Pick and Place Project Report

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I. Introduction

The goal of this project is to solve the inverse kinematic problem of the Kuka KR210 [1] robotic arm. A schematic drawing of this arm is shown in Fig. 1. It consists of six joints: the first three joints allow the arm to position its *wrist center* inside of its operating space, while the last three joints allow for positioning the tool (in this case a gripper) in any orientation in 3D space.

Accordingly, the inverse kinematic problem can be decoupled into two parts. After calculating the position of the wrist center from the required tool position and orientation, the angles θ_1, θ_2 and θ_3 are computed by an algebraic approach. After the first three joint angles have been established, the angles θ_4, θ_5 and θ_6 can be determined in order to achieve the required position and orientation of the tool.

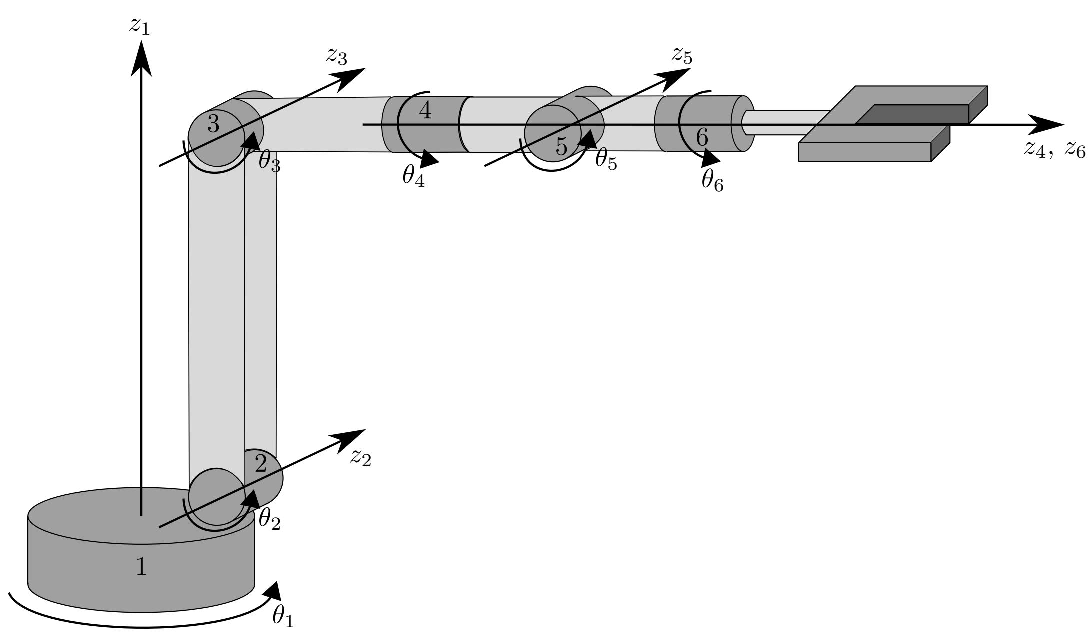

Figure 1: *Schematic of the Kuka KR210 arm.*

II. DH Parameters and Forward Kinematics

According to [2] the forward kinematic problem of a robot arm can be decomposed into consecutive *link frames*, each consisting of one *joint* and one *link*. For each of these segments, a homogenous transformation matrix can be defined, by adhering to a convention called the *modified Denavit-Hartenberg notation*:

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-i} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-i} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1} $$

Here, θ_i is the joint angle, which measures the rotation around the axis z_i. The parameters d_i and a_{i-1} measure the displacement from one link to the next in the direction of the axes z_i and x_i, respectively. Finally, the parameter α_{i-i} measures the angle between the axes z_{i-1} and z_i.

In this project, the geometry of the Kuka KR210 was provided in form of a URDF file that describes the dimensions and properties of the links and joints of the arm. From this description, the modified DH parameters can be inferred in a relatively straightforward manner and are summarized in Tab. 1.

Table 1: *Modified DH parameters of the Kuka KR210 arm.*

	α_{i-1}	a_{i-1}	d_i	θ_i
${}^0_1T$	0	0	0.750	θ_1
${}^1_2T$	$-\frac{\pi}{2}$	0.350	0	$\theta_2 - \frac{\pi}{2}$
${}^2_3T$	0	1.250	0	θ_3
${}^3_4T$	$-\frac{\pi}{2}$	-0.054	1.500	θ_4
${}^4_5T$	$\frac{\pi}{2}$	0	0	θ_5
${}^5_6T$	$-\frac{\pi}{2}$	0	0	θ_6
${}^6_GT$	0	0	0.303	0

Care has to be taken that the reference frames in the URDF differ from those in the DH convention. For example, the parameter d_1 results from summing the vertical displacement of the first two links in the URDF file. Another peculiarity is the joint angle θ_2 that includes an offset of $-90°$ because the arm has a right angle at its resting position for joint two.

