Statistical Methods
of Data Analysis

W. Waltenberger
wolfgang.waltenberger@gmail.com

based on the slides of the year 2018 by R. Frühwirth

VU 260014 (Uni) / SV 142.340 (TU)

October 2, 2025

Statistical Methods
of Data Analysis

W. Waltenberger

Part 1: Descriptive Statistics

Part 2: Probabilities

Part 3: Random Variables and Distributions

Part 4: Point Estimators

Statistical Methods
of Data Analysis

W. Waltenberger

Part 5: Confidence Intervals

Part 6: Testing Hypotheses

Part 7: Regression and Linear Models

Part 8: Bayes Statistics

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Part I

Descriptive Statistics

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

1. Introduction

2. One-dimensional Features

3. Two-dimensional Features

Subsection: Basic Terminology

Statistical Methods of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional Features

Basic Terminology

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional
Features

Definition of Statistics

- The collection and storage of data, e.g. by statistical offices.
- The mathematical analysis of data, e.g. the calculation of measures and ratios, the estimation of unknown parameters, the testing of hypotheses

Descriptive Statistics

- Description of existing data by measures, tables, graphs.

Basic Terminology

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional
Features

Inductive (or Inferential) Statistics

- Investigation of patterns and causes that are behind the data and (partially) explain the data.
- Exploratory data analysis: the goal is to generate hypotheses.
- Confirmatory data analysis: aim is to test existing theories, e.g. by estimating parameters or testing hypotheses.

Statistical Methods
of Data Analysis

W. Waltenberger

1 Introduction
- Basic Terminology
- Features and Scale Types
- Statements and Frequencies

2 One-dimensional Features

3 Two-dimensional Features

Features and Scale Types

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional
Features

Qualitative Features

- **binary** (yes/no). Example: EU citizenship.
- **categorical** (classification). Example:
  single/divorced/married/widowed.
- **ordinal** (rank). Example: Grades 1–5.

Quantitative Features

- **discrete** (integer). Example: counting.
- **continuous** (real-valued). Example: measuring process.

Scale Types

- **Nominal Scale**: Numerical values are only designations for mutually exclusive categories.
- **Ordinal Scale**: Order of numbers is essential.
- **Interval Scale**: Order and differences between values are interpretable in a meaningful way, the zero point is arbitrarily fixed.
- **Ratio Scale**: Order, differences and ratios are meaningfully interpretable, there is an absolute zero point.

Features and Scale Types

Example

- The marital status of a person is coded by numbers (1=single, 2=married, 3=divorced, 4=widowed). Nominal scale.

- A team's standing in the championship is indicated by its rank in the league. Ordinal scale.

- The years (2007, 2008, . . .) form an interval scale, since the zero point is arbitrary.

- The Celsius scale of temperature is an interval scale, since the zero point is arbitrarily fixed.

- The Kelvin scale of temperature is a ratio scale, since the zero point is physically fixed.

- The height of a person is given in cm. A ratio scale, the zero value is natural.

Features and Scale Types

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional
Features

Example

The following data matrix D compiles several features of eight individuals.

Number	gender	age	education
1	1	34	2
2	2	54	1
3	2	46	3
4	1	27	4
5	1	38	2
6	1	31	3
7	2	48	4
8	2	51	2

Gender: 1=F, 2=M, Age: in years. Education: 1=compulsory school,
2=high school, 3=bachelor's, 4=master's.

Statistical Methods
of Data Analysis

W. Waltenberger

1 Introduction
   - Basic Terminology
   - Features and Scale Types
   - Statements and Frequencies

2 One-dimensional Features

3 Two-dimensional Features

Statements and Frequencies

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Basic Terminology
Features and Scale Types
Statements and Frequencies

One-dimensional Features

Two-dimensional
Features

The Concept of a Statement

- We call the set of objects of study the **population** Ω.
- A **statement** $A(x)$ is a statement about properties of the elements $x \in \Omega$.
- For each $x \in \Omega$, $A(x)$ must be either **true** or **false**.

Example

Let $A(x)$ be the statement "x is female". Then $A(1)$ is true and $A(2)$ is false.

Example

Let $B(x)$ be the statement "x is over 50 years old". Then $B(2)$ is true and $B(6)$ is false.

Linking Statements

Let A and B be two statements.

Symbol	Name	Meaning
$A \cup B$	disjunction	A or B (or both)
$A \cap B$	conjunction	A and B (both A and B)
A'	negation	not A (the opposite of A)
$A \subseteq B$	implication	from A follows B

Definition (Absolute Frequency)

Let $A(x)$ be a statement about $x \in \Omega$. The **absolute frequency** $h(A)$ of A is the number of elements of Ω for which $A(x)$ is true.

Example

If $A(x)$ is the statement "The person $x \in D$ has at least a bachelor's degree". Then $h(A) = 4$.

Statements and Frequencies

Definition (Relative Frequency)

Let $A(x)$ be a statement about $x \in \Omega$. The **relative frequency**
$f(A) = h(A)/n$ of A is the absolute frequency $h(A)$ divided by
the total number $n = |\Omega| \in \mathbb{N}$ of elements.

Example

If $B(x)$ is the statement "The person $x \in D$ is older than thirty
years". Then $f(B) = 7/8$.

Statements and Frequencies

Special Statements

- $A = \emptyset$: A is never true, $h(A) = f(A) = 0$.
- $A = \Omega$: A is always true, $h(A) = n, f(A) = 1$.

Arithemetic Laws for Frequencies

Addition Law

$$A \cap B = \emptyset \implies \begin{cases} h(A \cup B) = h(A) + h(B) \\ f(A \cup B) = f(A) + f(B) \end{cases}$$

Statements and Frequencies

Sieve Formula

$$h(A \cup B) = h(A) + h(B) - h(A \cap B)$$

Example

33% of a bank's customers have a home loan, 24% have a loan to finance consumer goods, 11% have both. What is the percentage of customers who have neither a housing loan nor a consumer goods loan?

Section 2: One-dimensional Features

1 Introduction

2 One-dimensional Features
- Graphical Representation
- Kernel Density Estimator
- Empirical Measures
- Boxplot and Empirical Distribution Function
- Examples

3 Two-dimensional Features

Subsection: Graphical Representation

1. **Introduction**

2. **One-dimensional Features**
   - Graphical Representation
   - Kernel Density Estimator
   - Empirical Measures
   - Boxplot and Empirical Distribution Function
   - Examples

3. **Two-dimensional Features**

Graphical Representation

- A picture is worth more than a thousand words!
- Graphical representations of data sets are therefore extremely popular and much in use.
- Qualitative variable: frequency table, pie chart, bar chart.
- Quantitative variable: grouped frequency table, histogram, boxplot, empirical distribution function

Graphical Representation

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
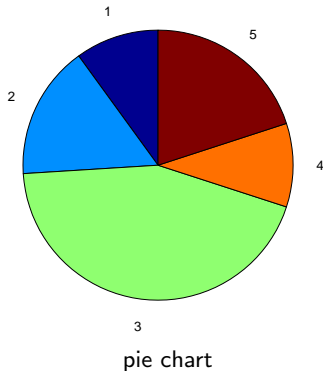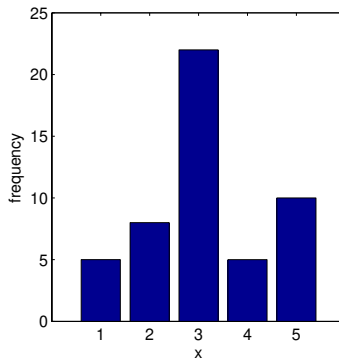Examples

Two-dimensional
Features

- Dataset 1 (500 normally distributed values):

histogram

Graphical Representation

- record 2 = record 1 + contamination (100 values):

histogram

Graphical Representation

- Dataset 3 (50 exam grades):

Grade k	$h(k)$	$f(k)$
1	5	0.10
2	8	0.16
3	22	0.44
4	5	0.10
5	10	0.20
	50	1.00

frequency table

- Dataset 3 (50 exam grades):

pie chart

Graphical Representation

- Dataset 3 (50 exam grades):

bar chart

1 **Introduction**

2 **One-dimensional Features**
- Graphical Representation
- Kernel Density Estimator
- Empirical Measures
- Boxplot and Empirical Distribution Function
- Examples

3 **Two-dimensional Features**

Kernel Density Estimator

- The frequency distribution (histogram) can be smoothed with a kernel density estimator.
- The density of the observed feature is thereby approximated by a sum of kernels $K(\cdot)$:

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

- h is the bandwidth of the kernel density estimator.
- A popular kernel is the Gaussian kernel:

$$K(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Kernel Density Estimator

- record 2:
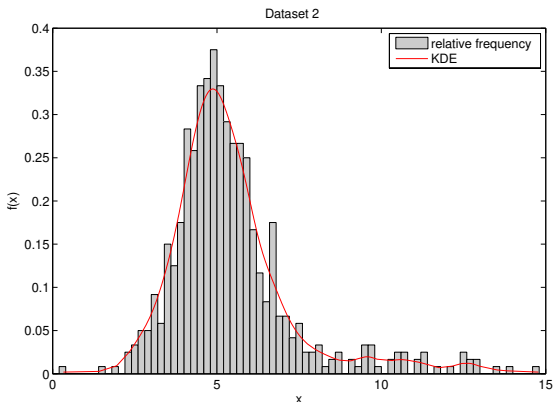
Smoothing of histogram by kernel density estimators.

🐍 PYTHON: sklearn.neighbors.KernelDensity

1 Introduction

2 One-dimensional Features
- Graphical Representation
- Kernel Density Estimator
- Empirical Measures
- Boxplot and Empirical Distribution Function
- Examples

3 Two-dimensional Features

Empirical Measures

- Data lists are often extensive enough that their content is to be summarized by a few empirical measures. Which measures are useful depends on the type of feature.
- Some measures start from the ordered data list $x_{(1)}, \ldots, x_{(n)}$.
- We distinguish between measures of location, dispersion, and skewness.
- A measure of location (or position) indicates the value around which the data is concentrated.
- A measure of dispersion indicates how large the fluctuations of the data are around their central value.
- A skewness measure indicates how symmetrical the data are about their central value.

Measures of Location

Definition (Positional Measure)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a data list. The function $\ell(\boldsymbol{x})$ is called a position measure for $\boldsymbol{x}$ if the following holds true:

- $\ell(a\boldsymbol{x} + b) = a\ell(\boldsymbol{x}) + b$
- $\min(\boldsymbol{x}) \leq \ell(\boldsymbol{x}) \leq \max(\boldsymbol{x})$

- Meaningful position measures indicate the "typical" or "central" value of the data list.
- Depending on the scale, different position measures may be meaningful.

Empirical Measures

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

The Mean Value

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- Meaningful for ordinal, interval and ratio scales.
- The mean minimizes the following function:

$$\bar{x} = \underset{x}{\operatorname{argmin}} \sum_{i=1}^{n} (x_i - x)^2$$

🐍 PYTHON: `xbar=numpy.mean(x)`

Median of Data

$$\tilde{x} = x_{(n/2)}$$

- The median divides the ordered data list into two equal parts.
- Useful for ordinal, interval and ratio scales.
- The median minimizes the following function:

$$\tilde{x} = \operatorname*{argmin}_x \sum_{i=1}^{n} |x_i - x|$$

🐍 PYTHON: xmed=numpy.median(x)

- The median is a special case of a more general term, the quantile.

α-quantile

$$Q_\alpha = x_{(\alpha n)}$$

- The α-quantile divides the ordered data list by the ratio $\alpha : 1 - \alpha$.
- Meaningful for ordinal, interval, and ratio scales.
- PYTHON: qa=numpy.quantile(x,alpha)
- Q_0 is the smallest value, Q_1 is the largest value in the data list. $Q_{0.5}$ is the median.
- The five **quartiles** $Q_0, Q_{0.25}, Q_{0.5}, Q_{0.75}, Q_1$ form the **five point summary** of the data list.
- PYTHON: fps=numpy.quantile(x,[0,0.25,0.5,0.75,1])

Empirical Measures

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical Distribution Function
Examples

Two-dimensional Features

LMS (Least Median of Squares)

LMS value is the midpoint of the shortest interval containing $h = \lfloor n/2 \rfloor + 1$ data points

.

- The LMS value is very insensitive to erroneous or atypical data ("outliers").
- The LMS value minimizes the following function:

$$\tilde{x} = \operatorname*{argmin}_{x} \operatorname{med}_{i=1}^{n}(x_i - x)^2$$

- A related measure of position is the "shorth", the mean of all data in the shortest interval containing h data points.

Mode

Mode is the most common value of a data list

- Useful mainly for qualitative features.
- For quantitative features, the mode can be determined from the kernel density estimator.
- PYTHON: xmode=scipy.stats.mode(x)

HSM (Half-sample mode)

- Determine the shortest interval containing $h = \lfloor n/2 \rfloor + 1$ data points.
- Repeat the process on the data in that interval until there are two data points left.
- The HSM value is the average of these last two data points.

Empirical Measures

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

Dispersion Measures

Definition (Measure of Dispersion)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a data list. A function $\sigma(\boldsymbol{x})$ is called a measure of dispersion for $\boldsymbol{x}$ if the following holds:

- $\sigma(\boldsymbol{x}) \geq 0$
- $\sigma(a\boldsymbol{x} + b) = |a| \, \sigma(\boldsymbol{x})$

- Meaningful measures of dispersion quantify the deviation of the data from their central value.
- Measures of dispersion are invariant under a shift of the data.
- Different measures of dispersion are useful depending on the scale and use case.

The Standard Deviation

$$s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

- Useful for interval and ratio scales.
- The standard deviation has the same dimension as the data.
- The square of the standard deviation is called **variance of the data**.
- PYTHON: `xstd=numpy.std(x)`
- PYTHON: `xvar=numpy.var(x)`

Interquartile Distance

$$IQR = Q_{0.75} - Q_{0.25}$$

- The interquartile range is the length of the interval containing the central 50% of the data.
- Useful for ordinal, interval and ratio scales.
- PYTHON: `xiqr=scipy.stats.iqr(x)`

Empirical Measures

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Graphical Representation

Kernel Density Estimator

Empirical Measures

Boxplot and Empirical Distribution Function

Examples

Two-dimensional Features

LoS (Length of the Shorth)

LoS is the length of the shortest interval containing $h = \lfloor n/2 \rfloor + 1$ data points

- Meaningful for ordinal, interval and ratio scales.

Empirical Measures

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

Skewness Measures

Definition (Measure of Skewness)

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a data list. The function $\mathfrak{s}(\boldsymbol{x})$ is called a skewness measure for $\boldsymbol{x}$ if the following holds:

- $\mathfrak{s}(a\boldsymbol{x} + b) = \mathrm{sgn}(a)\,\mathfrak{s}(\boldsymbol{x})$
- $\mathfrak{s}(\boldsymbol{x}) = 0$ if $\exists b : \boldsymbol{x} - b = b - \boldsymbol{x}$

- Sensible skewness measures quantify the asymmetry of the data.
- Skewness measures are invariant under translation of the data.
- Different skewness measures are useful depending on the scale.

Empirical Measures

The Skewness

$$\gamma = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3}{s^3}$$

- The skewness γ is equal to 0 for symmetric data.
- If $\gamma < 0$, the data is said to be **skewed towards the left**.
- If $\gamma > 0$, the data is said to be **right skewed**.
- Useful for interval and ratio scales.
- PYTHON: xgamma=scipy.stats.skew(x)

Empirical Measures

Skewness Coefficient

$$SK = \frac{R - L}{R + L}$$

where $R = Q_{0.75} - Q_{0.5}, L = Q_{0.5} - Q_{0.25}$.

- SK is between -1 ($R = 0$) and $+1$ ($L = 0$).
- The skewness coefficient is equal to 0 for symmetric data.
- If $SK < 0$, the data is called **left skewed**.
- If $SK > 0$, the data is called **right skewed**.
- Useful for ordinal, interval and ratio scales.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
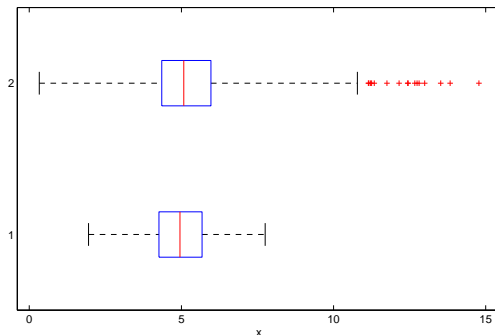Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

1. **Introduction**

2. **One-dimensional Features**
   - Graphical Representation
   - Kernel Density Estimator
   - Empirical Measures
   - Boxplot and Empirical Distribution Function
   - Examples

3. **Two-dimensional Features**

Boxplot and Empirical Distribution Function

- The **Boxplot** is the graphical representation of the five point summary.
- Comparison of data set 1 and data set 2:

 PYTHON: `matplotlib.pyplot.boxplot(x)`

Boxplot and Empirical Distribution Function

- For ordinal scales "onwards" it is useful to order the data.
- The frequency table can be supplemented by *cumulative* frequencies.
- Data set 3 (50 exam grades):

Grade k	$h(k)$	$H(k)$	$f(k)$	$F(k)$
1	5	5	0.10	0.10
2	8	13	0.16	0.26
3	22	35	0.44	0.70
4	5	40	0.10	0.80
5	10	50	0.20	1.00

frequency table with cumulative frequencies

Statistical Methods
of Data Analysis

W. Waltenberger

- The graphical representation of the cumulative frequencies is called the **empirical distribution function** of the data list.

Definition (Empirical Distribution Function)

The empirical distribution function $F_n(x)$ of the data list $\boldsymbol{x} = (x_1, \ldots, x_n)$ is the proportion of the data that are less than or equal to x:

$$F_n(x) = f(\vec{x} \leq x).$$

- If $x_i \leq x < x_{i+1}$, then:

$$F_n(x) = f(x_1) + \cdots + f(x_i).$$

- F_n is a step function. The step points are the data points, the step heights are the relative frequencies of the data points.

- record 3: (50 exam grades):

empirical distribution function

Boxplot and Empirical Distribution Function

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
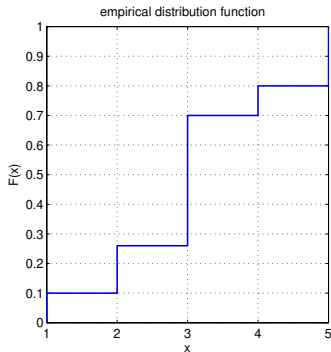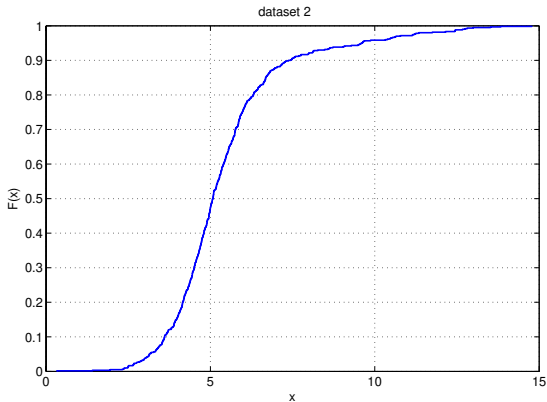Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical Distribution Function
Examples

Two-dimensional Features

- dataset 2 (500 values + contamination):

empirical distribution function

Boxplot and Empirical Distribution Function

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
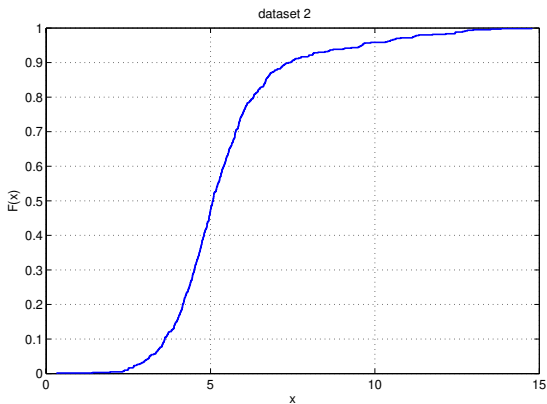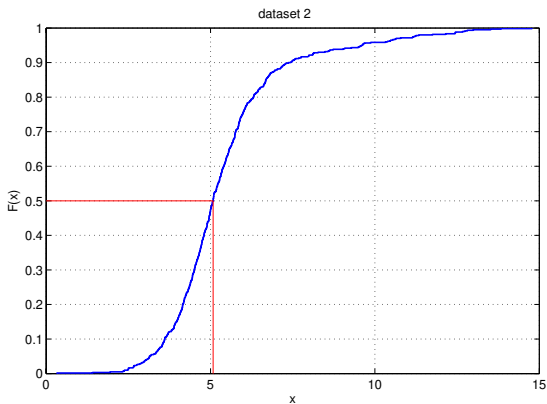Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

- Quantiles can be easily read off the empirical distribution function.
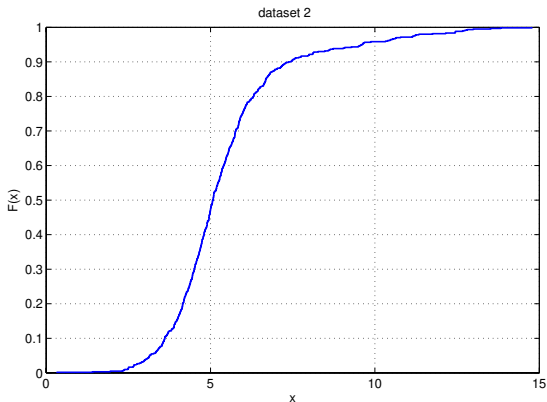- Median of data set 2:

empirical distribution function

- Quantiles can be easily read off the empirical distribution function.
- Median of data set 2:

empirical distribution function

Statistical Methods
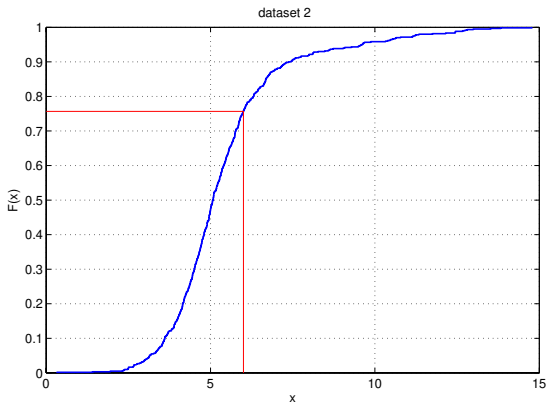of Data Analysis

W. Waltenberger

- Lower tail and upper tail frequencies can also be read off easily.
- What proportion of the data is less than or equal to 6?

empirical distribution function

Boxplot and Empirical Distribution Function

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

- Lower tail and upper tail frequencies can also be read off easily.
- What proportion of the data is less than or equal to 6?

empirical distribution function

1 Introduction

2 One-dimensional Features
- Graphical Representation
- Kernel Density Estimator
- Empirical Measures
- Boxplot and Empirical Distribution Function
- Examples

3 Two-dimensional Features

- dataset 1: Symmetric, 500 values

- Measures of location:

Mean:	4.9532
Median:	4.9518
LMS:	4.8080
Shorth:	4.8002
HSM:	5.0830

- Measures of dispersion:

Standard deviation:	1.0255
Interquartile range:	1.4168
Length of the Shorth:	1.3520

- Measures of Skewness:

Skewness:	0.0375
Skewness coefficient:	0.0258

dataset 1: mean, median, LMS, Shorth, HSM.

Statistical Methods
of Data Analysis

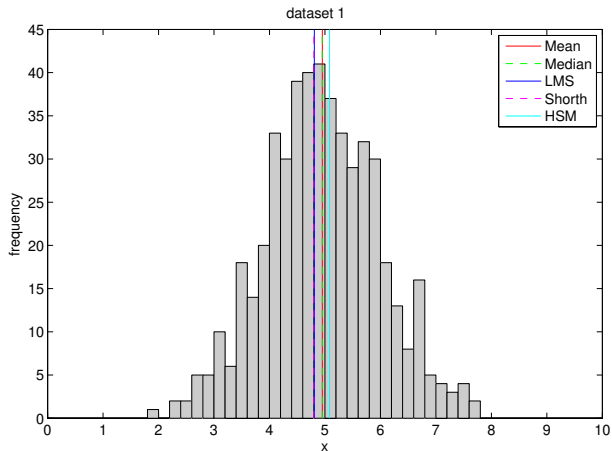W. Waltenberger

Introduction

One-dimensional Features
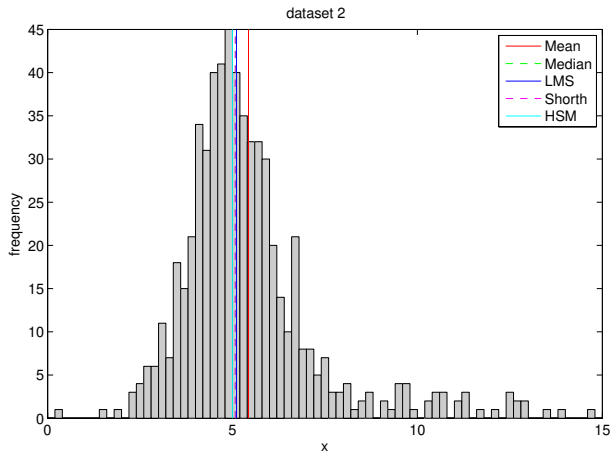Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

Examples

- dataset 2: dataset 1 + contamination (100 values)

- Measures of location:

Mean:	5.4343
Median:	5.0777
LMS:	5.1100
Shorth:	5.0740
HSM:	4.9985

- Measures of dispersion:

Standard deviation:	1.8959
Interquartile range:	1.6152
Length of the Shorth:	1.5918

- Measures of skewness:

Skewness:	1.7696
Skewness coefficient:	0.1046

dataset 2: mean, median, LMS, Shorth, HSM.

Examples

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features
Graphical Representation
Kernel Density Estimator
Empirical Measures
Boxplot and Empirical
Distribution Function
Examples

Two-dimensional
Features

- record 3: 50 exam grades

- Measures of position:

  Mean: 3.14

  Median: 3.0

  Mode: 3.0

- Measures of dispersion:

  Standard deviation: 1.20

  Interquartile range: 1.75

- Measures of skewness:

  Skewness: 0.0765

  Skewness coefficient: 0.14
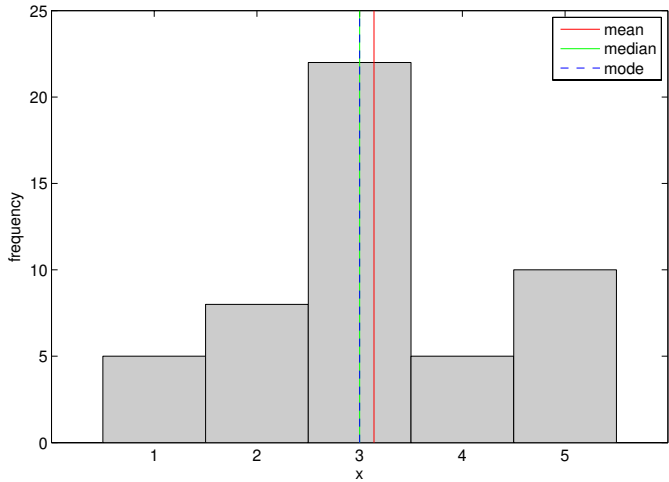
dataset 3: mean, median, mode.

Section 3: Two-dimensional Features

1. Introduction

2. One-dimensional Features

3. Two-dimensional Features
   - Qualitative Features
   - Quantitative Features
   - Correlations

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- Often two or more features of an object **are observed simultaneously**.
- Examples:
  - height and weight of a person
  - age and income of a person
  - education and gender of a person
- The correlation between the two features is additional information.

Qualitative Features

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- We first consider two binary features A and B.
- The frequency of the occurrence of A and B can be expressed in an **four-field table** or **contingency table**.
- example:

  $A="The\ person\ is\ female"$
  $B="The\ person\ is\ a\ smoker"$

- Four-field table for 1000 people:

	B	B'	
A	228	372	600
A'	136	264	400
	364	636	1000

Qualitative Features

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- General structure of a four-field table:

	B	B'	
A	$h(A \cap B)$	$h(A \cap B')$	$h(A)$
A'	$h(A' \cap B)$	$h(A' \cap B')$	$h(A')$
	$h(B)$	$h(B')$	n

- Row and column sums are the frequencies of the expressions A, A' and B, B'.

- The four-field table can be converted to **relative frequencies** using division by n:

$$\begin{array}{c|cc|c} & B & B' & \\ \hline A & f(A \cap B) & f(A \cap B') & f(A) \\ A' & f(A' \cap B) & f(A' \cap B') & f(A') \\ \hline & f(B) & f(B') & 1 \end{array}$$

- row and column sums are the relative frequencies of the expressions A, A' and B, B'.

Qualitative Features

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- The correlation of the two features can be measured by **four-field correlation**:

Four-field Correlation

$$\rho(A, B) = \frac{f(A \cap B) - f(A)f(B)}{\sqrt{f(A)f(A')f(B)f(B')}}$$

- It is always true: $-1 \leq \rho(A, B) \leq 1$
- If $\rho(A, B) > 0$, A and B are called **positively coupled**.
- If $\rho(A, B) < 0$, A and B are called **negatively coupled**.

Qualitative Features

- The sign of $\rho(A, B)$ indicates the **direction** of the coupling.
- The magnitude of $\rho(A, B)$ indicates the **strength** of the coupling.
- Specifically:

$$A = B \implies \rho(A, B) = 1$$
$$A = B' \implies \rho(A, B) = -1$$

- An existing coupling is no proof of a causal relationship!
- The coupling can also result from a common cause for both features.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features
Qualitative Features
Quantitative Features
Correlations

1. Introduction

2. One-dimensional Features

3. Two-dimensional Features
   - Qualitative Features
   - **Quantitative Features**
   - Correlations

Quantitative Features

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- Preferred representation of two-dimensional features: Scatter Plot.
- Each point corresponds to one object.
- The observed features determine the position of the point in the x-y-plane.
- Higher dimensional features can be represented by histograms and scatter plots. Of course, some of the information is lost in the process.

- dataset 4: height and weight of 100 individuals
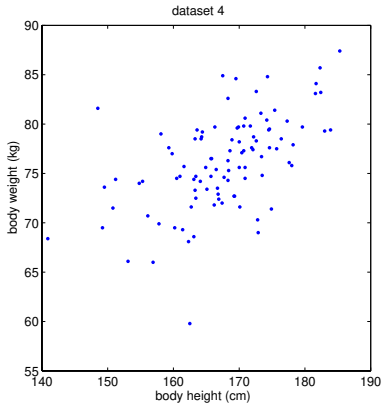
scatter plot

PYTHON: matplotlib.pyplot.scatter(x,y)

Quantitative Features

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional Features

Qualitative Features

Quantitative Features

Correlations

- dataset 5: height, weight and age of 100 people.

  Feature x_1:   height (in cm)

  Feature x_2:   weight (in kg)

  Feature x_3:   age (in years)

Quantitative Features

Statistical Methods
of Data Analysis

W. Waltenberger

1. Introduction

2. One-dimensional Features

3. Two-dimensional Features
   - Qualitative Features
   - Quantitative Features
   - Correlations

Properties of scatter plot

- $(\bar{x}, \bar{y})$ is the center of the point cloud.
- The projection of the scatter plot onto the x-axis results in the scatter plot of the data list $x_1, \ldots, x_n$.
- The projection of the point cloud onto the y-axis results in the scatter plot of the data list $y_1, \ldots, y_n$.

- From the scatter plot of dataset 4, it can be seen that **tendentially** larger height is associated with larger weight.
- There is obviously a relationship between the two features x and y, which intuitively is very obvious.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- We need a **measure** for this notion of correlation.
- A useful measure is the **empirical correlation coefficient**.
- Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a bivariate sample.
- We compute the **standard scores**:

$$z_{x,i} = \frac{x_i - \bar{x}}{s_x}, \quad z_{y,i} = \frac{y_i - \bar{y}}{s_y}$$

- We recall that

$$s_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{and} \quad s_y^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- The empirical correlation coefficient is the **mean of the products** of the standard scores.

Correlations

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional Features

Qualitative Features

Quantitative Features

Correlations

Definition (Empirical Correlation Coefficient)

The **empirical correlation coefficient** r_{xy} is defined as

$$r_{xy} = \frac{1}{n} \sum_{i=1}^{n} z_{x,i} z_{y,i} = \frac{1}{n} \left(z_{x,1} z_{y,1} + \cdots + z_{x,n} z_{y,n} \right)$$

- It is always true that:

$$-1 \leq r_{xy} \leq 1$$

- r_{xy} is positive if many products are positive, i.e. many pairs of of standard scores have the same sign.
- This is the case when the pairs of standard scores are predominantly in the 1st or 3rd quadrant.
- x and y are then called **positively correlated**.
- r_{xy} is negative if many products are negative, i.e. many pairs of standard scores have different sign.
- This is the case when the pairs of standard scores are predominantly in the 2nd or 4th quadrant.
- x and y are then called **negatively correlated**.

Correlations

Statistical Methods
of Data Analysis

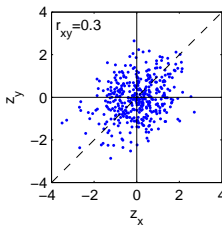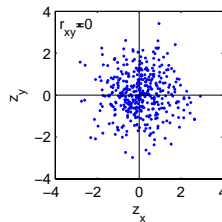W. Waltenberger

Introduction

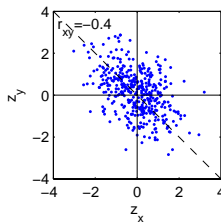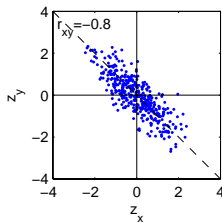One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- Scatter plot of standard scores from dataset 4:

Dataset 4
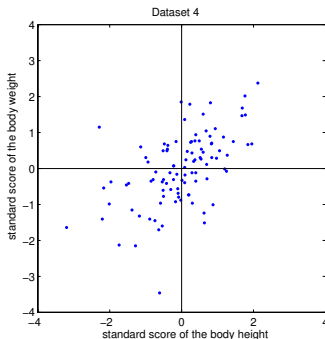
- Obviously x and y are positively correlated, since most of the points are in the 1st and 3rd quadrants.

- $r_{xy} = 0.5562$
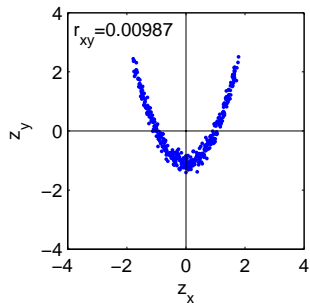
Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- A positive correlation does not necessarily imply a causal relationship.
- The positive correlation can also be caused by a common cause or a parallel trend.

Example

The number of children born correlates with the number of storks in Austria in the last 30 years. Why?

Example

We see a positive correlation between the price of butter and the price of bread in the last 50 years. Why?

Example

Over the last few hundred years, we observe an anti-correlation between the number of pirates on the world's oceans and global CO_2 emissions. Why?

Correlations

Statistical Methods
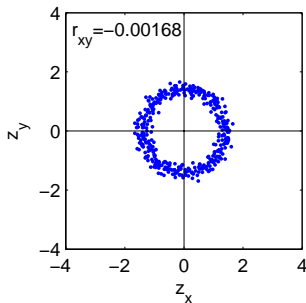of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features
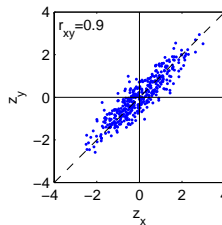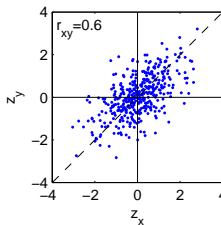
Correlations

standard scores with different correlation coefficients.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

- The correlation coefficient measures the **correlation** of the data.
- The correlation indicates the "binding" of the point cloud to a rising or falling **line**, the **major axis**.
- Thus, the correlation indicates the extent of the **linear** coupling.
- If there is a strong but **nonlinear** relationship between x and y, the correlation may still be very small.

Correlations

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

One-dimensional Features

Two-dimensional
Features

Qualitative Features

Quantitative Features

Correlations

Nonlinear relation between x and y

- The correlation coefficient can also be calculated directly from the sample:

$$r_{xy} = \frac{s_{xy}}{s_x s_y}$$

Definition (Covariance)

The quantity

$$s_{xy} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

is called the **covariance of the data**

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Part II

Probabilities

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Conditional Probability

4 Introduction

5 Events

6 Probabilities

7 Conditional Probability

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

4 **Introduction**

5 Events

6 Probabilities

7 Conditional Probability

Introduction

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

- The specific outcome of an experiment generally cannot be accurately predicted; however, the possible outcomes are known.
- Several reasons:
  - The observed objects are a random selection (sample) from a larger population.
  - The observed process is fundamentally indeterministic (quantum mechanics).
  - The observed process is practically indeterministic: lack of knowledge of the initial state (roulette), chaotic system (hydrodynamics, psychology).
- In addition, measurement errors can add stochasticity (randomness) to the result.

- The goal of probability theory is to assign **probabilities** to sets of outcomes, called events.
- Two interpretations of probabilities exist.

Introduction

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Frequentist Interpretation

- The probability of an event is its frequency, if the experiment repeat often enough, under the same conditions.
- The statistics based on this interpretation is called 'frequentist'.

Example

The probability of the output '1' when rolling a dice is the limit of the frequency for a large number of rolls.

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Subjective Interpretation

- The probability of an output is a statement about the belief of the person giving the probability.
- The statistics based on this interpretation is called 'Bayesian'.

Example

'The probability that it will rain tomorrow is 40 percent' is a statement about the belief of the person making this statement.

Introduction

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

- In practice, the transition between the two approaches is often fluid.
- In many cases the results are identical, only the interpretation is different.
- The Bayesian approach is more comprehensive and flexible.
- The frequentist approach is often simpler, but more limited. Its interpretation is typically clearer.

Section 5: Events

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

The Sample Space
The Event Algebra
Repeating Experiments

Probabilities

Conditional Probability

4 **Introduction**

5 **Events**
   - The Sample Space
   - The Event Algebra
   - Repeating Experiments

6 **Probabilities**

7 **Conditional Probability**

The Sample Space

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

The Sample Space
The Event Algebra
Repeating Experiments

Probabilities

Conditional Probability

- The notion of a (random) **event** is fundamental to statistics.
- Concretely: the **outcome of an experiment**, the result of which **cannot be predicted exactly**.
- The set Ω of all possible outcomes is called **sample space**.
- The sample space Ω can be discrete (finite or countably infinite) or continuous (uncountably infinite).

Example

- In roulette, there are 37 possible outcomes. The sample space is discrete and finite.

- If a radioactive source is monitored, the number of decays per second is in principle unbounded. The sample space is discrete and countably infinite.

- The waiting time between two decays can take any value. The sample space is continuous and uncountably infinite.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events
The Sample Space
The Event Algebra
Repeating Experiments

Probabilities

Conditional Probability

4 Introduction

5 Events
   - The Sample Space
   - The Event Algebra
   - Repeating Experiments

6 Probabilities

7 Conditional Probability

The Event Algebra

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events
The Sample Space
The Event Algebra
Repeating Experiments

Probabilities

Conditional Probability

Definition (Event)

An **event** E is a subset of the sample space Ω. An event E **occurs** if E contains the outcome $\omega \in \Omega$ of the experiment.

Example

The roll of a dice has the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. The event E (even number) is the subset

$$E = \{2, 4, 6\}$$

E occurs when an even number is thrown.

Definition (event algebra)

The set of all events of the sample space Ω is called the event algebra $\Sigma(\Omega)$.

