What is Model Theory?

Thomas Kern
Landmark College
`trkern.github.io/hrumc`

April 10, 2021

Model theory: logical languages

Definition (First-Order Logic for Graphs)

- Variables representing vertices of the graph
- We write $x \sim y$ if x is connected to y
- Equality $(=)$
- Quantifiers $\forall$, $\exists$
- Boolean logic (and, or, not, true, false)

Example:

- $\exists x : \forall y : \text{NOT } x \sim y$
  - (there is a vertex x which is not-connected to every other vertex, i.e. there is an isolated vertex)

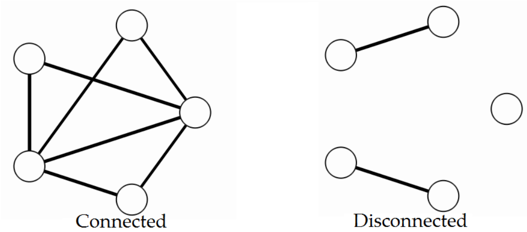
Connected Graphs

Question:

- Can we describe connected graphs?

Casually, yes:

- $\exists x_1, x_2, \ldots x_n : \forall i : x_i \sim x_{i+1}$
  - (there is a path of connected vertices from x_1 to x_n)



Can We Simplify?

We can say more complex things:

$$\exists x : \exists y : x \sim y \text{ AND } \forall z : \text{IF } z \sim x \text{ THEN } z = y$$

(there is a vertex x which has exactly one neighbor, y)
Question:

- Is there a simpler way of writing this?

E-F Games and Model Theory

We have the following correspondences:

Formulas which are true of one graph but not the other	$\rightleftharpoons$	Strategies for Spoiler
Formulas with n variables[1]	$\rightleftharpoons$	Strategies that take n turns to implement

[1] Actually *Quantifier Depth*: the maximum number of quantifiers active at once

No First-Order-Logic Formula

Proof:

- Suppose we had a formula with N variables that expressed connectivity
- Then spoiler has a universal strategy that takes only N turns to win on any pair (connected, disconnected)
- Produce a pair of graphs (connected, disconnected) that spoiler takes more than N turns to win on
- Contradiction!