# Probability and Algorithms - Sheet 6

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## Exercise 14i

We set $a_{ij} = \frac{w_{ij}}{2}$ and $b_{ij} = \frac{w'_{ij}}{2}$, so we get: $\underbrace{fads}_{fdas}$

$$
\begin{aligned}
& max \sum_{0 \leq i < j \leq n} a_{ij}(1 - y_i y_j) + b_{ij}(1 + y_i y_j) \\
= & max \sum_{0 \leq i < j \leq n} a_{ij}(1 - y_i y_j) + \sum_{0 \leq i < j \leq n} b_{ij}(1 + y_i y_j) \\
= & max \underbrace{\sum_{0 \leq i < j \leq n} w_{ij} \frac{(1 - y_i y_j)}{2}}_{\text{max cut with n+1 nodes}} + \underbrace{\sum_{0 \leq i < j \leq n} w'_{ij} \frac{(1 + y_i y_j)}{2}}_{\text{min cut with n+1 nodes}}
\end{aligned}
$$

Hence, we need to encode MAX2SAT as a maximum of a MAX-CUT and a MIN-CUT of n+1 nodes (the MAX-CUT and the MIN-CUT can be computed on different graphs). Note that the MAX2SAT can be solved with a MAX-CUT with 2n nodes on implication graphs.

**Idea:** use the node for $y_0$ to encode some kind of supersink/supersource to connect both graphs.

## Exercise 14ii

Define $\tilde{K}$ and $\hat{K}$ as follows:

$$
\tilde{K}(V) = \sum_{0 \leq i < j \leq n} a_{ij} \frac{arccos(v_i^T v_j)}{\pi} \hat{K}(V) = \sum_{0 \leq i < j \leq n} b_{ij}(1 - \frac{arccos(v_i^T v_j)}{\pi})
$$

Since $E(V) = \tilde{K}(V) + \hat{K}(V)$ and we know from lecture that

$$
\tilde{K}(V) = \sum_{0 \leq i < j \leq n} a_{ij}(1 - v_i^T v_j)
$$

holds, it remains to show:

$$\hat{K}(V) = \sum_{0 \le i < j \le n} b_{ij}(1 + v_i^T v_j)$$

So here we go:

$$\begin{aligned}
&\sum_{0 \le i < j \le n} b_{ij}(1 + v_i^T v_j) \\
&= \sum_{0 \le i < j \le n} b_{ij} Pr(sign(v_i^T r) = sign(v_j^T r)) \\
&= \sum_{0 \le i < j \le n} b_{ij}(1 - Pr(sign(v_i^T r) \neq sign(v_j^T r))) \\
&= \sum_{0 \le i < j \le n} b_{ij}(1 - 2Pr(v_i^T r \le 0 \wedge v_j^T r > 0)) \\
&= \sum_{0 \le i < j \le n} b_{ij}(1 - 2\tfrac{\theta}{2\pi}) \\
&= \sum_{0 \le i < j \le n} b_{ij}(1 - \tfrac{\theta}{\pi}) \\
&= \sum_{0 \le i < j \le n} b_{ij}(1 - \tfrac{arccos(v_i^T v_j)}{\pi}) \\
&= \hat{K}(V)
\end{aligned}$$

When we put things together we get:

$$E(V) = \tilde{K}(V) + \hat{K}(V) = \sum_{0 \le i < j \le n} a_{ij}(1 - v_i^T v_j) + b_{ij}(1 + v_i^T v_j)$$

Together with $\alpha \le 1$ we get:

$$E(V) = \alpha(\tilde{K}(V) + \hat{K}(V)) = \alpha(\sum_{0 \le i < j \le n} a_{ij}(1 - v_i^T v_j) + b_{ij}(1 + v_i^T v_j))$$