- In the finite or countably infinite case, **any** subset can be considered as an event. The event algebra is called **discrete**.
- In the uncountably infinite case, certain pathological (non-measurable) subsets must be excluded. The event algebra is called **continuous** or **real-valued**.
- Like statements, two events $A \in \Sigma$ and $B \in \Sigma$ can be logically **connected**.

The Event Algebra

Linking Events

Disjunction

Symbol	Name	Meaning
$A \cup B$	disjunction	A or B (or both)

conjunction

Symbol	Name	Meaning
$A \cap B$	conjunction	A and B (both A and B)

Negation

Symbol	Name	Meaning
A'	negation	not A (the opposite of A)

- With these operators, Σ is a **Boolean algebra**: distributive complementary lattice with zero- and one-elements.
- The null element $0 = \emptyset$ is the **impossible event**.
- The one element $1 = \Omega$ is the **sure event**.
- An event consisting of only one possible result is called an **elementary event**.

The Event Algebra

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events
The Sample Space
The Event Algebra
Repeating Experiments

Probabilities

Conditional Probability

- If Ω is (countably or uncountably) infinite, one requires that also countably many unions and intersections of events can be formed.

- The event algebra is then a so-called σ-algebra.

- If in the continuous case $\Omega = \mathbb{R}$, then the event algebra $\Sigma(\Omega)$ is the smallest σ-algebra containing all intervals.

4) Introduction

5) Events
   - The Sample Space
   - The Event Algebra
   - Repeating Experiments

6) Probabilities

7) Conditional Probability

Repeating Experiments

- The sample space of a dice roll is

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Consequently, the event algebra $\Sigma(\Omega)$ has six elementary events:

$$e_1 = \{1\}, e_2 = \{2\}, e_3 = \{3\}, e_4 = \{4\}, e_5 = \{5\}, e_6 = \{6\}$$

and a total of $2^6 = 64$ events (subsets of Ω).

- The sample space of a double roll is the Cartesian product $\Omega \times \Omega$:

$$\Omega \times \Omega = \{(i, j) | i, j = 1, \ldots, 6\}$$

The ordered pair (i, j) means i on the first throw, j on the second throw. Consequently, the event algebra $\Sigma(\Omega \times \Omega)$ has 36 elementary events e_{ij}:

$$e_{11} = \{(1, 1)\}, \ldots, e_{36} = \{(6, 6)\}$$

Repeating Experiments

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
    The Sample Space
    The Event Algebra
    Repeating Experiments
Probabilities
Conditional Probability

- Similarly, in the case of rolling n times, the sample space is n times the Cartesian product $\Omega \times \Omega \times \ldots \times \Omega$.

Example (event algebra of double throw)

Examples of events in the event algebra of the double roll are:

6 on the first throw:	$\{(6, 1), (6, 2), \ldots, (6, 6)\}$
6 on the second throw:	$\{(1, 6), (2, 6), \ldots, (6, 6)\}$
Both throws equal:	$\{(1, 1), (2, 2), \ldots, (6, 6)\}$
Sum of throws equal to 7:	$\{(1, 6), (2, 5), \ldots, (6, 1)\}$

Repeating Experiments

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
The Sample Space
The Event Algebra
Repeating Experiments
Probabilities
Conditional Probability

Example (Repeated Bernoulli Experiment)

An experiment that has only two possible outcomes is called a
Bernoulli experiment or **alternative experiment**. There are two
outcomes, 1 ("success") and 0 ("failure").

An alternative experiment that is executed n times is described by a
sample space with 2^n possible outcomes, namely all sequences of the
form $(i_1, \ldots, i_n)$ with $i_j = 0$ or 1.

Often, however, only the **frequency** of the occurrence of 1 (or 0) is of
interest. This situation is more comprehensively described by only
$n + 1$ outcomes: 1 occuring $0, 1, 2, \ldots$ or n times. If the event E_1
denotes a single occurrence of 1, then E_1 is the union of multiple
elementary events of the original event algebra:

$$E_1 = \{(e_1, e_0, \ldots, e_0), (e_0, e_1, e_0, \ldots, e_0), \ldots, (e_0, \ldots, e_0, e_1)\}$$

An example is an n-fold coin flip.

Section 6: Probabilities

Statistical Methods
of Data Analysis

W. Waltenberger

4 **Introduction**

5 **Events**

6 **Probabilities**
   - Probability Measures
   - Law of Large Numbers

7 **Conditional Probability**

Definition (probabilitymeasure)

Let Σ be an event algebra, A and B events in Σ, and P a mapping from Σ into $\mathbb{R}$. P is called a **probability measure** if the following holds:

1. positivity $\qquad P(A) \geq 0, \ \forall A \in \Sigma$
2. additivity: $\qquad A \cap B = \mathbf{0} \implies$

   $\qquad\qquad\qquad P(A \cup B) = P(A) + P(B)$
3. normalization $\quad P(\mathbf{1}) = 1$

Probability Measures

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Probability Measures
Law of Large Numbers
Conditional Probability

Definition (probability space)

If Σ is a σ-algebra, which must be assumed for infinite event spaces, one requires for countable J:

4. σ-additivity: $\quad A_i \in \sigma, \; i \in J; \; A_i \cap A_j = \mathbf{0}, \; i \neq j \implies$
$$P(\bigcup_{i \in J} A_i) = \sum_{i \in J} P(A_i)$$

Σ is then called normalized, and (Σ, P) a **probability space**. P is also called a **probability distribution**.

Calculating laws for probability

If (Σ, P) is a probability space, then:

- $P(\mathbf{0}) = 0$
- $P(\mathbf{1}) = 1$
- $0 \leq P(A) \leq 1, \ \forall A \in \Sigma$
- $P(A') = 1 - P(A), \ \forall A \in \Sigma$
- $A \subseteq B \implies P(A) \leq P(B), \ \forall A, B \in \Sigma$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B), \ \forall A, B \in \Sigma$
- If Σ has at most countably many elementary events $\{e_i \mid i \in I\}$, then $\sum_{i \in I} P(e_i) = 1$.

- In a discrete event algebra, the probability of an event is equal to the sum of the probabilities of the elementary events whose union it is.

- Therefore, in this case, a probability measure is **uniquely determined** by the values it assigns to the elementary events.

- On the other hand, any positive function defined on the set of elementary events that satisfies the normalization condition can be uniquely continued to a probability measure.

- Thus, one can define infinitely many distributions on a discrete event algebra Σ.

- In a continuous event algebra, the probability of each elementary event is 0.
- Therefore, the probability of an event can no longer be determined by summation.
- Instead, a **density function** $f(x)$ is given which assigns a nonnegative value $f(x)$ to each elementary event x.
- The probability of an event A is determined by **integration** over the density:

$$P(A) = \int_A f(x)\,\mathrm{d}x$$

- The density function is **normalized** to 1:

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1$$

Law of Large Numbers

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Probability Measures
Law of Large Numbers
Conditional Probability

- We consider a simple random experiment: coin toss.
- Two possible outcomes: Heads (H), Tails (T).
- Assumption: symmetric coin, H and T equally probable
- Experiment is repeated n times

n	$h_n(H)$	$f_n(H)$	$|f_n(H) - 0.5|$
10	6	0.6	0.1
100	51	0.51	0.01
500	252	0.504	0.004
1000	488	0.488	0.012
5000	2533	0.5066	0.0066

frequency table

Law of Large Numbers
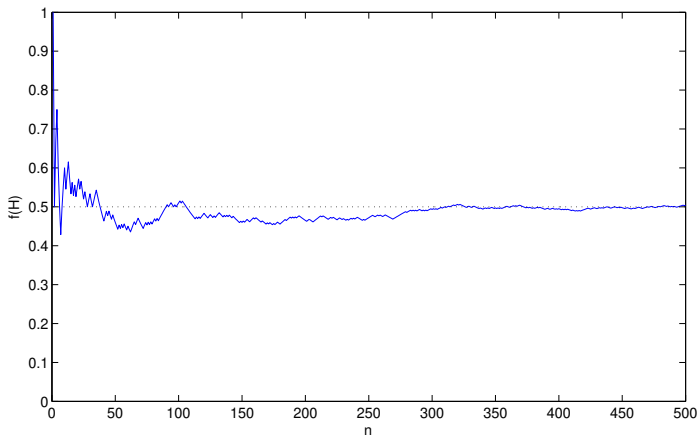
evolution of the relative frequency of H

Law of Large Numbers

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Probability Measures
Law of Large Numbers
Conditional Probability

- The relative frequency of the event K seems to converge towards the limit value 0.5.
- This limit is called the **probability** $P(K)$.

Empirical Law of Large Numbers

$$\lim_{n \to \infty} f_n(K) = P(K)$$

- The mathematical problem of this definition lies in the fact that the existence of the limit cannot be seen a priori and in fact need not exist in the classical analytic sense, but only in a broader, statistical sense.

Section 7: Conditional Probability

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

4. Introduction

5. Events

6. Probabilities

7. Conditional Probability
   - Coupling and Conditional Probability
   - Bayes' Theorem
   - Independence

Coupling and Conditional Probability

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- We now consider two events A and B that can occur in an experiment.
- **Question**: Is there a relationship between the events?
- Such a connection is called a **coupling**.
- **Positive Coupling**: The more often A occurs, the more often B tends to occur.
- **Negative Coupling**: The more often A occurs, the less often B also tends to occur.
- We can quantify the meanings of 'often' and 'rarely' with the frequency table.

Coupling and Conditional Probability

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Conditional Probability
Coupling and Conditional
Probability
Bayes' Theorem
Independence

- The frequency of the occurrence of A and B can be summarized in a **four-field table** or **contingency table**.
- Example:

  $A=$'The person under study is female'
  $B=$'The person under study has diabetes'

- Four-field table for 1000 persons:

	B	B'	
A	19	526	545
A'	26	429	455
	45	955	1000

- **Ordinary relative frequencies** are relative to the size n of the entire data set:

$$f(A \cap B) = \frac{h(A \cap B)}{n}$$

- **Conditional relative frequencies** are relative to the occurrence of the other feature:

$$f(A|B) = \frac{h(A \cap B)}{h(B)} = \frac{f(A \cap B)}{f(B)}$$

- $f(A|B)$ is called the conditional relative frequency of A given B.

- The four-field table results in the following conditional relative frequencies:

$$f(A|B) = \frac{19}{45} = 0.422, \quad f(A|B') = \frac{526}{955} = 0.551$$

- Thus, it can be assumed that the two features are coupled.
- $f(A|B) > f(A)$ indicates a positive coupling, $f(A|B) < f(A)$ indicates a negative coupling.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- If the data originate from a random experiment, then combinations of events can also be assigned probabilities.
- Probability table:

	B	B'	
A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
	$P(B)$	$P(B')$	1

- By the empirical law of large numbers, these probabilities are the limits of the respective relative frequencies.

- The conditional relative frequencies converge towards a limit with $n \to \infty$:

$$f_n(A|B) = \frac{f_n(A \cap B)}{f_n(B)} \to P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Definition (Conditional probability)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

is called the **conditional probability of A under condition B**, provided that $P(B) \neq 0$.

Example (The symmetric cube)

If the cube is completely symmetric, the elementary events $e_i = \{i\}$ are assigned equal probabilities:

$$P(e_i) = \frac{1}{6}, \ 1 \le i \le 6$$

We define the following events:

$$O = \{1, 3, 5\}, \ E = \{2, 4, 6\}$$

Then for example

$$P(e_1|O) = \frac{P(e_1 \cap O)}{P(O)} = \frac{P(e_1)}{P(O)} = \frac{1}{3}$$

$$P(e_1|E) = \frac{P(e_1 \cap E)}{P(E)} = \frac{P(\mathbf{0})}{P(E)} = 0$$

Example (Continuation)

$$P(O|e_1) = \frac{P(e_1 \cap O)}{P(e_1)} = \frac{P(e_1)}{P(e_1)} = 1$$

$$P(e_1 \cup e_3|O) = \frac{P((e_1 \cup e_3) \cap O)}{P(O)} = \frac{P(e_1 \cup e_3)}{P(O)} = \frac{2}{3}$$

$$P(e_1 \cup e_2|O) = \frac{P((e_1 \cup e_2) \cap O)}{P(O)} = \frac{P(e_1)}{P(O)} = \frac{1}{3}$$

- The

Product Formula

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

follows immediately from the definition of conditional probabilities; as well as the formula for

Inverse Probability

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

- Both formulas are also valid for relative frequencies!

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

Definition (Decomposition)

The events $B_1, B_2, \ldots, B_m$ form a **decomposition** of the sample space Ω if the following holds:

1. Non-overlapping: $B_i \cap B_j = \emptyset, i \neq j$
2. Completeness: $B_1 \cup B_2 \cup \ldots \cup B_m = \Omega$

Theorem

If the events $B_1, B_2, \ldots, B_m$ form a decomposition of the sample space Ω, it the follows that:

$$P(B_1) + P(B_2) + \ldots + P(B_m) = P(\Omega) = \mathbf{1}$$

- Let $B_1, \ldots, B_m$ be a decomposition. It follows that:

Total Probability

$$P(A) = P(A|B_1)P(B_1) + \ldots + P(A|B_m)P(B_m)$$

Example

A factory produces resistors with $10\,\text{k}\Omega$ (35% of production), with $22\,\text{k}\Omega$ (45%) and with $47\,\text{k}\Omega$ (20%). After two years of continuous operation, 98% of the $10\,\text{k}\Omega$ resistors are still functional, 96% of the $22\,\text{k}\Omega$ resistors, and 92% of the $47\,\text{k}\Omega$ resistors. What proportion of all resistors is still functional after two years?

- Let $B_1, \ldots, B_m$ be a decomposition. It follows that:

Bayes' Theorem

$$
\begin{aligned}
P(B_i|A) &= \frac{P(A|B_i)P(B_i)}{P(A)} \\
&= \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + \ldots + P(A|B_m)P(B_m)}
\end{aligned}
$$

- $P(B_i)$ will be the **a-priori** probability of B, $P(B_i|A)$ is called the **a-posteriori** probability.

Example

A company buys components from two suppliers, where the share of the first one is 65%. The rate of defective components of supplier 1 is equal to 3%, that of supplier 2 is equal to 4%.

1. What is the total rate of defective components?

2. What is the probability that a flawless part comes from supplier 2?

3. What is the probability that a defective component comes from supplier 1?

Bayes' Theorem

Statistical Methods of Data Analysis

W. Waltenberger

Introduction
Events
Probabilities
Conditional Probability
Coupling and Conditional Probability
Bayes' Theorem
Independence

Example

A component is supplied by four companies, and 20% comes from company 1, 30% comes from company 2, 35% comes from company 3, and 15% comes from company 4. The probability that the component fails in test operation within 24 hours is 0.02 for company 1, 0.015 for company 2, 0.025 for company 3, and 0.02 for company 4. A component fails in test operation after 16 hours. The probability that it comes from company i is to be calculated by means of Bayes' theorem.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- Two events are **positively coupled** if

$$P(A|B) > P(A) \quad or \quad P(A \cap B) > P(A)P(B)$$

- Two events are **negatively coupled** if

$$P(A|B) < P(A) \quad P(A \cap B) < P(A)P(B)$$

- If neither positive nor negative coupling is present, then A and B are mutually **independent**.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

Definition (Independence)

Two events A and B are called **stochastically independent** if

$$P(A \cap B) = P(A)P(B)$$

The events $A_1, A_2, \ldots, A_n$ are called independent if:

$$P(A_1 \cap \ldots \cap A_n) = P(A_1) \cdot \ldots \cdot P(A_n)$$

For $n > 2$, pairwise independence of any two events A_i and A_j is not generally sufficient!

Example

We consider flipping a fair coin twice (heads H/tails T). The possible outcomes are $\Omega = \{HH, HT, TH, TT\}$. We define the events:

$$E_1 = \{HH, HT\} \dots \text{head on first throw}$$
$$E_2 = \{HH, TH\} \dots \text{head on second throw}$$
$$E_3 = \{HH, TT\} \dots \text{even number of heads}$$

Then, for all $i \neq j$:

$$P(E_i \cap E_j) = \frac{1}{4} = P(E_i) \cdot P(E_j)$$

but

$$P(E_1 \cap E_2 \cap E_3) = \frac{1}{4} \neq \frac{1}{8} = P(E_1) \cdot P(E_2) \cdot P(E_3)$$

Independence

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- If A and B are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$.

- The four-field table for two independent events:

	B	B'	
A	$P(A)P(B)$	$P(A)P(B')$	$P(A)$
A'	$P(A')P(B)$	$P(A')P(B')$	$P(A')$
	$P(B)$	$P(B')$	1

- The coupling can be measured by the **four-field correlation**:

Four-field Correlation

$$\rho(A, B) = \frac{P(A \cap B) - P(A)P(B)}{\sqrt{P(A)P(A')P(B)P(B')}}$$

Properties of Four-field Correlation

- $-1 \leq \rho(A, B) \leq 1$
- $\rho(A, B) = 0 \iff A$ and B independent
- $\rho(A, B) > 0 \iff A$ and B positively coupled
- $\rho(A, B) < 0 \iff A$ and B negatively coupled

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- The sign of $\rho(A, B)$ indicates the **direction** of the coupling.
- The magnitude of $\rho(A, B)$ indicates the **strength** of the coupling.
- Specifically:

$$A = B \implies \rho(A, B) = 1$$
$$A = B' \implies \rho(A, B) = -1$$

- An existing coupling is no proof for a causal relationship!
- The coupling can also result from a common cause for both events.

Independence

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

- Two events can be considered independent if there is no connection whatsoever between them. The occurrence of one event cannot affect the probability of the other.

- Two elementary events are never independent, since their ∩-connection is always the impossible event.

- Two elementary events are even highly 'mutually dependent' since the occurrence of one precludes the occurrence of the other with complete certainty.

- If E_1 and E_2 are two independent events of a probability space (Σ, P), then E_1 and E_2', E_1' and E_2, and E_1' and E_2' are also independent.

Example (rolls with two distinguishable dice)

There are 36 elementary events $e_{ij} = \{(i,j)\}$, $1 \le i, j \le 6$. The event E_i^1 of rolling an i on the first roll is thus composed of:

$$E_i^1 = e_{i1} \cup e_{i2} \cup \ldots \cup e_{i6} \text{ and analogously}$$

$$E_j^2 = e_{1j} \cup e_{2j} \cup \ldots \cup e_{6j}$$

Clearly, $E_i^1 \cap E_j^2 = e_{ij}$.
If we can assume that all elementary events are **equally probable**, then:

$$P(E_i^1) = \frac{1}{6}, \ P(E_j^2) = \frac{1}{6}$$

$$P(E_i^1 \cap E_j^2) = P(e_{ij}) = \frac{1}{36} = P(E_i^1) \cdot P(E_j^2)$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Events

Probabilities

Conditional Probability

Coupling and Conditional
Probability

Bayes' Theorem

Independence

Example (Continuation)

Thus, in this case, the elementary events of the simple throw are also **equally probable** and the two partial throws are **independent**. Conversely, if we assume that for both partial throws the elementary events are equally probable, and that E_i^1 and E_j^2 are independent for all i and j, then the e_{ij} are equally probable. If the partial rolls are not independent, then the e_{ij} **are no longer** equally probable, **despite** e_i and e_j being equally probable! An example of this is the 'rolling' a very large dice, which can be rotated by a mere 90° each time. E.g. the elementary event e_{34} is then no longer possible and must therefore be assigned a probability of 0.

Example (Repetition of an alternative experiment)

The event algebra has 2^n elementary events, namely sequences of the form $(i_1, \ldots, i_n)$, $i_j = 0$ or 1. If the repetitions are independent, and p denotes the probability of 1 occurring, the probability of a sequence

$$P(\{(i_1, \ldots, i_n)\}) = p^{n_1}(1-p)^{n_0}$$

where n_0 or n_1 denotes the number of occurrences of 0 or 1, respectively. Clearly, $n_0 + n_1 = n$.

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables

Moments

Functions of Random
Samples

Part III

Random Variables and Distributions

8 Random Variables

9 Moments

10 Functions of Random Samples

8 Random Variables
- Basic Terms
- Discrete Random Variable
- Continuous Random Variable
- Independence
- Convolution

9 Moments

10 Functions of Random Samples

8 Random Variables
- Basic Terms
- Discrete Random Variable
- Continuous Random Variable
- Independence
- Convolution

9 Moments

10 Functions of Random Samples

Basic Terms

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

Definition (Random Variable)

A mapping X:

$$\omega \in \Omega \mapsto x = X(\omega) \in \mathbb{R}$$

that assigns a real number to each element ω of the sample space Ω is called a (one-dimensional or univariate) **random variable**.

- If Ω is finite or countably infinite, any mapping X is allowed.
- If Ω is uncountably infinite, X must be a **measurable** mapping.
- Since the value of a random variable depends on the outcome of the experiment, we can assign probabilities to the possible values.

Basic Terms

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

- If the random variable X takes only finitely or countably infinitely many values, it is called **discrete**.
- If the random variable X takes a continuum of values, it is called **continuous** or **real-valued**.

Example

The mapping that assigns the number of eyes i to the elementary event e_i when rolling the dice is a discrete random variable. Of course, the mapping $e_i :\longrightarrow 7 - i$ would also be a discrete random variable.

Example

The mapping assigning a lifetime x to the decay of a particle is a continuous random variable.

Statistical Methods
of Data Analysis

W. Waltenberger
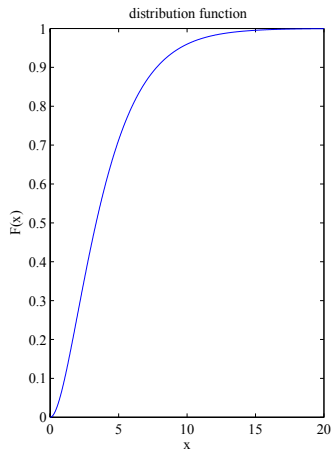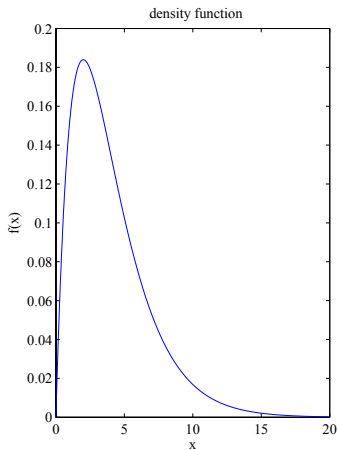
Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

- Discrete random variables are often the result of **counting operations**.

- Discrete random variables occur in many areas: for example, counting events in a fixed time interval (Poisson distribution) or counting successes in repeated alternative trials (binomial distribution),

- In the following, we assume that the values of a discrete random variable are nonnegative integers. This is not a restriction because any countable set of real numbers can be mapped bijectively to (a subset of) $\mathbb{N}_0$.

- The event algebra is the power set (set of all subsets) P of $\mathbb{N}_0$.

Discrete Random Variable

- If a probability measure P is defined on the event algebra $\Sigma(\Omega)$, then a probability measure can also be defined on the power set P of $\mathbb{N}_0$ using the random variable X.

Definition (Distribution of a Discrete Random Variable)

Let $\Sigma(\Omega)$ be a discrete event algebra. The discrete random variable $X : \Omega \mapsto \mathbb{N}_0$ induces a probability measure on $\mathbb{N}_0$ by means of.

$$P_X(\{k\}) = P(X^{-1}(k)) = P(\{\omega | \boldsymbol{X}(\omega) = k\})$$

P_X is called the **distribution** of X, and it is discrete.

Discrete Random Variable

Statistical Methods of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random Samples

Example

We assign the number 0 to the even numbers of the cube, and the number 1 to the odd numbers:

$$X : \omega \mapsto \quad \mod (\omega, 2)$$

The distribution of X is then given by

$$P_X(0) = P(X^{-1}(0)) = P(\{2, 4, 6\}) = \frac{1}{2}$$

$$P_X(1) = P(X^{-1}(1)) = P(\{1, 3, 5\}) = \frac{1}{2}$$

Example

We assign the sum of the numbers of eyes to the outcome of a double roll:

$$X : (i, j) \mapsto i + j$$

The values of X are the natural numbers from 2 to 12. The distribution of X is then given by

$$P_X(k) = P(X^{-1}(k)) = \sum_{i+j=k} P(\{(i,j)\}) = \begin{cases} \dfrac{k-1}{36}, & k \leq 7 \\ \dfrac{13-k}{36}, & k \geq 7 \end{cases}$$

Discrete Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

- The numbers $P_X(k)$ can be regarded as function values of a spectral function f_X:

$$f_X(x) = \begin{cases} P_X(k), & \text{if } x = k \\ 0, & \text{else} \end{cases}$$

Definition (Discrete Density Function)

The function $f_X(x)$ is called **probability density function** or density of a random variable X for short

.

Discrete Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

- The density of the random variable $X = i + j$:
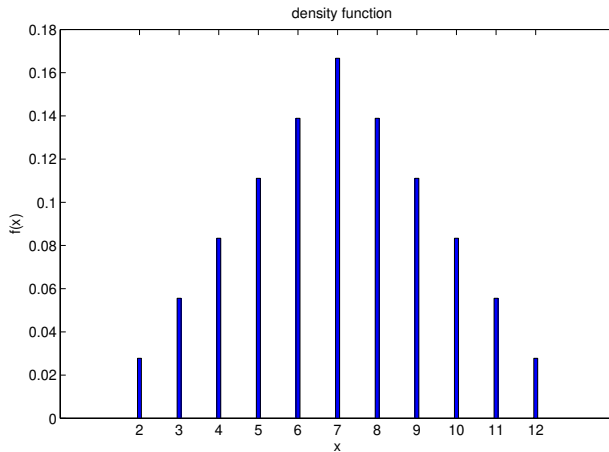
Discrete Random Variable

Statistical Methods of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random Samples

- The probability $P_X(E)$ of an event E can be conveniently calculated using the density of X:

$$P_X(E) = \sum_{k \in E} f_X(k)$$

Definition (Discrete Distribution Function)

If X is a discrete random variable, the **distribution function** F_X of X is defined by:

$$F_X(x) = P(X \le x)$$

Obviously,

$$F_X(x) = \sum_{k \le x} f_X(k) = \sum_{k \le x} P_X(\{k\})$$

Discrete Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

- The distribution function of the random variable $X = i + j$:
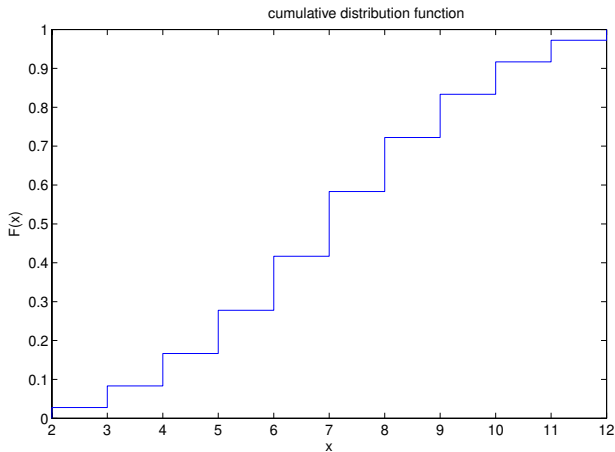
cumulative distribution function

Discrete Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

Properties of a Discrete Distribution Function F

- F has a step in all points within the range of values.
- The height of the step at point k is equal to $f_X(k)$
- $0 \leq F(x) \leq 1, \ \forall x \in \mathbb{R}$
- $x \leq y \implies F(x) \leq F(y), \ \forall x, y \in \mathbb{R}$
- $\lim_{x \to -\infty} F(x) = 0; \ \lim_{x \to \infty} F(x) = 1$
- The probability that r falls within the interval $(a, b]$ is $F(b) - F(a)$:

$$P(r \leq a) + P(a < r \leq b) = P(r \leq b) \implies$$
$$P(a < r \leq b) = F(b) - F(a)$$

Discrete Random Variable

Important Discrete Distribution Families

- Poisson distribution $\mathrm{Po}(\lambda)$
- Bernoulli or alternative distribution $\mathrm{Al}(p)$
- Binomial distribution $\mathrm{Bi}(n, p)$
- Hypergeometric distribution $\mathrm{Hy}(N, M, n)$

8 Random Variables
- Basic Terms
- Discrete Random Variable
- Continuous Random Variable
- Independence
- Convolution

9 Moments

10 Functions of Random Samples

Continuous Random Variable

- So far, only random variables defined on discrete event algebras were treated.

- Let us now drop this restriction, i.e. let us permit uncountably many elementary events. This is necessary, if we wish to describe any types of measurements.

- A function X defined on such an uncountable set of elementary events can take on arbitrary real values.

Continuous Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

Definition (Continuous Distribution Function)

Let (Σ, P) be a probability space over a countable outcome space Ω. Let X be a random variable, i.e., a (measurable) function of Ω in $\mathbb{R}$. The function F_X, defined by:

$$F_X(x) = P(X \leq x)$$

is called the **distribution** function of X. The probability that X falls within an interval $(x, x + \delta x]$ is then:

$$P(x < X \leq x + \Delta x) = F_X(x + \Delta x) - F_X(x) = \Delta F_X.$$

Continuous Random Variable

Properties of a Continuous Distribution Function

- $0 \leq F(x) \leq 1, \ \forall x \in \mathbb{R}$
- $x_1 \leq x_2 \implies F(x_1) \leq F(x_2), \ \forall x_1, x_2 \in \mathbb{R}$
- $\lim_{x \to -\infty} F(x) = 0; \ \lim_{x \to \infty} F(x) = 1$

Definition (quantile)

Let $F_X(x)$ be a continuous distribution function. The value x_α, for which

$$F_X(x_\alpha) = \alpha, \quad 0 < \alpha < 1,$$

is called the α-**quantile** of the distribution of X. The function

$$x = Q_X(\alpha) = F_X^{-1}(\alpha), \quad 0 < \alpha < 1$$

is called the **quantile function** of the distribution of X.

Continuous Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

Definition (quartile)

The quantiles to the values $\alpha = 0.25, 0.5, 0.75$ are called
quartiles. The quantile to the value $\alpha = 0.5$ is called the
median of the distribution.

- Quantiles can also be defined for discrete distributions, in which
  case they are not always unique.

Continuous Random Variable

Definition (Continuous density function)

If F_X is differentiable, X is called a **continuous random variable**. For the distribution of X, according to the main theorem of integral calculus:

$$P_X(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\, \mathrm{d}x$$

where $f_X(x) = F_X'(x)$. The derivative of the distribution function, the function f_X, is called the **probability density function** or, again, the density of X for short.

Continuous Random Variable

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution

Moments

Functions of Random
Samples

density function

distribution function

Continuous Random Variable

- The probability measure P_X is called the distribution of X. It is defined on an event algebra Σ consisting of sets of real numbers and containing at least all intervals and their unions as elements.

- In analogy with discrete random variables, the probability P_X of a set $M \in \Sigma$ can be easily specified using density:

$$P_X(M) = \int_M f_X(x)\, \mathrm{d}x$$

- The probability of a single point is always equal 0:

$$P_X(\{x\}) = \int_x^x f_X(x)\, \mathrm{d}x = 0$$

Continuous Random Variable

- Therefore

$$P_X((x_1, x_2]) = P_X((x_1, x_2)) = P_X([x_1, x_2]).$$

- More generally, a statement about continuous random variables is obtained by replacing summation with integration in a statement about discrete random variables.

- E.g. for a discrete density f:

$$\sum_{k \in \mathbb{N}_0} f(k) = 1$$

similarly, for a continuous density f:

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 1$$

Continuous Random Variable

Important continuous distribution families

- Normal distribution $\mathrm{No}(\mu, \sigma^2)$
- Gamma distribution $\mathrm{Ga}(a, b)$, with the special cases
  - Exponential distribution $\mathrm{Ex}(\tau)$
  - Chisquare distribution $\chi^2(n)$
- Student's or t-distribution $\mathrm{t}(n)$
- Beta distribution $\mathrm{Be}(a, b)$
- Uniform distribution $\mathrm{Un}(a, b)$

8 Random Variables
- Basic Terms
- Discrete Random Variable
- Continuous Random Variable
- Independence
- Convolution

9 Moments

10 Functions of Random Samples

Definition (Independent Random Variable)

Two random variables X and Y are called independent if for all pairs (A, B) of events:

$$P(X \in A \cap Y \in B) = P_X(A) \cdot P_Y(B)$$

Definition (Joint Distribution Function)

The joint distribution function F_{XY} of X and Y is defined by

$$F_{XY}(x, y) = P(X \leq x \cap Y \leq y)$$

Definition (Common Density)

If X and Y are discrete random variables, their joint density is defined by:

$$f_{XY}(x, y) = P(X = x \cap Y = y)$$

If X and Y are continuous random variable, their joint density is defined by

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}}{\partial x \ \partial y}$$

Independence

Theorem

Two random variables X and Y are independent if and only if their joint distribution function is the product of their individual distribution functions:

$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$$

Theorem

Two random variables X and Y are independent if and only if their joint density is the product of their individual density functions:

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables
Basic Terms
Discrete Random Variable
Continuous Random Variable
Independence
Convolution
Moments
Functions of Random
Samples

8 Random Variables
- Basic Terms
- Discrete Random Variable
- Continuous Random Variable
- Independence
- Convolution

9 Moments

10 Functions of Random Samples

Definition (Convolution)

Let X and Y be two **independent** random variables and $Z = X + Y$ be their sum. The distribution of Z is called the **convolution** of the distributions of X and Y.

Example

An experiment measures the lifetime X of an unstable particle, with a measurement error Y. If X and Y are independent, the distribution of the observation $Z = X + Y$ is the convolution of the distributions of X and Y, respectively.

Theorem

Let X and Y be two **independent** random variables and $Z = X + Y$ be their sum. Then the density of Z is the **convolution product** of the densities of X and Y.

- If X and Y are discrete, then:

$$f_Z(n) = \sum_k f_X(n - k) f_Y(k)$$

- If X and Y are continuous, then:

$$f_Z(z) = \int_{\mathbb{R}} f_X(z - y) f_Y(y) \, \mathrm{d}y$$

It should be noted that the effective integration range may depend on z.

8 Random Variables

9 Moments
- Expectation
- Variance
- Skewness
- Error Propagation

10 Functions of Random Samples

8 Random Variables

9 Moments
  - Expectation
  - Variance
  - Skewness
  - Error Propagation

10 Functions of Random Samples

Definition (Expectation)

Let X be a (discrete or continuous) random variable with density $f(x)$. Further, let g be any continuous real- or complex-valued function. Let $\mathsf{E}_X[g] = \mathsf{E}[g(X)]$ be defined by:

$$\mathsf{E}[g(X)] = \sum_{k \in \mathbb{N}_0} g(k) f(k) \quad \text{or} \quad \mathsf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) \, \mathrm{d}x$$

$\mathsf{E}_X[g] = \mathsf{E}[g(X)]$ is called the **expectation** of $g(X)$.

Definition (expectation of random variables)

If $g(x) = x$, then $\mathsf{E}[g(X)] = \mathsf{E}[X]$ is called the **expectation** or the **mean** of X.

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x \text{ but } \mathsf{E}[X] = \sum_{k \in \mathbb{N}_0} k \, f(k)$$

Properties of the Expectation

- $\mathsf{E}[c] = c, \ c \in \mathbb{R}$
- $\mathsf{E}[aX + b] = a\mathsf{E}[X] + b$
- $\mathsf{E}[X_1 + X_2] = \mathsf{E}[X_1] + \mathsf{E}[X_2]$
- X_1 and X_2 independently $\implies \mathsf{E}[X_1 X_2] = \mathsf{E}[X_1] \cdot \mathsf{E}[X_2]$

Expectation

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables

Moments
Expectation
Variance
Skewness
Error Propagation

Functions of Random
Samples

- The expectation is a **location parameter**.
- The expectation may not exist. An example is the Cauchy distribution with the density

$$f(x) = \frac{1}{\pi(1 + x^2)}, \ x \in \mathbb{R}$$

In this case, the median is a suitable location parameter.

Definition (moments)

Let X be a random variable. The expectation of $g(x) = (x - a)^k$, if it exists, is called k-**th moment of** X **around** a. The k-th moment around 0 is denoted by μ'_k. The k-th moment around the expected value $\mathsf{E}[X]$ is called **central moment** μ_k.

- The expectation is the first moment around 0.

- The central moments $\mu_1, \ldots, \mu_k$ can be calculated from the moments around 0 $\mu'_1, \ldots, \mu'_k$, and vice versa.

- Even if all moments of a distribution exist, it is still **not** uniquely determined.

8 Random Variables

9 Moments
- Expectation
- Variance
- Skewness
- Error Propagation

10 Functions of Random Samples

Variance

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables

Moments

Expectation

Variance

Skewness

Error Propagation

Functions of Random
Samples

Definition (Variance)

The second central moment μ_2 is called the **variance** of X, denoted by $\mathrm{var}[X]$. The root of the variance is called the **standard deviation** of X, denoted by $\sigma[X]$.

- The standard deviation is a scale parameter that describes the width of the distribution.
- The standard deviation has the same dimension as the values of the random variables.
- Variance and standard deviation are (like all central moments) invariant against translations.
- The variance may not exist. An example is the distribution with density
$$f(x) = \frac{1}{(2 + x^2)^{3/2}}, \ x \in \mathbb{R}$$

Properties of the Variance

- $\text{var}[X] = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$
- $\text{var}[aX + b] = a^2 \text{var}[X]$
- For independent $X_1, \ldots, X_n$:

$$\text{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \text{var}[X_i]$$

8 Random Variables

9 Moments
- Expectation
- Variance
- Skewness
- Error Propagation

10 Functions of Random Samples

Statistical Methods of Data Analysis

W. Waltenberger

Random Variables

Moments
Expectation
Variance
Skewness
Error Propagation

Functions of Random Samples

Definition (Skewness)

The reduced third central moment $\gamma = \mu_3/\sigma^3$ is called the **skewness**.

- The skewness measures the asymmetry of a distribution. If the skewness is positive (negative), the distribution is called right skewed (left skewed). For symmetric distributions, it is 0.

Definition (Kurtosis)

The reduced fourth central moment $\kappa = \mu_4/\sigma^4$ is called the **curvature** or **kurtosis**.

- The kurtosis of the normal distribution is 3.
- Distributions with higher kurtosis have more pronounced tails than distributions with lower kurtosis.

Subsection: Error Propagation

Statistical Methods
of Data Analysis

W. Waltenberger

Random Variables

Moments
Expectation
Variance
Skewness
Error Propagation

Functions of Random
Samples

Affine Transformations

- Let X be a random variable with density $f(x)$ and $Y = aX + b$.
- Then the density $g(y)$ of Y is equal to

$$g(y) = \frac{1}{|a|} f\left(\frac{y - b}{a}\right)$$

- Furthermore,

$$\begin{aligned}
\mathsf{E}[Y] &= a\mathsf{E}[X] + b \\
\mathsf{var}[Y] &= a^2 \mathsf{var}[X] \\
\gamma[Y] &= \operatorname{sgn}(a)\gamma[X] \\
\kappa[Y] &= \kappa[X]
\end{aligned}$$

Nonlinear Transformations

- Let X be a random variable with density $f(x)$ and $Y = h(X)$.
- If $h(x)$ is bijective, the density $g(y)$ of Y is equal to

$$g(y) = \left| \frac{\mathrm{d}h^{-1}}{\mathrm{d}y} \right| f\left(h^{-1}(y)\right)$$

- The expectation and variance of $Y = h(X)$ can be calculated **approximately** using the Taylor expansion of $h(x)$.

