Permutation Models: Staring into the Formless Void

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Assume ZFC is consistent.

Axioms of ZF

- Extensionality: two sets are $=$ iff they have the same elements
- Foundation: every nonempty set has an $\in$-minimal element
- Comprehension: can filter a set using a formula ϕ
- Pairing: closed under constructing pairs $\{x, y\}$
- Union: closed under unary union operation
- Replacement: closed under definable functions ϕ
- Infinity: ω exists
- Power Set: closed under $\mathcal{P}$

Axiom of Choice

AC

Given a set of nonempty sets X, there is a function

$$f : X \to \bigcup X$$

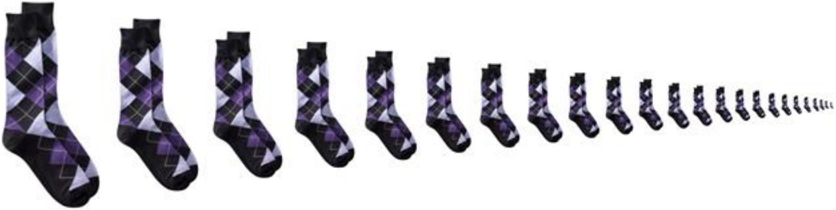
with

$$f(x) \in x.$$

Minimal Counterexample for AC

Any finite set can be shown to have a choice function (manually make the choices and combine them with pairing/union).

We're looking to produce a countable set of pairs with no choice function.



ZFA

In line with the naive uses of set theory notation, we can add extra objects to our set theory called urelements or atoms, which aren't sets, but can be contained in sets.

- Atoms are exempt from extensionality
- Atoms don't contain anything
- There is a set A of atoms (optional)

For now, let

$$A = \{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}.$$

Our set of pairs with no choice function will be:

$$F = \{\{a_0, b_0\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots\}.$$

Context

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\boxed{\text{Larger Universe}} \arrow[r, "\text{laughs at}"] \arrow[d, "\text{satisfies}"] & \boxed{\text{Our universe } (\mathcal{M}, \in)} \arrow[r, "\text{contains}", "(\mathcal{M}', \in') \in \mathcal{M}"'] \arrow[d, "\text{satisfies}"] & \boxed{(\mathcal{M}', \in')} \arrow[r, "\text{substructure}", "\mathcal{N} \subseteq \mathcal{M}'"'] \arrow[d, "\text{satisfies}"] & \boxed{(\mathcal{N}, \in')} \arrow[r, "\text{contains}", "F \in \mathcal{N}"'] \arrow[d, "\text{satisfies}"] \arrow[dr, "\text{doesn't contain}"] & \boxed{F} \\
\boxed{\text{Actual truth, not whatever nonsense is going on in } \mathcal{M}} & \boxed{\text{ZFAC}} & \boxed{\text{ZFAC}} & \boxed{\text{ZFA}} & \boxed{\text{Choice Function on } F}
\end{tikzcd}
$$

Fooling Models

A model may see things differently than we do.

- A model may contain a set that has no choice function in the model.
- What we think is countable and what the model thinks is countable could be different.
  - Model might not be aware of the bijection between ω and the set
  - May have a different notion of ω
- May have a different notion of $\mathcal{P}$
- A model may have an infinite descending chain $\cdots x_2 \in x_1 \in x_0$, but not contain the set $\{x_0, x_1, x_2, \ldots\}$, thus still satisfying the foundation axiom.

Context

There is a structure $(\mathcal{M}', \in')$ which forms a model of ZFA + C, with our particular choice of atoms A. We are going to find a substructure $\mathcal{N} \subseteq \mathcal{M}'$ so that $(\mathcal{N}, \in')$ is also a model of ZFA, contains

$$F = \{\{a_0, b_0\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots\},$$

and doesn't have a choice function on F.

Attempt 1

We want our model to contain our atoms, and be closed under basic set-theoretic constructions (nested finite set construction).

Keep in $\mathcal{N}$ any set such that the set of atoms that are contained (at any depth) in that set is finite.

Example

$$\{a_0, b_0, \{a_1, \{b_3\}\}, \{\{\{b_4\}\}\}, \omega, \{\emptyset\}\}$$

Refers to: a_0, b_0, a_1, b_3, b_4.

Attempt 1

Any set
Keep in $\mathcal{N}$ any set such that the set of atoms that are contained (at any depth) in that set is finite.

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But F (and A) don't exist.

Allowing Infinitely Many Atoms

$$F = \{\{a_0, b_0\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots\}.$$

- a_i, b_i occuring in the same set is OKAY
- a_i not paired with a b_i is LESS OKAY

Permuting the Atoms

Given a function $f : A \to A$, we can extend it to all sets via:

$$f(X) = \{f(x) | x \in X\}$$

For instance, if f is the identity, except swapping a_1 and b_1, then:

$$f(\{a_0, a_1, b_3, \{a_0, \{b_1, b_2\}\}\})$$

$$= \{a_0, b_1, b_3, \{a_0, \{a_1, b_2\}\}\}$$

We will focus for now on the group of permutations f which either swap or don't swap each pair (a_i, b_i).

Attempt 2

Keep in $\mathcal{N}$ any set which is closed under all swapping permutations.

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Attempt 3

Keep in $\mathcal{N}$ any set, for which there is a finite collection $I \subset \omega$ of indices, so that any permutation that fixes those indices fixes the set. In this case, we say the set has finite support.

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Extensionality Woes

Suppose $\mathcal{M}'$ contains the sets:

$$\{a, b, c, d, \ldots, z\} \qquad \{a, c, d, \ldots, z\}$$

But, in filtering down to $\mathcal{N}$, we remove b.

By extensionality, these two sets should now be equal.

Attempt 3

Keep in $\mathcal{N}$ any set, for which there is a finite collection $I \subset \omega$ of indices, so that any permutation that fixes those indices fixes the set. In this case, we say the set has finite support.

- $\mathcal{P}(A)$, the set of subsets of the atoms needs zero support!
- But it contains tons of elements that don't have finite support, like $\{a_i | i \in \omega\}$

Attempt 4

Keep in $\mathcal{N}$ any set with hereditary finite support.

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Counterexample to the Axiom of Choice

$$F = \{\{a_0, b_0\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots\},$$

This model $\mathcal{N}$ contains A and F, but no choice function on F.

- A has no well-ordering
- F witnesses that a countable union of countable sets isn't countable
- With $a_i = -b_i$, (and $b_i = -a_i$) can use to construct a vector space without a basis

More General Permutation Model

- Let $A = \{a_0, a_1, a_2, \ldots\}$
- Construct a model $\mathcal{M}'$ of ZFA+C with atoms A
- A set X has finite support if there is a finite collection of indices I such that if a permutation on A fixes all a_i with $i \in I$, then it fixes X.
- Create a submodel $\mathcal{N} \subseteq \mathcal{M}'$ consisting of the sets with hereditary finite support.
- In $\mathcal{N}$, A has only finite or cofinite subsets (amorphous set).