1 Question: What is an expected distance between two points randomly allocated in the unit square

Let's say that the first random point is $A = (X_A, Y_A)$ and the second one is $B = (X_B, Y_B)$. Of course every x and y is between 0 and 1 with the uniform distribution. The distance between those two points is

$$D = \sqrt{(X_A - X_B)^2 + (Y_A - Y_B)^2} \tag{1}$$

We will be have to calculate the integral of the distance so we need to simplify it a little bit. As we can see we have four independent variables which is way too much. We can change the differents to other variables

$$DX = X_A - X_B \tag{2}$$
$$DY = Y_A - Y_B \tag{3}$$

So the distance will look like this

$$D = \sqrt{(DX)^2 + (DY)^2} \tag{4}$$

The only problem is that variables DX and DY have lo longer uniform distribution becuase it's easy to tell that it's way more probable that the difference will be close to the 0.5 rather than close to one or zero. DX and DY have now triangular distribution and their density function is

$$f_{DX} = 2(1 - DX) \tag{5}$$

So with this knowledge we can find the average distance using expected value formula of D variable using DX and DY variables.

$$ED = 4\int_0^1 \int_0^1 \sqrt{(DX)^2 + (DY)^2}(1 - DX)(1 - DY)\, dDX\, dDY \tag{6}$$

Now let's change the variables to polar coordinates and integrate over only one half of the unit square and double the value later.

$$DX = rcos(\theta) \tag{7}$$
$$DY = rsin(\theta) \tag{8}$$

where

$$0 \leq \theta \leq \frac{\pi}{4} \tag{9}$$
$$0 \leq r \leq sec(\theta) \tag{10}$$

Now the expected value of the distance will look like this

$$ED = 4\int_0^{\frac{\pi}{4}} 2\int_0^{sec(\theta)} f(\theta, r)r\ dr\ d\theta \tag{11}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \int_0^{sec(\theta)} \sqrt{r^2cos^2(\theta) + r^2sin^2(\theta)}(1 - rcos(\theta))(1 - rsin(\theta))r\ dr\ d\theta \tag{12}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \int_0^{sec(\theta)} \sqrt{r^2(cos^(\theta) + sin^2(\theta))}(1 - rcos(\theta))(1 - rsin(\theta))r\ dr\ d\theta \tag{13}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \int_0^{sec(\theta)} r(1 - rcos(\theta))(1 - rsin(\theta))r\ dr\ d\theta \tag{14}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \int_0^{sec(\theta)} r^2(1 - rcos(\theta))(1 - rsin(\theta))\ dr\ d\theta \tag{15}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \int_0^{sec(\theta)} r^2 - r^3cos(\theta) - r^3sin(\theta) + r^4cos(\theta)sin(\theta)\ dr\ d\theta \tag{16}$$

$$ED = 8\int_0^{\frac{\pi}{4}} \frac{sec^3(\theta)}{12} - \frac{sec^3(\theta)tan(\theta)}{20}\ d\theta \tag{17}$$

This is an integral with the famous clever substitution of $ln(sec\theta + tan\theta)$ so I will just take the result of this integral.

$$ED = 8\left(\frac{sec\theta tan\theta + ln|sec\theta tan\theta|}{24} - \frac{sec^3\theta}{60}\right)_0^{\frac{\pi}{4}} \tag{18}$$

$$ED = \frac{2 + \sqrt{2} + 5ln(\sqrt{2} + 1)}{15} \tag{19}$$