Error Propagation

- Expanding around x_0, the linear approximation reads

$$h(x) \approx h(x_0) + h'(x_0)(x - x_0)$$

- With a choice of $x_0 = \mathsf{E}[X]$ we obtain

Theorem

$$\mathsf{E}[h(X)] \approx h(\mathsf{E}[X])$$
$$\mathsf{var}[h(X)] \approx h'(\mathsf{E}[X])^2 \cdot \mathsf{var}[X] \quad \text{(linear error propagation)}$$

- If the expansion is extended to the 2nd order, the improved approximation reads:

Theorem

$$\mathsf{E}[h(X)] \approx h(\mathsf{E}[X]) + \frac{1}{2}h''(\mathsf{E}[X]) \cdot \mathsf{var}[X]$$

8 Random Variables

9 Moments

10 Functions of Random Samples
- Basic Terminology
- Sample Mean
- Sample Variance
- Sample Median

8 Random Variables

9 Moments

10 Functions of Random Samples
- Basic Terminology
- Sample Mean
- Sample Variance
- Sample Median

Basic Terminology

- Let $X_1, \ldots, X_n$ be independent random variables that all have the same distribution F.

- They then form a **random sample** of the distribution F.

- A random variable

$$Y = h(X_1, \ldots, X_n)$$

is called a **sample function**.

- In many cases, we wish to determine the moments of the distribution of Y.

8 Random Variables

9 Moments

10 Functions of Random Samples
- Basic Terminology
- Sample Mean
- Sample Variance
- Sample Median

Definition (Sample Means)

The **sample mean** $\overline{X}$ of the sample $X_1, \ldots, X_n$ is defined by

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Moments of the Sample Mean

If F has the mean μ and the variance σ^2, the following holds true

- $\mathsf{E}[\overline{X}] = \mu$
- $\mathsf{var}[\overline{X}] = \dfrac{\sigma^2}{n}$

8. Random Variables

9. Moments

10. Functions of Random Samples
   - Basic Terminology
   - Sample Mean
   - Sample Variance
   - Sample Median

Definition (Sample Variance)

The **sample variance** S^2 of the sample $X_1, \ldots, X_n$ is defined by

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Moments of the Sample Variance

If F has a variance σ^2, then:

$$\mathsf{E}[S^2] = \sigma^2$$

If the fourth central moment μ_4 of F exists, then:

$$\mathsf{var}[S^2] = \frac{\mu_4}{n} - \frac{\sigma^4(n-3)}{n(n-1)}$$

8 Random Variables

9 Moments

10 Functions of Random Samples
- Basic Terminology
- Sample Mean
- Sample Variance
- Sample Median

Definition (Sample Median)

The **sample median** $\tilde{X}$ of the sample $X_1, \ldots, X_n$ is defined by

$$\tilde{X} = \begin{cases} X_{((n+1)/2)}, & n \text{ odd} \\ \frac{1}{2}\left(X_{(n/2)} + X_{(n/2+1)}\right), & n \text{ even} \end{cases}$$

Moments of the Sample Median

If F has median m and density f, then:

- $\displaystyle \lim_{n \to \infty} \mathsf{E}[\tilde{X}] = m$

- $\displaystyle \lim_{n \to \infty} \mathsf{var}[\tilde{X}] = \frac{1}{4\,nf^2(m)}, \quad \text{if } f(m) > 0$

Part IV

Point Estimators

Overview Part 4

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Basic Terminology

Sample Moments and Sample
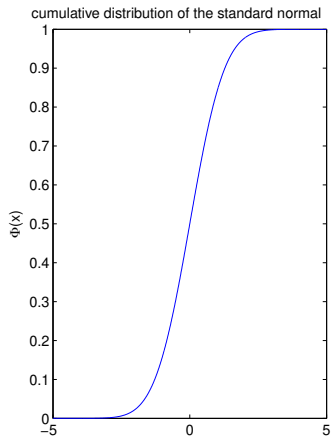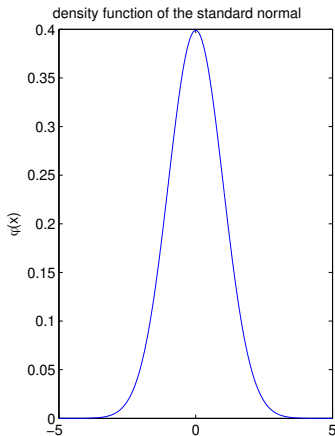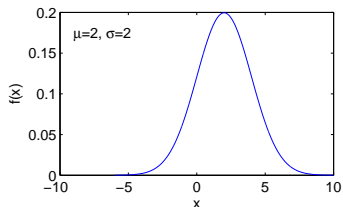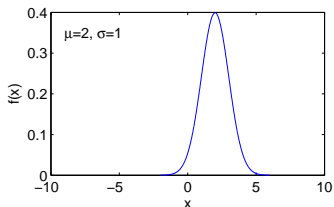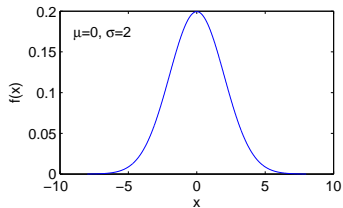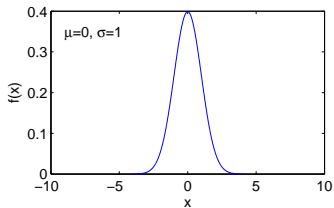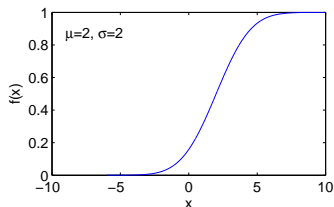Median

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Statistical Methods
of Data Analysis

W. Waltenberger

- A point estimator is a sample function designed to provide as accurate an approximation as possible for an unknown distribution parameter ϑ:

$$T = g(X_1, \ldots, X_n)$$

- The function $g(x_1, \ldots, x_n)$ is called the estimator.
- The construction of sensible point estimators for a parameter ϑ is the central task of estimation theory.
- Many point estimators are conceivable for a parameter ϑ. However, a 'good' point estimator should meet certain requirements.

Definition (Bias)

A point estimator T for the parameter ϑ is called **unbiased** if for all values of ϑ:

$$\mathsf{E}_\vartheta[T] = \vartheta$$

T is called **asymptotically unbiased** if:

$$\lim_{n \to \infty} \mathsf{E}_\vartheta[T] = \vartheta$$

- If the unknown parameter is ϑ, then the expectation of the point estimator is equal to ϑ.
- An unbiased point estimator has random deviations from the true value ϑ, but no systematic bias.

Definition (MSE)

The **mean squared error (MSE)** of a point estimator T for parameter ϑ is defined by:

$$\mathsf{MSE}[T] = \mathsf{E}_\vartheta[(T - \vartheta)^2]$$

Definition (MSE-consistency)

A point estimate T for parameter ϑ is called **mean squared error consistent (MSE-consistent)** if:

$$\lim_{n \to \infty} \mathsf{MSE}[T] = 0$$

Definition (MSE-efficiency)

A point estimator T_1 is called more **MSE-efficient** than the point estimator T_2 if for all allowed ϑ:

$$\mathsf{MSE}[T_1] \leq \mathsf{MSE}[T_2]$$

Definition (Efficiency)

An unbiased point estimator T_1 is called **efficient** than the unbiased point estimator T_2 if for all possible ϑ:

$$\mathsf{var}[T_1] \leq \mathsf{var}[T_2]$$

An unbiased point estimator T is called **efficient** if its variance is minimal.

Definition (Fisher Information)

Let $X_1, \ldots, X_n$ be a sample with a joint density $g(x_1, \ldots, x_n | \vartheta)$. The expectation

$$I_\vartheta = \mathsf{E}\left[-\frac{\partial^2 \ln g(X_1, \ldots, X_n | \vartheta)}{\partial \vartheta^2} \right]$$

is called the **Fisher information** of the sample.

Theorem of Rao and Cramèr

Let $X_1, \ldots, X_n$ be a sample with joint density $g(x_1, \ldots, x_n | \vartheta)$. The variance of an unbiased estimator T for the parameter ϑ is bounded from below by:

$$\mathsf{var}[T] \geq 1/I_\vartheta$$

11 Point Estimators
- Basic Terminology
- Sample Moments and Sample Median

12 Normally Distributed Data

13 Exponentially Distributed Data

14 Poisson Distributed Data

15 Data from Bernoulli and Drawing Experiments

16 Maximum Likelihood Estimator

17 Mixture Models and the EM Algorithm

Theorem

- Let $X_1, \ldots, X_n$ be a sample from the distribution F with expectation μ. Then the sample mean $\overline{X}$ is an unbiased point estimator of μ.
- If F has finite variance σ^2, then:

$$\mathsf{var}(\overline{X}) = \frac{\sigma^2}{n}$$

- In this case, $\overline{X}$ is MSE-consistent.

Theorem

- Let $X_1, \ldots, X_n$ be a sample from the distribution F with expectation μ and variance σ^2. Then the sample variance S^2 is an unbiased point estimator of σ^2.

- If F has a finite fourth central moment μ_4, then:

$$\mathsf{var}(S^2) = \frac{\mu_4}{n} - \frac{(n-3)\sigma^4}{n(n-1)}$$

- In this case S^2 is MSE-consistent.

Theorem

- Let $X_1, \ldots, X_n$ be a sample from the continuous distribution F with median m and density f. Then the sample median $\tilde{X}$ is an asymptotically unbiased point estimator of m.

- For symmetric F, $\tilde{X}$ is unbiased.

- Asymptotically, the sample median $\tilde{X}$ has a variance

$$\mathsf{var}(\tilde{X}) \approx \frac{1}{4nf(m)^2}$$

- The sample median is MSE-consistent provided that $f(m) > 0$.

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

The Gauß or Normal
Distribution

Estimate of the Mean

Estimate of the Variance

Estimate of the Median

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Statistical Methods
of Data Analysis

W. Waltenberger

The Normal Distribution $\mathrm{No}(\mu, \sigma^2)$

- The **normal distribution** is one of the most important families of distributions in science and engineering. We denote it by $\mathrm{No}(\mu, \sigma^2)$.
- Its density is:

$$f_{\mathrm{No}}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The normal distribution with $\mu = 0$, $\sigma = 1$ is called **standard normal distribution**. Its density is often denoted as $\varphi(x)$.
- Its distribution function $\Phi(x)$ can not be written in closed form.
- The mode M (the maximum of the density function) and the median m are at $x = \mu$.

The moments and CF of the normal distribution

Let $X \sim \mathrm{No}(\mu, \sigma^2)$. Then:

- $\mathsf{E}[X] = \mu$
- $\mathsf{var}[X] = \sigma^2$
- $\gamma[X] = 0$
- $\kappa[X] = 3$

- The α quantile of the standard normal distribution is denoted by z_α.
- The α quantile of $\mathrm{No}(\mu, \sigma^2)$ is equal to $\mu + \sigma z_\alpha$.

The Gauß or Normal Distribution

The Gauß or Normal Distribution

four normal distributions (density functions)

four normal distributions (distribution functions)

The Gauß or Normal Distribution

- One of the reasons for why the normal distribution is so important, is the **central limit theorem**. In its simplest formulation it reads:

Central Limit Theorem

Let $X_1, \ldots, X_n$ be a sample from the distribution F with mean μ and variance σ^2. Then the distribution of

$$U_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

for $n \to \infty$ converges to the standard normal distribution.

- Clearly, if F is a normal distribution, then U_n is exactly standard normally distributed for all n.

The Gauß or Normal Distribution

Definition (Standard Score)

Let X be a normally distributed random variable with mean μ and variance σ^2. Then

$$Z = \frac{X - \mu}{\sigma}$$

is the **standard score** of X.

Theorem

If X is normally distributed, then the standard score Z is standard-normally distributed.

The Gauß or Normal Distribution

- Probability contents of the standard normal distribution
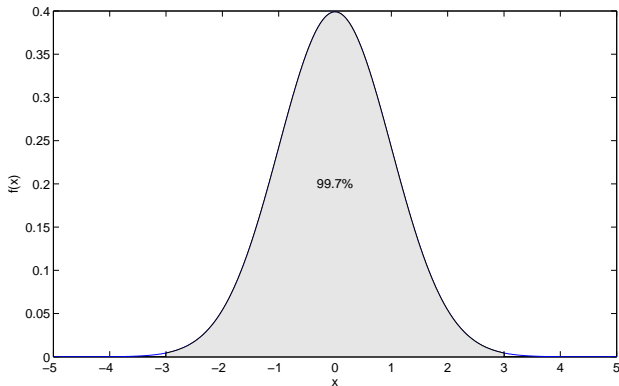
$$P(-1 \leq X \leq 1)$$

- Probability contents of the standard normal distribution

$$P(-2 \leq X \leq 2)$$

- Probability contents of the standard normal distribution

$$P(-3 \leq X \leq 3)$$

Fisher information of a normally distributed variable

Let $X_1, \ldots, X_n$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$. Then:

- $I_\mu = \dfrac{n}{\sigma^2}$

- $I_{\sigma^2} = \dfrac{n}{2\sigma^4}$

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

The Gauß or Normal
Distribution

Estimate of the Mean

Estimate of the Variance

Estimate of the Median

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Theorem

Let $X_1, \ldots, X_n$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$ and $\overline{X}$ be the sample mean. Then:

- $\overline{X}$ is normally distributed according to $\mathrm{No}(\mu, \sigma^2/n)$.
- $\overline{X}$ is an unbiased and consistent estimator of μ.
- $\overline{X}$ is efficient.

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

The Gauß or Normal
Distribution

Estimate of the Mean

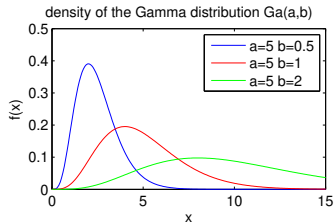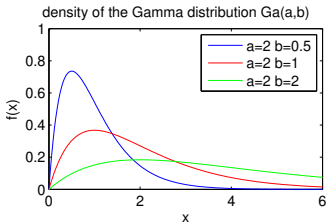Estimate of the Variance
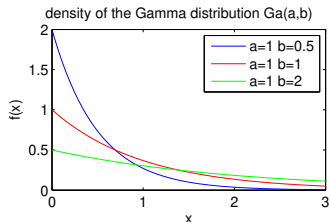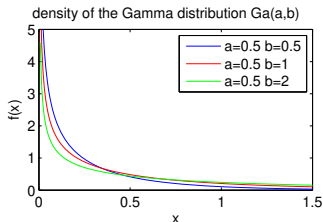
Estimate of the Median

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Subsection: Estimate of the Variance

Estimate of the Variance

Theorem

Let $X_1, \ldots, X_n$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$ and S^2 be the sample variance. Then:

- $(n-1)S^2/\sigma^2$ is χ^2-distributed with $n-1$ degrees of freedom.
- $\mathsf{E}[S^2] = \sigma^2$
- $\mathrm{var}[S^2] = \dfrac{2\sigma^4}{n-1}$
- S^2 is an unbiased and consistent estimator of σ^2.
- S^2 is asymptotically efficient.

Digression: The χ^2 distribution $\chi^2(n)$

- The density of the $\chi^2(n)$ distribution with n degrees of freedom is:

$$f_{\chi^2}(x;n) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} \, x^{n/2-1} e^{-x/2} \cdot I_{[0,\infty)}(x)$$

- It is the gamma distribution $\mathrm{Ga}(n/2,2)$.
- The $\chi^2(2)$ distribution is the exponential distribution $\mathrm{Ex}(2)$.
- The mode of the density is at $x = \max(n-2,0)$.
- The density of the $\chi^2(1)$ distribution has a pole at $x = 0$.
- For $n \to \infty$, the $\chi^2(n)$ distribution converges to the normal distribution $\mathrm{No}(n,2n)$.
- If $X_1, \ldots, X_n$ are independent and standard normally distributed, then $Y = \sum_{i=1}^{n} X_i^2$ is $\chi^2(n)$-distributed with n degrees of freedom.

density of the χ^2-distribution $\chi^2(n)$

The moments of the $\chi^2(n)$ distribution

Let $X \sim \chi^2(n)$. Then:

- $\mathrm{E}[X] = n$
- $\mathrm{var}[X] = 2n$
- $\gamma[X] = \sqrt{8/n}$
- $\kappa[X] = 3 + 12/n$

- The α quantile of the $\chi^2(n)$ distribution is denoted by $\chi^2_{\alpha;n}$.

Digression: The gamma distribution $\mathrm{Ga}(a,b)$

- The density of the gamma distribution is:

$$f_{\mathrm{Ga}}(x; a, b) = \frac{x^{a-1} e^{-x/b}}{b^a \Gamma(a)} \cdot I_{[0,\infty)}(x)$$

- Its distribution function is the regularized incomplete gamma function:

$$F_{\mathrm{Ga}}(x; a, b) = \int_0^x \frac{t^{a-1} e^{-t/b}}{b^a \Gamma(a)} \, \mathrm{d}t = \frac{\gamma(a, x/b)}{\Gamma(a)}$$

- The mode M is at $m = (a-1)b$ when $a \geq 1$.
- The α quantile of the $\mathrm{Ga}(a,b)$ distribution is denoted by $\gamma_{\alpha;a,b}$.

Estimate of the Variance

Estimate of the Variance

The moments of the $\mathrm{Ga}(a, b)$ distribution

With $X \sim \mathrm{Ga}(a, b)$:

- $\mathsf{E}[X] = ab$
- $\mathsf{var}[X] = ab^2$
- $\gamma[X] = 2/\sqrt{a}$
- $\kappa[X] = 3 + 6/a$

Other properties of the $\mathrm{Ga}(a, b)$ distribution

- If $X \sim \mathrm{Ga}(a, b)$, then $cX \sim \mathrm{Ga}(a, cb)$
- $\gamma_{\alpha;a,b} = b\,\gamma_{\alpha;a,1}$
- If $X_1 \sim \mathrm{Ga}(a_1, b)$ and $X_2 \sim \mathrm{Ga}(a_2, b)$ are independent, $Y = X_1 + X_2 \sim \mathrm{Ga}(a_1 + a_2, b)$

11 Point Estimators

12 Normally Distributed Data
- The Gauß or Normal Distribution
- Estimate of the Mean
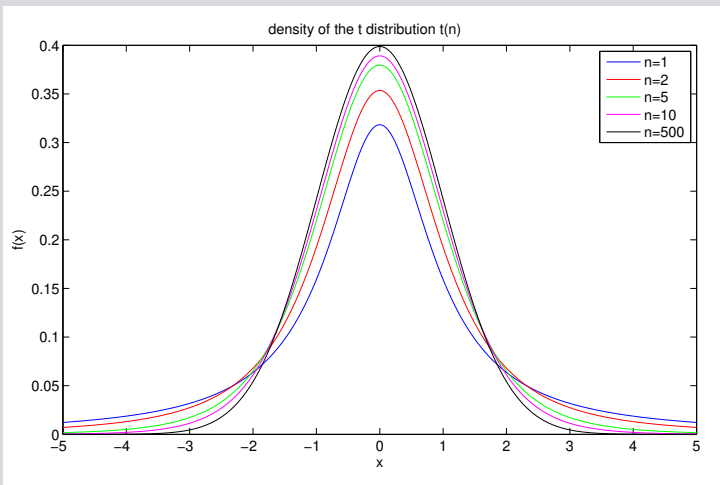- Estimate of the Variance
- Estimate of the Median

13 Exponentially Distributed Data

14 Poisson Distributed Data

15 Data from Bernoulli and Drawing Experiments

16 Maximum Likelihood Estimator

Theorem

Let $X_1, \ldots, X_n$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$ and $\tilde{X}$ be the sample median. Then:

- $\mathsf{E}[\tilde{X}] = \mu$

- $\mathrm{var}[\tilde{X}] \approx \dfrac{2\,\pi\sigma^2}{4\,n} \approx 1.57 \cdot \dfrac{\sigma^2}{n}$

Thus, for large n, the variance of $\tilde{X}$ is more than 50 percent larger than the variance of $\overline{X}$.

- However, there are distributions for which the sample median has a smaller variance than the sample mean, as shown in the following example.

Digression: The t distribution $t(n)$

- The density of the $t(n)$ distribution with n degrees of freedom is:

$$f_t(x; n) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

- The $t(1)$ distribution is also called the Cauchy or (in particle physics) Breit–Wigner distribution.
- The k-th moment μ_k exists only for $k < n$.
- For $n \to \infty$, the $t(n)$ distribution converges to the standard normal distribution.

density of the t distribution t(n)

The moments of the $t(n)$ distribution

Let $X \sim t(n)$. Then:

- $E[X] = 0, \quad n > 1$
- $\text{var}[X] = \dfrac{n}{n-2}, \quad n > 2$
- $\gamma[X] = 0, \quad n > 3$
- $\kappa[X] = \dfrac{3n-6}{n-4}, \quad n > 4$

- The α quantile of the $t(n)$ distribution is denoted by $t_{\alpha;n}$.

Example

Let $X_1, \ldots, X_n$ be a sample from the t distribution $t(3)$. The variance of $\overline{X}$ is equal to.

$$\mathsf{var}(\overline{X}) = \frac{3}{n}$$

The variance of $\tilde{X}$ for large n is equal to

$$\mathsf{var}(\tilde{X}) = \frac{1}{4\,nf(0)^2} = \frac{1.8506}{n} \approx 0.62 \cdot \frac{3}{n}$$

Thus, it is almost 40 percent smaller than the variance of $\overline{X}$.

Section 13: Exponentially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

**Exponentially Distributed
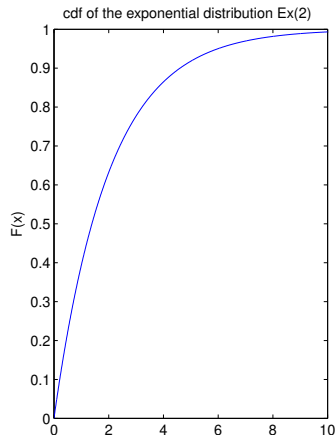Data**

The Exponential Distribution

Estimate of the Mean

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

11 Point Estimators

12 Normally Distributed Data

13 Exponentially Distributed Data
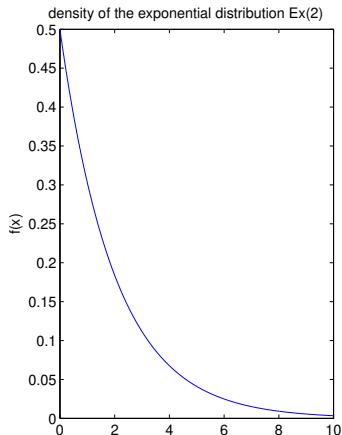   - The Exponential Distribution
   - Estimate of the Mean

14 Poisson Distributed Data

15 Data from Bernoulli and Drawing Experiments

16 Maximum Likelihood Estimator

17 Mixture Models and the EM Algorithm

Statistical Methods
of Data Analysis

W. Waltenberger

The Exponential Distribution $\mathrm{Ex}(\tau)$

- The **exponential** or **waiting time distribution** is an important family of distributions in science and engineering. We denote it by $\mathrm{Ex}(\tau)$.
- Its density is:

$$f_{\mathrm{Ex}}(x; \tau) = \frac{1}{\tau}\, e^{-x/\tau}$$

- Its distribution function is:

$$F_{\mathrm{Ex}}(x; \tau) = 1 - e^{-x/\tau}$$

- The exponential distribution is a special case of the **gamma distribution**: $\mathrm{Ex}(\tau)$ is equal to $\mathrm{Ga}(1, \tau)$.
- The mode M is at $x = 0$.

The Exponential Distribution

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

The Exponential Distribution

Estimate of the Mean

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

- The α quantile of the exponential distribution $\mathrm{Ex}(\tau)$ is:

$$\gamma_{\alpha;1,\tau} = -\tau \, \ln(1 - \alpha)$$

- Consequently, the median of the distribution is at $x = \tau \ln 2$.
- The exponential distribution is the distribution of a waiting time **without memory**: The distribution is independent of the time at which the timing starts.
- This behavior is typical for physical processes like the radioactive decay of an atom or the decay of an unstable elementary particle.

The Exponential Distribution

The moments of the exponential distribution

Let $X \sim \mathrm{Ex}(\tau)$. Then:

- $\mathsf{E}[X] = \tau$
- $\mathsf{var}[X] = \tau^2$
- $\gamma[X] = 2$
- $\kappa[X] = 9$

Fisher information of an exponentially distributed sample

Let $X_1, \ldots, X_n$ be a sample from the exponential distribution $\mathrm{Ex}(\tau)$. Then:

$$I_\tau = \frac{n}{\tau^2}$$

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

The Exponential Distribution

Estimate of the Mean

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Theorem

Let $X_1, \ldots, X_n$ be a sample from the exponential distribution $\mathrm{Ex}(\tau)$ and $\overline{X}$ be the sample mean. Then:

- $\overline{X}$ is gamma distributed according to $\mathrm{Ga}(n, \tau/n)$.
- $\overline{X}$ is an unbiased and consistent estimator of τ.
- $\overline{X}$ is efficient.
- $\overline{X}$ is asymptotically normally distributed with mean $\mu = \tau$ and variance $\sigma^2 = \tau^2/n$.

Statistical Methods
of Data Analysis

W. Waltenberger

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

The Poisson Process

Estimation of the Mean

Data from Bernoulli and
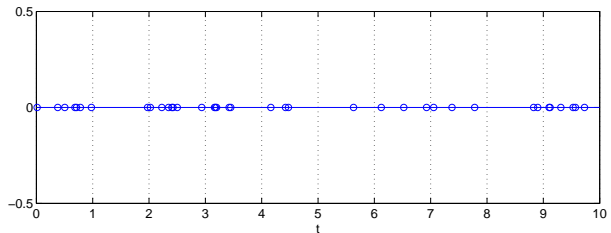Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

- We observe a process in which certain events occur at random times.

- If the waiting times between events are independent and exponentially distributed with mean τ, we speak of a **Poisson process** with intensity $\lambda = 1/\tau$.

- The number X of events per time unit is then independent and **Poisson-distributed** according to $\mathrm{Po}(\lambda)$.

- Conversely, if the number X of events per unit time is independent and Poisson-distributed according to $\mathrm{Po}(\lambda)$, then the waiting times are exponentially distributed with mean $\tau = 1/\lambda$.

- In a Poisson process of intensity λ, the number of events per time interval of length T is again Poisson distributed according to $\mathrm{Po}(\lambda T)$.

- The sum of two Poisson processes is again a Poisson process. Its intensity is the sum of the intensities of the summands.

Examples of Poisson processes

- The number of decays per unit time in a (large) radioactive source.
- The number of particles per unit time in cosmic radiation.
- The number of pixel errors on a LCD display.
- The number of rare events per time unit (insurance claims, suicides, accidents).

The Poisson distribution $\mathrm{Po}(\lambda)$

- The density of the Poisson distribution is:

$$f_{\mathrm{Po}}(k; \lambda) = \frac{\lambda^k}{k!} \cdot e^{-\lambda}, \quad k \in \mathbb{N}_0$$

- The mode M is equal to

$$M = \begin{cases} \lfloor \lambda \rfloor, & \text{if } \lambda \notin \mathbb{N} \\ \lambda \text{ and } \lambda - 1, & \text{if } \lambda \in \mathbb{N} \\ 0, & \text{if } \lambda = 0 \end{cases}$$

- The distribution function can be expressed by the distribution function of the gamma distribution $\mathrm{Ga}(k+1, 1)$:

$$F_{\mathrm{Po}}(k; \lambda) = 1 - F_{\mathrm{Ga}}(\lambda; k+1, 1)$$

The Poisson Process

The moments of the Poisson distribution

let $X \sim \mathrm{Po}(\lambda)$. Then:

- $\mathsf{E}[X] = \lambda$
- $\mathsf{var}[X] = \lambda$
- $\gamma[X] = 1/\sqrt{\lambda}$
- $\kappa[X] = 3 + 1/\lambda$

Fisher information of a Poisson distributed sample

Let $X_1, \ldots, X_n$ be a sample from the Poisson distribution $\mathrm{Po}(\lambda)$.
Then:

$$I_\lambda = \frac{n}{\lambda}$$

The Poisson Process

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
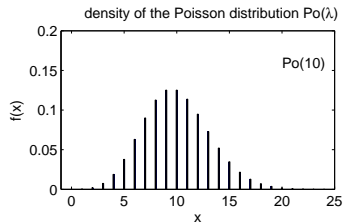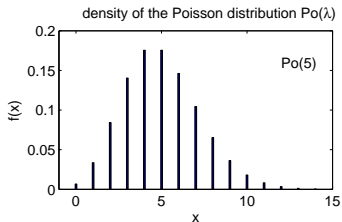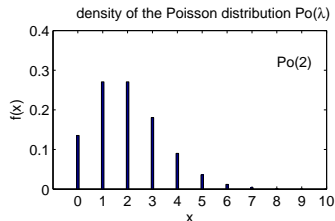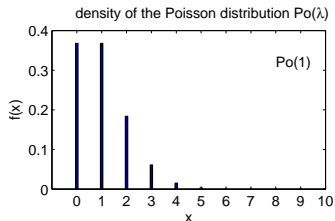Data

Poisson Distributed Data

The Poisson Process
Estimation of the Mean

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

The Poisson distribution for large λ

- A random variable distributed according to $\mathrm{Po}(\lambda)$ can be represented as a sum of λ $\mathrm{Po}(1)$-distributed random variables.

- Therefore, according to the central limit theorem, the Poisson distribution for $\lambda \to \infty$ must tend toward a normal distribution.

- The following figure shows the distribution function of the Poisson distribution $\mathrm{Po}(\lambda)$ with $\lambda = 25$, and the distribution function of the normal distribution $\mathrm{No}(\mu, \sigma^2)$ with $\mu = \lambda = 25$ and $\sigma^2 = \lambda = 25$.

distribution functions of the Poisson and the normal distributions

Theorem

Let $X_1, \ldots, X_n$ be a sample from the Poisson distribution $\mathrm{Po}(\lambda)$ and $\overline{X}$ be the sample mean. Then:

- $\overline{X}$ is an unbiased and consistent estimator of λ.
- $\overline{X}$ is efficient.
- $\overline{X}$ is asymptotically normally distributed with mean $\mu = \lambda$ and variance $\sigma^2 = \lambda/n$.

The alternative or Bernoulli distribution $\mathrm{Al}(p)$

- A Bernoulli experiment or alternative experiment has two possible outcomes, "success" and "failure" respectively.
- The random variable X assigns the value 1 to success and 0 to failure.
- If p is the probability of success, the density $f_{\mathrm{Al}}(x; p)$ is as follows:

$$f_{\mathrm{Al}}(0; p) = 1 - p, \ \ f_{\mathrm{Al}}(1; p) = p$$

or

$$f_{\mathrm{Al}}(x; p) = p^x (1-p)^{1-x}, \quad x \in \{0, 1\}$$

- If the alternative test is performed n times independently, there are 2^n possible outcomes, namely all sequences of the form
$$\boldsymbol{e} = (i_1, \ldots, i_n), \quad i_j \in \{0, 1\}.$$

- The discrete random variable Y maps the sequence $\boldsymbol{e}$ to the frequency of 1:
$$Y(\boldsymbol{e}) = \sum_{j=1}^{n} i_j$$

- The range of values of Y is the set $\{0, 1, \ldots, n\}$. Mapped to the number k ($0 \leq k \leq n$) are all sequences where 1 occurs exactly k times. There are C_k^n such sequences, and each has probability $p^k(1-p)^{n-k}$.

Repeating Bernoulli Experiments

The binomial distribution $\mathrm{Bi}(n, p)$

- Therefore, the density f of Y is:

$$f_{\mathrm{Bi}}(k; p, n) = \binom{n}{k} p^k (1-p)^{n-k},\ 0 \le k \le n$$

- The distribution of Y is called **binomial distribution** $\mathrm{Bi}(n, p)$ with parameters n and p.

- The following holds:

$$\sum_{k=0}^{n} f_{\mathrm{Bi}}(k; p, n) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

This is just the binomial theorem.

Repeating Bernoulli Experiments

- The mode M is equal to

$$M = \begin{cases} \lfloor (n+1)p \rfloor, & \text{if } p = 0 \text{ or } (n+1)p \notin \mathbb{N} \\ (n+1)p \text{ and} \\ (n+1)p - 1, & \text{if } (n+1)p \in \{1, \ldots, n\} \\ n, & \text{if } p = 1 \end{cases}$$

- The distribution function can be expressed by the distribution function of the beta distribution $\mathrm{Be}(x; n-k, k+1)$:

$$F_{\mathrm{Bi}}(k; n, p) = F_{\mathrm{Be}}(1-p; n-k, k+1)$$

Repeating Bernoulli Experiments

Digression: The beta distribution $\mathrm{Be}(a, b)$

- The density of the beta distribution is:

$$f_{\mathrm{Be}}(x; a, b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)} \cdot I_{[0,1]}(x)$$

- Its distribution function is the regularized incomplete beta function:

$$F_{\mathrm{Be}}(x; a, b) = \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{B(a, b)} \, \mathrm{d}t$$

- The mode M is at $x = (a-1)/(a+b-2)$ when $a, b > 1$.

The moments of the $\mathrm{Be}(a, b)$ distribution

let $X \sim \mathrm{Be}(a, b)$. Then:

- $\mathsf{E}[X] = \dfrac{a}{a + b}$

- $\mathsf{var}[X] = \dfrac{ab}{(a + b)^2(a + b + 1)}$

- $\gamma[X] = \dfrac{2(b - a)\sqrt{a + b + 1}}{(a + b + 2)\sqrt{ab}}$

- $\kappa[X] = \dfrac{6(a - b)^2 + 3ab(a + b + 2)}{ab(a + b + 2)(a + b + 3)}$

- The α quantile of the $\mathrm{Be}(a, b)$ distribution is denoted by $\beta_{\alpha;a,b}$.

Repeating Bernoulli Experiments

The moments of the binomial distribution

Let $X \sim \mathrm{Bi}(n, p)$. Then:

- $\mathsf{E}[X] = np$
- $\mathsf{var}[X] = np(1-p)$
- $\gamma[X] = \dfrac{1 - 2p}{\sqrt{np(1-p)}}$
- $\kappa[X] = 3 - \dfrac{6}{n} + \dfrac{1}{np(1-p)}$

Fisher information of a binomial distributed observation

Let X be an observation from the binomial distribution $\mathrm{Bi}(n, p)$. Then:

$$I_p = \frac{n}{p(1-p)}$$

The Binomial Distribution for Large n

- A random variable distributed according to $\mathrm{Bi}(n, p)$ can be represented as a sum of n Bernoulli distributed random variables.

- Therefore, according to the central limit theorem, the binomial distribution for $n \to \infty$ and fixed p must tend toward a normal distribution.

- The following figure shows the distribution function of the binomial distribution $\mathrm{Bi}(n, p)$ with $n = 200$ and $p = 0.1$, and the distribution function of the normal distribution $\mathrm{No}(\mu, \sigma^2)$ with $\mu = np = 20$ and $\sigma^2 = np(1 - p) = 18$.

Repeating Bernoulli Experiments

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments
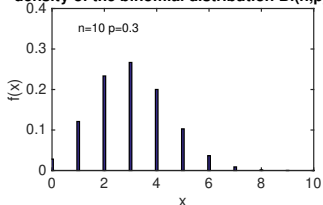
Repeating Bernoulli
Experiments

Estimation of Probability of
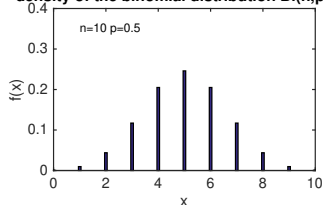Success

Sampling Experiments

Maximum Likelihood
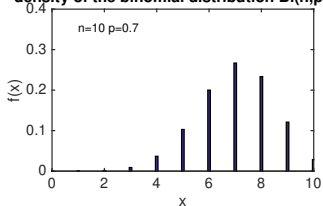Estimator

Mixture Models and the
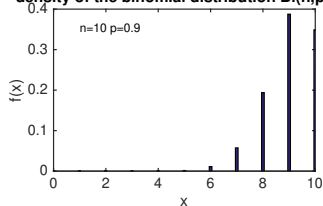EM Algorithm

distribution functions of the binomial and normal distributions

Repeating Bernoulli Experiments

The binomial distribution for large n and $p = \lambda/n$

- In the limit $n \to \infty$ and $p = \lambda/n \to 0$ the binomial distribution $\mathrm{Bi}(n, p)$ tends to the Poisson distribution $\mathrm{Po}(\lambda)$:

$$\lim_{n \to \infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k} = \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

- Thus, if the number of trials n is increased and the probability of success is decreased in inverse proportion, the number of successes is asymptotically Poisson distributed.

Repeating Bernoulli Experiments

distribution functions of the binomial and Poisson distributions

11 Point Estimators

12 Normally Distributed Data

13 Exponentially Distributed Data

14 Poisson Distributed Data

15 Data from Bernoulli and Drawing Experiments
   ● Repeating Bernoulli Experiments
   ● Estimation of Probability of Success
   ● Sampling Experiments

16 Maximum Likelihood Estimator

17 Mixture Models and the EM Algorithm

Estimation of Probability of Success

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Repeating Bernoulli
Experiments

Estimation of Probability of
Success

Sampling Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Theorem

Let k be an observation from the binomial distribution $\mathrm{Bi}(n, p)$. Then:

- $\hat{p} = k/n$ is an unbiased estimator of p.
- $\hat{p}$ is efficient.
- $\hat{p}$ is asymptotically normally distributed with mean $\mu = p$ and variance $\sigma^2 = p(1 - p)/n$.

11 Point Estimators

12 Normally Distributed Data

13 Exponentially Distributed Data

14 Poisson Distributed Data

15 Data from Bernoulli and Drawing Experiments
- Repeating Bernoulli Experiments
- Estimation of Probability of Success
- Sampling Experiments

16 Maximum Likelihood Estimator

17 Mixture Models and the EM Algorithm

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
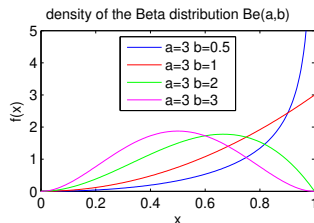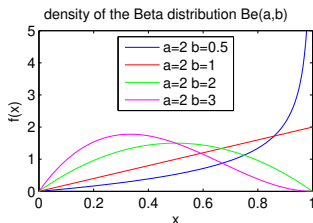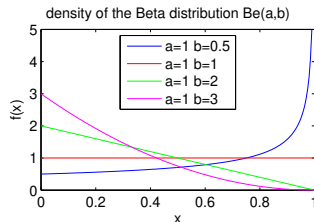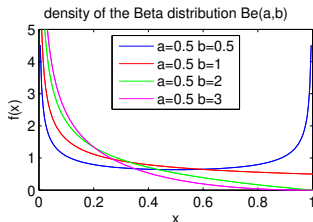Drawing Experiments

Repeating Bernoulli
Experiments

Estimation of Probability of
Success

Sampling Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Sampling Experiments

Drawing and Placing Back

- Given a population of N objects, of which M objects exhibit a certain **property** E.
- n objects are drawn, each object having the same probability of being drawn.
- Each object drawn is immediately put back.
- The number of objects drawn exhibiting feature E is a random variable X.
- X is then **binomially distributed** according to $\text{Bi}(n, M/N)$.
- If N and M are much larger than n, X is binomially distributed to a good approximation even if drawn objects are **not** put back.
- An example of this is a survey in which the sample size n is much smaller than N and M.

Pulling without Putting Back

- Population of N objects, of which M objects exhibit **feature** E.
- n objects are drawn, each object having the same probability of being drawn.
- Once drawn, objects are **not** put back.
- The number of objects with feature E is a random variable X.
- The distribution of X is called a **hypergeometric distribution** $\mathrm{Hy}(N, M, n)$.

The hypergeometric distribution $\mathrm{Hy}(N, M, n)$

- Its density is:

$$f_{\mathrm{Hy}}(m) = \frac{\dbinom{M}{m}\dbinom{N-M}{n-m}}{\dbinom{N}{n}}, \quad 0 \le m \le \min(n, M)$$

- The mode D is equal to

$$D = \left\lfloor \frac{(n+1)(M+1)}{N+2} \right\rfloor$$

The moments of the hypergeometric distribution

Let $X \sim \mathrm{Hy}(N, M, n)$ and $p = M/N$. Then:

- $\mathsf{E}[X] = \dfrac{nM}{N} = np$

- $\mathsf{var}[X] = \dfrac{nM}{N}\dfrac{N-M}{N}\dfrac{N-n}{N-1} = np(1-p)\dfrac{N-n}{N-1}$

- The term $(N-n)(N-1)$ in the variance is called **correction for finiteness**. It is close to 1 if $N \gg n$.