With the help of (1), the following transformation matrices for the Kuka arm joints can be established that fully describe its forward kinematics:

$${}^0_1T = \begin{bmatrix} \mathrm{c}\theta_1 & -\mathrm{s}\theta_1 & 0 & 0 \\ \mathrm{s}\theta_1 & \mathrm{c}\theta_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2) \qquad {}^1_2T = \begin{bmatrix} \mathrm{c}(\theta_2 - \frac{\pi}{2}) & -\mathrm{s}(\theta_2 - \frac{\pi}{2}) & 0 & a_1 \\ 0 & 0 & 1 & 0 \\ -\mathrm{s}(\theta_2 - \frac{\pi}{2}) & \mathrm{c}(\theta_2 - \frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{s}\theta_2 & \mathrm{c}\theta_2 & 0 & a_1 \\ 0 & 0 & 1 & 0 \\ \mathrm{c}\theta_2 & \mathrm{s}\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

$${}^2_3T = \begin{bmatrix} \mathrm{c}\theta_3 & -\mathrm{s}\theta_3 & 0 & a_2 \\ \mathrm{s}\theta_3 & \mathrm{c}\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4) \qquad {}^3_4T = \begin{bmatrix} \mathrm{c}\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -\mathrm{s}\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5) \qquad {}^4_5T = \begin{bmatrix} \mathrm{c}\theta_5 & -\mathrm{s}\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mathrm{s}\theta_5 & \mathrm{c}\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$${}^5_6T = \begin{bmatrix} \mathrm{c}\theta_6 & -\mathrm{s}\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{s}\theta_6 & -\mathrm{c}\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7) \qquad {}^6_GT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_G \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8)$$

In addition, the following correction matrix is needed to convert from the DH frame of the gripper to the URDF frame:

$$T_{corr} = \begin{bmatrix} \mathrm{c}\pi & -\mathrm{s}\pi & 0 & 0 \\ \mathrm{s}\pi & \mathrm{c}\pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{c}\frac{\pi}{2} & 0 & \mathrm{s}\frac{\pi}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\mathrm{s}\frac{\pi}{2} & 0 & \mathrm{c}\frac{\pi}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

By multiplying these matrices, the total homogeneous transformation from the base link to the gripper link can be expressed as:

$${}^0_GT = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T \cdot {}^3_4T \cdot {}^4_5T \cdot {}^5_6T \cdot {}^6_GT \cdot T_{corr} \quad (10)$$

III. Calculation of the Wrist Center Position

The orientation of the gripper is supplied to the inverse kinematic server as a quaternion [3]. From this, the rotation matrix relative to the base frame can be computed using the function `tf.transformations.quaternion_matrix`[4], which is equivalent to the following:

$$R_{gripper} = \begin{bmatrix} 1-2q_y^2-2q_z^2 & 2q_xq_y-2q_zq_w & 2q_xq_z+2q_yq_w \\ 2q_xq_y+2q_zq_w & 1-2q_x^2-2q_z^2 & 2q_yq_z-2q_xq_w \\ 2q_xq_z-2q_yq_w & 2q_yq_z+2q_xq_w & 1-2q_x^2-2q_y^2 \end{bmatrix} \cdot R_{corr} \tag{11}$$

Alternatively, the roll (ϕ), pitch (θ), and yaw (ψ) angles can be computed with the function `transformations.euler_from_quaternion`[4], by specifiying the `'rzyx'` option. This computes the Euler angles relative to the local frame in the order of a rotation about the z, y and x axes in this order. With these Euler angles, the total rotation matrix can be computed as:

$$R_{gripper} = R_z(\psi)R_y(\theta)R_x(\phi)R_{corr} = \begin{bmatrix} \mathrm{c}\theta\mathrm{c}\psi & \mathrm{s}\phi\mathrm{s}\theta\mathrm{c}\psi-\mathrm{s}\psi\mathrm{c}\phi & \mathrm{s}\phi\mathrm{s}\psi+\mathrm{s}\theta\mathrm{c}\phi\mathrm{c}\psi \\ \mathrm{s}\psi\mathrm{c}\theta & \mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi+\mathrm{c}\phi\mathrm{c}\psi & -\mathrm{s}\phi\mathrm{c}\psi+\mathrm{s}\theta\mathrm{s}\psi\mathrm{c}\phi \\ -\mathrm{s}\theta & \mathrm{s}\phi\mathrm{c}\theta & \mathrm{c}\phi\mathrm{c}\theta \end{bmatrix} \cdot R_{corr} \tag{12}$$

