential or derive sin and integrate exponential.

$$\int sin(x)e^{-tx}dx = -cos(x)e^{-tx} - t\int cos(x)e^{-tx}dx$$
$$\int sin(x)e^{-tx}dx = -cos(x)e^{-tx} - t\int cos(x)e^{-tx}dx$$
$$\int sin(x)e^{-tx}dx = -cos(x)e^{-tx} - t(sin(x)e^{-tx} + t\int sin(x)e^{-tx}dx)$$
$$\int sin(x)e^{-tx}dx = -cos(x)e^{-tx} - tsin(x)e^{-tx} - t^2\int sin(x)e^{-tx}dx$$
$$\int sin(x)e^{-tx}dx + t^2\int sin(x)e^{-tx} = -cos(x)e^{-tx} - tsin(x)e^{-tx}$$
$$(1+t^2)\int sin(x)e^{-tx}dx = -e^{-tx}(cos(x) + tsin(x))$$
$$\int sin(x)e^{-tx}dx = \frac{-e^{-tx}(cos(x) + tsin(x))}{t^2+1}$$

Now we go back to (7) and change the integral to its calculated value.

$$\frac{d}{dt}I(t) = -\frac{-e^{-tx}(cos(x) + tsin(x))}{t^2+1}\Bigg|_{x=0}^{x=\infty} \tag{8}$$
$$= 0 - \frac{e^{-t\cdot 0}(cos(0) + tsin(0))}{t^2+1} \tag{9}$$
$$= -\frac{1}{1+t^2} \tag{10}$$

Now we need to calculate I(t) back so we need to take an integral of both sides with respect to **t** which is easy (it's just an atan(t)).

$$I(t) = -atan(t) + C \tag{11}$$

Now the last part is to find what the C value is. We shouldn't forget that at the beginning we set

$$I(t) = \int_0^\infty \frac{sin(x)e^{-tx}}{x}dx \tag{12}$$

And therefore

$$-atan(t) + C = \int_0^\infty \frac{sin(x)e^{-tx}}{x}dx \tag{13}$$

Now we need to find t value that deletes the whole integral at the right side. So in other words we need to make the exponential function goes to zero. That means that t has to be infinity.

$$-atan(\infty) + C = \int_0^\infty \frac{sin(x)e^{-\infty x}}{x} dx \tag{14}$$

$$-\frac{\pi}{2} + C = 0 \tag{15}$$

$$C = \frac{\pi}{2} \tag{16}$$

atan of infinity is $\frac{\pi}{2}$ so we were able to find the C constant.

$$I(t) = -atan(t) + \frac{\pi}{2} \tag{17}$$

Our main integral has the value of I(0) so

$$\int_0^\infty \frac{sin(x)}{x} dx = I(0) = -atan(0) + \frac{\pi}{2} = \frac{\pi}{2} \tag{18}$$