- If $N \gg n$ and $M \gg n$, the hypergeometric distribution can be approximated by the binomial distribution since in this case drawing from the sample does not significantly change the composition of the population.

Sampling Experiments

Estimating the Number of Feature Carriers

Theorem

Let k be an observation from the hypergeometric distribution $\mathrm{Hy}(N, M, n)$ with known N but unknown M. Then:

- $\hat{M} = \dfrac{kN}{n}$ is an unbiased estimator of M.

- $\mathsf{var}[\hat{M}] = M \dfrac{N - M}{n} \dfrac{N - n}{N - 1}$

Estimating the size of the population

Theorem

Let k be an observation from the hypergeometric distribution $\mathrm{Hy}(N, M, n)$ with known M but unknown N. Then:

- $\hat{N} = \dfrac{nM}{k}$ is a biased estimator of N.

- $1/\hat{N} = \dfrac{k}{nM}$ is an unbiased estimator of $1/N$.

- With linear error propagation we obtain:

$$\mathsf{E}[\hat{N}] \approx N + \frac{N-M}{M}\frac{N-n}{n}$$

$$\mathsf{var}[\hat{N}] \approx N \frac{N-M}{M}\frac{N-n}{n}$$

Maximum Likelihood Estimator

Definition (ML Estimator)

- Let $X_1, \ldots, X_n$ be a sample with joint density $g(x_1, \ldots, x_n | \vartheta)$. The function

$$L(\vartheta | X_1, \ldots, X_n) = g(X_1, \ldots, X_n | \vartheta)$$

is called the **likelihood function** of the sample.

- The **plausible** or **maximum likelihood estimator** $\hat{\vartheta}$ is that value of ϑ that maximizes the likelihood function of the sample.

- Often, instead of the likelihood function, we maximize its logarithm, the log-likelihood function $\ell(\vartheta) = \ln L(\vartheta)$.

- In simple cases we can maximize analytically. If this is not possible, we maximize numerically.

- The ML estimator is invariant under (differentiable) transformations of the parameter: if $\hat{\vartheta}$ is the ML estimator of ϑ, then $h(\hat{\vartheta})$ is the ML estimator of $h(\vartheta)$.

Example (ML estimator of a Bernoulli parameter)

Let $X_1, \ldots, X_n$ be a sample from the Bernoulli distribution $\mathrm{Al}(p)$. The joint density is:

$$g(x_1, \ldots, x_n | p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

Therefore, the log-likelihood function is:

$$\ell(p) = \sum_{i=1}^{n} X_i \ln p + \left(n - \sum_{i=1}^{n} X_i\right) \ln(1-p)$$

Example (Continuation)

Deriving w.r.t. p results in:

$$\frac{\partial \ell(p)}{\partial p} = \frac{1}{p} \sum_{i=1}^{n} X_i - \frac{1}{1-p} \left(n - \sum_{i=1}^{n} X_i \right)$$

Setting the derivative to zero and solving for p yields:

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

The ML estimator is unbiased and efficient.

Maximum Likelihood Estimator

Example (ML estimator of a Poisson parameter)

Let $X_1, \ldots, X_n$ be a sample from the Poisson distribution $\mathrm{Po}(\lambda)$. The joint density is:

$$g(x_1, \ldots, x_n | \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

Therefore, the log-likelihood function is:

$$\ell(\lambda) = \sum_{i=1}^{n} [X_i \ln \lambda - \lambda - \ln(x_i!)]$$

Deriving to λ yields:

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} X_i - n$$

Maximum Likelihood Estimator

Example (Continuation)

Setting the derivative to zero and solving for λ yields:

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

. The ML estimator is unbiased and efficient.

Example (ML estimator of a mean lifetime)

Let $X_1, \ldots, X_n$ be a sample from the exponential distribution $\mathrm{Ex}(\tau)$. The joint density is:

$$g(x_1, \ldots, x_n | \tau) = \prod_{i=1}^{n} \frac{e^{-x_i/\tau}}{\tau}$$

Therefore, the log-likelihood function is:

$$\ell(\tau) = \sum_{i=1}^{n} [-\ln \tau - \frac{1}{\tau} X_i]$$

Deriving to τ yields:

$$\frac{\partial \ell(\tau)}{\partial \tau} = -\frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{n} X_i$$

Maximum Likelihood Estimator

Example (Continuation)

Setting the derivative to zero and solving for τ yields:

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

The ML estimator is unbiased and efficient.

Maximum Likelihood Estimator

Statistical Methods of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed Data

Exponentially Distributed Data

Poisson Distributed Data

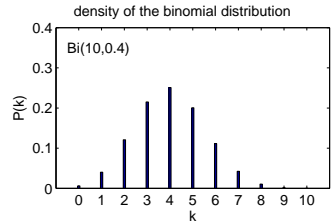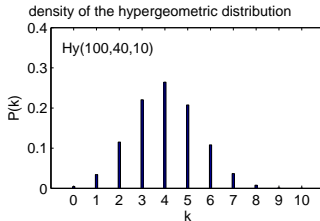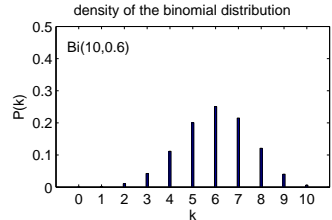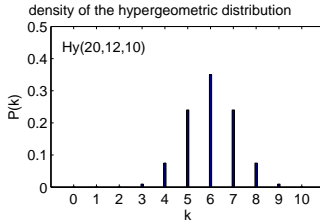Data from Bernoulli and Drawing Experiments

Maximum Likelihood Estimator

Mixture Models and the EM Algorithm

Example (ML estimator of the parameters of a normal distribution)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$. The joint density is:

$$g(x_1, \ldots, x_n | \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x_i - \mu)^2}{2\,\sigma^2}\right]$$

Therefore, the log-likelihood function is:

$$\ell(\mu, \sigma^2 | \boldsymbol{X}) = \sum_{i=1}^{n} \left[-\ln\sqrt{2\pi} - \frac{1}{2}\ln\sigma^2 - \frac{(X_i - \mu)^2}{2\,\sigma^2}\right]$$

Deriving by μ and σ^2 yields:

$$\frac{\partial\ell(\mu, \sigma^2)}{\partial\mu} = \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma^2}, \quad \frac{\partial\ell(\mu, \sigma^2)}{\partial\sigma^2} = \sum_{i=1}^{n} \left[-\frac{1}{2\,\sigma^2} + \frac{(X_i - \mu)^2}{2\,\sigma^4}\right]$$

Maximum Likelihood Estimator

Example (Continuation)

Zeroing the derivatives and solving for μ and σ^2 yields:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X}$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{n-1}{n} S^2$$

The ML estimator of μ is unbiased and efficient. The ML estimator of σ^2 is asymptotically unbiased and asymptotically efficient.
The ML estimator of σ is equal to

$$\hat{\sigma} = \sqrt{\frac{n-1}{n}} S$$

It is also asymptotically unbiased and asymptotically efficient.

- The normalized likelihood function can be interpreted as an a posteriori distribution of the estimated parameter.
- For large n, the variance of the likelihood estimate $\hat{\vartheta}$ can therefore be determined from the second central moment of the normalized likelihood function.
- If the estimated parameter ϑ is the mean of a normal distribution, then this procedure is exact for any n:

$$L(\vartheta) = \frac{1}{\sigma^n \sqrt{2\,\pi}^n} \, \exp\left[-\frac{n}{2\,\sigma^2}\left((\hat{\vartheta}-\vartheta)^2 + \frac{1}{n}\sum(x_i-\hat{\vartheta})^2\right)\right]$$

- If $L(\vartheta)$ is normalized, it is the 'density' of a normal distribution with mean $\hat{\vartheta}$ and variance $\frac{\sigma^2}{n}$, i.e. the variance of the estimate $\hat{\vartheta} = \frac{1}{n}\sum x_i$.

Example (Estimation of the parameter a of a gamma distribution)

The sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ consists of $n = 200$ values drawn independently from a $\mathrm{Ga}(a, 1)$ distribution:

$$f(x_i | a) = \frac{x_i^{a-1} e^{-x_i}}{\Gamma(a)}, \quad i = 1, \ldots, n$$

The (unknown) true value of a is $a_\mathrm{t} = 2$. The log-likelihood function is

$$\ln L(a | \boldsymbol{X}) = \sum_{i=1}^{n} \ln f(X_i | a) = (a-1) \sum_{i=1}^{n} \ln X_i - \sum_{i=1}^{n} X_i - n \ln \Gamma(a)$$

Maximum Likelihood Estimator

Example (Continuation)

Numerical maximization of $\ln L(a)$ results in the maximum likelihood estimate $\hat{a}$. The experiment is repeated N times and the estimates of each experiment $(\hat{a}^{(k)}, k = 1, \ldots, N)$ are histogrammed. Comparison of the individual (normalized) likelihood function with the histogram $(N = 500)$ shows good agreement between the standard deviations.

Maximum Likelihood Estimator

Example (Continuation)

histogram: σ=0.0858
likelihood function: σ=0.085

Maximum Likelihood Estimator

Example (Continuation)

With known b, the Fisher information of the sample with respect to a is equal to

$$I_a = n \frac{\mathrm{d}^2 \ln \Gamma(a)}{\mathrm{d}a^2} \approx 128.99$$

This corresponds to a standard deviation $\sigma = 0.088$. Thus, the ML estimator is practically efficient.

Because of $\mathrm{E}[\overline{X}] = a$, $\overline{X}$ is an unbiased estimator of a for all n. However, it has a larger standard deviation than the ML estimator:

$$\sigma[\overline{X}] = \sqrt{\frac{a}{n}} = 0.1$$

On the other hand, it is much easier to calculate.

Maximum Likelihood Estimator

- The ML estimator has the following important property:

Theorem

If the first two derivatives of $L(\vartheta)$ exist, the information $I_g(\vartheta)$ exists for all ϑ, and if $E\left[(\ln L)'\right] = 0$, then the maximum likelihood estimate $\hat{\vartheta}$ is asymptotically normally distributed with mean ϑ and variance $1/I_g(\vartheta)$. $\hat{\vartheta}$ is therefore asymptotically unbiased and asymptotically efficient.

- From this, the following follows immediately:

Theorem

The likelihood estimator $\hat{\vartheta}$ is consistent (under the same assumptions).

Example (ML estimation of the location parameter of a Cauchy distribution)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from the Cauchy distribution $\mathrm{t}(1)$ with location parameter μ. The joint density is:

$$g(x_1, \ldots, x_n | \mu) = \prod_{i=1}^{n} \frac{1}{\pi[1 + (x_i - \mu)^2]}$$

Therefore, the log-likelihood function is:

$$\ell(\mu | \boldsymbol{X}) = -n \ln \pi - \sum_{i=1}^{n} \ln[1 + (X_i - \mu)^2]$$

The maximum $\hat{\mu}$ of $\ell(\mu | \boldsymbol{X})$ must be found numerically.

Maximum Likelihood Estimator

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Example (Continuation)

It can be shown that the Fisher information of the sample is equal to

$$I_\mu = \frac{n}{2}$$

Therefore, for large samples, the variance of the ML estimator $\hat{\mu}$ must be approximately equal to $2/n$.

The sample median $\tilde{X}$ is also a consistent estimator for μ. Its variance is asymptotically equal to $\pi^2/(4n) \approx 2.47/n$. Thus, it is about 23 percent larger than the variance of the ML estimator.

The sample mean $\overline{X}$, on the other hand, is **no** consistent estimator for μ. Indeed, one can show that $\overline{X}$ has the same distribution as a single observation.

Maximum Likelihood Estimator

Example (Continuation)

Simulation of 10000 samples of size $n = 100$:
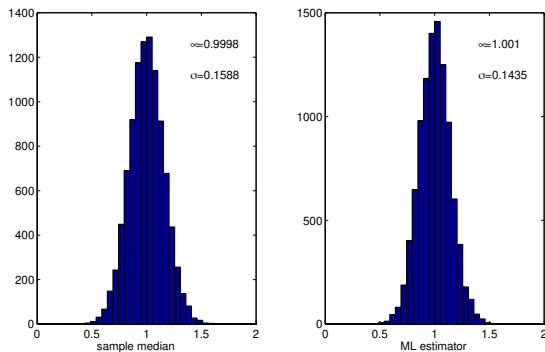
The correlation between $\tilde{x}$ and $\hat{\mu}$ is about 90%.

Maximum Likelihood Estimator

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

Example (Continuation)

The standard deviation of the ML estimator can again be approximately estimated from the normalized likelihood function of a sample:
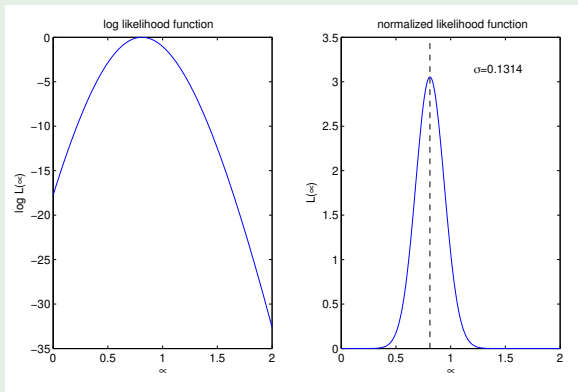
Example (ML estimation of the upper limit of a uniform distribution)

Let $X_1, \ldots, X_n$ be a sample from the uniform distribution $\mathrm{Un}(0, b)$ with upper limit b. The joint density is:

$$g(x_1, \ldots, x_n | b) = \frac{1}{b^n}, 0 \leq x_1, \ldots, x_n \leq b$$

Therefore, the largest value of the likelihood function is at:

$$\hat{b} = \max_i X_i$$

Since the maximum sits at the boundary of the variable's domain, the usual asymptotic properties do not apply.

Example (Continuation)

The density of $\hat{b} = \max_i X_i$ is:

$$f(x) = \frac{nx^{n-1}}{b^n}$$

From this, expectation and variance can be calculated:

$$\mathsf{E}[\hat{b}] = \frac{bn}{n+1}, \quad \mathsf{var}[\hat{b}] = \frac{b^2 n}{(n+2)(n+1)^2}$$

The estimator is asymptotically unbiased, but the variance approaches zero, as does $1/n^2$! The estimator is also not asymptotically normally distributed.

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators
Normally Distributed
Data
Exponentially Distributed
Data
Poisson Distributed Data
Data from Bernoulli and
Drawing Experiments
Maximum Likelihood
Estimator
Mixture Models and the
EM Algorithm

Maximum Likelihood Estimator

Example (Continuation)

Simulation of 10000 samples ($b = 1$) of size $n = 25$ or $n = 100$:
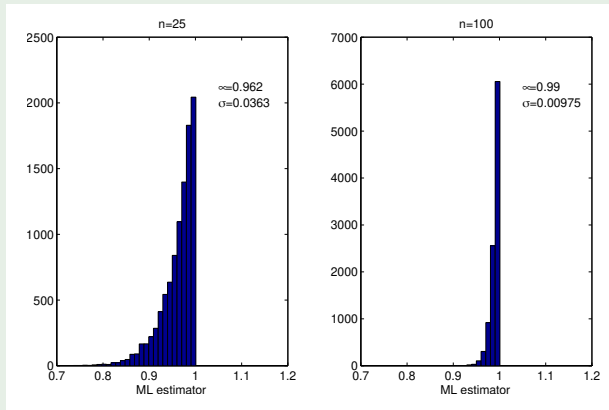
Maximum Likelihood Estimator

- If the likelihood function is (approximately) normal, the log-likelihood function is (approximately) parabolic.
- In this case, error intervals of the ML estimator can be read from the log-likelihood function.

Width of the log-likelihood function

Let $\ell(\vartheta)$ be the log-likelihood function of the parameter ϑ, further $\hat{\vartheta}$ be the ML estimator and $\sigma[\hat{\vartheta}]$ its standard deviation. Then, approximately:

$$\ell(\hat{\vartheta} - k\sigma[\hat{\vartheta}]) = \ell(\hat{\vartheta} + k\sigma[\hat{\vartheta}]) = \ell(\hat{\vartheta}) - k^2/2$$

Maximum Likelihood Estimator

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

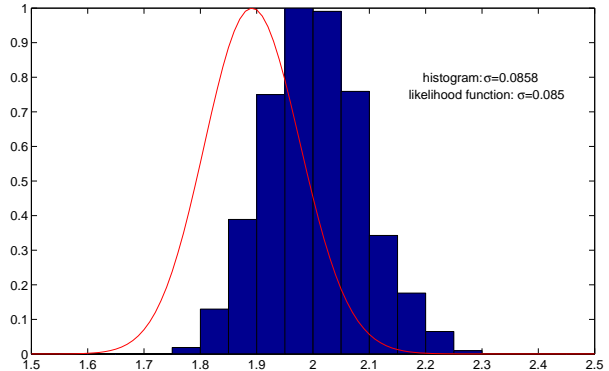Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

- If two parameters $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2)$ are estimated simultaneously, the log-likelihood function $\ell(\boldsymbol{\vartheta})$ is asymptotically an elliptic paraboloid.

- The contour lines of $\ell(\boldsymbol{\vartheta})$ are then ellipses.

- If a contour line is to have the probability content $1 - \alpha$, then its height z is equal to

$$z = \ell(\hat{\boldsymbol{\vartheta}}) - \frac{1}{2}\chi^2_{1-\alpha;2} = \ell(\hat{\boldsymbol{\vartheta}}) + \ln(\alpha)$$

- For example, the ellipse with $1 - \alpha = 0.95$ is the contour line to the height $z = \ell(\hat{\boldsymbol{\vartheta}}) - 2.996$.

- The covariance matrix of the ML estimator $\hat{\boldsymbol{\vartheta}}$ can be approximated by the inverse negative Hessian matrix at the maximum.

Maximum Likelihood Estimator

Example

Mean μ and standard deviation σ of a normal distribution are estimated from a sample of size $n = 500$. The true values are $\mu = 5, \sigma = 1.5$, the estimated values are $\hat{\mu} = 4.980, \hat{\sigma} = 1.529$. The inverse negative Hessian matrix in the maximum of log-likelihood function is equal to.

$$\mathbf{V} = \begin{pmatrix} 0.0047 & 0 \\ 0 & 0.0023 \end{pmatrix}$$

This corresponds to a standard error of $\sigma[\hat{\mu}] = 0.0684$ and $\sigma[\hat{\sigma}] = 0.0483$. The inverse Fisher information matrix is equal to

$$\mathbf{I}_{\mu,\sigma} = \begin{pmatrix} 0.0045 & 0 \\ 0 & 0.0022 \end{pmatrix}$$

Example (Continuation)

Contour lines of log-likelihood function with probability contents $0.3935, 0.9, 0.95, 0.99$.

Maximum Likelihood Estimator

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm
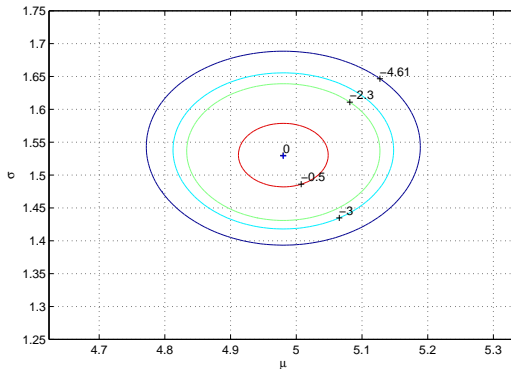
Numerical Calculation of the Hessian matrix

- Let $\ell(\vartheta_1, \vartheta_2)$ be the log-likelihood function, with maximum in $(\hat{\vartheta}_1, \hat{\vartheta}_2)$.

- Furthermore let $\varepsilon_1, \varepsilon_2 > 0$ and

$$\mathbf{L} = \begin{pmatrix} \ell(\hat{\vartheta}_1 - \varepsilon_1, \hat{\vartheta}_2 - \varepsilon_2) & \ell(\hat{\vartheta}_1, \hat{\vartheta}_2 - \varepsilon_2) & \ell(\hat{\vartheta}_1 + \varepsilon_1, \hat{\vartheta}_2 - \varepsilon_2) \\ \ell(\hat{\vartheta}_1 - \varepsilon_1, \hat{\vartheta}_2) & \ell(\hat{\vartheta}_1, \hat{\vartheta}_2) & \ell(\hat{\vartheta}_1 + \varepsilon_1, \hat{\vartheta}_2) \\ \ell(\hat{\vartheta}_1 - \varepsilon_1, \hat{\vartheta}_2 + \varepsilon_2) & \ell(\hat{\vartheta}_1, \hat{\vartheta}_2 + \varepsilon_2) & \ell(\hat{\vartheta}_1 + \varepsilon_1, \hat{\vartheta}_2 + \varepsilon_2) \end{pmatrix}$$

- The Hessian matrix is then equal to

$$\mathbf{H} = \begin{pmatrix} \dfrac{\mathbf{L}_{21} + \mathbf{L}_{23} - 2\,\mathbf{L}_{22}}{\varepsilon_1^2} & \dfrac{\mathbf{L}_{11} + \mathbf{L}_{33} - \mathbf{L}_{13} - \mathbf{L}_{31}}{4\,\varepsilon_1 \varepsilon_2} \\ \dfrac{\mathbf{L}_{11} + \mathbf{L}_{33} - \mathbf{L}_{13} - \mathbf{L}_{31}}{4\,\varepsilon_1 \varepsilon_2} & \dfrac{\mathbf{L}_{12} + \mathbf{L}_{32} - 2\,\mathbf{L}_{22}}{\varepsilon_2^2} \end{pmatrix}$$

- Experimental observations often come from different distributions, e.g. signal and background.
- Such data can be described by a **mixture model**.

Definition (Mixture Model)

A **mixture model** with k components is a distribution whose density has the following form:

$$f(x) = \sum_{j=1}^{k} w_j f_j(x), \quad w_j \geq 0, \quad \sum_{j=1}^{k} w_j = 1$$

The w_j are nonnegative and are called the **weights** of the components.

- The components f_j are typically normal distributions or other simple distributions.

- The l-th moment μ'_l around 0 of the mixture is the mixture of the corresponding moments μ'_{lj} around 0 of the components:

$$\mu'_l = \sum_{j=1}^{k} w_j \mu'_{lj}$$

expectation and variance of a mixture distribution

Let a mixture distribution consist of k components with means μ_j and variances σ_j^2. Then the mean μ and variance σ^2 of the mixture distribution are given by:

$$\mu = \sum_{j=1}^{k} w_j \mu_j, \qquad \sigma^2 = \sum_{j=1}^{k} w_j \left(\mu_j^2 + \sigma_j^2 \right) - \mu^2$$

Example

- A mixture distribution of k normally distributed components has $3k - 1$ parameters:

$$\boldsymbol{\vartheta} = (\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2, w_1, \ldots, w_{k-1})$$

- A mixture distribution of k exponentially distributed components has $2k - 1$ parameters:

$$\boldsymbol{\vartheta} = (\tau_1, \ldots, \tau_k, w_1, \ldots, w_{k-1})$$

- The estimation of the parameters can be performed with the maximum likelihood method.
- In the case of a mixture of normal distributions, the log-likelihood function of a sample X of size n is:

$$\ell(\boldsymbol{\vartheta}) = \sum_{i=1}^{n} \ln \left(\sum_{j=1}^{k} w_j \cdot \varphi(x_i; \mu_j, \sigma_j^2) \right)$$

- The log-likelihood function must be maximized numerically, subject to the constraint $\sum_{j=1}^{k} w_j = 1$.
- The log-likelihood function may have several local maxima. In this case, repeated maximization with different initial values is recommended.

Statistical Methods
of Data Analysis

W. Waltenberger

Point Estimators

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Data from Bernoulli and
Drawing Experiments

Maximum Likelihood
Estimator

Mixture Models and the
EM Algorithm

- One method for maximizing the log-likelihood function is the **EM algorithm**.
- The EM algorithm is iterative. The choice of initial values can influence which local maximum is reached.
- In each iteration, Bayes' theorem is used to calculate the a posteriori probabilities p_{ij} that observation i comes from component j.
- The means and variances of the components are then estimated by weighted sample means and weighted sample variances, respectively.
- These two steps are iterated until the estimate stabilizes.
- The log-likelihood function cannot get smaller in any iteration; however, reaching the global maximum is **not** guaranteed.

Mixture Models and the EM Algorithm

EM algorithm for a mixture of normal distributions

1. Choice of initial values for mixture parameters
$$\boldsymbol{\vartheta} = (\mu_1, \ldots, \mu_k, \sigma_1^2, \ldots, \sigma_k^2, w_1, \ldots, w_k)$$

2. Calculation of p_{ij} and p_j for observation X_i, component φ_j:

$$p_{ij} = \frac{w_j \, \varphi(X_i; \mu_j, \sigma_j^2)}{\sum_{l=1}^{k} w_l \, \varphi(X_i; \mu_l, \sigma_l^2)}, \quad p_j = \sum_{i=1}^{n} p_{ij}$$

3. Estimation of weights and moments:

$$w_j = \frac{p_j}{n}, \ \mu_j = \frac{\sum_{i=1}^{n} p_{ij} X_i}{p_j}, \ \sigma_j^2 = \frac{\sum_{i=1}^{n} p_{ij} (X_i - \mu_j)^2}{p_j}$$

4. Repeat steps 2 and 3 until convergence.

Mixture Models and the EM Algorithm

Example

Data set 2 is a mixture of 500 values from $No(5, 1)$ and 100 values from $No(8, 9)$. The EM algorithm yields the following estimates of the mixture parameters:

$$\mu_1 = 4.946, \sigma_1 = 1.053, w_1 = 0.850$$
$$\mu_2 = 8.206, \sigma_2 = 2.944, w_2 = 0.150$$

Maximizing the log-likelihood function with the simplex algorithm of Nelder and Mead yields almost identical values:

$$\mu_1 = 4.946, \sigma_1 = 1.052, w_1 = 0.850$$
$$\mu_2 = 8.195, \sigma_2 = 2.946, w_2 = 0.150$$

Example (Continuation)

Densities of components and mixture density:
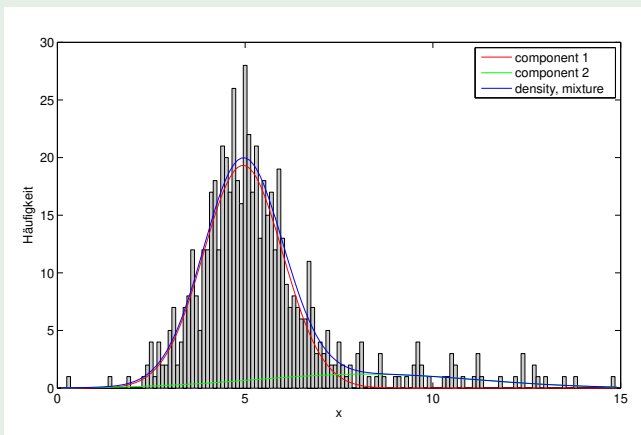
Example (Continuation)

A-posteriori probabilities p_{ij}:

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Part V

Confidence Intervals

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

- Apart from the estimated value itself, its distribution around the true value is also of interest.
- We want to determine an interval from a sample that contains the true value with a certain probability.

Definition (Confidence Interval)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from the distribution F with unknown parameter ϑ and $0 < \alpha < 1$. An interval with boundaries $G_1 = g_1(\boldsymbol{X})$ and $G_2 = g_2(\boldsymbol{X})$ is called an **confidence interval** with a certainty $1 - \alpha$ if:

$$P(G1 \leq G2) = 1$$
$$P(G1 \leq \vartheta \leq G2) \geq 1 - \alpha$$

Such an interval is called a $(1 - \alpha)$ confidence interval. α is called the **significance level**.

Basic Terminology

- For each value of certainty $1 - \alpha$ there are many different confidence intervals.
- If F is continuous, there are infinitely many confidence intervals with certainty $1 - \alpha$.
- If F is discrete, the confidence is usually greater than $1 - \alpha$.
- The confidence interval is called **symmetric** if:

$$P(\vartheta \leq G_1) = P(\vartheta \geq G_2)$$

- The confidence interval is called **one-sided** if:

$$\text{lower tail:} \quad P(\vartheta \leq G_2) \geq 1 - \alpha \quad \text{or}$$
$$\text{upper tail:} \quad P(G_1 \leq \vartheta) \geq 1 - \alpha$$

- Let $Y = h(\boldsymbol{X})$ be a sample function. The distribution G of Y depends on the unknown parameter ϑ.

- For each value of ϑ, we determine a **prediction interval** $[y_1(\vartheta), y_2(\vartheta)]$ of significance level α:

$$P(y_1(\vartheta) \leq Y \leq y_2(\vartheta)) \geq 1 - \alpha$$

- If the observation is equal to $Y = y_0$, the confidence interval $[G_1(Y), G_2(Y)]$ is given by:

$$G1 = \min_{\vartheta}\{\vartheta | y_1(\vartheta) \leq y_0 \leq y_2(\vartheta)\}$$
$$G2 = \max_{\vartheta}\{\vartheta | y_1(\vartheta) \leq y_0 \leq y_2(\vartheta)\}$$

Example

Let $\boldsymbol{X}$ be a sample of $\mathrm{No}(\mu, \sigma^2)$ with unknown variance σ^2. Then $(n-1)S^2/\sigma^2$ is χ^2-distributed with $n-1$ degrees of freedom. Therefore:

$$P\left(\frac{\sigma^2 \chi^2_{\alpha/2;n-1}}{n-1} \le S^2 \le \frac{\sigma^2 \chi^2_{1-\alpha/2;n-1}}{n-1}\right) = 1 - \alpha$$

The expression in the parenthesis can be transformed to:

$$\frac{(n-1)S^2}{\chi^2_{1-\alpha/2;n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{\alpha/2;n-1}}$$

From this follows

$$G_1(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{1-\alpha/2;n-1}}, \quad G_2(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{\alpha/2;n-1}}$$

Blue: prediction interval for $\sigma^2 = 3$; red: confidence interval for $S^2 = 5$.

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Mean

Variance

Difference between Two Mean
Values

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Mean

Statistical Methods of Data Analysis

W. Waltenberger

Basic Terminology

General Construction According to Neyman

Normally Distributed Data

Mean

Variance

Difference between Two Mean Values

Exponentially Distributed Data

Poisson Distributed Data

Binomially Distributed Data

Data from Other Distributions

- Let X be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$.
- $\overline{X}$ is normally distributed according to $\mathrm{No}(\mu, \sigma^2/n)$.
- If σ^2 is known, the standard score is

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

standard normally distributed. From

$$P(-z_{1-\alpha/2} \leq Z \leq z_{1-\alpha/2}) = 1 - \alpha$$

then follows

$$P(\overline{X} - z_{1-\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \overline{X} + z_{1-\alpha/2}\sigma/\sqrt{n}) = 1 - \alpha$$

Symmetric CI for the Mean, Known Variance

$$G_1(\boldsymbol{X}) = \overline{X} - z_{1-\alpha/2}\sigma/\sqrt{n}, \quad G_2(\boldsymbol{X}) = \overline{X} + z_{1-\alpha/2}\sigma/\sqrt{n}$$

- The lower and upper tail confidence intervals are constructed, analogously.

Lower tail CI for the mean, known variance

$$G_1(\boldsymbol{X}) = -\infty, \quad G_2(\boldsymbol{X}) = \overline{X} + z_{1-\alpha}\sigma/\sqrt{n}$$

Upper tail CI for the mean, known variance

$$G_1(\boldsymbol{X}) = \overline{X} - z_{1-\alpha}\sigma/\sqrt{n}, \quad G_2(\boldsymbol{X}) = \infty$$

Mean

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Mean

Variance

Difference between Two Mean
Values

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Quantiles of the standard normal distribution

α	$z_{1-\alpha/2}$	$z_{1-\alpha}$
0.001	3.29	3.09
0.002	3.09	2.88
0.005	2.81	2.58
0.01	2.58	2.33
0.02	2.33	2.05
0.05	1.96	1.64
0.1	1.64	1.28

PYTHON: `scipy.stats.norm.ppf`

- If σ^2 is unknown, σ^2 is estimated by the sample variance S^2. The standard score

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

is then t-distributed with $n-1$ degrees of freedom. From

$$P(-t_{1-\alpha/2;n-1} \leq T \leq t_{1-\alpha/2;n-1}) = 1 - \alpha$$

follows

Symmetric CI for the Mean, Unknown Variance

$$G_1(\boldsymbol{x}) = \overline{X} - t_{1-\alpha/2;n-1}S/\sqrt{n}, \ G_2(\boldsymbol{X}) = \overline{X} + t_{1-\alpha/2;n-1}S/\sqrt{n}$$

Example

A sample of size $n = 50$ from the standard normal distribution has
sample mean $\overline{X} = 0.0540$ and sample variance $S^2 = 1.0987$. If the
variance is assumed to be known, the symmetric 95%-confidence
interval for μ is:

$$G_1 = 0.0540 - 1.96/\sqrt{50} = -0.2232$$
$$G_2 = 0.0540 + 1.96/\sqrt{50} = 0.3312$$

If the variance is assumed to be unknown, the symmetric
95%-confidence interval for μ is:

$$G_1 = 0.0540 - 2.01 \cdot 1.0482/\sqrt{50} = -0.2439$$
$$G_2 = 0.0540 + 2.01 \cdot 1.0482/\sqrt{50} = 0.3519$$

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Mean

Variance

Difference between Two Mean
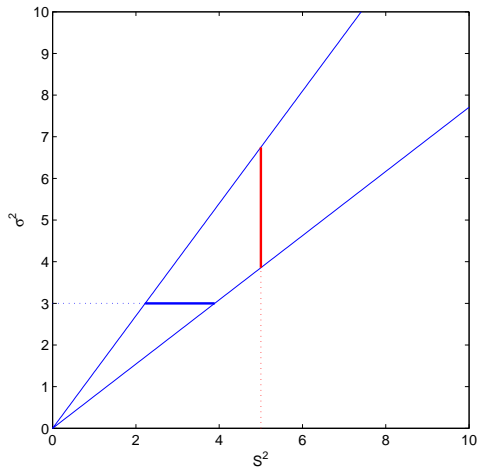Values

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

- Let $X_1, \ldots, X_n$ be a sample from the normal distribution $\mathrm{No}(\mu, \sigma^2)$.
- $(n-1)S^2/\sigma^2$ is χ^2-distributed with $n-1$ degrees of freedom. From

$$P\left(\chi^2_{\alpha/2;n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2;n-1}\right) = 1 - \alpha$$

follows

Symmetric CI for the Variance

$$G_1(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{1-\alpha/2;n-1}}, \quad G_2(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{\alpha/2;n-1}}$$

Variance

Lower Tail CI for the Variance

$$G_1(\boldsymbol{X}) = 0, \quad G_2(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{\alpha;n-1}}$$

Upper Tail CI for the Variance

$$G_1(\boldsymbol{X}) = \frac{(n-1)S^2}{\chi^2_{1-\alpha;n-1}}, \quad G_2(\boldsymbol{X}) = \infty$$

Example

A sample of size $n = 50$ from the normal distribution $\mathrm{No}(0, 4)$ has sample variance $S^2 = 4.3949$. The symmetric 95%-confidence interval for σ^2 is:

$$G_1 = 49 \cdot 4.3949/70.2224 = 3.0667$$
$$G_2 = 49 \cdot 4.3949/31.5549 = 6.8246$$

If the quantiles of the χ^2 distribution $\chi^2(n - 1)$ are replaced by the quantiles of the normal distribution $\mathrm{No}(n - 1, 2(n - 1))$, the confidence interval is:

$$G_1 = 49 \cdot 4.3949/68.4027 = 3.1483$$
$$G_2 = 49 \cdot 4.3949/29.5973 = 7.2760$$

18 Basic Terminology

19 General Construction According to Neyman

20 Normally Distributed Data
   ● Mean
   ● Variance
   ● Difference between Two Mean Values

21 Exponentially Distributed Data

22 Poisson Distributed Data

23 Binomially Distributed Data

24 Data from Other Distributions

Difference between Two Mean Values

- Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples from the normal distributions $\mathrm{No}(\mu_x, \sigma_x^2)$ respectively, $\mathrm{No}(\mu_y, \sigma_y^2)$.

- We are looking for a confidence interval for $\mu_x - \mu_y$. The difference $D = \overline{X} - \overline{Y}$ is normally distributed according to $\mathrm{No}(\mu_x - \mu_y, \sigma^2)$, with $\sigma_D^2 = \sigma_x^2/n + \sigma_y^2/m$.

- If the variances are known, the standard score of D is standard normally distributed. From

$$P\left(-z_{1-\alpha/2} \leq \frac{D - (\mu_x - \mu_y)}{\sigma_D} \leq z_{1-\alpha/2}\right) = 1 - \alpha$$

follows

Symmetric CI for the Difference between Two Mean Values

$$G_1(\boldsymbol{X}) = D - z_{1-\alpha/2}\sigma_D, \quad G_2(\boldsymbol{X}) = D + z_{1-\alpha/2}\sigma_D$$

Difference between Two Mean Values

- If the variances are unknown and equal, then

$$S^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$$

and $(n + m - 2)S^2/\sigma^2$ is χ^2-distributed with $k = n + m - 2$ degrees of freedom.

- The standard score

$$T = \frac{D - (\mu_x - \mu_y)}{S_D}$$

with $S_D = S\sqrt{1/n + 1/m}$ is therefore t-distributed with $k = n + m - 2$ degrees of freedom.

Difference between Two Mean Values

- From

$$P\left(-t_{1-\alpha/2;k} \leq T \leq t_{1-\alpha/2;k}\right) = 1 - \alpha$$

follows

Symmetric CI for the Difference between Two Means, Unknown Variance

$$G_1(\boldsymbol{X}) = D - t_{1-\alpha/2;k}S_D, \quad G_2(\boldsymbol{X}) = D + t_{1-\alpha/2;k}S_D$$

with $k = n + m - 2$.