With this rotation matrix, the total transformation matrix from the base frame to the gripper frame can be expressed as:

$$T_{gripper} = \begin{bmatrix} \mathrm{s}\phi\mathrm{s}\psi+\mathrm{s}\theta\mathrm{c}\phi\mathrm{c}\psi & -\mathrm{s}\phi\mathrm{s}\theta\mathrm{c}\psi+\mathrm{s}\psi\mathrm{c}\phi & \mathrm{c}\theta\mathrm{c}\psi & p_x \\ -\mathrm{s}\phi\mathrm{c}\psi+\mathrm{s}\theta\mathrm{s}\psi\mathrm{c}\phi & -\mathrm{s}\phi\mathrm{s}\theta\mathrm{s}\psi-\mathrm{c}\phi\mathrm{c}\psi & \mathrm{s}\psi\mathrm{c}\theta & p_y \\ \mathrm{c}\phi\mathrm{c}\theta & -\mathrm{s}\phi\mathrm{c}\theta & -\mathrm{s}\theta & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

The third column of this transformation matrix represents the z-axis of the griper frame $N_{z,gripper}$. With the help of this vector, it now becomes possible to calculate the position of the wrist center from that of the gripper tip:

$$\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} - \begin{bmatrix} \mathrm{c}\theta\mathrm{c}\psi \\ \mathrm{s}\psi\mathrm{c}\theta \\ -\mathrm{s}\theta \end{bmatrix} \cdot d_G \tag{14}$$

IV. Inverse Kinematics of the Wrist Center

The next step in the inverse kinematic problem lies in determining the angles θ_1, θ_2, and θ_3 in such a way that the wrist center of the Kuka arm lies at the coordinates w_x, w_y, and w_z. To achieve this, the angle θ_1 can easily be computed in the xy-plane of the base frame from:

$$\theta_1 = \mathrm{atan2}(w_y, w_x) \tag{15}$$

After computing θ_1, the problem of solving for θ_2 and θ_3 can be transformed into a second 2D problem in the xz-plane of the Kuka arm, as shown in Fig. 2. For this, the wrist center is transformed into the reference frame of link 1:

$$\begin{aligned} w_{tx} &= w_x\mathrm{c}\theta_1 + w_y\mathrm{s}\theta_1 \\ w_{ty} &= -w_x\mathrm{s}\theta_1 + w_y\mathrm{c}\theta_1 = 0 \\ w_{tz} &= w_z - d_1 \end{aligned} \tag{16}$$

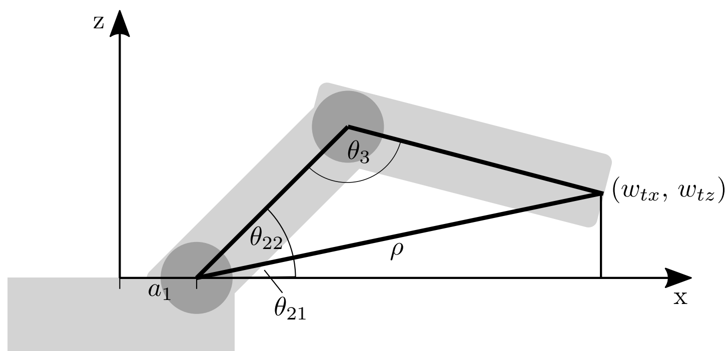

Figure 2: *Sideview of the Kuka arm.*

The basis for determining the angles θ_2 and θ_3 is the transformation from joint one to the wrist center, which can be expressed by the matrix ${}^1_4T$:

$$
{}^4_1T = \begin{bmatrix} s(\theta_2+\theta_3)c\theta_4 & -s\theta_4 s(\theta_2+\theta_3) & c(\theta_2+\theta_3) & a_1 + a_2 s(\theta_2) + a_3 s(\theta_2+\theta_3) + d_4 c(\theta_2+\theta_3) \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ c\theta_4 c(\theta_2+\theta_3) & -s\theta_4 c(\theta_2+\theta_3) & -s(\theta_2+\theta_3) & a_2 c(\theta_2) + a_3 c(\theta_2+\theta_3) - d_4 s(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \\ & & & \end{bmatrix} \tag{17}
$$