Example

A sample of $\mathrm{No}(2,4)$ of size $n = 50$ has sample mean $\overline{X} = 2.1080$ and sample variance $S_x^2 = 4.3949$; a second sample of $\mathrm{No}(1,4)$ of size $m = 25$ has sample mean $\overline{X} = 1.6692$ and sample variance $S_x^2 = 5.2220$. If the variances are assumed to be known, the 95%-confidence interval for $\mu_x - \mu_y$ is:

$$G_1 = 0.4388 - 1.96 \cdot 0.4899 = -0.5213$$
$$G_2 = 0.4388 + 1.96 \cdot 0.4899 = 1.3990$$

Example (Continuation)

If the variances are assumed to be unknown, $S^2 = 4.6668$ and $S_D = 0.5292$. The 95%-confidence interval for $\mu_x - \mu_y$ is then:

$$G_1 = 0.4388 - 1.993 \cdot 0.5292 = -0.6158$$
$$G_2 = 0.4388 + 1.993 \cdot 0.5292 = 1.4935$$

18 Basic Terminology

19 General Construction According to Neyman

20 Normally Distributed Data

21 Exponentially Distributed Data

22 Poisson Distributed Data

23 Binomially Distributed Data

24 Data from Other Distributions

- Let X be a sample from the exponential distribution $\mathrm{Ex}(\tau)$.
- Then the sample mean $\overline{X}$ is gamma distributed according to $\mathrm{Ga}(n, \tau/n)$ and has the following density:

$$f(x) = \frac{x^{n-1}}{(\tau/n)^n \Gamma(n)} \exp\left(-\frac{x}{\tau/n}\right)$$

- Therefore, for any τ:

$$P\left(\gamma_{\alpha/2;n,\tau/n} \leq \overline{X} \leq \gamma_{1-\alpha/2;n,\tau/n}\right) = 1 - \alpha$$

Exponentially Distributed Data

- From this follows

$$P\left(\gamma_{\alpha/2;n,1/n} \leq \frac{\overline{X}}{\tau} \leq \gamma_{1-\alpha/2;n,1/n}\right) = 1 - \alpha$$

and

$$P\left(\frac{\overline{X}}{\gamma_{1-\alpha/2;n,1/n}} \leq \tau \leq \frac{\overline{X}}{\gamma_{\alpha/2;n,1/n}}\right) = 1 - \alpha$$

- Thus:

Symmetric CI for the Mean

$$G_1(\boldsymbol{X}) = \frac{\overline{X}}{\gamma_{1-\alpha/2;n,1/n}}, \quad G_2(\boldsymbol{X}) = \frac{\overline{X}}{\gamma_{\alpha/2;n,1/n}}$$

Exponentially Distributed Data

Lower tail CI for the Mean Value

$$G_1(\boldsymbol{X}) = 0, \quad G_2(\boldsymbol{X}) = \frac{\overline{X}}{\gamma_{\alpha;n,1/n}}$$

Upper tail CI for the Mean Value

$$G_1(\boldsymbol{X}) = \frac{\overline{X}}{\gamma_{1-\alpha;n,1/n}}, \quad G_2(\boldsymbol{X}) = \infty$$

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Statistical Methods
of Data Analysis

W. Waltenberger

- Let K be an observation from the Poisson distribution $\mathrm{Po}(\lambda)$.
- The quantiles of the Poisson distribution can be calculated using the distribution function of the gamma distribution.
- For the calculation of the confidence interval, therefore, the **quantiles of the gamma distribution** can be used.

Symmetric CI for the Mean

$$G_1(K) = \gamma_{\alpha/2;K,1}, \quad G_2(K) = \gamma_{1-\alpha/2;K+1,1}$$

- If n observations $K_1, \ldots, K_n$ are available, then $L = \sum K_i$ is Poisson distributed with mean $n\lambda$. The symmetric confidence interval for λ is then:

Symmetric CI for the Mean

$$G_1(L) = \gamma_{\alpha/2;L,1/n}, \quad G_2(L) = \gamma_{1-\alpha/2;L+1,1/n}$$

- This interval is **conservative** in the sense that the coverage is practically always greater than $1 - \alpha$.

Poisson Distributed Data

coverage of the symmetric confidence interval

Poisson Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

- The one-sided confidence intervals are determined in an analogous way.

Lower tail CI for the Mean Value

$$G_1(L) = 0, \quad G_2(L) = \gamma_{1-\alpha;L+1,1/n}$$

Upper tail CI for the Mean Value

$$G_1(L) = \gamma_{\alpha;L,1/n}, \quad G_2(L) = \infty$$

Poisson Distributed Data

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

coverage of the lower tail confidence interval

Poisson Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

- Let K be an observation from the binomial distribution $\mathrm{Bi}(n, p)$.
- The quantiles of the binomial distribution can be calculated using the distribution function of the beta distribution.

Symmetric CI according to Clopper and Pearson

$$G_1(K) = \beta_{\alpha/2; K, n-K+1}, \quad G_2(K) = \beta_{1-\alpha/2; K+1, n-K}$$

- Special cases: for $K = 0$, $G_1(0) = 0$, for $K = n$, $G_2(n) = 1$.
- This interval is **conservative** in the sense that the certainty is practically always greater than $1 - \alpha$.

Binomially Distributed Data

- The one-sided confidence intervals are determined in an analogous way.

Lower tail CI according to Clopper and Pearson

$$G_1(K) = 0, \quad G_2(K) = \beta_{1-\alpha;K+1,n-K}$$

- If $K = 0$, the left-hand confidence interval extends from 0 to $\beta_{1-\alpha;1,n} = 1 - \sqrt[n]{\alpha}$.

Upper tail CI according to Clopper and Pearson

$$G_1(K) = \beta_{\alpha;K,n-K+1}, \quad G_2(K) = 1$$

- If $K = n$, the right-hand confidence interval extends from $\beta_{\alpha;n,1} = \sqrt[n]{\alpha}$ to 1.

- For sufficiently large n, $\hat{p} = K/n$ is approximately normally distributed according to $\mathrm{No}(p, p(1-p)/n)$.
- The standard score

$$Z = \frac{\hat{p} - p}{\sigma[p]}$$

is then approximately standard normally distributed.
- From

$$P(-z_{1-\alpha/2} \leq Z \leq z_{1-\alpha/2}) = 1 - \alpha$$

follows

$$P(\hat{p} - z_{1-\alpha/2}\sigma[p] \leq p \leq \hat{p} + z_{1-\alpha/2}\sigma[p]) = 1 - \alpha$$

Binomially Distributed Data

- Since p is not known, $\sigma[p]$ must be approximated.
- **bootstrap method**: p is approximated by $\hat{p}$:

$$\sigma[p] \approx \sigma[\hat{p}] = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Symmetric bootstrapped CI

$$G_1(K) = \hat{p} - z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$G_2(K) = \hat{p} + z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

- **Robust method**: p is chosen so that $\sigma[p]$ is maximally large, i.e. $p = 0.5$ and

$$\sigma[p] \approx \frac{1}{2\sqrt{n}}$$

Symmetric CI using the Robust Method

$$G_1(K) = \hat{p} - z_{1-\alpha/2}\frac{1}{2\sqrt{n}}$$

$$G_2(K) = \hat{p} + z_{1-\alpha/2}\frac{1}{2\sqrt{n}}$$

Binomially Distributed Data

- procedure according to **Agresti–Coull**: We set

$$c = z_{1-\alpha/2}, K' = K + c^2/2, n' = n + c^2, p' = K'/n'$$

and

$$\sigma[p'] = \sqrt{\frac{p'(1-p')}{n'}}$$

Symmetric CI according to Agresti-Coull

$$G_1(K) = p' - z_{1-\alpha/2}\sqrt{\frac{p'(1-p')}{n}}$$

$$G_2(K) = p' + z_{1-\alpha/2}\sqrt{\frac{p'(1-p')}{n}}$$

Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger
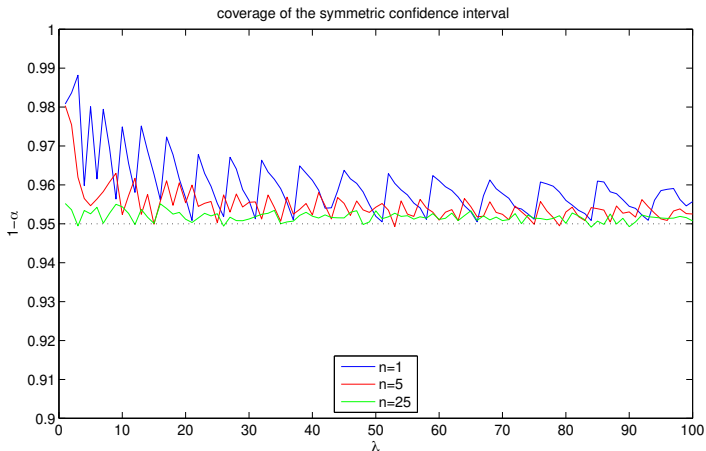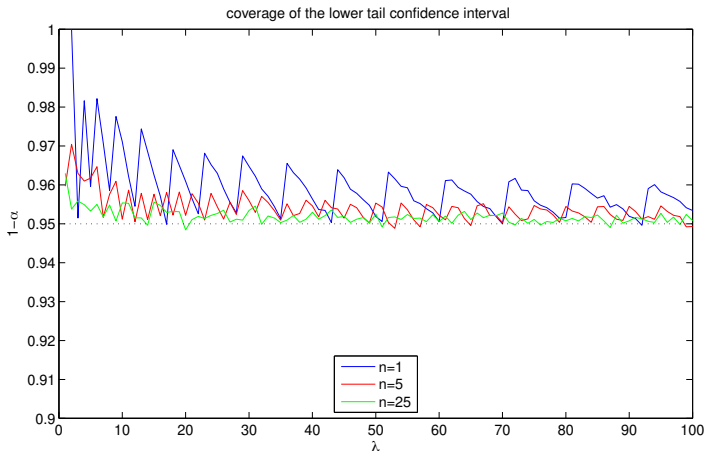
Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Example

In a survey of $n = 400$ people, $k = 157$ people claim to know product X. We are looking for a 95% confidence interval for the degree of familiarity, p.

Clopper-Pearson:

$$G_1(k) = \beta_{0.025;157,244} = 0.3443$$
$$G_2(k) = \beta_{0.975;158,243} = 0.4423$$

Approximation by normal distribution:

$\hat{p} = 0.3925$, $z_{0.975} = 1.96$. Using the bootstrap procedure, $\sigma[\hat{p}] = 0.0244$ is obtained. The limits of the confidence interval are therefore

$$G_1 = 0.3925 - 1.96 \cdot 0.0244 = 0.3446$$
$$G_2 = 0.3925 + 1.96 \cdot 0.0244 = 0.4404$$

Example (Continuation)

Using the robust procedure, we get $\sigma[\hat{p}] = 0.025$ and the bounds

$$G_1 = 0.3925 - 1.96 \cdot 0.025 = 0.3435$$
$$G_2 = 0.3925 + 1.96 \cdot 0.025 = 0.4415$$

The robust interval is only marginally larger than the bootstrap interval.

With the Agresti-Coull correction, we get $\hat{p} = 0.3936$. The limits of the confidence interval are then

$$G_1 = 0.3936 - 1.96 \cdot 0.0244 = 0.3457$$
$$G_2 = 0.3936 + 1.96 \cdot 0.0244 = 0.4414$$

Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

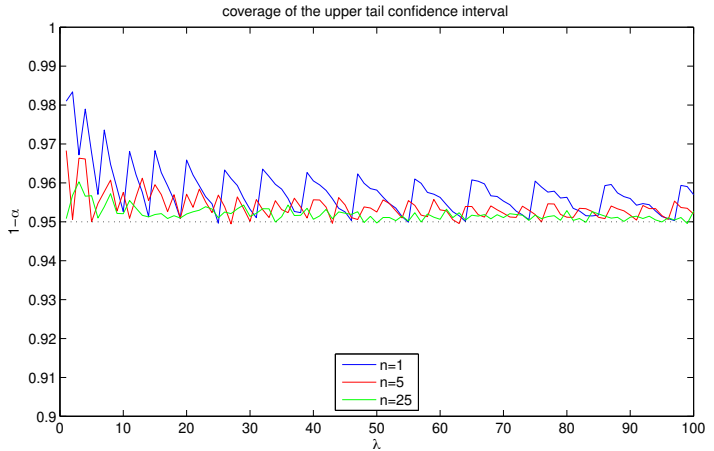Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

coverage of the confidence intervals

Statistical Methods
of Data Analysis

W. Waltenberger

Basic Terminology

General Construction
According to Neyman

Normally Distributed
Data

Exponentially Distributed
Data

Poisson Distributed Data

Binomially Distributed
Data

Data from Other
Distributions

Data from Other Distributions

- Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from distribution F with mean μ and variance σ^2.

- Given the central limit theorem, the standard score Z of the sample mean

$$Z = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

converges against a normal distribution for large samples. Thus, **approximately**

$$P(\overline{X} - z_{1-\alpha/2}S/\sqrt{n} \leq \mu \leq \overline{X} + z_{1-\alpha/2}S/\sqrt{n}) \approx 1 - \alpha$$

Approximate CI for the Mean

$$G_1(\boldsymbol{X}) = \overline{X} - z_{1-\alpha/2}S/\sqrt{n}, \quad G_2(\boldsymbol{X}) = \overline{X} + z_{1-\alpha/2}S/\sqrt{n}$$

Data from Other Distributions

Example

For exponentially distributed samples of size n, the following table lists the confidence of 95%-confidence interval approximated by the normal distribution, estimated from $N = 20000$ samples:

n	25	50	100	200	400
$1 - \alpha$	0.9112	0.9289	0.9408	0.9473	0.9476

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

Part VI

Testing Hypotheses

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Parametric Tests
Non-Parametric Tests

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

- We observe a sample $X_1, \ldots, X_n$ from a distribution F.
- We wish to determine whether the observations are compatible with some assumption about F.
- The assumption is called the **null hypothesis** H_0.
- If F is specified up to one or more parameters, the test is called **parametric**.
- If the form of F is not specified, the test is called **nonparametric** or **parameter free**.
- The test determines whether the sample is consistent with the hypothesis, not whether the hypothesis is correct!
- If the sample is consistent with the hypothesis, the hypothesis may still deviate from the truth, albeit only to a limited extent.

Introduction

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

General Procedure

- A test statistic T is calculated from the sample.
- The range of values of T is divided, depending on H_0, into a **rejection region** (critical region) C and an **acceptance region** C'.
- The acceptance region is usually a **prediction interval** for T.
- If the value of T falls within the rejection range, H_0 is discarded.
- Otherwise, H_0 is retained for the time being.
- However, this is not a confirmation of H_0. It simply means that the data are consistent with the hypothesis.

One-sided and Two-sided Tests

- If the acceptance region is a two-sided prediction interval for T, the test is called **two-sided**. The critical region then breaks down into two disjoint intervals.
- If the acceptance region is an interval of the form $T \leq c$ or $T \geq c$, the test is called **one-sided**. The critical region is then an interval of the form $T > c$ or $T < c$.

Tests and Confidence Intervals

- In many cases, the acceptance region is a confidence interval for the tested parameter.
- The null hypothesis is rejected if the hypothesized value is outside the confidence interval, as it is considered to be insufficiently plausible.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

The p-value

- A test can alternatively be formulated using the p-value $P(T)$.
- The p-value indicates how likely it is to observe the value T or an even more extreme value, given the null hypothesis.
- One-sided test: if $F_0(x)$ is the distribution function of T under the null hypothesis, then the p-value is equal to

$$P(T) = F_0(T) \text{ or } p = 1 - F_0(T)$$

- Two-sided test: If $F_0(x)$ is the distribution function of T under the null hypothesis, then the p-value is equal to

$$P(T) = 2\min(F_0(T), 1 - F_0(T))$$

- The null hypothesis is rejected if $P(T) < \alpha$.

Significance and Power

- In any hypothesis test, we distinguish between two types of errors.
  - **Type I error**: The hypothesis H_0 is rejected although it is true.
  - **Type II error**: The hypothesis H_0 is retained, although it is not true.
- The distribution of T under H_0 is determined.
- The rejection region is determined such that the probability of a type I error is at most equal to a value of α.
- α is called the **significance level**, or the **test size**. Common values are $\alpha = 0.05, 0.01, 0.005$.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

- Once the rejection region is defined, the probability of a type II error of an „alternative hypothesis" H_1, $\beta(H_1)$ can be calculated.

- $1 - \beta(H_1)$ is called the **power** of the test for H_1.

- The power should never be smaller than α.

- The test is called **unbiased**, if the power is never less than α.

- A goal of test theory is to construct unbiased tests with maximum power (uniformly most powerful unbiased test, UMPU).

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics
Tests for Normally Distributed
Data
Tests for Poisson Distributed
Data
Tests for Binomially
Distributed Data
Likelihood-Ratio Tests

Non-Parametric Tests

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

- We consider a sample $X_1, \ldots, X_n$ from a distribution F specified up to one or more parameters.
- Hypothesis tests about F are called **parametric**.
- A null hypothesis H_0 can be thought of as a subset of the parameter space Θ.
- The test decides whether the sample is consistent with the hypothesis.
- Before testing, be sure that the assumed parametric form is reasonable.

Basics

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

- First, the test statistic T and the significance level α are chosen.
- The critical region C is then defined such that

$$P(T \in C | \vartheta \in H_0) \leq \alpha$$

- An **alternative hypothesis** H_1 to the null hypothesis H_0 can be formulated.
- H_1 can also be constructed as a subset of the parameter space Θ.
- If the significance level α is fixed, the power of the test can be computed for each $\vartheta \in H_1$:

$$1 - \beta(\vartheta) = P(T \in C | \vartheta \in H_1)$$

- $1 - \beta(\vartheta)$ is called the **power function** of the test.

Example with Exponential Distribution

- $X_1, \ldots, X_n$ is an exponentially distributed sample of $\mathrm{Ex}(\tau)$.
- The hypothesis $H_0 : \tau = \tau_0$ is to be tested given the sample.
- As test statistic T we choose the sample mean: $T = \overline{X}$.
- Under H_0, T has the following density:

$$f(t) = \frac{t^{n-1}}{(\tau_0/n)^n \Gamma(n)} \exp\left(-\frac{t}{\tau_0/n}\right)$$

- T is thus distributed according to $\mathrm{Ga}(n, \tau_0/n)$.
- The symmetric prediction interval $[y_1(\tau_0), y_2(\tau_0)]$ for T at confidence $1 - \alpha$ is given by:

$$y_1(\tau_0) = \gamma_{\alpha/2;n,\tau_0/n}, \quad y_2(\tau_0) = \gamma_{1-\alpha/2;n,\tau_0/n}$$

Basics

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics
Tests for Normally Distributed
Data
Tests for Poisson Distributed
Data
Tests for Binomially
Distributed Data
Likelihood-Ratio Tests

Non-Parametric Tests

- Therefore, the rejection region at significance α is the set

$$C = [0, y_1(\tau_0)] \cup [y_2(\tau_0), \infty[$$

- So H_0 is rejected if T is too "far away" from the hypothetical value τ_0.

- The power of the test for any given value of τ is given by:

$$1 - \beta(\tau) = P(T \in C) = G(y_1(\tau)) + 1 - G(y_2(\tau))$$

where G is the distribution function of the $\mathrm{Ga}(n, \tau/n)$-distribution.

- The test is biased because, e.g. for $\tau_0 = 1$ and $n = 25$.

$$1 - \beta(0.986) = 0.0495 < \alpha$$

density of sample mean ($\tau_0 = 1$) and critical regions

power ($\tau_0 = 1$)

Expected Value with Known Variance

- $X_1, \ldots, X_n$ is a normally distributed sample of $\mathrm{No}(\mu, \sigma^2)$ with known σ^2.
- The hypothesis $H_0 : \mu = \mu_0$ is to be tested against the alternative hypothesis $H_1 : \mu \neq \mu_0$.
- As the test statistic T we choose the standard score of the sample mean:
$$T = \frac{\sqrt{n}(\overline{X} - \mu_0)}{\sigma}$$
- Under H_0, $T \sim \mathrm{No}(0, 1)$.
- H_0 is rejected if T is not within the prediction interval at significance α of the standard normal distribution.

Tests for Normally Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

Two-sided Test

- The hypothesis H_0 is rejected if

$$|T| = \frac{\sqrt{n}\,|\overline{X} - \mu_0|}{\sigma} > z_{1-\alpha/2}$$

- The power function for a value μ is given by:

$$1 - \beta(\mu) = P(T \in C) = G(z_{\alpha/2}) + 1 - G(z_{(1-\alpha)/2})$$

where G is the distribution function of the $\mathrm{No}(\sqrt{n}(\mu - \mu_0)/\sigma, 1)$-distribution.

- The test is unbiased.

Tests for Normally Distributed Data

power of two-sided test ($\mu_0=1$)

One-sided Test

- The hypothesis $H_0 : \mu \leq \mu_0$ is to be tested with the test statistic T against the alternative hypothesis $H_1 : \mu > \mu_0$.
- H_0 is rejected if T is "too large".
- A rejection region with significance level α is

$$C = [z_{1-\alpha}, \infty[$$

- Thus, the hypothesis H_0 is rejected if

$$T = \frac{\sqrt{n}\left(\overline{X} - \mu_0\right)}{\sigma} > z_{1-\alpha}$$

- The power function for a value $\mu > \mu_0$ is given by:

$$1 - \beta(\mu) = P(T \in C) = 1 - G(z_{1-\alpha})$$

where G is the distribution function of the $\mathrm{No}(\sqrt{n}(\mu - \mu_0)/\sigma, 1)$-distribution.

- Analogously, one can test for $H_0 : \mu \geq \mu_0$ and $H_1 : \mu < \mu_0$.
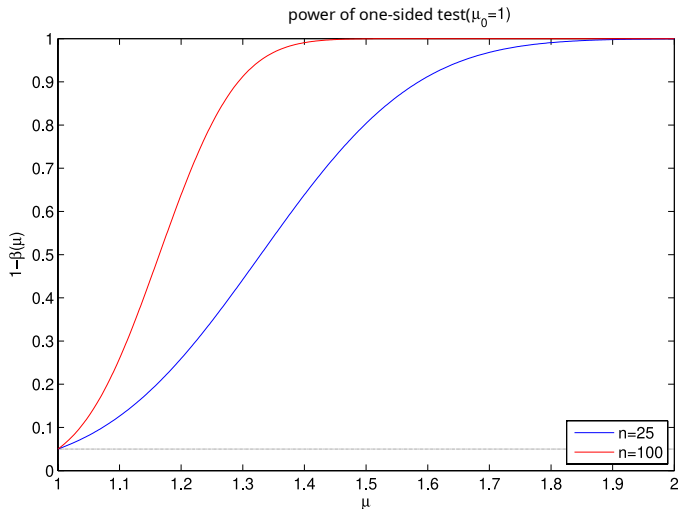
Expected Value with Unknown Variance: t-Test

- $X_1, \ldots, X_n$ is a normally distributed sample of $\mathrm{No}(\mu, \sigma^2)$ with unknown σ^2.
- The hypothesis $H_0 : \mu = \mu_0$ is to be tested against the alternative hypothesis $H_1 : \mu \neq \mu_0$.
- As test statistic T we choose the standard score of the sample mean, using the sample variance S^2:

$$T = \frac{\sqrt{n}(\overline{X} - \mu_0)}{S}$$

- Under H_0, $T \sim \mathrm{t}(n-1)$.

- H_0 is rejected if T does not fall within the prediction interval at significance α of the t-distribution with $n-1$ degrees of freedom.

- The rejection region with significance level α is given as

$$C =\,] -\infty, t_{\alpha/2;n-1}] \cup [t_{1-\alpha/2;n-1}, \infty[$$

where $t_{p;n}$ is the quantile of the t-distribution with n degrees of freedom at significance p.

- Thus, the hypothesis H_0 is rejected if

$$|T| = \frac{\sqrt{n}\,\left|\overline{X} - \mu_0\right|}{S} > t_{1-\alpha/2;n-1}$$

- The power for a value μ is given by:

$$1 - \beta(\mu) = P(T \in C) = G(z_{\alpha/2}) + 1 - G(z_{(1-\alpha)/2})$$

where G is the distribution function of the non-central $\mathrm{t}(n-1, \delta)$-distribution with

$$\delta = \sqrt{n}(\mu - \mu_0)/\sigma$$

- The test is unbiased.

Tests for Normally Distributed Data

Statistical Methods
of Data Analysis

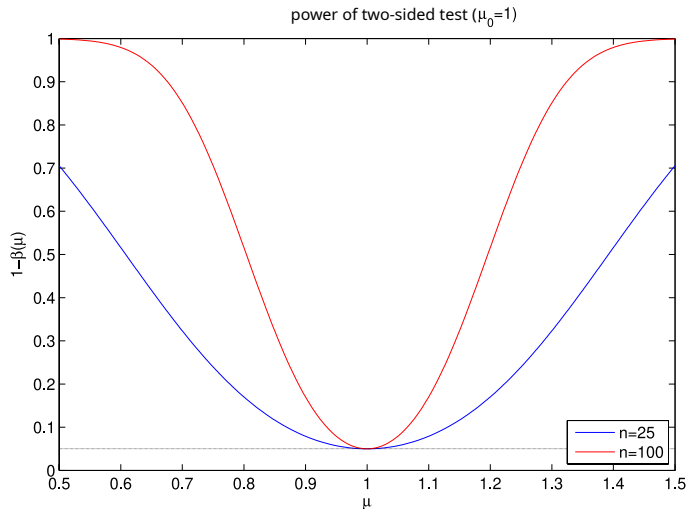W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

power of two-sided t-test ($\mu_0=1$)

Tests for Normally Distributed Data

Equality of Two Expected Values

- $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are two independent normally distributed samples of $\mathrm{No}(\mu_x, \sigma_x^2)$ and $\mathrm{No}(\mu_y, \sigma_y^2)$.
- The hypothesis $H_0 : \mu_x = \mu_y$ is to be tested against the alternative hypothesis $H_1 : \mu_x \neq \mu_y$.
- If the **variances are known**, we choose as test statistic T the difference of the sample means:

$$T = \overline{X} - \overline{Y}$$

- Assuming H_0, $T \sim \mathrm{No}(0, \sigma_x^2/n + \sigma_y^2/m)$.

- The default score

$$Z = \frac{T}{\sqrt{\sigma_x^2/n + \sigma_y^2/m}}$$

is then standard normally distributed.

- Thus, the hypothesis H_0 is rejected if

$$|Z| > z_{1-\alpha/2}$$

or

$$\frac{|\overline{X} - \overline{Y}|}{\sqrt{\sigma_x^2/n + \sigma_y^2/m}} > z_{1-\alpha/2}$$

Tests for Normally Distributed Data

- If the **variances are unknown and equal**, the variance can be estimated from the combined ('pooled') sample:

$$S^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$$

- Under H_0,

$$T = \frac{\overline{X} - \overline{Y}}{\sqrt{S^2(1/n + 1/m)}}$$

  t-distributed with $n + m - 2$ degrees of freedom.

- Thus, the hypothesis H_0 is rejected if

$$|T| > t_{1-\alpha/2;n+m-2}$$

  where $t_{1-\alpha/2;n+m-2}$ is the quantile of the t-distribution with $n + m - 2$ degrees of freedom.

t-Test for Paired Samples

- Paired samples $(X_1, Y_1), \ldots, (X_n, Y_n)$ occur when the same quantity is measured twice for each observed object, before and after a given intervention.

- The effect of the intervention is described by the differences $W_i = Y_i - X_i, i = 1, \ldots, n$.

- We assume that $W_1, \ldots, W_n$ is normally distributed with mean μ_w and unknown variance σ_w^2.

- The hypothesis $H_0 : \mu_w = 0$ (the intervention had no effect) is to be tested against the alternative hypothesis $H_1 : \mu_w \neq 0$.

- This is performed via the t-test for individual samples.

Tests for Normally Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

Test of Variance

- $X_1, \ldots, X_n$ is a normally distributed sample with unknown expected value μ and unknown variance σ^2.
- The hypothesis $H_0 : \sigma^2 = \sigma_0^2$ is to be tested against the alternative hypothesis $H_1 : \sigma^2 \neq \sigma_0^2$.
- For the test statistic T we choose:

$$T = \frac{(n-1)S^2}{\sigma_0^2}$$

- Assuming H_0, T is χ^2-distributed with $n-1$ degrees of freedom.

- Thus, the hypothesis H_0 is rejected if

$$T < \chi^2_{\alpha/2;n-1} \quad \text{or} \quad T > \chi^2_{1-\alpha/2;n-1}$$

where $\chi^2_{p;k}$ is the p-quantile of the χ^2-distribution with k degrees of freedom.

- The power function for a value σ^2 is given by:

$$1 - \beta(\sigma^2) = G(\sigma_0^2/\sigma^2 \cdot \chi^2_{\alpha/2;n-1}) + 1 - G(\sigma_0^2/\sigma^2 \cdot \chi^2_{1-\alpha/2;n-1})$$

where G is the distribution function of the $\chi^2(n-1)$-distribution.

- The test is not unbiased.

Tests for Normally Distributed Data

power function of two-sided test ($\sigma_0^2=1$)

Statistical Methods
of Data Analysis

W. Waltenberger

25 Introduction
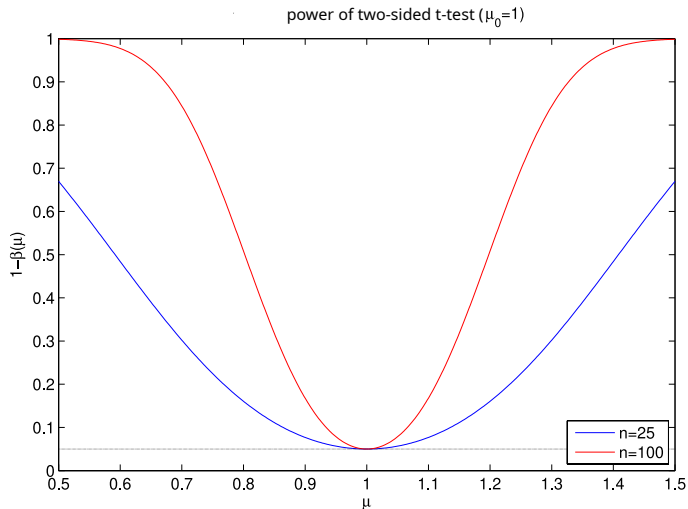
26 Parametric Tests
  - Basics
  - Tests for Normally Distributed Data
  - Tests for Poisson Distributed Data
  - Tests for Binomially Distributed Data
  - Likelihood-Ratio Tests

27 Non-Parametric Tests

Test for Two-sided Expected Value

- Let $X_1, \ldots, X_n$ be a Poisson distributed sample $\mathrm{Po}(\lambda)$.
- The hypothesis $H_0 : \lambda = \lambda_0$ is to be tested against the alternative hypothesis $H_1 : \lambda \neq \lambda_0$.
- As test statistic T we choose the sample sum:

$$T = \sum_{i=1}^{n} X_i$$

- T is Poisson distributed according to $\mathrm{Po}(n\lambda)$.
- H_0 is rejected if T is "too small" or "too large", i.e. if

$$\sum_{k=0}^{T} \frac{(n\lambda_0)^k e^{-n\lambda_0}}{k!} < \alpha/2 \text{ or } \sum_{k=T}^{\infty} \frac{(n\lambda_0)^k e^{-n\lambda_0}}{k!} < \alpha/2$$

- If the distribution function of the Poisson distribution is expressed by the distribution function of the gamma distribution, the following critical region is obtained:

$$n\lambda_0 < \gamma_{\alpha/2;T,1} \quad \text{or} \quad n\lambda_0 > \gamma_{1-\alpha/2;T+1,1}$$

- H_0 is thus rejected if $n\lambda_0$ is not within the confidence interval based on observation T.

Tests for Poisson Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

One-sided Test for Expected Value

- The hypothesis $H_0 : \lambda \leq \lambda_0$ is rejected if T is "too large" and thus the p-value is too small:

$$P(T) = \sum_{k=T}^{\infty} \frac{(n\lambda_0)^k e^{-n\lambda_0}}{k!} < \alpha \quad \text{or} \quad n\lambda_0 < \gamma_{\alpha;T,1}$$

- The hypothesis $H_0 : \lambda \geq \lambda_0$ is rejected if T is "too small" and thus the p-value is too small:

$$P(T) = \sum_{k=0}^{T} \frac{(n\lambda_0)^k e^{-n\lambda_0}}{k!} < \alpha \quad \text{or} \quad n\lambda_0 > \gamma_{1-\alpha;T+1,1}$$

Tests for Poisson Distributed Data

Statistical Methods of Data Analysis
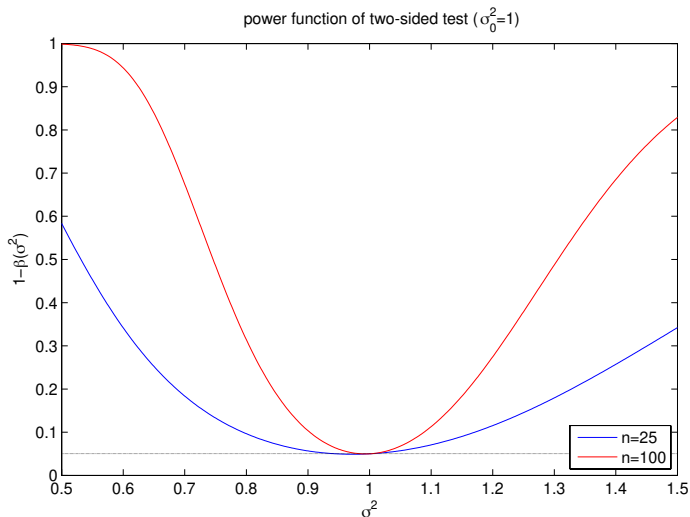
W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed Data

Tests for Poisson Distributed Data

Tests for Binomially Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

Example

A manufacturer aims to produce no more than 25 defective components per day on average in a factory. A sample of 5 days results in 28,34,32,38 and 22 defective components. Has the manufacturer achieved his goal?

The test statistic $T = 154$. Therefore,

$$P(T) = \sum_{k=T}^{\infty} \frac{(125)^k e^{-125}}{k!} = 0.0067 < 0.01$$

$$\gamma_{0.01;154,1} = 126.61 < 154$$

Thus, the hypothesis can be refuted at a significance level of 1 percent.

Approximation by Normal Distribution

- If n is sufficiently large, the distribution of T can be approximated by a normal distribution $\mathrm{No}(n\lambda, n\lambda)$.
- H_0 is rejected if the standard score

$$Z = \frac{T - n\lambda_0}{\sqrt{n\lambda_0}}$$

is not within a prediction interval of level $1 - \alpha$ of the standard normal distribution.

Example

Given the last example, the approximation yields:

$$Z = 2.5938 > z_{0.99} = 2.3263$$

Thus, the hypothesis can be rejected at a significance level of 1 percent.

Tests for Binomially Distributed Data

Bipartite Test for Parameter p

- k is an observation from the binomial distribution $\mathrm{Bi}(n, p)$.
- The hypothesis $H_0 : p = p_0$ is to be tested against the alternative hypothesis $H_1 : p \neq p_0$.
- H_0 is rejected if k assuming H_0 is not in the symmetric prediction interval $[y_1(p_0), y_2(p_0)]$, i.e., is "too small" or "too large".

Tests for Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

- This is the case if either

$$\sum_{i=0}^{k} \binom{n}{i} p_0^i (1-p_0)^{n-i} = F_{\mathrm{Be}}(p_0; k, n-k+1) < \alpha/2$$

or

$$\sum_{i=k}^{n} \binom{n}{i} p_0^i (1-p_0)^{n-i} = F_{\mathrm{Be}}(1-p_0; n-k, k+1) < \alpha/2$$

holds where $F_{\mathrm{Be}}(x; a, b)$ is the distribution function of $\mathrm{Be}(a, b)$.

Tests for Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

One-sided Test for Parameter p

- The hypothesis $H_0 : p \leq p_0$ is to be tested on the basis of the observation k against the alternative hypothesis $H_1 : p > p_0$.
- H_0 is rejected if k is "too large" and thus the p-value is too small:

$$P(k) = \sum_{i=k}^{n} \binom{n}{i} p_0^i (1 - p_0)^{n-i} = F_{\mathrm{Be}}(p_0; k, n - k + 1) < \alpha$$

- The hypothesis $H_0 : p \geq p_0$ is rejected if k is "too small" and thus the p-value is too small:

$$P(k) = \sum_{i=0}^{k} \binom{n}{i} p_0^i (1 - p_0)^{n-i} = F_{\mathrm{Be}}(1 - p_0; n - k, k + 1) < \alpha$$

Example

A manufacturer claims that no more than 2 percent of a certain component are defective. In a sample of size 300, 9 pieces are defective. Can the manufacturer's claim be refuted?

$$P(k) = \sum_{i=9}^{300} \binom{300}{i} 0.02^i 0.98^{300-i} = 0.1507$$

The manufacturer's claim cannot be refuted at a 5 percent significance level.

Tests for Binomially Distributed Data

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed Data

Tests for Poisson Distributed Data

Tests for Binomially Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

Approximation by Normal Distribution

- If n is sufficiently large, the distribution of k can be approximated by a normal distribution $\mathrm{No}(np, np(1-p))$.
- H_0 is rejected if the standard score is

$$Z = \frac{k - np_0}{\sqrt{np(1-p_0)}}$$

does not lie within a prediction interval of level $1 - \alpha$ of the standard normal distribution.

- Two-sided test: H_0 is rejected if

$$Z < z_{\alpha/2} \text{ or } Z > z_{1-\alpha/2}$$

- One-sided test: H_0 is rejected if

$$Z < z_{\alpha} \text{ or } Z > z_{1-\alpha}$$

Example

Given the last example, the approximation yields:

$$Z = 1.2372 < z_{0.95} = 1.6449$$

Thus, the hypothesis cannot be rejected.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests
Basics
Tests for Normally Distributed
Data
Tests for Poisson Distributed
Data
Tests for Binomially
Distributed Data
Likelihood-Ratio Tests

Non-Parametric Tests

Likelihood-Ratio Tests

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics
Tests for Normally Distributed
Data
Tests for Poisson Distributed
Data
Tests for Binomially
Distributed Data
Likelihood-Ratio Tests

Non-Parametric Tests

- A hypothesis is called **simple** if it has no free parameters, i.e. if the statistical model is completely defined.

- Given two **simple** hypotheses $H_0 : \theta = \theta_0$, and $H_1 : \theta = \theta_1$, and data $\boldsymbol{X}$, a test can be constructed from the ratio of the likelihoods.

$$\Lambda(\boldsymbol{X}) := \frac{L(\theta_0|\boldsymbol{X})}{L(\theta_1|\boldsymbol{X})}$$

- The null hypothesis H_0 is rejected if the test statistic is below a certain value, $\Lambda(\boldsymbol{X}) \leq \eta$.

- Here η is chosen such that $\alpha = P(\Lambda(\boldsymbol{X}) \leq \eta | H_0)$. As usual, the significance level (,,test size") α is again freely chosen. Common values are $\alpha = 0.05, 0.1$ or 0.01.

Likelihood-Ratio Tests

Example (Likelihood-Ratio Test)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from a normal distribution $\mathsf{N}(\mu, \sigma^2)$ with fixed μ. We test for the *variance* of the distribution, with $H_0 : \sigma^2 = \sigma_0^2$ and $H_1 : \sigma^2 = \sigma_1^2, \sigma_1^2 > \sigma_0^2$.

- The likelihood ratio is

$$\Lambda(\boldsymbol{X}) = \frac{L(\sigma_0^2 | \boldsymbol{X})}{L(\sigma_1^2 | \boldsymbol{X})} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{-\frac{n}{2}} \exp\left\{-\frac{\sigma_0^{-2} - \sigma_1^{-2}}{2} \sum_{i=1}^{n}(X_i - \mu)^2\right\}$$

- $\sum_{i=1}^{n}(X_i - \mu)^2$ is the only term of the likelihood ratio that depends on the data, therefore H_0 is rejected if the sum is extreme enough under the null hypothesis. Since we required that $\sigma_1^2 > \sigma_0^2$, we can restrict ourselves to test whether the sum is too large (one-sided test).

Neyman-Pearson lemma

The likelihood-ratio test is the test with asymptotically maximum power, for a given significance level α.

- A hypothesis that is not *simple* is called **composite**.
- Often the alternative hypothesis H_1 is constructed as the complement of the null hypothesis, i.e.
  $H_0 : \theta \in \Theta_0, H_1 : \theta \in \Theta_0^C$.
- The alternative hypothesis $H_1 : \theta \in \Theta$ can also be formulated to *contain* the null hypothesis $H_0 : \theta \in \Theta_0$, with $\Theta_0 \in \Theta$. In this case we speak of a **nested model**. The Neyman-Pearson lemma holds for the likelihood-ratio test:

$$\Lambda(\boldsymbol{X}) = \frac{\sup\{L(\theta|\boldsymbol{X}) : \theta \in \Theta_0\}}{\sup\{L(\theta|\boldsymbol{X}) : \theta \in \Theta\}}$$

Likelihood-Ratio Tests

Wilks' Theorem

For nested models, under H_0 the test statistic $T = -2 \ln \Lambda(\boldsymbol{X})$ converges asymptotically towards a $\chi^2(n)$-distribution with n degrees of freedom, where n is given as the difference of the dimensionalities of Θ and Θ_0: $n = \dim(\Theta) - \dim(\Theta_0)$.

- Wilks' theorem can be used to specify η: $\eta = \chi^2_{1-\alpha;n}$, where α is again the significance level and n is the number of degrees of freedom of the test.

Likelihood-Ratio Tests

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Basics

Tests for Normally Distributed
Data

Tests for Poisson Distributed
Data

Tests for Binomially
Distributed Data

Likelihood-Ratio Tests

Non-Parametric Tests

Example (Likelihood Ratio Test for P_λ)

We construct a test for the intensity parameter λ of a Poisson distribution, $H_0 : \lambda = \lambda_0, H_1 : \lambda \neq \lambda_0$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample of the distribution.

- The supremum of the alternative hypothesis H_1 (the ,,maximum likelihood") is at $\hat{\lambda} = \overline{X}$.

- From this follows for the likelihood ratio $\Lambda(\boldsymbol{X})$:

$$\Lambda(\boldsymbol{X}) = e^{n(\overline{X} - \lambda_0)} \prod_{i=1}^{n} \left(\frac{\lambda_0}{\overline{X}} \right)^{X_i}$$

- The test statistic $T = -2 \ln \Lambda(\boldsymbol{X})$:

$$T = -2n \left[\overline{X} - \lambda_0 + \overline{X} \ln \left(\frac{\lambda_0}{\overline{X}} \right) \right]$$

Likelihood-Ratio Tests

Example (Likelihood Ratio Test for P_λ, continued)

- The test statistic $T = -2 \ln \Lambda(\boldsymbol{X})$:

$$T = 2n \left[\lambda_0 - \overline{X} + \overline{X} \ln \left(\frac{\overline{X}}{\lambda_0} \right) \right]$$

- According to Wilks' theorem T is asymptotically χ^2-distributed with one degree of freedom. The 95% quantile of $\chi^2(1)$ is 3.84. The critical region of the test at a significance level of 5% is therefore

$$C = [3.84, \infty)$$

.

- A test that checks the hypothesis whether the data can come from a certain distribution is called a **goodness-of-fit test**.
- The distribution may be completely determined or determined up to unknown parameters.
- A goodness-of-fit test may precede a parametric test to check its applicability.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Parametric Tests
Non-Parametric Tests
The Chi-squared Test
The Kolmogorov-Smirnov Test

The Chi-squared Test

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

The Chi-squared Test

The Kolmogorov-Smirnov Test

The Chi-squared Test for Discrete Data

- The sample $X_1, \ldots, X_n$ comes from a discrete distribution with finite range of values $\{1, \ldots, k\}$.

- We test the hypothesis H_0 that the density f has values $f(j) = p_j, j = 1, \ldots, k$:

$$H_0 : P(X_i = j) = p_j, j = 1, \ldots, k$$

vs.

$$H_1 : P(X_i = j) \neq p_j, \text{for at least one } j$$

- Let Y_j be the number of observations equal to j.

- Under the null hypothesis, $Y_1, \ldots, Y_k$ is multinomially distributed according to $\mathrm{Mu}(n, p_1, \ldots, p_k)$ and $\mathsf{E}[Y_j] = np_j$.

Excursion: the Multinomial Distribution $\mathrm{Mu}(n, p_1, \ldots, p_d)$

- The Bernoulli experiment can be generalized to allow not only two, but d elementary events $e_1, \ldots, e_d$, to which are assigned the probabilities $p_1, \ldots, p_d$, which only must satisfy

$$\sum_{i=1}^{d} p_i = 1$$

- Performing the **generalized alternative experiment** n times, the elementary events are the sequences of the form:

$$(e_{i_1}, \ldots, e_{i_n}),\ 1 \le i_j \le d$$

- If the n observations are independent, then:

$$P(e_{i_1}, \ldots, e_{i_n}) = \prod_{j=1}^{n} P(e_{i_j}) = \prod_{j=1}^{n} p_{i_j} = \prod_{i=1}^{d} p_i^{n_i}$$

Where n_i is the number of occurrences of e_i. Therefore, the sum of n_i is n.

- The d-dimensional random variable $\boldsymbol{X} = (X_1, \ldots, X_d)$ maps the sequence $(e_{i_1}, \ldots, e_{i_n})$ to the vector $(n_1, \ldots, n_d)$. In this way $n!/(n_1! \cdots n_d!)$ sequences are mapped onto the same vector.

- Therefore, the density of $\boldsymbol{X}$ is:

$$f_{\boldsymbol{X}}(n_1, \ldots, n_d) = \frac{n!}{n_1! \ldots n_d!} \prod_{i=1}^{d} p_i^{n_i}, \ \sum_{i=1}^{d} n_i = n, \ \sum_{i=1}^{d} p_i = 1$$

The Chi-squared Test

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

The Chi-squared Test

The Kolmogorov-Smirnov Test

- The distribution of X is called a **multinomial distribution** with parameters n and $p_1, \ldots, p_d$: $P_X = \mathrm{Mu}(n, p_1, \ldots, p_d)$

- A simple example of a multinomially distributed random vector is a histogram (grouped frequency distribution), which is used to graphically represent the (absolute) **experimental frequencies**.

- X_i is the number of times the outcome of the random variable R falls into group i.

- Let the probability that R falls into group i be equal to p_i.

- If n outcomes are filled into the histogram, the group contents $(X_1, \ldots, X_d)$ are multinomially distributed according to $\mathrm{Mu}(n, p_1, \ldots, p_d)$.

The Chi-squared Test

- A histogram
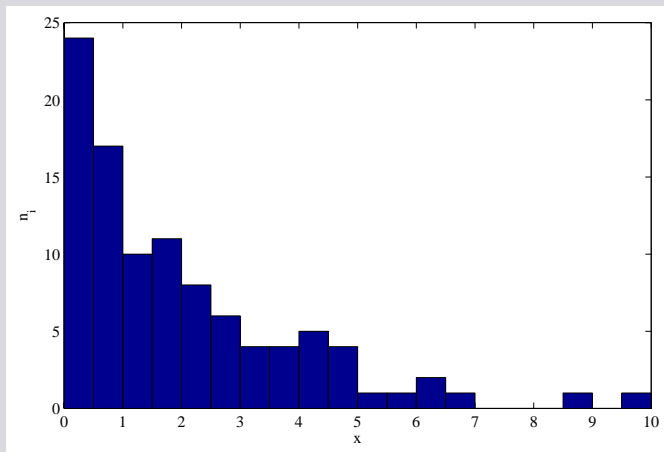
The Moments of the $\mathrm{Mu}(n, p_1, \ldots, p_d)$-distribution

Let $\boldsymbol{X} = (X_1, \ldots, X_d) \sim \mathrm{Mu}(n, p_1, \ldots, p_d)$. Then:

- $\mathsf{E}[X_i] = np_i$
- $\mathsf{var}[X_i] = np_i(1 - p_i)$
- $\mathsf{cov}[X_i, X_j] = -np_i p_j$

Definition (Covariance)

The **covariance** of two random variables X and Y is defined by:

$$\mathsf{cov}[X, Y] = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$$

If $\mathsf{cov}[X, Y] = 0$, X and Y are called **uncorrelated**.

The Chi-squared Test

Properties of Covariance

- $\text{cov}[aX + b, cY + d] = ac \cdot \text{cov}[X, Y]$
- $\text{var}[X + Y] = \text{var}[X] + \text{var}[Y] + 2\text{cov}[X, Y]$
- $\text{cov}[X, X] = \text{var}[X]$

Definition (Correlation)

The **correlation coefficient** $\rho[X, Y]$ is defined by:

$$\rho[X, Y] = \frac{\text{cov}[X, Y]}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

Properties of Correlation Coefficient

- $-1 \leq \rho[X, Y] \leq 1$
- $\rho[aX + b, cY + d] = \rho[X, Y]$
- $\rho[X, X] = 1$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction
Parametric Tests
Non-Parametric Tests
The Chi-squared Test
The Kolmogorov-Smirnov Test

The Chi-squared Test

Definition (Covariance Matrix)

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector. If all variances and covariances exist, they are combined in the **covariance matrix** $\mathbf{C} = \mathrm{Cov}[\boldsymbol{X}]$:

$$\mathbf{C}_{ij} = \mathrm{cov}[X_i, X_j]$$

Similarly, the correlations are summarized in the **correlation matrix** $\mathbf{R} = \mathrm{Cor}[X, Y]$:

$$\mathbf{R}_{ij} = \rho[X_i, X_j]$$

- The covariance matrix is always **symmetric** and **positive definite**. All eigenvalues are real and positive.
- The correlation matrix is also **symmetric** and **positive definite**. All diagonal elements are equal to 1.

- The test statistic compares the observed frequencies Y_j with their expected values:

$$T = \sum_{j=1}^{k} \frac{(Y_j - np_j)^2}{np_j}$$

- The null hypothesis is rejected if T is large.
- The critical region can be determined according to the following result.

Theorem

Under the null hypothesis, the random variable T is asymptotically (i.e. for $n \to \infty$) χ^2-distributed with $k-1$ degrees of freedom.

- If the test is to have significance level α, H_0 is rejected if

$$T \geq \chi^2_{1-\alpha;k-1}$$

  where $\chi^2_{1-\alpha;k-1}$ is the $(1-\alpha)$ quantile of the χ^2-distribution with $k-1$ degrees of freedom.

- The reason that T has only $k-1$ degrees of freedom is because:

$$\sum_{j=1}^{k} Y_j = n$$

- As a rule of thumb, n should be large enough so that $np_j > 5, j = 1, \ldots, k$.

- If this is not the case, the rejection range should be determined by simulation.

Example

We use a sample of size 50 to test whether a die is fair, that is, whether the number of eyes X has the following distribution:

$$P(X = 1) = \ldots = P(X = 6) = \frac{1}{6}$$

A simulation of $N = 100000$ samples yields:

$$\overline{T} = 5.000, \quad S_T^2 = 9.789$$

The 0.95 quantile of the χ^2-distribution with five degrees of freedom is $\chi^2_{0.95;5} = 11.07$, and

$$P(T \geq 11.07) = 0.048$$

The Chi-squared Test for Continuous Data

- Let the sample $X_1, \ldots, X_n$ come from a continuous distribution F.
- We test the hypothesis $H_0 : F(x) = F_0(x)$.
- To this end, we divide the range of values of X into k groups $G_1, \ldots, G_k$.
- Let Y_j be the number of observations in group G_j.
- Under the null hypothesis, $Y_1, \ldots, Y_k$ is multinomially distributed according to $\mathrm{Mu}(n, p_1, \ldots, p_k)$ and $\mathsf{E}[Y_j] = np_j$, with.

$$p_j = P(X \in G_j | H_0)$$

- The test continues as in the discrete case.

The Chi-squared Test

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

The Chi-squared Test

The Kolmogorov-Smirnov Test

Unknown Parameters

- The null hypothesis need not be fully specified. We consider the case where the p_j still depend on unknown parameters $\boldsymbol{\vartheta}$:

$$P(X \in G_j) = p_j(\boldsymbol{\vartheta})$$

- The statistic T is now a function of the unknown parameters:

$$T(\boldsymbol{\vartheta}) = \sum_{j=1}^{k} \frac{(Y_j - np_j(\boldsymbol{\vartheta}))^2}{np_j(\boldsymbol{\vartheta})}$$

- First, the parameters are estimated, by ML estimation or minimization of T:

$$\tilde{\boldsymbol{\vartheta}} = \arg \min_{\boldsymbol{\vartheta}} T(\boldsymbol{\vartheta})$$

- The critical region can be determined according to the following result.

Theorem

If m parameters are estimated from the sample, $T(\tilde{\boldsymbol{\vartheta}})$ is asymptotically χ^2-distributed with $k - 1 - m$ degrees of freedom.

- If the test is to have significance level α, H_0 is rejected if

$$T \geq \chi^2_{1-\alpha;k-1-m}$$

where $\chi^2_{1-\alpha;k-1-m}$ is the $(1-\alpha)$ quantile of the χ^2-distribution with $k - 1 - m$ degrees of freedom.

The Chi-squared Test

Example

Angabe: The number of work-related accidents was surveyed in a large company over a period of 30 weeks. The following shows the number of accidents in each week:

$$\boldsymbol{X} = \{8, 0, 0, 1, 3, 4, 0, 2, 12, 5, 1, 8, 0, 2, 0,$$
$$1, 9, 3, 4, 5, 3, 3, 4, 7, 4, 0, 1, 2, 1, 2\}$$

We test the hypothesis that the observations are Poissonian, i.e. distributed according to $\mathrm{Po}(\lambda)$.

Lösung: The observations are divided into five groups:

group	1	2	3	4	5
X	0	1	2–3	4–5	> 5

The frequencies of the groups are:

$$Y_1 = 6, Y_2 = 5, Y_3 = 8, Y_4 = 6, Y_5 = 5$$

Example (Continuation)

The estimated value for λ is the sample mean:

$$\tilde{\lambda} = 3.1667$$

The expected values of Y_j assuming $H_0 = \mathrm{Po}(\tilde{\lambda})$ are:

j	1	2	3	4	5
$\mathsf{E}[Y_1]$	1.2643	4.0037	13.0304	8.6522	3.0493

The test statistic T is equal to $T = 21.99$. The 99% quantile of the χ^2-distribution with three degrees of freedom is equal to $\chi^2_{0.99;3} = 11.35$. Thus, the hypothesis that the observations are Poisson distributed must be rejected.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests
The Chi-squared Test
The Kolmogorov-Smirnov Test

One Sample

- Let the sample $X_1, \ldots, X_n$ be realizations of a continuous distribution with distribution function F.
- We test the hypothesis $H_0 : F(x) = F_0(x)$.
- The test statistic D_n is the maximum absolute deviation of the empirical distribution function $F_n(x)$ of the sample from the hypothesized distribution function $F_0(x)$:

$$D_n = \max_x |F_n(x) - F_0(x)|$$

- For samples from F_0, the distribution function of $\sqrt{n}D$ for $n \to \infty$ tends to:

$$K(x) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2 x^2}$$

- The quantiles $K_{1-\alpha}$ can be calculated from the asymptotic distribution function.
- The null hypothesis is rejected if

$$\sqrt{n}D_n > K_{1-\alpha}$$

- If parameters of F_0 are estimated before the test, the quantiles are no longer valid.
- In this case, the rejection range must be determined by simulation.

The Kolmogorov-Smirnov Test

D_n, the test statistic of the Kolmogorov-Smirnov test.

The Kolmogorov-Smirnov Test

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Parametric Tests

Non-Parametric Tests

The Chi-squared Test

The Kolmogorov-Smirnov Test

Two Samples

- We test whether two samples of size n or m are from the same distribution F.
- The test size is the maximum absolute difference of the empirical distribution functions:

$$D_{n,m} = \max_x \left| F_n^1(x) - F_m^2(x) \right|$$

- The null hypothesis is rejected if

$$\sqrt{\frac{nm}{n+m}} D_{n,m} > K_{1-\alpha}$$

Part VII

Regression and Linear Models

Overview Part 7

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

Introduction

- Regression analysis examines the dependence of observations on various variables.
- **influence (predictor) variable** (independent variable) $\boldsymbol{x} = (x_1, \ldots, x_r)$.
- **outcome variable** (dependent variable) Y.
- **regression model**:

$$Y = f(\boldsymbol{\beta}, \boldsymbol{x}) + \varepsilon$$

with **regression coefficients** $\boldsymbol{\beta}$ and **error term** ε.

- The objective is to **estimate** $\boldsymbol{\beta}$ using observations $Y_1, \ldots, Y_n,\ n \geq r$.
- One influence variable: single regression; Multiple influence variables: multiple regression.

Introduction

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

- Each observation Y_i has an error term ε_i.
- The error terms need not all have the same distribution, nor need they be independent.
- It is often assumed that the random vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ has a **multivariate normal distribution**.
- However, other distributions of ε are also conceivable.

Section 29: Multidimensional Random Variable

28 Introduction

29 Multidimensional Random Variable
- Basic Terminology
- Marginal Distributions and Conditional Distributions
- The Multivariate Normal Distribution
- Multivariate Error Propagation

30 Simple Regression

31 Multiple Regression

32 Data Reconciliation

Definition (Multidimensional Random Variable)

A mapping $\boldsymbol{X}$:

$$\omega \in \Omega \mapsto \boldsymbol{x} = \boldsymbol{X}(\omega) \in \mathbb{R}^d$$

that assigns a real vector $\boldsymbol{x} \in \mathbb{R}^d$ to each element ω of the sample space Ω is called a d-dimensional random variable.

- Each component of a d-dimensional random variable is itself a random variable.
- Each component can be discrete or continuous.

Definition (distribution function)

If $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a d-dimensional random variable, then its distribution function $F_{\boldsymbol{X}}$ is given by

$$F_{\boldsymbol{X}}(x_1, \ldots, x_d) = P(X_1 \leq x_1 \cap \ldots \cap X_d \leq x_d)$$

Definition (density function)

If $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a d-dimensional discrete random variable, then its density function $f_{\boldsymbol{X}}$ is given by

$$f_{\boldsymbol{X}}(x_1, \ldots, x_d) = P(X_1 = x_1 \cap \ldots \cap X_d = x_d)$$

defined.

Example

The bivariate (two-dimensional) random variable $\boldsymbol{X} = (X_1, X_2)$ assigns the numbers of eyes (i, j) to the outcome (e_i, e_j) of the roll with two dice. If all outcomes are equally probable, $P_{\boldsymbol{X}}$ is given by:

$$P_{\boldsymbol{X}}\big(\{(i,j)\}\big) = \frac{1}{36}$$

The density $f_{\boldsymbol{X}}$ is:

$$f_{\boldsymbol{X}}(x_1, x_2) = . \begin{cases} \frac{1}{36}, & x_1 \in \{1, \ldots, 6\} \cap x_2 \in \{1, \ldots, 6\} \\ 0, & \text{other} \end{cases}$$
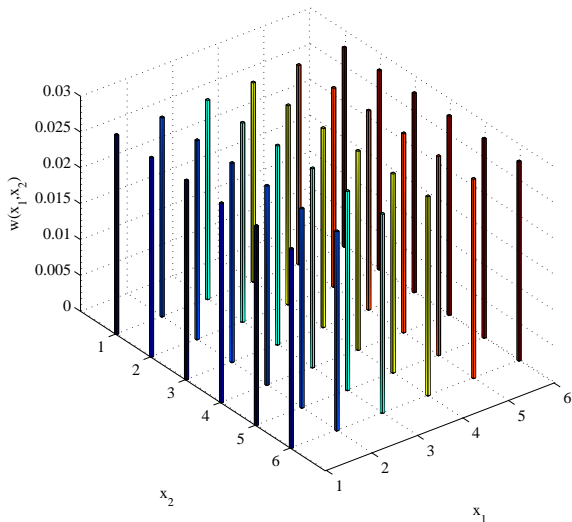
Example (Continuation)

Therefore, the distribution function $F_{\boldsymbol{X}}$ is:

$$F_{\boldsymbol{X}}(x_1, x_2) = P(X_1 \leq x_1 \cap X_2 \leq x_2) = \sum_{i \leq x_1 \cap j \leq x_2} f(i, j)$$

For example, $F_{\boldsymbol{X}}(3, 4) = \sum_{i \leq 3 \cap j \leq 4} \frac{1}{36} = \frac{12}{36} = \frac{1}{3}$.

- Given the countability of the elementary events, they can also be uniquely mapped onto $\mathbb{R}$ by a univariate random variable Y, e.g.:

$$Y : (e_i, e_j) \longrightarrow 6i + j - 6$$

The set of values of Y are the natural numbers between 1 and 36, and P_Y is given by:

$$P_Y\big(\{k\}\big) = \frac{1}{36}, \ 1 \leq k \leq 36$$

Definition (density function)

If $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a d-dimensional continuous random variable, then its density function $f_{\boldsymbol{X}}$ is given by

$$f_{\boldsymbol{X}}(x_1, \ldots, x_d) = \frac{\partial^d F_{\boldsymbol{X}}}{\partial x_1 \ldots \partial x_d}$$

defined.

Definition (moments)

- Expectation: $\mathsf{E}[\boldsymbol{X}] = (\mathsf{E}[X_1] \cdots \mathsf{E}[X_d])$
- Covariance matrix: $\mathsf{Cov}[\boldsymbol{X}]_{ij} = \mathsf{cov}[X_i, X_j]$

- If X_1 and X_2 are two (discrete or continuous) univariate random variables, then $\boldsymbol{X} = (X_1, X_2)$ is a bivariate random variable. The distribution (distribution function, density) of $\boldsymbol{X}$ is also called the **joint distribution** of X_1 and X_2.

- The following problem now arises: can we calculate the distribution of X_1 or X_2 from the joint distribution?

- Let F be the distribution function and f be the density of continuous random variables $\boldsymbol{X} = (X_1, X_2)$. Then the distribution function F_1 of X_1 is given by:

$$F_1(x_1) = P(X_1 \leq x_1) = P(X_1 \leq x_1 \cap -\infty < X_2 < \infty) =$$
$$= \int_{-\infty}^{x_1} \int_{-\infty}^{\infty} f(x_1, x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1$$

- It follows that

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, \mathrm{d}x_2$$

is the density of X_1.

Marginal Distributions and Conditional Distributions

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Multidimensional Random Variable

Basic Terminology

Marginal Distributions and Conditional Distributions

The Multivariate Normal Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

Definition (marginal distribution)

Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate continuous random variable with distribution function F and density f. The distribution of X_1 is called the **marginal distribution** of X_1 with respect to $\boldsymbol{X}$. Its density f_1 is:

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) \, \mathrm{d}x_2.$$

Analogously, if $\boldsymbol{X} = (X_1, X_2)$ is discrete with density f, then the density f_1 of the marginal distribution of X_1 with respect to $\boldsymbol{X}$ is given by:

$$f_1(k_1) = \sum_{k_2} f(k_1, k_2)$$

- The distributions of X_1 and X_2 can be calculated from the joint distribution of X_1 and X_2.

- The reverse process is generally not possible because the joint distribution also contains information about correlations (couplings) between X_1 and X_2.

- Let X_1 and X_2 be two discrete random variables with a joint density $f(k_1, k_2)$ and marginal densities $f_1(k_1)$ and $f_2(k_2)$. Then the conditional probability of the event $X_1 = k_1$ under the condition $X_2 = k_2$ is given by:

$$P(X_1 = k_1 | X_2 = k_2) = \frac{P(X_1 = k_1 \cap X_2 = k_2)}{P(X_2 = k_2)} = \frac{f(k_1, k_2)}{f_2(k_2)}$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Basic Terminology

Marginal Distributions and
Conditional Distributions

The Multivariate Normal
Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

Definition (Conditional Density)

Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate discrete random variable with density $f(k_1, k_2)$ and marginal distribution densities $f_1(k_1)$ and $f_2(k_2)$, respectively. The function $f(k_1|k_2)$, defined by:

$$f(k_1|k_2) = \frac{f(k_1, k_2)}{f_2(k_2)}$$

is called the conditional density of X_1 **given** X_2.

- For fixed k_2, the conditional density is the density of a distribution, the **conditional distribution** of X_1 given $X_2 = k_2$.

- If $\boldsymbol{X} = (X_1, X_2)$ is continuous, then analogously $f(x_1|x_2)$ is defined by:

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} \ (f_2(x_2) \neq 0)$$

- $f(x_1|x_2)$ is for fixed x_2 the density of a distribution, the distribution of X_1 conditioned by $X_2 = x_2$.

- It can easily be verified that $f(x_1|x_2)$ is indeed a density:

$$\int_{-\infty}^{\infty} f(x_1|x_2) \, \mathrm{d}x_1 = \int_{-\infty}^{\infty} \frac{f(x_1, x_2)}{f_2(x_2)} \, \mathrm{d}x_1 = \frac{f_2(x_2)}{f_2(x_2)} = 1$$

and analogously for discrete $\boldsymbol{X}$.

Marginal Distributions and Conditional Distributions

Conditional Density Properties

$$f(x_1, x_2) = f(x_1|x_2) \cdot f_2(x_2)$$

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1|x_2) \cdot f_2(x_2) \, \mathrm{d}x_2$$

and analogously for discrete densities.

Definition (independence of random variables)

If the (unconditional) density of the marginal distribution of X_1 is equal to the density conditioned by X_2, then X_1 and X_2 are called **independent**.

$$X_1 \text{ and } X_2 \text{ independent} \iff f(x_1|x_2) = f_1(x_1).$$

Marginal Distributions and Conditional Distributions

- For independent random variables X_1 and X_2:

$$f(x_1|x_2) = f_1(x_1) \iff f(x_2|x_1) = f_2(x_2)$$
$$\iff f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

and analogously for discrete X.

Properties of Independent Random Variables

- $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$
- $\mathsf{E}[X_1 X_2] = \mathsf{E}[X_1] \cdot \mathsf{E}[X_2]$
- $\mathrm{cov}[X_1, X_2] = \rho[X_1, X_2] = 0$

- Uncorrelated random variables are not necessarily also independent!

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Basic Terminology

Marginal Distributions and
Conditional Distributions

The Multivariate Normal
Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

- If $\boldsymbol{X} = (X_1, \ldots, X_d)$, $d > 2$, then the definitions of the marginal distribution, conditional densities and independence must be generalized accordingly.

- The density $f_{i_1,\ldots,i_m}$ of the marginal distribution of $X_{i_1}, \ldots, X_{i_m}$ is obtained by integrating or summing over all other variables.

- The density of X_i given X_j is given by:

$$f(x_i | x_j) = \frac{f_{i,j}(x_i, x_j)}{f_j(x_j)}$$

where $f_{i,j}(x_i, x_j)$ is the marginal distribution density of X_i, X_j.

- $X_{i_1}, \ldots, X_{i_k}$ are called independent if the densities of their joint distributions are the product of the densities of the marginal distributions of each X_{i_j}.

Example (The Acceptance or Detection Probability)

Let X be a random variable with density $f(x)$. If X takes the value x, then there is a probability $a(x)$ that x is actually observed. One now defines a random variable I that is 1 if x is observed and 0 otherwise. Then I is Bernoulli distributed under the condition $X = x$ according to $\mathrm{Al}(a(x))$:

$$P(I = 1 | X = x) = a(x)$$
$$P(I = 0 | X = x) = 1 - a(x)$$

Therefore, the joint density of X and I is:

$$f(x, 1) = a(x) f(x)$$
$$f(x, 0) = [1 - a(x)] f(x)$$

Example (Continuation)

Because experimentalists can only work with observed quantities, they restrict their sample to the detected events, i.e. they need the density of X under the condition that X is observed:

$$f_A(x) = f(x|I = 1) = \frac{f(x, 1)}{f_2(1)} = \frac{a(x)f(x)}{\int a(x)f(x)\,\mathrm{d}x}$$

As a concrete example, consider the measurement of a lifetime. Let the measurement start at $t_{\min}$ and end at $t_{\max}$. Then $a(t)$ has the following shape:

$$a(t) = \begin{cases} 0, & \text{for } t \leq t_{\min} \\ 1, & \text{for } t_{\min} < t \leq t_{\max} \\ 0, & \text{for } t > t_{\max} \end{cases}$$

Example (Continuation)

For the measured probability density:

$$f_A(t) = \begin{cases} 0, \ t \leq t_{\min} \\ \dfrac{\frac{1}{\tau}\exp(-t/\tau)}{\exp(-t_{\min}/\tau) - \exp(-t_{\max}/\tau)}, \ t_{\min} \leq t < t_{\max} \\ 0, \ t > t_{\max} \end{cases}$$

The denominator $[\exp(-t_{\min}/\tau) - \exp(-t_{\max}/\tau)]$ corrects for those particles that decay before $t_{\min}$ or after $t_{\max}$.

The detection probability $a(t)$ can also have a much more complicated dependence on t. For example, whether or not a decay can be observed at t may depend on the nature and geometric configuration of the decay products.

Statistical Methods
of Data Analysis

W. Waltenberger

The Multivariate Normal Distribution

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Multidimensional Random Variable

Basic Terminology

Marginal Distributions and Conditional Distributions

The Multivariate Normal Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

The multivariate normal distribution $\mathrm{No}(\boldsymbol{\mu}, \mathbf{V})$

- Its density is:

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{|\mathbf{V}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}}\mathbf{V}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$$

Moments

Let $\boldsymbol{X} = (X_1, \ldots, X_d) \sim \mathrm{No}(\boldsymbol{\mu}, \mathbf{V})$. Then holds:

- $\mathsf{E}[\boldsymbol{X}] = \boldsymbol{\mu}$
- $\mathrm{Cov}[\boldsymbol{X}] = \mathbf{V}$
- $\mathbf{V}^{-1}$ is also symmetric and positive definite, and is called a weight or information matrix.

The Multivariate Normal Distribution

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Multidimensional Random Variable

Basic Terminology

Marginal Distributions and Conditional Distributions

The Multivariate Normal Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

Linear Transformations

Let $X \sim \mathrm{No}(\boldsymbol{\mu}, \mathbf{V})$ and $\mathbf{H}$ be a $m \times d$ matrix. Then $Y = \mathbf{H}X \sim \mathrm{No}(\mathbf{H}\boldsymbol{\mu}, \mathbf{H}\mathbf{V}\mathbf{H}^{\mathrm{T}})$.

Marginal Distributions

Each marginal distribution of a normal distribution is itself a normal distribution. The mean and matrix of the marginal distribution are obtained by removing the corresponding columns and rows.

Conditional Distributions

Each conditional distribution of a normal distribution is again a normal distribution.

- If $X \sim \mathrm{No}(\boldsymbol{\mu}, \mathbf{V})$, then $\mathbf{V}$ can be put to diagonal form as a positive definite symmetric matrix using an orthogonal transformation (rotation):

$$\mathbf{U}\mathbf{V}\mathbf{U}^{\mathrm{T}} = \mathbf{D}^2$$

- All diagonal elements of $\mathbf{D}^2$ are positive. The random variable $\boldsymbol{Z} = \mathbf{D}\mathbf{U}(\boldsymbol{X} - \boldsymbol{\mu})$ is then distributed according to a multivariate standard normal distribution, i.e.:

$$\mathrm{E}[\boldsymbol{Z}] = \mathbf{0}, \quad \mathrm{Cov}[\boldsymbol{Z}] = \mathbf{I}$$

The rotation $\mathbf{U}$ is called a **principle axis transformation**.

The Multivariate Normal Distribution

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Basic Terminology

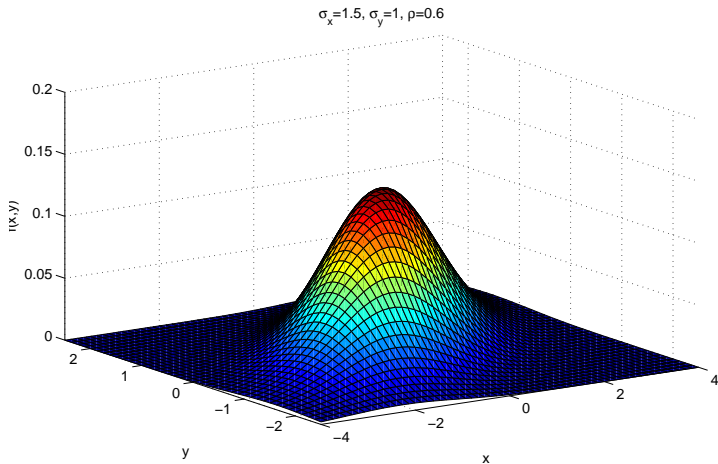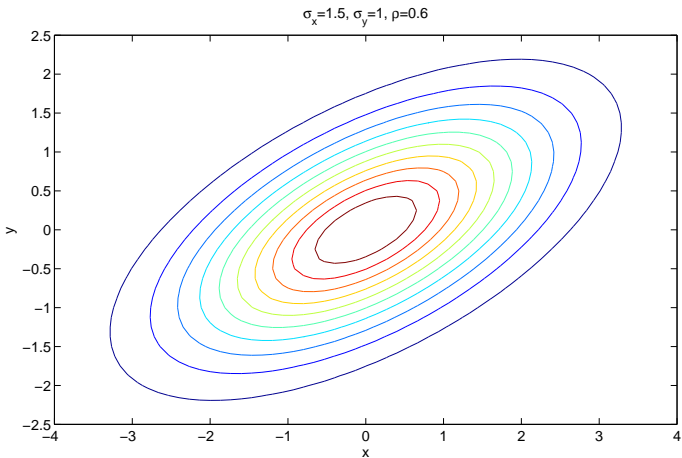Marginal Distributions and
Conditional Distributions

The Multivariate Normal
Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

The bivariate normal distribution

- For $d = 2$ and $\boldsymbol{\mu} = \mathbf{0}$, the density can be written as follows:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2\,\rho\,x\,y}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right]$$

- $\rho = \sigma_{xy}/(\sigma_x\sigma_y)$ is the correlation coefficient. If X and Y are uncorrelated, i.e. $\rho = 0$, it follows:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] = f_1(x) \cdot f_2(y)$$

- Two uncorrelated normally distributed random variables with a joint normal distribution are therefore **independent**.

The Multivariate Normal Distribution

$\sigma_x = 1.5$, $\sigma_y = 1$, $\rho = 0.6$

density function of a bivariate normal distribution

$\sigma_x=1.5$, $\sigma_y=1$, $\rho=0.6$

contour lines of a bivariate normal distribution

- The **conditional density** $f(y|x)$ is given by

$$
f(y|x) = \frac{f(x,y)}{f(x)} =
$$
$$
= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\,\sigma_y^2\,(1-\rho^2)}\left(y - \frac{\rho\,y\,\sigma_y}{\sigma_x}\right)^2\right]
$$

- $Y|X = x$ is therefore a normally distributed random variable with expectation

$$
\mathsf{E}[Y|X] = \rho\,x\,\sigma_y/\sigma_x
$$

- $\mathsf{E}[Y|X]$ is called the **conditional expectation** or **regression of** y **on** x.

- Depending on the sign of ρ, the conditional expectation of Y falls or grows as X grows.
- If $\rho = 1$, X and Y are proportional: $Y = X \, \sigma_y / \sigma_x$.
- The contour lines of the density function are **ellipses**.
- The **principle axis transformation** is that rotation which aligns the ellipses with the axes.
- In the case $d = 2$ it only depends on ρ. If $\rho = 0$, X and Y are already independent, and the rotation angle is equal to 0. If $\rho \neq 0$, the rotation matrix U is equal to

$$U = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \text{ with } \varphi = -\frac{1}{2} \operatorname{arccot} \frac{\sigma_y^2 - \sigma_x^2}{2\rho\sigma_x\sigma_y}$$

Principal axis and regression lines of a bivariate normal distribution

Affine Transformations

- Let X be a random variable with density $f(x)$ and $Y = AX + b$ with regular A.
- The density $g(y)$ of Y is then equal to

$$g(y) = \frac{1}{|A|} f\left(A^{-1}(y - b)\right)$$

- Furthermore, for any A:

$$E[Y] = A \cdot E[X] + b$$
$$Cov[Y] = A \cdot Cov[X] \cdot A^T$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Basic Terminology
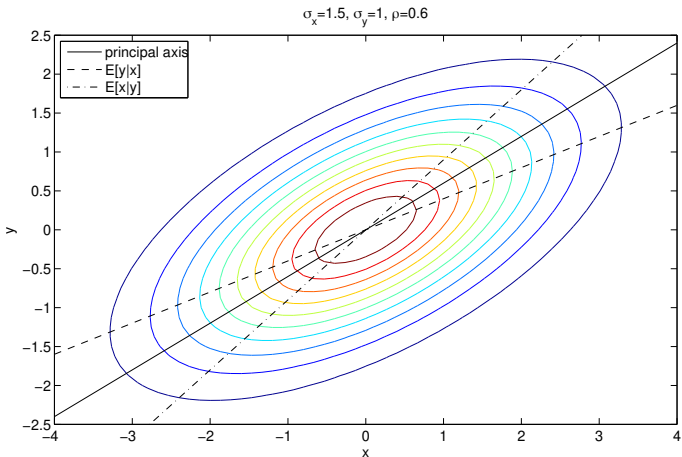
Marginal Distributions and
Conditional Distributions

The Multivariate Normal
Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

Nonlinear Transformations

- Let X be a random variable with density $f(x)$ and $Y = h(X)$.
- If $h(x)$ is bijective, the density $g(y)$ of Y is equal to

$$g(y) = \left| \frac{\partial h^{-1}}{\partial y} \right| f\left(h^{-1}(y) \right)$$

- The expectation and variance of $Y = h(X)$ can be calculated **approximately** using the Taylor expansion of $h(x)$, even if $h(x)$ is not bijective.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Basic Terminology

Marginal Distributions and
Conditional Distributions

The Multivariate Normal
Distribution

Multivariate Error Propagation

Simple Regression

Multiple Regression

Data Reconciliation

- With the expansion point $\boldsymbol{x}_0$ we obtain in linear approximation

$$\boldsymbol{h}(\boldsymbol{x}) \approx \boldsymbol{h}(\boldsymbol{x}_0) + \mathbf{H}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0), \quad \mathbf{H} = \frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}$$

- With the choice $\boldsymbol{x}_0 = \mathsf{E}[\boldsymbol{X}]$ it follows that:

Theorem

$$\mathsf{E}[\boldsymbol{h}(\boldsymbol{X})] \approx \boldsymbol{h}(\mathsf{E}[\boldsymbol{X}])$$

$$\mathsf{Cov}[\boldsymbol{h}(\boldsymbol{X})] \approx \mathbf{H} \cdot \mathsf{Cov}[X] \cdot \mathbf{H}^{\mathrm{T}} \quad \text{(Linear Error Propagation)}$$

- The simplest regression model is a straight line:

$$Y = \alpha + \beta x + \varepsilon, \quad \mathsf{E}[\varepsilon] = 0, \ \mathsf{var}[\varepsilon] = \sigma^2$$

- Let $Y_1, \ldots, Y_n$ be the results for the values $x_1, \ldots, x_n$ of the predictor variable x.
- α and β can be estimated using the principle of least sum of squared errors.
- The following objective function is minimized:

$$SS = \sum_{i=1}^{n}(Y_i - \alpha - \beta x_i)^2$$

- gradient of SS:

$$\frac{\partial SS}{\partial \alpha} = -2\sum_{i=1}^{n}(Y_i - \alpha - \beta x_i), \ \frac{\partial SS}{\partial \beta} = -2\sum_{i=1}^{n} x_i(Y_i - \alpha - \beta x_i)$$

- Zeroing the gradient results in the **normal equations**:

$$\sum_{i=1}^{n} Y_i = n\alpha + \beta \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i Y_i = \alpha \sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} x_i^2$$

- The estimated regression coefficients are:

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i - \bar{x} \sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$\hat{\alpha} = \overline{Y} - \hat{\beta}\bar{x}$$

- It is true that:

$$\mathsf{E}[\hat{\alpha}] = \alpha, \quad \mathsf{E}[\hat{\beta}] = \beta$$

- The following is an unbiased estimator of the variance of the error term:

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^{n} r_i^2$$

with

$$r_i = Y_i - \hat{Y}_i, \quad \hat{Y}_i = \hat{\alpha} + \hat{\beta} x_i$$

- The r_i are called the **residuals** of the regression.

Linear Regression

- The covariance matrix of the estimated regression coefficients is obtained by linear error propagation:

$$
\mathsf{Cov}[\hat{\alpha}, \hat{\beta}] = \sigma^2 \begin{pmatrix} \dfrac{\sum x_i^2}{n\left(\sum x_i^2 - n\bar{x}^2\right)} & -\dfrac{\sum x_i}{n\left(\sum x_i^2 - n\bar{x}^2\right)} \\[2em] -\dfrac{\sum x_i}{n\left(\sum x_i^2 - n\bar{x}^2\right)} & \dfrac{1}{\sum x_i^2 - n\bar{x}^2} \end{pmatrix}
$$

Linear Regression

Example

Data set 4:

$$\bar{x} = 167.60 \qquad r_{xy} = 0.5562$$
$$\bar{y} = 76.16 \qquad \hat{a} = 23.37$$
$$s_x = 8.348 \qquad \hat{b} = 0.3150$$
$$s_y = 4.727$$

Linear Regression

Example (Continuation)

scatter plot with regression line

- The spread of the values Y_i is due to various reasons:
- There are systematic differences due to different values of x.
- In addition, there is the random scattering of the data.