The entries of the rightmost column of the matrix ${}^4_1T$ represent the translation of the wrist center relative to the origin of link 1, so we can extract the following two equations from this:

$$
\begin{aligned}
w_{tx} - a_1 &= a_2 s(\theta_2) + a_3 s(\theta_2+\theta_3) + d_4 c(\theta_2+\theta_3) && (18) \\
w_{tz} &= a_2 c(\theta_2) + a_3 c(\theta_2+\theta_3) - d_4 s(\theta_2+\theta_3) && (19)
\end{aligned}
$$

The angle θ_3 effectively defines the reach or radius of the robot arm, which has to be equal to the distance between link one and the transformed wrist center position in the plane of the arm. Thus by computing the squared distance we get:

$$
(w_{tx} - a_1)^2 + w_{tz}^2 = a_2^2 + a_3^2 + d_4^2 + 2a_2a_3c\theta_3 - 2a_2d_4s\theta_3 \tag{20}
$$

$$
\frac{\rho^2 - \sigma^2}{2a_2} = a_3c\theta_3 - d_4s\theta_3 = \sqrt{a_3^2 + d_4^2} \cdot \sin(\theta_3 + \operatorname{atan2}(a_3, -d_4)) \tag{21}
$$

$$
\rho^2 = (w_{tx} - a_1)^2 + w_{tz}^2 \tag{22}
$$

$$
\sigma^2 = a_2^2 + a_3^2 + d_4^2 \tag{23}
$$

Solving (21) for θ_3 finally yields [5]:

$$
\theta_3 = \operatorname{asin}\left(\frac{\rho^2 - \sigma^2}{2a_2\sqrt{a_3^2 + d_4^2}}\right) - \operatorname{atan2}(a_3, -d_4) \tag{24}
$$

With θ_3 determined, it now becomes possible to solve for θ_2. This sub-problem can be further divided into the two angles θ_{21} and θ_{22} (cf. Fig. 2). The angle θ_{21} can be thought of as the angle required to place the tip of a straight arm with length ρ at the wrist center. It be can easily computed as:

$$
\theta_{21} = \operatorname{atan2}(w_{tz}, w_{tx} - a_1) \tag{25}
$$

The angle θ_{22} can now be solved by utilizing the matrix element ${}^{1}_{4}T_{33}$ in (17). Here, the vertical displacement w_{tz} has already been accounted for by θ_{21}. Thus, this matrix element must be equal to zero for θ_{22}:

$$a_2 \mathrm{c}\theta_{22} + a_3 \mathrm{c}(\theta_{22} + \theta_3) - d_4 \mathrm{s}(\theta_{22} + \theta_3) = \alpha_{22} \sin(\theta_{22} + \phi_{22}) = 0 \tag{26}$$

Solving this equation for θ_{22}, using the linear combination of the sinusoids [5], yields (with $n = 0$):

$$\theta_{22} = -\mathrm{atan2}(-a_3 \mathrm{s}\theta_3 - d_4 \mathrm{c}\theta_3, a_2 + a_3 \mathrm{c}\theta_3 - d_4 \mathrm{s}\theta_3) + n\pi \tag{27}$$

Finally, the two angles θ_{21} and θ_{22} can be combined, taking into account the offset angle at that joint:

$$\theta_2 = \frac{\pi}{2} - \theta_{21} - \theta_{22} \tag{28}$$

V. Inverse Kinematics of the Gripper Orientation

Once the first three joint angles have been determined, the inverse kinematics problem can be completed by computing the orientation of the tool relative to the wrist center. To achieve this, the rotation matrix ${}^{3}_{6}R$ is calculated by premultiplying the matrix $R_{gripper}$ with the inverse of the composite rotation ${}^{0}_{3}R$. This matrix is then compared to the product of the rotations ${}^{3}_{4}R$, ${}^{4}_{5}R$ and ${}^{5}_{6}R$:

$${}^{3}_{6}R = ({}^{0}_{1}R \cdot {}^{1}_{2}R \cdot {}^{2}_{3}R)^{-1} R_{gripper} = {}^{3}_{4}R \cdot {}^{4}_{5}R \cdot {}^{5}_{6}R = \begin{bmatrix} -\mathrm{s}\theta_4 \mathrm{s}\theta_6 + \mathrm{c}\theta_4 \mathrm{c}\theta_5 \mathrm{c}\theta_6 & -\mathrm{s}\theta_4 \mathrm{c}\theta_6 - \mathrm{s}\theta_6 \mathrm{c}\theta_4 \mathrm{c}\theta_5 & -\mathrm{s}\theta_5 \mathrm{c}\theta_4 \\ \mathrm{s}\theta_5 \mathrm{c}\theta_6 & -\mathrm{s}\theta_5 \mathrm{s}\theta_6 & \mathrm{c}\theta_5 \\ -\mathrm{s}\theta_4 \mathrm{c}\theta_5 \mathrm{c}\theta_6 - \mathrm{s}\theta_6 \mathrm{c}\theta_4 & \mathrm{s}\theta_4 \mathrm{s}\theta_6 \mathrm{c}\theta_5 - \mathrm{c}\theta_4 \mathrm{c}\theta_6 & \mathrm{s}\theta_4 \mathrm{s}\theta_5 \end{bmatrix} \tag{29}$$