Explained Sum of Squares $\quad SS^* = \displaystyle\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2 = r_{xy}^2 n s_Y^2$

Residual Sum of Squares $\quad SS_R = \displaystyle\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = (1 - r_{xy}^2) n s_Y^2$

Total Sum of Squares $\quad SS_T = \displaystyle\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = n s_Y^2$

Decomposition of Variability

$$SS_T = SS^* + SS_R$$

- The goodness of the regression line can be determined by the **coefficient of determination**:

Coefficient of determination of the regression

$$B = \frac{SS^*}{SS_T} = r_{xy}^2$$

- It indicates what proportion of the total variability can be explained by the correlation of x and Y.

- If $\beta = 0$, the result does not depend on the influence variables at all.
- A test of the null hypothesis $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ is based on the following theorem.

Theorem

If ε is normally distributed, then:

$$\frac{\hat{\alpha} - \alpha}{\hat{\sigma}_{\hat{\alpha}}}, \quad \frac{\hat{\beta} - \beta}{\hat{\sigma}_{\hat{\beta}}}$$

is t-distributed with $n - 2$ degrees of freedom, where

$$\hat{\sigma}_{\hat{\alpha}}^2 = \frac{\hat{\sigma}^2 \sum x_i^2}{n\left(\sum x_i^2 - n\bar{x}^2\right)}, \quad \hat{\sigma}_{\hat{\beta}}^2 = \frac{\hat{\sigma}^2}{\sum x_i^2 - n\bar{x}^2}$$

- The null hypothesis $H_0 : \beta = 0$ is rejected, if the test statistic

$$T = \frac{\hat{\beta}}{\hat{\sigma}_{\hat{\beta}}}$$

is relatively small or relatively large, i.e. if

$$\frac{|\hat{\beta}|}{\hat{\sigma}_{\hat{\beta}}} > t_{1-\alpha/2;n-2}$$

where $t_{p;n-2}$ is the p-quantile of the t-distribution with $n-2$ degrees of freedom.

- An analogous test can be performed for the null hypothesis $H_0 : \alpha = 0$.

Symmetric Confidence Intervals

$$\hat{\alpha} \pm \hat{\sigma}_{\hat{\alpha}} \cdot t_{1-\alpha/2;n-2}, \quad \hat{\beta} \pm \hat{\sigma}_{\hat{\beta}} \cdot t_{1-\alpha/2;n-2}$$

- For $n \gtrsim 50$, the quantiles of the t-distribution can be replaced by quantiles of the standard normal distribution.
- We want to predict the outcome $Y_0 = Y(x_0)$ for a given value x_0 of the influence variable x.
- The expected value of Y_0 is

$$\mathsf{E}[Y_0] = \hat{\alpha} + \hat{\beta}x_0$$

- The variance of $\mathsf{E}[Y_0]$ is obtained by error propagation:

$$\mathsf{var}[\mathsf{E}[Y_0]] = \sigma^2 \left[\frac{1}{n} + \frac{(\bar{x} - x_0)^2}{\sum x_i^2 - n\bar{x}^2} \right]$$

- Since Y_0 scatters around its expected value with variance σ^2, we get:

$$\text{var}[Y_0] = \sigma^2 \left[\frac{n+1}{n} + \frac{(\bar{x} - x_0)^2}{\sum x_i^2 - n\bar{x}^2} \right]$$

- The symmetric prediction interval for Y_0 with certainty α is therefore the same:

$$\hat{\alpha} + \hat{\beta} x_0 \pm t_{1-\alpha/2; n-2} \hat{\sigma} \sqrt{\frac{n+1}{n} + \frac{(\bar{x} - x_0)^2}{\sum x_i^2 - n\bar{x}^2}}$$

prediction bands for $E[Y]$ and Y

- The adequacy of the model can be checked by examining the **studentized residuals** (residual errors).

- The residual r_k has the variance

$$\text{var}[r_k] = \sigma^2 \left[1 - \frac{1}{n} - \frac{(x_k - \bar{x})^2}{\sum x_i^2 - n\bar{x}^2} \right]$$

- The studentized residual is then:

$$r'_k = \frac{r_k}{\hat{\sigma}\sqrt{1 - \frac{1}{n} - \frac{(x_k - \bar{x})^2}{\sum x_i^2 - n\bar{x}^2}}}$$

- It has expectation 0 and variance 1.

regression line and studentized residuals

regression line and studentized residuals

- As an LS estimator, the regression line is not robust, i.e. sensitive to outliers.
- Single outliers in the outcome variable y usually result in a slight distortion of the regression line.
- Single outliers in the influence variable x, so-called leverage points, can cause catastrophic distortions.

Robust Regression

linear regression with outliers

- **LMS (Least Median of Squares)**: Instead of the sum of the error squares, the **median** of the error squares is minimized.
- "Exact fit property": The LMS straight line passes through two data points.
- Calculation is performed combinatorially.
- **LTS (Least Trimmed Squares)**: The sum of a fixed number $h \leq n$ of squared errors is minimized.
- Calculation iterative (FAST-LTS).
- Both methods are authored by P. Rousseeuw.

Robust Regression

- If the relationship between x and Y is not approximately linear, one can try to fit a polynomial.
- The model is then:

$$Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_r x^r + \varepsilon, \quad \mathsf{E}[\varepsilon] = 0, \ \mathsf{var}[\varepsilon] = \sigma^2$$

- Let again $Y_1, \ldots, Y_n$ be the results for the values $x_1, \ldots, x_n$ of the influence variables x.
- In matrix vector notation:

$$\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathsf{E}[\boldsymbol{\varepsilon}] = \mathbf{0}, \quad \mathsf{Cov}[\boldsymbol{\varepsilon}] = \sigma^2 \mathbf{I}$$

with

$$\mathbf{X} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^r \\ 1 & x_2 & x_2^2 & \cdots & x_2^r \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^r \end{pmatrix}$$

- The following target function is minimized:

$$SS = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})^{\mathrm{T}}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$$

- gradient of SS:

$$\frac{\partial SS}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^{\mathrm{T}}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$$

- Zeroing the gradient results in the **normal equations**:

$$\mathbf{X}^{\mathrm{T}}\boldsymbol{Y} = \mathbf{X}^{\mathrm{T}}\mathbf{X}\boldsymbol{\beta}$$

- The solution is:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{Y}$$

- $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$.
- An unbiased estimate of the variance of the error term is given by:

$$\hat{\sigma}^2 = \frac{1}{n - r - 1} \sum_{i=1}^{n} r_i^2$$

with the vector of residuals

$$\boldsymbol{r} = \boldsymbol{Y} - \hat{\boldsymbol{Y}}, \quad \hat{\boldsymbol{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$$

- The covariance matrix of estimated regression coefficients:

$$\mathrm{Cov}[\hat{\boldsymbol{\beta}}] = \sigma^2 \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}$$

- The covariance matrix of residuals $\boldsymbol{r}$:

$$\mathrm{Cov}[\boldsymbol{r}] = \sigma^2 \left[\mathbf{I} - \mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\right]$$

Polynomial Regression

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression
Linear Regression
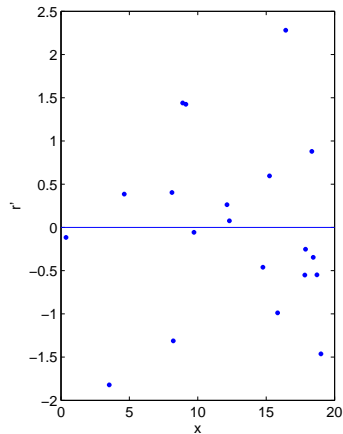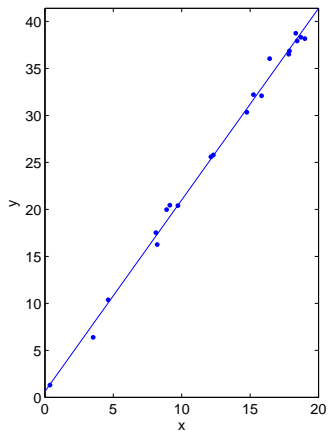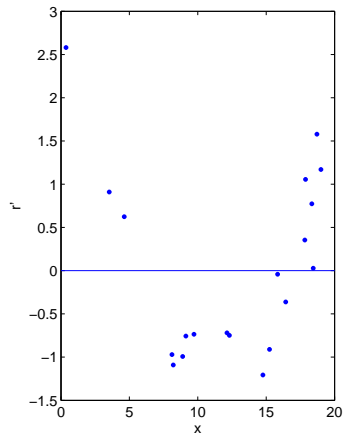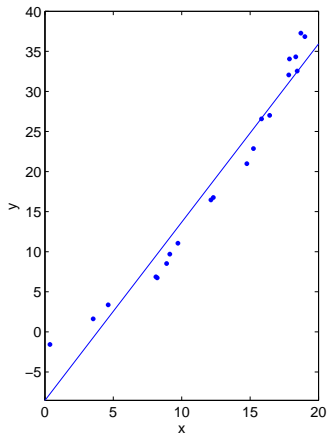Tests, Confidence and
Prediction Intervals
Robust Regression
Polynomial Regression

Multiple Regression

Data Reconciliation
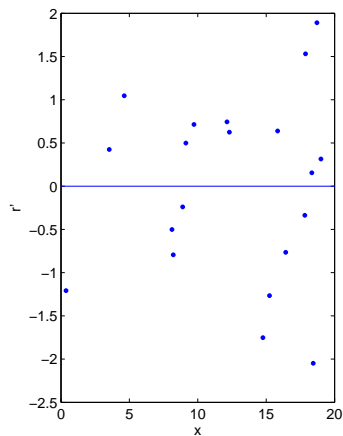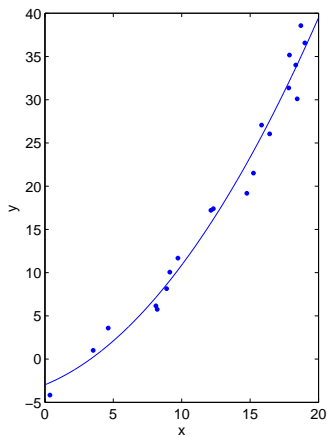
regression parabola and studentized residuals

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

The Linear Model
Estimation, Tests, and
Prediction Intervals
Weighted Regression
Nonlinear Regression

Data Reconciliation

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

The Linear Model

Estimation, Tests, and
Prediction Intervals

Weighted Regression

Nonlinear Regression

Data Reconciliation

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
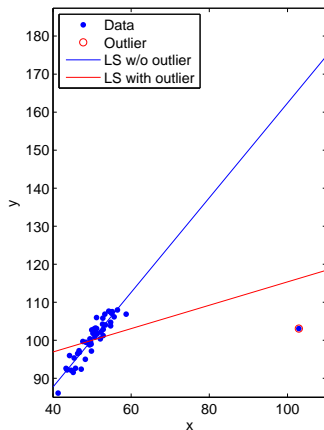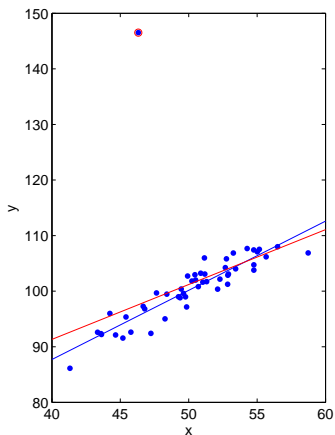Random Variable

Simple Regression

Multiple Regression

The Linear Model

Estimation, Tests, and
Prediction Intervals

Weighted Regression

Nonlinear Regression

Data Reconciliation

The Linear Model

- If the outcome Y depends on several influence variables, the simplest linear regression model is:

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_r x_r + \varepsilon, \quad \mathsf{E}[\varepsilon] = 0, \ \mathsf{var}[\varepsilon] = \sigma^2$$

- Let again $Y_1, \ldots, Y_n$ be the results for n values $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ of the influence variables $\boldsymbol{x} = (x_1, \ldots, x_r)$.

- In matrix vector notation:

$$\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathsf{E}[\varepsilon] = \mathbf{0}, \quad \mathsf{Cov}[\varepsilon] = \sigma^2 \mathbf{I}$$

with

$$\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \cdots & x_{1,r} \\ 1 & x_{2,1} & x_{2,2} & \cdots & x_{2,r} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & x_{n,1} & x_{n,2} & \cdots & x_{n,r} \end{pmatrix}$$

28 Introduction

29 Multidimensional Random Variable

30 Simple Regression

31 Multiple Regression
- The Linear Model
- Estimation, Tests, and Prediction Intervals
- Weighted Regression
- Nonlinear Regression

32 Data Reconciliation

- The following objective function is minimized:

$$SS = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})^{\mathrm{T}}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$$

- gradient of SS:

$$\frac{\partial SS}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^{\mathrm{T}}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$$

- Zeroing the gradient results in the **normal equations**:

$$\mathbf{X}^{\mathrm{T}}\boldsymbol{Y} = \mathbf{X}^{\mathrm{T}}\mathbf{X}\boldsymbol{\beta}$$

- The solution is:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\boldsymbol{Y}$$

- $\hat{\boldsymbol{\beta}}$ is an expectation-trusted estimator of $\boldsymbol{\beta}$.
- The variance of the error term is estimated expectation-trusted by:

$$\hat{\sigma}^2 = \frac{1}{n-r-1} \sum_{i=1}^{n} r_i^2$$

with

$$\boldsymbol{r} = \boldsymbol{Y} - \hat{\boldsymbol{Y}}, \quad \hat{\boldsymbol{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$$

- The covariance matrix of estimated regression coefficients:

$$\mathrm{Cov}[\hat{\boldsymbol{\beta}}] = \sigma^2 \left(\mathbf{X}^{\mathrm{T}}\mathbf{X} \right)^{-1}$$

- The covariance matrix of the residuals $\boldsymbol{r}$:

$$\mathrm{Cov}[\boldsymbol{r}] = \sigma^2 \left[\mathbf{I} - \mathbf{X} \left(\mathbf{X}^{\mathrm{T}}\mathbf{X} \right)^{-1}\mathbf{X}^{\mathrm{T}} \right]$$

- If $\beta_k = 0$, the result does not depend at all on the influence variables x_k.
- A test of the null hypothesis $H_0 : \beta_k = 0$ against $H_1 : \beta_k \neq 0$ is based on the following theorem.

Theorem

If ε is normally distributed, then

$$\frac{\hat{\beta}_k - \beta_k}{\hat{\sigma}_{\hat{\beta}_k}}$$

t-distributed with $n - r - 1$ degrees of freedom, where $\hat{\sigma}^2_{\hat{\beta}_k}$ is the k-th diagonal element of the estimated covariance matrix.

$$\hat{\sigma}^2 \left(\mathbf{X}^{\mathrm{T}} \mathbf{X} \right)^{-1}$$

- The null hypothesis $H_0 : \beta_k = 0$ is rejected, if the test statistic

$$T = \frac{\hat{\beta}_k}{\hat{\sigma}_{\hat{\beta}_k}}$$

is relatively small or relatively large, i.e. if

$$\frac{|\hat{\beta}_k|}{\hat{\sigma}_{\hat{\beta}_k}} > t_{1-\alpha/2;n-r-1}$$

- The symmetric confidence interval for β_k with 95% confidence is:

$$\hat{\beta}_k \pm \hat{\sigma}_{\hat{\beta}_k} \cdot t_{1-\alpha/2;n-r-1}$$

- We now want to predict the outcome $Y_0 = Y(\boldsymbol{x}_0)$ for a given value $\boldsymbol{x}_0 = (x_{01}, \ldots, x_{0r})$ of the influence variable.
- We extend $\boldsymbol{x}_0$ by the value 1: $\boldsymbol{x}_+ = (1, x_{01}, \ldots, x_{0r})$.
- The expected value of Y_0 is then

$$\mathsf{E}[Y_0] = \boldsymbol{x}_+ \cdot \hat{\boldsymbol{\beta}}$$

- The variance of $\mathsf{E}[Y_0]$ is obtained by error propagation:

$$\mathsf{var}[\mathsf{E}[Y_0]] = \sigma^2 \boldsymbol{x}_+ \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1} \boldsymbol{x}_+{}^{\mathrm{T}}$$

- Since Y_0 scatters around its expected value with variance σ^2, we get:

$$\mathrm{var}[Y_0] = \sigma^2 \left[1 + \boldsymbol{x}_+ \left(\mathbf{X}^{\mathrm{T}} \mathbf{X} \right)^{-1} \boldsymbol{x}_+^{\mathrm{T}} \right]$$

- The symmetric prediction interval for Y_0 with certainty α is therefore the same:

$$\boldsymbol{x}_+ \cdot \hat{\boldsymbol{\beta}} \pm t_{1-\alpha/2;n-k-1} \hat{\sigma} \sqrt{1 + \boldsymbol{x}_+ \left(\mathbf{X}^{\mathrm{T}} \mathbf{X} \right)^{-1} \boldsymbol{x}_+^{\mathrm{T}}}$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

The Linear Model

Estimation, Tests, and
Prediction Intervals

Weighted Regression

Nonlinear Regression

Data Reconciliation

28 Introduction

29 Multidimensional Random Variable

30 Simple Regression

31 Multiple Regression
- The Linear Model
- Estimation, Tests, and Prediction Intervals
- Weighted Regression
- Nonlinear Regression

32 Data Reconciliation

- In the general case, the error term can have any covariance matrix:

$$\boldsymbol{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathsf{Cov}[\boldsymbol{\varepsilon}] = \mathbf{V}$$

- If $\mathbf{V}$ is known, the objective function is:

$$SS = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})^{\mathrm{T}}\mathbf{G}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}), \quad \mathbf{G} = \mathbf{V}^{-1}$$

- gradient of SS:

$$\frac{\partial SS}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^{\mathrm{T}}\mathbf{G}(\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta})$$

- Zeroing the gradient results in the **normal equations**:

$$\mathbf{X}^{\mathrm{T}}\mathbf{G}\boldsymbol{Y} = \mathbf{X}^{\mathrm{T}}\mathbf{G}\mathbf{X}\boldsymbol{\beta}$$

- The solution is:

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{G}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{G}\boldsymbol{Y}$$

- Covariance matrix of the estimated regression coefficients:

$$\text{Cov}[\hat{\boldsymbol{\beta}}] = (\mathbf{X}^{\text{T}}\mathbf{G}\mathbf{X})^{-1}$$

- covariance matrix of residuals $\boldsymbol{r}$:

$$\text{Cov}[\boldsymbol{r}] = \mathbf{V} - \mathbf{X}(\mathbf{X}^{\text{T}}\mathbf{G}\mathbf{X})^{-1}\mathbf{X}^{\text{T}}$$

- Tests and prediction intervals can be modified accordingly.

28. Introduction

29. Multidimensional Random Variable

30. Simple Regression

31. Multiple Regression
   - The Linear Model
   - Estimation, Tests, and Prediction Intervals
   - Weighted Regression
   - Nonlinear Regression

32. Data Reconciliation

- In practice, the dependence of the results on the regression coefficients is often nonlinear:

$$\boldsymbol{Y} = \boldsymbol{h}(\boldsymbol{\beta}) + \boldsymbol{\varepsilon}, \quad \mathrm{Cov}[\boldsymbol{\varepsilon}] = \mathbf{V}$$

- If $\mathbf{V}$ is known, the objective function is:

$$SS = [\boldsymbol{Y} - \boldsymbol{h}(\boldsymbol{\beta})]^{\mathrm{T}} \mathbf{G} [\boldsymbol{Y} - \boldsymbol{h}(\boldsymbol{\beta})], \quad \mathbf{G} = \mathbf{V}^{-1}$$

- SS can be minimized using the Gauss-Newton method.
- For this $\boldsymbol{h}$ is linearized at a point $\boldsymbol{\beta}_0$:

$$\boldsymbol{h}(\boldsymbol{\beta}) \approx \boldsymbol{h}(\boldsymbol{\beta}_0) + \mathbf{H}(\boldsymbol{\beta} - \boldsymbol{\beta}_0) = \boldsymbol{c} + \mathbf{H}\boldsymbol{\beta}, \quad \mathbf{H} = \left. \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\beta}} \right|_{\boldsymbol{\beta}_0}$$

- The $\boldsymbol{\beta}$ estimate is:

$$\hat{\boldsymbol{\beta}} = (\mathbf{H}^{\mathrm{T}}\mathbf{G}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{G}(\boldsymbol{Y} - \boldsymbol{c})$$

- $\boldsymbol{h}$ is linearized again at the point $\boldsymbol{\beta}_1 = \hat{\boldsymbol{\beta}}$.
- The procedure is iterated until the estimate does not change significantly.
- Many other methods for minimizing SS are available.

Linear Constraints

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^{\mathrm{T}}$ be a vector of observations of the n unknown quantities $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$:

$$\boldsymbol{Y} = \boldsymbol{\beta} + \boldsymbol{\epsilon}, \mathsf{Cov}[\boldsymbol{\epsilon}] = \mathbf{V},$$

with known $\mathbf{V}$. Moreover, assume that the quantities $\boldsymbol{\beta}$ m must satisfy the following independent linear constraints with r unknown parameters $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_r)^{\mathrm{T}}$:

$$\mathbf{A}\boldsymbol{\beta} + \mathbf{B}\boldsymbol{\vartheta} = \boldsymbol{c}$$

Where $\mathbf{A}$ is a matrix of dimension $m \times n$, $\mathbf{B}$ is a matrix of dimension $m \times r$, and $\boldsymbol{c}$ a $m \times 1$ dimensional column vector.

Statistical Methods of Data Analysis

W. Waltenberger

Introduction

Multidimensional Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

Data Reconciliation

Example (Continuation)

$\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}$ are estimated according to the principle of least sum of squares of error. The objective function is

$$S(\boldsymbol{\beta}) = (\boldsymbol{Y} - \boldsymbol{\beta})^{\mathrm{T}} \mathbf{G} (\boldsymbol{Y} - \boldsymbol{\beta}), \mathbf{G} = \mathbf{V}^{-1}$$

under the condition:

$$\mathbf{A}\boldsymbol{\beta} + \mathbf{B}\boldsymbol{\vartheta} = \boldsymbol{c}.$$

The problem is determined if $r \leq m$ and consistent if $m \leq n + r$. The minimization of S can be done using m Lagrangians $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)^{\mathrm{T}}$. The extended objective function is

$$L(\boldsymbol{\beta}, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) = (\boldsymbol{Y} - \boldsymbol{\beta})^{\mathrm{T}} \mathbf{G} (\boldsymbol{Y} - \boldsymbol{\beta}) + 2\boldsymbol{\lambda}^{\mathrm{T}} (\mathbf{A}\boldsymbol{\beta} + \mathbf{B}\boldsymbol{\vartheta} - \boldsymbol{c})$$

with gradients

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = -2\mathbf{G}(\boldsymbol{Y} - \boldsymbol{\beta}) + 2\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}, \frac{\partial L}{\partial \boldsymbol{\vartheta}} = 2\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}, \frac{\partial L}{\partial \boldsymbol{\lambda}} = 2(\mathbf{A}\boldsymbol{\beta} + \mathbf{B}\boldsymbol{\vartheta} - \boldsymbol{c})$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

Data Reconciliation

Example (Continuation)

Zeroing the gradient leads to the following system of linear equations:

$$\mathbf{G}\boldsymbol{\beta} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{G}\boldsymbol{Y}$$

$$\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}$$

$$\mathbf{A}\boldsymbol{\beta} + \mathbf{B}\boldsymbol{\vartheta} = \boldsymbol{c}$$

With the additional assumption $m \leq n$ and the labels
$\mathbf{G_A} = .(\mathbf{AVA}^{\mathrm{T}})^{-1}, \mathbf{V_B} = (\mathbf{B}^{\mathrm{T}}\mathbf{G_A}\mathbf{B})^{-1}$ we obtain the following
closed-form solution:

$$\hat{\boldsymbol{\beta}} = \boldsymbol{Y} + \mathbf{VA}^{\mathrm{T}}\mathbf{G_A}[\mathbf{I} - \mathbf{BV_B}^{\mathrm{T}}\mathbf{G_A}](\boldsymbol{c} - \mathbf{A}\boldsymbol{Y})$$

$$\hat{\boldsymbol{\vartheta}} = \mathbf{V_B}\mathbf{B}^{\mathrm{T}}\mathbf{G_A}(\boldsymbol{c} - \mathbf{A}\boldsymbol{Y})$$

The joint covariance matrix of $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\vartheta}}$ is:

$$\mathsf{Cov}\left[\begin{pmatrix}\hat{\boldsymbol{\beta}}\\ \hat{\boldsymbol{\vartheta}}\end{pmatrix}\right] = \begin{pmatrix}\mathbf{V} + \mathbf{VA}^{\mathrm{T}}\mathbf{G_A}(\mathbf{BV_B}\mathbf{B}^{\mathrm{T}}\mathbf{G_A} - \mathbf{I})\mathbf{AV} & -\mathbf{VA}^{\mathrm{T}}\mathbf{G_A}\mathbf{BV_B}\\ -\mathbf{V_B}\mathbf{V}^{\mathrm{T}}\mathbf{G_A}\mathbf{AV} & \mathbf{V_B}\end{pmatrix}$$

Example (Continuation)

If the constraints do not contain unknown parameters, the following applies:

$$\hat{\boldsymbol{\beta}} = \boldsymbol{Y} + \mathbf{V}\mathbf{A}^{\mathrm{T}}\mathbf{G_A}(\boldsymbol{c} - \mathbf{A}\boldsymbol{Y}), \mathrm{Cov}[\hat{\boldsymbol{\beta}}] = \mathbf{V} - \mathbf{V}\mathbf{A}^{\mathrm{T}}\mathbf{G_A}\mathbf{A}\mathbf{V}$$

The χ^2 statistic of the balance is defined by:

$$\chi^2 = (\boldsymbol{Y} - \hat{\boldsymbol{\beta}})^{\mathrm{T}}\mathbf{G}(\boldsymbol{Y} - \hat{\boldsymbol{\beta}})$$

If $\boldsymbol{Y}$ is normally distributed with covariance matrix $\mathbf{V} = \mathbf{G}^{-1}$, then χ^2 χ^2-distributed with $m - r$ degrees of freedom.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Multidimensional
Random Variable

Simple Regression

Multiple Regression

Data Reconciliation

Example (balancing angles in triangle)

In a triangle, the angles $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ are measured. Let the values in degrees be $\boldsymbol{Y} = (34.26, 86.07, 59.52)$, the measurement errors $\sigma_1 = 0.1, \sigma_2 = 0.08, \sigma_3 = 0.12$. The measured angles must obey the constraint $\beta_1 + \beta_2 + \beta_3 = 180$.

- $\mathbf{A} = (1, 1, 1), c = 180$, and $\mathbf{V} = \mathrm{diag}(.1^2, .08^2, .12^2)$.

- Furthermore, $\mathbf{G_A} = (\mathbf{AVA}^{\mathrm{T}})^{-1} = 32.468$.

- Thus $\hat{\boldsymbol{\beta}} = \boldsymbol{Y} + \mathbf{VA}^{\mathrm{T}}\mathbf{G_A}(c - \mathbf{AY}) = (34.31, 86.10, 59.59)$

- $\mathrm{Cov}[\hat{\boldsymbol{\beta}}] = \mathbf{V} - \mathbf{VA}^{\mathrm{T}}\mathbf{G_A}\mathbf{AV} = 10^{-2} \cdot \begin{pmatrix} .68 & -.21 & -.47 \\ -.21 & .51 & -.30 \\ -.47 & -.30 & .77 \end{pmatrix}$.

  All estimated values are negatively correlated.

- $\chi^2 = (\boldsymbol{Y} - \hat{\boldsymbol{\beta}})^T \mathbf{G}(\boldsymbol{Y} - \hat{\boldsymbol{\beta}}) = 0.846, p = 0.36$. The errors seem to be realistic.

Nonlinear Constraints

If the constraints are nonlinear, they can be written in the following general form:

$$h(\boldsymbol{\beta}, \boldsymbol{\vartheta}) = \mathbf{0}$$

First, h is linearized at appropriate point $(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0)$:

$$h(\boldsymbol{\beta}, \boldsymbol{\vartheta}) \approx h(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0) + \mathbf{A}_0(\boldsymbol{\beta} - \boldsymbol{\beta}_0) + \mathbf{B}_0(\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0) = \mathbf{A}_0\boldsymbol{\beta} + \mathbf{B}_0\boldsymbol{\vartheta} - \boldsymbol{c}_0$$

with

$$\mathbf{A}_0 = \frac{\partial h}{\partial \boldsymbol{\beta}}, \mathbf{B}_0 = \frac{\partial h}{\partial \boldsymbol{\vartheta}}, \boldsymbol{c}_0 = \mathbf{A}_0\boldsymbol{\beta}_0 + \mathbf{B}_0\boldsymbol{\vartheta}_0 - h(\boldsymbol{\beta}_0, \boldsymbol{\vartheta}_0).$$

The estimation follows as in the linear case. Then h is estimated at the point $\boldsymbol{\beta}_1, \boldsymbol{\vartheta}_1 = \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\vartheta}}$ is re-linearized. It is repeated until the constraints are satisfied well enough.

Part VIII

Bayes Statistics

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

- In Bayesian statistics probabilities are interpreted as rational assessments or degrees of belief about the true state of the world.
- Both data and unknown parameters of distributions are considered as random variables.
- Prior information about the unknown parameters is summarized by the **prior distribution**.
- The information contained in the data is used to update the information expressed by the prior distribution. Specifically, application of Bayes' theorem produces the posterior distribution of the parameters, conditional on the observations.
- The posterior distribution can be used for estimation of parameters, calculation of credible intervals, for testing and forecasting, for model choice etc.

- Choice of the prior introduces a subjective component into data analysis.
- This can have a noticeable effect in particular for small samples.
- There are ways of constructing priors that minimize this effect.
- For large samples the effect of the prior is usually negligible.
- In this case, Bayesian analysis gives approximately the same results as frequentist methods.
- The posterior contains more information than frequentist point and interval estimators.

- The **sample space** $\mathcal{Y}$ is the set of all possible samples $\boldsymbol{y}$.

- The **parameter space** Θ is the set of all possible values of the parameters ϑ.

- The **prior density** $\pi(\vartheta)$ reflects our belief whether a value ϑ describes the true distribution of the data.

- The **likelihood function** $p(\boldsymbol{y}|\vartheta)$ gives the probability of observing data $\boldsymbol{y}$ if ϑ is the true parameter.

- After observation of $\boldsymbol{y}$ our belief about ϑ can be updated, by using Bayes' theorem to compute the **posterior density**:

$$p(\vartheta|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\vartheta)\pi(\vartheta)}{\int_{\Theta} p(\boldsymbol{y}|\vartheta)\pi(\vartheta)\,\mathrm{d}\vartheta}$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

- The posterior distribution describes our belief about ϑ given the observation $\boldsymbol{y}$.

- If new data become available, the posterior can be used as again as the prior.

- If the normalizing integral cannot be computed analytically, one has to resort to numerical or Monte-Carlo methods.

- Common point estimators of ϑ are the **posterior mean**, the **posterior mode**, and the **posterior median** of $p(\vartheta|\boldsymbol{y})$.

- In addition, **credible intervals** and **HPD intervals** can be constructed from the posterior density.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

33 Introduction

34 Prior distributions

35 Binomially Distributed Data

36 Poisson Distributed Data

37 Normally Distributed Data

38 Exponentially Distributed Data

39 Markov Chain Monte Carlo

Prior distributions

- The choice of the prior can have a large effect for small samples.
- We distinguish between **informative** and **non-informative** priors.
- An informative prior describes actual information about ϑ. The information can be objective or subjective.
- A non-informative prior describes the absence of such information.
- In all cases the sensitivity of the result on the prior has to be studied.

- The prior does not have to be normalized, as the normalizing constant cancels out in Bayes' theorem.
- Priors that can be normalized are called **proper**; otherwise they are called **improper**.
- An improper prior **may** lead to a proper posterior.
- The choice of the prior is frequently based on computational convenience.
- For some types of likelihood functions there exist priors that give a posterior from the same family of distributions.
- Such priors are called **conjugate** priors.
- In a few cases a non-informative prior is also a conjugate prior.

Prior distributions

- There are several proposals for choosing a non-informative prior:

  - Principle of **maximal entropy**
  - Invariance under parameter transformations: **Jeffrey's prior**
  - Minimal influence of the prior on the posterior: **Reference prior**

- For univariate ϑ Jeffrey's prior and Reference prior are identical in most cases; this is no longer true in the multivariate case.

Prior distributions

Continuous distributions with maximal entropy

- $E[X]$ and $\text{var}[X]$ fixed: normal distribution
- $X \geq 0$, $E[\ln X]$ and $\text{var}[\ln X]$ fixed: lognormal distribution
- $X \geq 0$, $E[X]$ and $E[\ln X]$ fixed: Gamma distribution
- $X \geq 0$, $E[X]$ fixed: exponential distribution
- $X \in [a, b]$: uniform distribution in $[a, b]$

Discrete distributions with maximal entropy

- $X \in \{1, \ldots, n\}$: uniform distribution on $\{1, \ldots, n\}$
- $X \in \mathbb{N}$, $E[X]$ fixed: geometric distribution

Prior distributions

- **Jeffrey's prior** is constructed such that it is invariant under a transformation of ϑ.

- Let $\tau = h(\vartheta)$ and $\pi_{\mathrm{J}}(\vartheta)$ Jeffrey's prior in ϑ.

- The transformed prior in τ is equal to

$$\pi(\tau) = \pi_{\mathrm{J}}(\vartheta) \left| \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} \right|$$

- The transformed Fisher information is equal to

$$I(\tau) = I(\vartheta) \left(\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} \right)^2$$

- Jeffrey's prior is therefore chosen as

$$\pi_{\mathrm{J}}(\vartheta) \propto \sqrt{I(\vartheta)}$$

Statistical Methods
of Data Analysis

W. Waltenberger

33 Introduction

34 Prior distributions

35 Binomially Distributed Data

36 Poisson Distributed Data

37 Normally Distributed Data

38 Exponentially Distributed Data

39 Markov Chain Monte Carlo

- We consider a Bernoulli experiment repeated n times with probability of success ϑ.
- The number Y of successes is then binomially distributed according to $\mathrm{Bi}(n, \vartheta)$.
- We want to obtain a statement about the success probability ϑ from an observation y.
- To do this, we need an a-priori distribution of ϑ.
- The maximum entropy principle yields the uniform distribution $\mathrm{Un}(0, 1) = \mathrm{Be}(1, 1)$ as the a-priori distribution:

$$\pi(\vartheta) = I_{[0,1]}(\vartheta)$$

- The likelihood function is equal to

$$p(y|\vartheta) = \binom{n}{y} \vartheta^y (1 - \vartheta)^{n-y}$$

- Bayes' theorem provides the a-posteriori density:

$$p(\vartheta|y) \propto \vartheta^y (1-\vartheta)^{n-y} I_{[0,1]}(\vartheta)$$

- Since the a-posteriori density is proportional to the density of the beta distribution $\mathrm{Be}(y+1, n-y+1)$, it must be identical to it.

- The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\vartheta|y] = \frac{y+1}{n+2}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\vartheta} = \frac{y}{n}$$

and thus the maximum likelihood estimator of ϑ.

Binomially Distributed Data

a-priori and a-posteriori densities of ϑ with n=20

Binomially Distributed Data

Statistical Methods
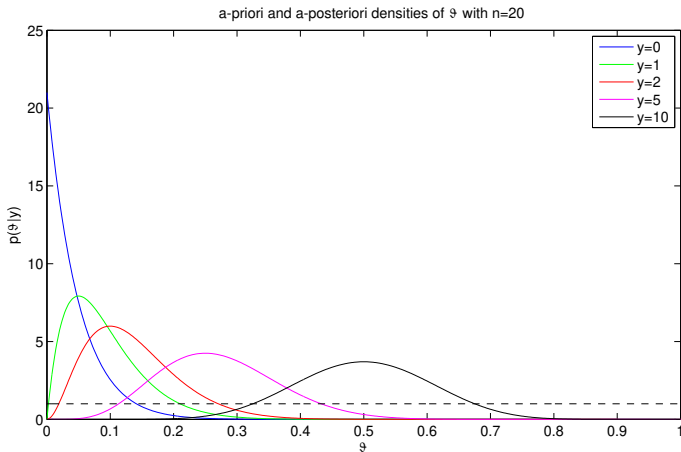of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed Data

Poisson Distributed Data

Normally Distributed Data

Exponentially Distributed Data

Markov Chain Monte Carlo

- The Fisher information of an observation is equal to

$$I(\vartheta) = \frac{1}{\vartheta(1-\vartheta)}$$

- Jeffrey's prior is thus $\mathrm{Be}(0.5, 0.5)$.
- Bayes' theorem again provides the a-posteriori density:

$$p(\vartheta|y) \propto \vartheta^{y-0.5}(1-\vartheta)^{n-y-0.5}I_{[0,1]}(\vartheta)$$

- Since the a-posteriori density is proportional to the density of the beta distribution $\mathrm{Be}(y + 0.5, n - y + 0.5)$, it must be identical to it.
- The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\vartheta|y] = \frac{y + 0.5}{n + 1}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\vartheta} = \frac{y - 0.5}{n - 1}$$

Binomially Distributed Data

a-prior and a-posteriori densities of ϑ with n=20
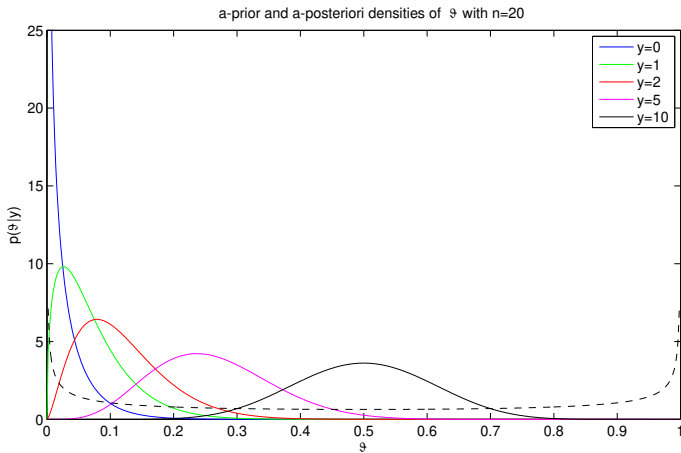
- If prior information about ϑ is available, it can be included via a suitable a-priori density.
- It is computationally the simplest if the a-priori distribution is also a beta distribution.
- Let

$$\pi(\vartheta) = \frac{\vartheta^{a-1}(1-\vartheta)^{b-1}}{B(a, b)}$$

- Then the a-posteriori density is equal to

$$p(\vartheta|y) \propto \vartheta^{y}(1-\vartheta)^{n-y}\vartheta^{a-1}(1-\vartheta)^{b-1}$$
$$= \vartheta^{y+a-1}(1-\vartheta)^{n-y+b-1}$$

- The a-posteriori distribution is again a beta distribution, namely $\mathrm{Be}(y + a, n - y + b)$.