From this matrix, θ_5 can be directly computed as:

$$\theta_5 = \pm\mathrm{acos}({}^{3}_{6}R_{23}) \tag{30}$$

To guarantee that the gripper will point into the correct direction and not its opposite, it is important to compute θ_4 by satisfying *two* entries of the matrix ${}^{3}_{6}R$.

$$\frac{{}^{3}_{6}R_{33}}{-{}^{3}_{6}R_{13}} = \frac{\mathrm{s}\theta_4 \mathrm{s}\theta_5}{\mathrm{c}\theta_4 \mathrm{s}\theta_5} = \tan(\theta_4) \tag{31}$$

From this, θ_4 can be computed as follows. Here, I set θ_4 to zero for very small θ_5. In this case, the correct orientation of the gripper is achieved by θ_6.

$$\theta_4 \begin{cases} \mathrm{atan2}\left({}^{3}_{6}R_{33}, -{}^{3}_{6}R_{13}\right) & \text{if } |\theta_5| > \epsilon \\ 0 & \text{if } |\theta_5| \le \epsilon \end{cases} \tag{32}$$

Finally, θ_6 can be computed from one of two cases: If θ_5, and consequently θ_4 are close to zero, the first entry ${}^{3}_{6}R_{11}$ simplifies to $\mathrm{c}\theta_6$. Otherwise, θ_6 can easily be computed from ${}^{3}_{6}R_{21}$. Thus, we get the following expressions for computing θ_6:

$$\theta_6 = \begin{cases} \mathrm{acos}\left(\frac{{}^{3}_{6}R_{21}}{\sin(\theta_5)}\right) & \text{if } |\theta_5| > \epsilon \\ \mathrm{acos}\left({}^{3}_{6}R_{11}\right) & \text{if } |\theta_5| \le \epsilon \end{cases} \tag{33}$$

VI. Results

With the presented inverse kinematics procedure, the robot arm is able to reliably pick up the object and place it in the receptacle. In order to measure the accuracy of the procedure, I calculate the root-mean-square error (RMSE) of the gripper position in the following way: at each time step I perform a forward kinematics pass with the calculated joint angles in order to compute the predicted gripper position $p_{pred,i}$, which is then compared to the ground truth position $p_{gt,i}$ (the original gripper coordinates provided by the path planner):

$$\mathrm{RMSE(p)} = \sqrt{\frac{1}{n}\sum_i (p_{pred,i} - p_{gt,i})^2} \tag{34}$$

The resulting RMSE values for three typical pick and place cycles are summarized in Tab. 2. As can be seen, the RMSE values are roughly of the same magnitude as floating point rounding errors, thus supporting the correctness and high accuracy of the presented inverse kinematics procedure.

Table 2: *RMSE values of the predicted gripper position for three typical pick and place cycles.*

n	RMSE(p_x)$/m$	RMSE(p_y)$/m$	RMSE(p_z)$/m$
1	$1.38372517 \cdot 10^{16}$	$2.50313645 \cdot 10^{16}$	$1.01349032 \cdot 10^{16}$
2	$8.69068332 \cdot 10^{15}$	$1.21391017 \cdot 10^{10}$	$4.44089210 \cdot 10^{16}$
3	$1.64697269 \cdot 10^{15}$	$2.23592115 \cdot 10^{11}$	$3.59867566 \cdot 10^{16}$

References

[1] "Kr 210 f exclusive — kuka ag." `https://www.kuka.com/en-us/products/robotics-systems/industrial-robots/kr-210-2-f-exclusive`. (Accessed on 07/26/2017).

[2] J. J. Craig, *Introduction to robotics: mechanics and control*. Pearson, 3rd ed., 2005.

[3] "Quaternion - wikipedia." `https://en.wikipedia.org/wiki/Quaternion`. (Accessed on 07/26/2017).

[4] "tf.transformations." `https://w3.cs.jmu.edu/spragunr/CS354_S14/labs/tf_lab/html/tf.transformations-module.html`. (Accessed on 07/26/2017).

[5] "List of trigonometric identities - wikipedia." `https://en.wikipedia.org/wiki/List_of_trigonometric_identities#Linear_combinations`. (Accessed on 07/26/2017).