- Obviously, an a-priori beta distribution with a binomial likelihood function again gives an a-posteriori beta distribution.
- Therefore, the beta distribution is the **conjugate** a-priori distribution to the binomial distribution.
- The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\vartheta|y] = \frac{a + y}{a + b + n}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\vartheta} = \frac{a + y - 1}{a + b + n - 2}$$

- The expected value of the a-posteriori distribution can be written as a weighted average of a-priori information and data:

$$
\begin{aligned}
\mathsf{E}[\vartheta|y] &= \frac{a+y}{a+b+n} = \frac{a+b}{a+b+n}\frac{a}{a+b} + \frac{n}{a+b+n}\frac{y}{n} \\
&= \frac{a+b}{a+b+n} \times \text{a-priori expectation} \\
&\quad + \frac{n}{a+b+n} \times \text{average of data}
\end{aligned}
$$

- a and b can be interpet as "a-priori data":

a	number of successes a-priori
b	number of failures a-priori
$a+b$	number of attempts a-priori

- If the number of trials a-priori is set equal to 0, we get **Haldane's prior**:

$$\pi_{\mathrm{H}=} \frac{1}{\vartheta(1-\vartheta)}$$

- Haldane's prior can be interpreted as $\mathrm{Be}(0,0)$, but is improper.
- The a-posteriori mean is then equal to the ML estimator.
- Paradox: The a-priori distribution without any prior information gives the values $\vartheta = 0$ and $\vartheta = 1$ the highest probability!

a–priori and a–posteriori densities of ϑ with n=10, y=3

Statistical Methods
of Data Analysis

W. Waltenberger
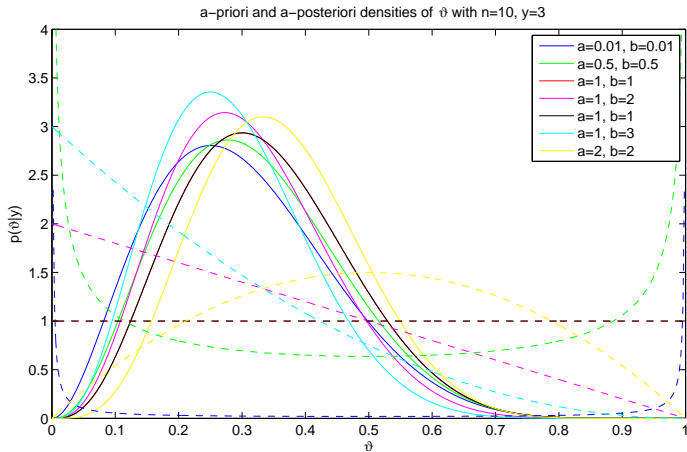
Introduction

Prior distributions
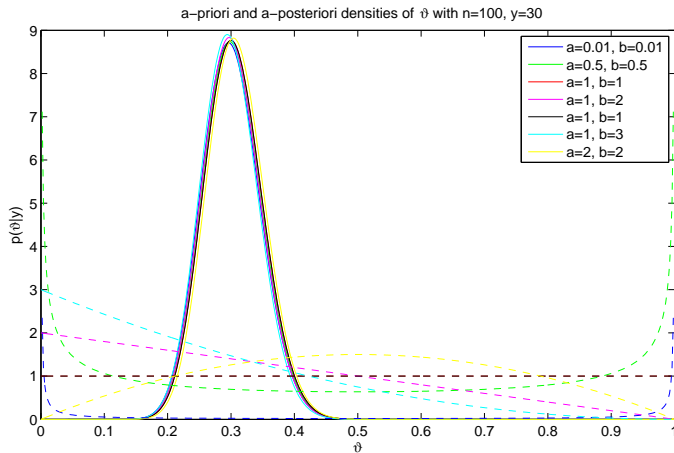
Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Binomially Distributed Data

a–priori and a–posteriori densities of ϑ with n=100, y=30

Legend:
- a=0.01, b=0.01
- a=0.5, b=0.5
- a=1, b=1
- a=1, b=2
- a=1, b=1
- a=1, b=3
- a=2, b=2

- We now want to construct ranges of the parameter space Θ that contain the true value of ϑ with a certain confidence $1 - \alpha$.
- Such a range is called a **credible (or credibility) region**. It is usually an interval (credible interval).
- The simplest construction of a credibile interval $[\vartheta_1(y), \vartheta_2(y)]$ uses the quantiles of the a-posteriori distribution.
- The symmetric credible interval is:

$$\vartheta_1(y) = q_{\alpha/2}, \quad \vartheta_1(y) = q_{1-\alpha/2}$$

where q_p is the p-quantile of the a-posteriori distribution $p(\vartheta|y)$.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Binomially Distributed Data

Example

Let $y = 4$ be the number of successes in $n = 20$ independent
alternative trials with probability of success ϑ. With the uniform
distribution as an a-priori distribution, the a-posteriori distribution of ϑ
is a $\mathrm{Be}(5, 17)$ distribution. The symmetric credible interval with
$1 - \alpha = 0.95$ then has the limits

$$\vartheta_1(y) = \beta_{0.025;5,17} = 0.0822$$
$$\vartheta_2(y) = \beta_{0.975;5,17} = 0.4191$$

The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\vartheta|y] = \frac{5}{22} = 0.2273$$

The mode of the a-posteriori distribution is

$$\hat{\vartheta} = \frac{4}{20} = 0.2$$

a–posteriori density of ϑ with n=20, y=4

Statistical Methods
of Data Analysis
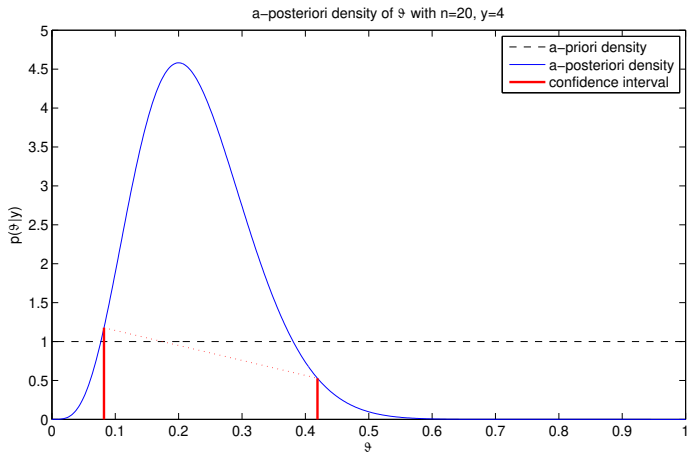
W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Binomially Distributed Data

Example (Continuation)

Using Jeffrey's prior, the a-posteriori distribution of ϑ is a $\mathrm{Be}(4.5, 16.5)$ distribution. The symmetric credible interval with $1 - \alpha = 0.95$ then has the limits

$$\vartheta_1(y) = \beta_{0.025;4.5,16.5} = 0.0715$$
$$\vartheta_2(y) = \beta_{0.975;4.5,16.5} = 0.4082$$

The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\vartheta|y] = \frac{4.5}{21} = 0.2143$$

The mode of the a-posteriori distribution is the same

$$\hat{\vartheta} = \frac{3.5}{19.5} = 0.1795$$

Binomially Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger
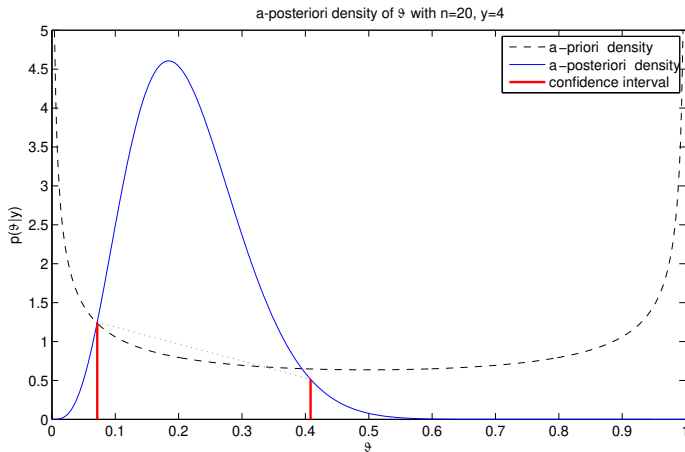
Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

a-posteriori density of ϑ with n=20, y=4

- The symmetric credible interval contains values of ϑ that have lower a-posteriori probabilities than some points outside the interval.

- A range in which all values of ϑ have higher a-posteriori probability than all values outside the range is called a high-posterior density range, or **HPD** range for short.

- If the a-posteriori density is unimodal, the HPD range is an HPD interval.

- In this case, the ϑ_1, ϑ_2 bounds of the HPD interval are obtained as the solution of the system of equations:

$$p(\vartheta_2|y) - p(\vartheta_1|y) = 0$$
$$P(\vartheta_2|y) - P(\vartheta_1|y) = 1 - \alpha$$

Here $P(\vartheta|y)$ is the a-posteriori distribution function.

- The system of equations must be solved numerically.

Example (Continuation)

The HPD interval with $1 - \alpha = 0.95$ has the limits

$$\vartheta_1(y) = 0.06921, \quad \vartheta_1(y) = 0.3995$$

With a length of 0.3303, it is shorter than the symmetric credible interval, which has a length of 0.3369.

Binomially Distributed Data

Statistical Methods
of Data Analysis

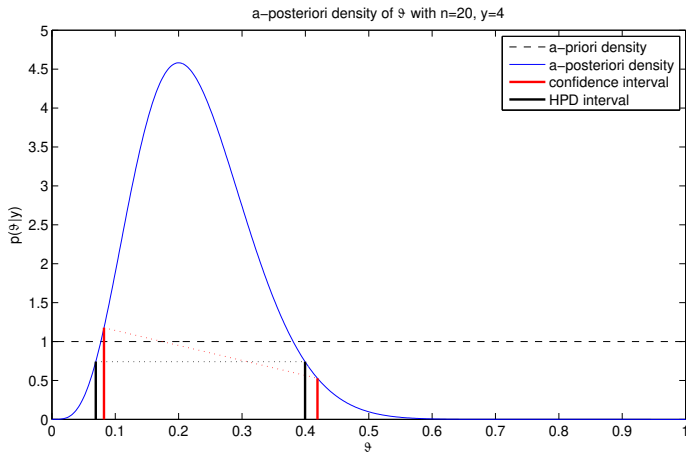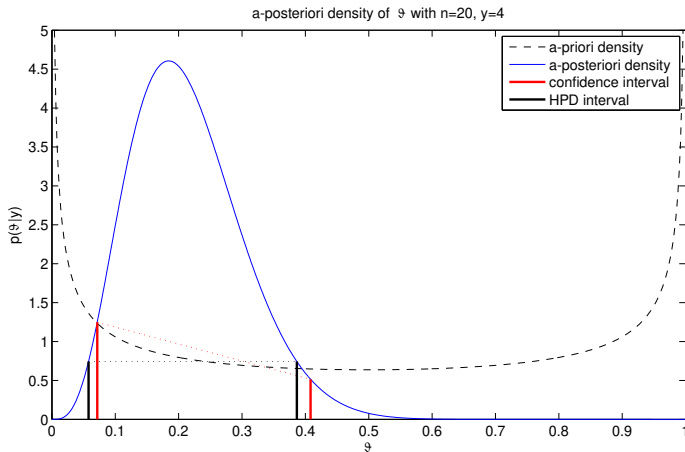W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

a–posteriori density of ϑ with n=20, y=4

Binomially Distributed Data

a-posteriori density of ϑ with n=20, y=4

Example

An alternative trial is repeated $n = 20$ times and does not yield a single success ($k = 0$). What can be said about the probability of success ϑ?

- With a-priori density $\pi(\vartheta) = 1$, the a-posteriori distribution is $\mathrm{Be}(1, 21)$. The mode is equal to 0, and the expected value is equal to 0.0455. The HPD interval with $1 - \alpha = 0.95$ is equal to $[0, 0.1329]$.

- With Jeffrey's prior, the a-posteriori distribution is $\mathrm{Be}(0.5, 20.5)$. The mode is equal to 0, and the expected value is equal to 0.0238. The HPD interval with $1 - \alpha = 0.95$ is equal to $[0, 0.0905]$.

- If it is known that ϑ is rather small, e.g. $\mathrm{Be}(0.5, 5)$ can be chosen as a-priori distribution. The a-posteriori distribution is then $\mathrm{Be}(0.5, 25)$. The mode is equal to 0, the expected value is equal to 0.0196, and the HPD interval is equal to $[0, 0.0747]$.

Example (Continuation)

- The likelihood estimator of ϑ is equal to 0.

- The one-sided Clopper-Pearson confidence interval is equal to $[0, 0.1391]$.

- The approximation by normal distribution is only useful with the correction according to Agresti-Coull, otherwise the confidence interval shrinks to zero. The estimate is $\hat{\vartheta} = 0.0833$, the symmetric confidence interval is $[-0.0378, 0.2045]$, the left-hand confidence interval is $[0, 0.1850]$.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

- Observing a Poisson process yields the values $\boldsymbol{y} = y_1, \ldots, y_n$.
- We want to obtain an estimate of the intensity λ of the process from the data.
- The likelihood function is:

$$p(\boldsymbol{y}|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \propto \lambda^{\sum y_i} e^{-n\lambda}$$

- It depends only on $s = \sum y_i$.
- As non-informative a-priori distributions we may consider:
  - The improper density $\pi(\lambda) = 1$
  - Jeffrey's prior $\pi_{\mathrm{J}}(\lambda) = \lambda^{-1/2}$, also improper

Poisson Distributed Data

- With $\pi(\lambda) = 1$, the a-posteriori density is proportional to the likelihood function:

$$p(\lambda|s) \propto \lambda^s e^{-n\lambda}$$

- Since the a-posteriori density is proportional to the density of the gamma distribution $\mathrm{Ga}(s+1, 1/n)$, it must be identical to it:

$$p(\lambda|s) = \frac{\lambda^s e^{-n\lambda}}{n^{-(s+1)}\Gamma(s+1)}$$

- The expected value of the a-posteriori distribution is equal to:

$$\mathsf{E}[\lambda|s] = \frac{s+1}{n}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\lambda} = \frac{s}{n}$$

  and thus the maximum likelihood estimator of λ.

Poisson Distributed Data

Poisson Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

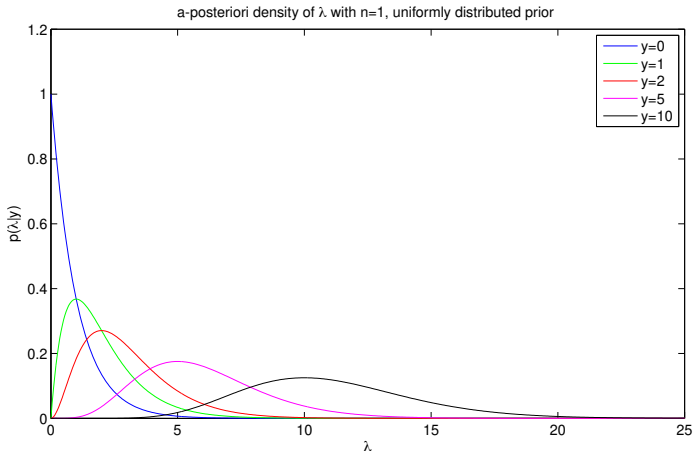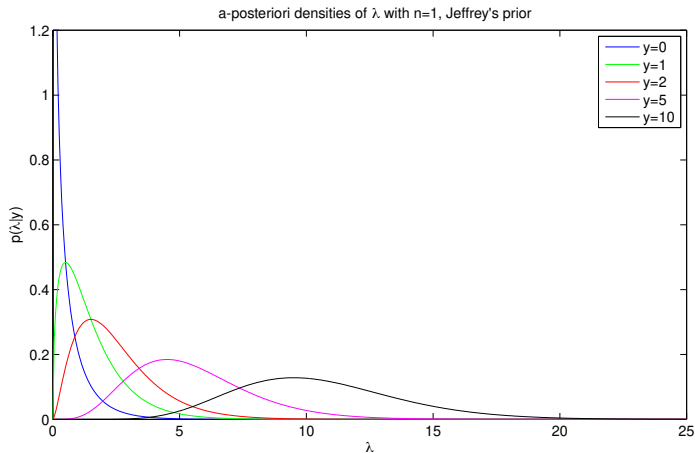Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

- With $\pi(\lambda) = \lambda^{-1/2}$ the a-posteriori density is:

$$p(\lambda|s) \propto \lambda^{s-0.5} e^{-n\lambda}$$

- Since the a-posteriori density is proportional to the density of the gamma distribution $\mathrm{Ga}(s + 0.5, 1/n)$, it must be identical to it:

$$p(\lambda|s) = \frac{\lambda^{s-0.5} e^{-n\lambda}}{n^{-(s+0.5)} \Gamma(s + 0.5)}$$

- The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\lambda|s] = \frac{s + 0.5}{n}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\lambda} = \frac{s - 0.5}{n}$$

a-posteriori densities of λ with n=1, Jeffrey's prior

Poisson Distributed Data

- If prior information about λ is available, it can be included by a suitable a-priori density.
- It is computationally the simplest if the a-priori distribution is a gamma distribution.
- Let

$$\pi(\lambda) = \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}$$

This is the density of the gamma distribution $\mathrm{Ga}(a, 1/b)$.

- Then the a-posteriori density is equal to

$$p(\lambda|s) \propto \lambda^s e^{-n\lambda} \lambda^{a-1} e^{-b\lambda}$$
$$= \lambda^{s+a-1} e^{-(b+n)\lambda}$$

- The a-posteriori distribution is the gamma distribution $\mathrm{Ga}(a + s, 1/(b + n))$. Thus, the gamma distribution is conjugate to the Poisson distribution.

Statistical Methods
of Data Analysis

W. Waltenberger
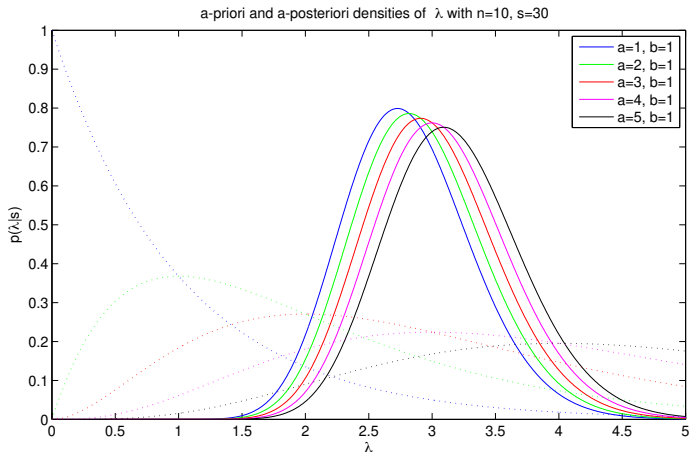
Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Poisson Distributed Data

a-priori and a-posteriori densities of λ with n=10, s=30

- The expected value of the a-posteriori distribution is equal to

$$E[\lambda|s] = \frac{a+s}{b+n}$$

- The mode of the a-posteriori distribution is equal to

$$\hat{\lambda} = \frac{a+s-1}{b+n}$$

- The expected value of the a-posteriori distribution can be written as a weighted average of a-priori information and data:

$$E[\lambda|s] = \frac{a+s}{b+n} = \frac{b}{b+n}\frac{a}{b} + \frac{n}{b+n}\frac{s}{n}$$

$$= \frac{b}{b+n} \times \text{a-priori expectation}$$

$$+ \frac{n}{b+n} \times \text{average of data}$$

Poisson Distributed Data

Statistical Methods of Data Analysis
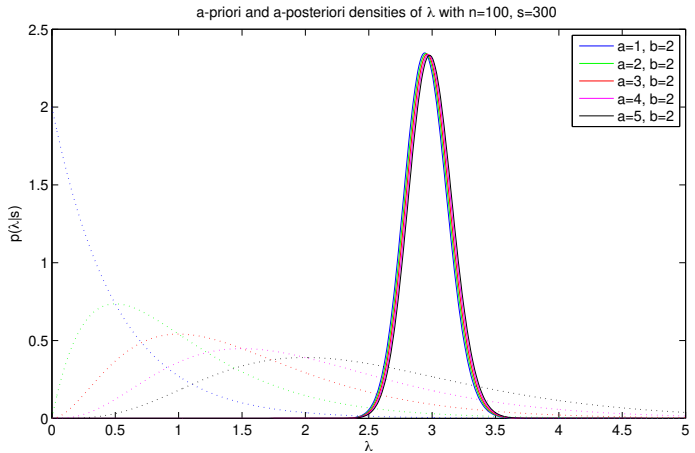
W. Waltenberger

Introduction

Prior distributions

Binomially Distributed Data

Poisson Distributed Data

Normally Distributed Data

Exponentially Distributed Data

Markov Chain Monte Carlo

- The a-priori parameters a and b can be interpetted as follows:

  a sum of data a-priori

  b number of data a-priori

- If $b \ll n$, dominate the data:

$$\mathsf{E}[\lambda|s] \approx \frac{s}{n}$$

Poisson Distributed Data

a-priori and a-posteriori densities of λ with n=100, s=300

- Credible intervals $[\lambda_1(s), \lambda_2(s)]$ can be easily constructed using the quantiles of the a-posteriori gamma distribution.
- The symmetric credible interval is equal to

$$\left[\gamma_{\alpha/2;a+s,1/(b+n)}, \gamma_{1-\alpha/2;a+s,1/(b+n)}\right]$$

- The one-sided credible intervals are

$$[0, \gamma_{\alpha;a+s,1/(b+n)}] \quad \text{and} \quad [\gamma_{\alpha;a+s,1/(b+n)}, \infty]$$

- HPD ranges can be determined using numerical methods.

Example

You measure the background radiation in a laboratory over a period of 20 seconds. The count values are

$$6, 2, 6, 1, 6, 8, 5, 3, 8, 4, 2, 5, 7, 8, 5, 4, 7, 9, 4, 4$$

Their sum is equal to $s = 104$. Using Jeffrey's prior, the a-posteriori distribution is the gamma distribution $\mathrm{Ga}(104.5, 0.05)$. Its expectation is 5.225, and its mode is 5.1750. The symmetric credible interval at $1 - \alpha = 0.95$ is $[4.2714, 6.2733]$, the HPD interval is $[4.2403, 6.2379]$. Since the distribution is almost symmetric, the two intervals are practically the same length.

Poisson Distributed Data

a-posteriori density of λ with n=20, s=104, Jeffrey's prior

Statistical Methods
of Data Analysis

W. Waltenberger
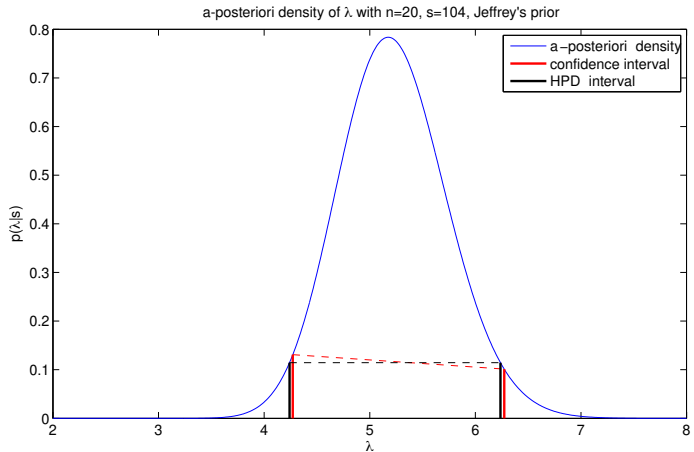
Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Poisson Distributed Data

Example

An observation from a Poisson distribution has the value $k = 0$. What can be said about the mean λ?

- With a-priori density $\pi(\lambda) = 1$, the a-posteriori distribution is $\mathrm{Ga}(1, 1)$. The mode is 0, and the mean is 1. The HPD interval with $1 - \alpha = 0.95$ is $[0, 2.9957]$.

- With Jeffrey's prior, the a-posteriori distribution is $\mathrm{Ga}(0.5, 1)$. The mode is 0, and the mean is 0.5. The HPD interval with $1 - \alpha = 0.95$ is $[0, 1.9207]$.

- If it is known that λ is significantly smaller than 1, for example, $\mathrm{Ga}(0.5, 0.5)$ can be chosen as the a-priori distribution. The a-posteriori distribution is then $\mathrm{Ga}(0.5, 0.6667)$. The mode is 0, the mean is 0.3333, the HPD interval is $[0, 1.2805]$.

- The likelihood estimator of λ is 0.

- The lower confidence interval is $[0, 2.9957]$, so it is identical to the HPD interval with improper a-priori density.

- We consider a normally distributed sample $\boldsymbol{y} = (y_1, \ldots, y_n)$.
- We want to obtain an estimate of the mean and variance of the distribution from which the data are drawn.
- We first assume that the variance σ^2 is known.
- Then the likelihood function is:

$$p(\boldsymbol{y}|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_i - \mu)^2}{2\,\sigma^2}\right]$$

- In the absence of any prior information about the actual value of μ, we choose the improper a-priori density $\pi(\mu) = 1$, which is also Jeffrey's prior.

- Then the a-posteriori density is equal to

$$p(\mu|\boldsymbol{y}, \sigma^2) \propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(y_i - \mu)^2}{2\,\sigma^2}\right]$$

$$\propto \exp\left[-\frac{n(\mu - \bar{y})^2}{2\,\sigma^2}\right]$$

- Since the a-posteriori density is proportional to the density of the normal distribution $\mathrm{No}(\bar{y}, \sigma^2/n)$, it must be identical to it.
- The expected value of the a-posteriori distribution is equal to

$$\mathsf{E}[\mu|\boldsymbol{y}] = \bar{y}$$

and thus the maximum likelihood estimator of μ.

- The mode is also equal to $\bar{y}$.

- If prior information about μ is available, it can be included by a suitable a-priori density.
- The computationally simplest case is if the a-priori distribution is also a normal distribution.
- Let

$$\pi(\mu|\sigma^2) = \frac{1}{\sqrt{2\pi}\tau_0} \exp\left[-\frac{(\mu - \mu_0)^2}{2\,\tau_0^2}\right]$$

- Then the a-posteriori density is equal to

$$p(\mu|\boldsymbol{y}, \sigma^2) \propto \exp\left[-\frac{n(\mu - \bar{y})^2}{2\,\sigma^2}\right] \exp\left[-\frac{(\mu - \mu_0)^2}{2\,\tau_0^2}\right]$$

$$\propto \exp\left[-\frac{a(\mu - b/a)^2}{2}\right]$$

with

$$a = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2} \quad \text{und} \quad b = \frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}$$

- The a-posteriori distribution is therefore the normal distribution $\mathrm{No}(\mu_n, \tau_n^2)$ with

$$\mu_n = \frac{b}{a} = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}}, \quad \tau_n^2 = \frac{1}{a} = \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}}$$

- The mean μ_n is the weighted average of the prior information μ_0 and the mean of the data $\bar{y}$, where the weights are given by the **precision**, i.e. the inverse variance.

- The precision $1/\tau_n^2$ is the sum of the precision of the prior information μ_0 and the precision of the mean value of the data $\bar{y}$.
- The smaller the precision of the prior information is, i.e. the larger τ_0^2 is, the smaller is the influence of the prior information on the a-posteriori distribution.

- If the variance σ^2 is unknown, we need a joint a-priori density for μ and σ^2.
- The calculation is simpler if we use the precision $\zeta = 1/\sigma^2$ instead of the variance σ^2.
- A possible choice is the improper a-priori density

$$\pi(\mu, \zeta) = \frac{1}{\zeta}$$

- Then:

$$p(\mu, \zeta | \boldsymbol{y}) \propto \frac{1}{\zeta} \prod_{i=1}^{n} \sqrt{\zeta} \exp\left[-\frac{\zeta}{2} \sum_{i=1}^{n} (y_i - \mu)^2 \right]$$

$$\propto \zeta^{n/2 - 1} \exp\left[-\frac{\zeta}{2} \sum_{i=1}^{n} (y_i - \mu)^2 \right]$$

- Integration over μ gives the a-posteriori density of ζ:

$$p(\zeta|\boldsymbol{y}) \propto \int \zeta^{n/2-1} \exp\left[-\frac{\zeta}{2}\sum_{i=1}^{n}(y_i - \mu)^2\right] \mathrm{d}\mu$$

$$\propto \zeta^{(n-3)/2} \exp\left[-\frac{\zeta}{2}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]$$

- ζ is therefore a-posteriori gamma distributed according to $\mathrm{Ga}((n-1)/2, 2/\sum(y_i - \bar{y})^2)$. Because of

$$\sum(y_i - \bar{y})^2 = \sum y_i^2 - n\bar{y}^2$$

the distribution depends only on $s_1 = \sum y_i$ and $s_2 = \sum y_i^2$ i.e. on the sample mean $\bar{y}$ and the sample variance S^2.

- The a-posteriori density of σ^2 is obtained by the transformation $\sigma^2 = 1/\zeta$:

$$g(\sigma^2) = \frac{p(1/\sigma^2|\boldsymbol{y})}{\sigma^4}$$

- The distribution of σ^2 is an inverse gamma distribution.

Normally Distributed Data

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

The Inverse Gamma Distribution $\mathrm{IG}(a,b)$

- The density of the inverse gamma distribution is:

$$f_{\mathrm{IG}}(x; a, b) = \frac{(1/x)^{a+1} e^{-1/(xb)}}{b^a \Gamma(a)} \cdot I_{[0,\infty)}(x)$$

- Its distribution function is:

$$F_{\mathrm{IG}}(x; a, b) = 1 - F_{\mathrm{Ga}}(1/x; a, b)$$

- The mean is $1/(b(a-1))$ when $a > 1$; the mode is $m = 1/(b(a+1))$.

- We obtain the a-posteriori distribution of μ conditioned by ζ from

$$
\begin{aligned}
p(\mu|\zeta, \boldsymbol{y}) &= \frac{p(\mu, \zeta|\boldsymbol{y})}{p(\zeta|\boldsymbol{y})} \\
&\propto \zeta^{0.5} \exp\left\{-\frac{\zeta}{2}\left[\sum(y_i - \mu)^2 - \sum(y_i - \bar{y})^2\right]\right\} \\
&\propto \zeta^{0.5} \exp\left[-\frac{n\zeta}{2}(\mu - \bar{y})^2\right] \\
&= \frac{1}{\sigma} \exp\left[-\frac{n}{2\sigma^2}(\mu - \bar{y})^2\right]
\end{aligned}
$$

- This is the normal distribution $\mathrm{No}(\bar{y}, \sigma^2/n)$.

- Finally, we obtain the a-posteriori distribution of μ by integrating over ζ:

$$p(\mu|\boldsymbol{y}) \propto \int \zeta^{n/2-1} \exp\left[-\frac{\zeta}{2}\sum_{i=1}^{n}(y_i - \mu)^2\right] \mathrm{d}\zeta$$

$$\propto \frac{1}{\left[\sum(\mu - y_i)^2\right]^{n/2}}$$

- The default score

$$t = \frac{\mu - \bar{y}}{S/\sqrt{n}}$$

is a-posteriori t-distributed with $n - 1$ degrees of freedom is.

- If one wants an actual a-priori density for ζ, the gamma distribution is a good choice, since it is the conjugate distribution.

- The a-posteriori distribution $p(\zeta|\boldsymbol{y})$ is then again a gamma distribution.

Example

From a normally distributed data set of size $n = 100$ from $\mathrm{No}(10, \sqrt{2})$, $\bar{y} = 9.906$ and $S = 1.34981$ are given. The a-posteriori density of μ is a $\mathrm{t}(n-1)$ distribution scaled by $S/\sqrt{n}$ and shifted by $\bar{y}$. The HPD interval with $1 - \alpha = 0.95$ is equal to $[9.6382, 10.1739]$.

A-posteriori density of μ
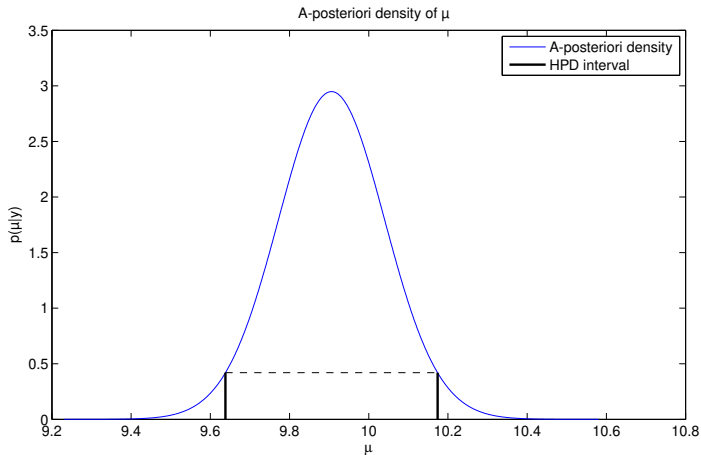
Example (Continuation)

The a-posteriori density $p(\sigma^2|\boldsymbol{y})$ of the variance σ^2 is the inverse gamma distribution

$$\mathrm{IG}((n-1)/2, 2/(S^2(n-1))) = \mathrm{IG}(49.5, 0.0110879)$$

The HPD interval with $1 - \alpha = 0.95$ is equal to $[1.3645, 2.4002]$.

Normally Distributed Data

A posteriori density of σ^2

Statistical Methods
of Data Analysis
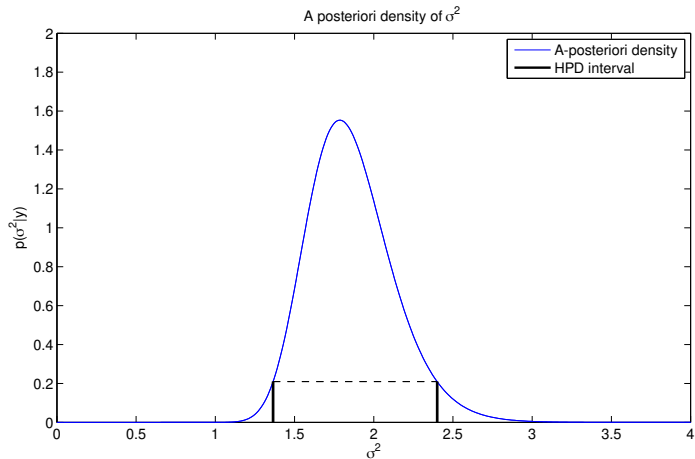
W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

- We consider an exponentially distributed sample $\boldsymbol{y} = (y_1, \ldots, y_n), \ n > 1$.
- We want to obtain an estimate of the mean τ of the exponential distribution from which the data are drawn.
- The likelihood function is:

$$p(\boldsymbol{y}|\tau) = \prod_{i=1}^{n} \frac{1}{\tau} \exp\left(-\frac{y_i}{\tau}\right) = \frac{1}{\tau^n} \exp\left(-\frac{\sum y_i}{\tau}\right)$$

- The Fisher information is equal to $I_\tau = n/\tau^2$; Jeffrey's prior is therefore $\pi_{\mathrm{J}}(\tau) = \tau^{-1}$ and hence improper.
- The a-posteriori distribution is then proportional to

$$p(\tau|\boldsymbol{y}) \propto \frac{1}{\tau^{n+1}} \exp\left(-\frac{\sum y_i}{\tau}\right) = \frac{1}{\tau^{n+1}} \exp\left(-\frac{s}{\tau}\right)$$

- The a-posteriori distribution is therefore the inverse gamma distribution $\mathrm{IG}(n, 1/s)$.
- The mean is $s/(n-1)$, the mode is $s/(n+1)$.

Example

There is an exponentially distributed sample of size $n = 50$. The sum of the data is $s = 102.58$. The a-posteriori mean is 2.0935, the a-posteriori mode is 2.0114. The HPD interval with $1 - \alpha = 0.95$ is $[1.5389, 2.6990]$.

Exponentially Distributed Data

a–posteriori density of τ with n=50, s=102.58, Jeffrey's prior
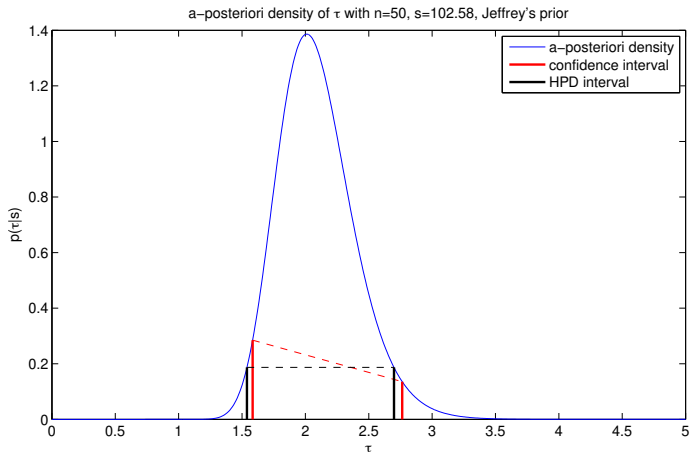
Exponentially Distributed Data

- The likelihood function can also be parameterized with the mean rate λ:

$$p(\boldsymbol{y}|\lambda) = \prod_{i=1}^{n} \lambda \exp\left(-\lambda y_i\right) = \lambda^n \exp\left(-\lambda \sum y_i\right) = \lambda^n \exp\left(-\lambda s\right)$$

- Jeffrey's prior is $\pi_{\mathrm{J}}(\lambda) = \lambda^{-1}$.

- The a-posteriori distribution is then proportional to

$$p(\lambda|\boldsymbol{y}) \propto \lambda^{n-1} \exp\left(-\lambda s\right)$$

- The a-posteriori distribution is therefore the gamma distribution $\mathrm{Ga}(n, 1/s)$ with mean n/s and mode $(n-1)/s$. The mean is equal to the maximum likelihood estimator of λ.

- The gamma distribution is also the conjugate a-priori distribution.

- The sequence $X_0, X_1, \ldots$ of discrete random variables is called a **discrete Markov chain** if the **Markov property** holds:

$$P(X_{n+1} = j | X_1 = i_1, \ldots, X_n = i_n) = P(X_{n+1} = j | X_n = i_n)$$

- The transition probabilities are calculated in a **transition matrix** $p_{ij}^{(n)}$, with:

$$p_{ij}^{(n)} = P(X_{n+1} = j | X_n = i)$$

$p_{ij}^{(n)}$ is an **stochastic matrix**: the row sums are 1.

- In a **time homogeneous** Markov chain, $p_{ij}^{(n)} = p_{ij}$ holds for all n.
- Let $\boldsymbol{\pi}^{(0)}$ be the density of X_0. Then the density of X_n is.

$$\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(0)} \, p_{ij}^{(n)}$$

- Under certain circumstances (the chain must be irreducible, all states must be positively recurrent), a temporally homogeneous Markov chain has a **stationary distribution π**, with

$$\pi = \pi \, p_{ij}$$

- Obviously, π is a left eigenvector of p_{ij} with the eigenvalue $\lambda = 1$.

Example

Simple weather model. States: sunny=1, rainy=2. Transition matrix:

$$p_{ij} = \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix}$$

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Markov Chain Monte Carlo

Example (Continuation)

Evolution over time, starting from a sunny day:

n	$P(1)$	$P(2)$
0	1.0	0.0
1	0.9	0.1
2	0.85	0.15
3	0.825	0.175
4	0.813	0.188
5	0.806	0.194
...	...	...

Example (Continuation)

Time evolution, starting with a rainy day:

n	$P(1)$	$P(2)$
0	0.0	1.0
1	0.4	0.6
2	0.6	0.4
3	0.7	0.3
4	0.75	0.25
5	0.775	0.225
$\ldots$	$\ldots$	$\ldots$

Stationary distribution:

$$\boldsymbol{\pi} = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix}$$

- A discrete Markov chain is called **reversible** if the condition of **detailed balance** holds, i.e. if there is a distribution p_{ij} for which holds:

$$\pi_i p_{ij}^{(n)} = \pi_j p_{ji}^{(n)}$$

- In that case p_{ij} is a stationary distribution of the chain.
- For **continuous** random variables X_i, the transition matrix is replaced by a **transition kernel** $q(x, y)$ with:

$$\int q(x, y) \, \mathrm{d}y = 1$$

- If at time n the state of the chain is equal to x, then the State y at time $n + 1$ is given by $q(x, y)$.

- **Markov Chain Monte Carlo (MCMC)** proceeds in reverse: given a target distribution $\pi(x)$, we look for a transition kernel $q(x, y)$ that has $\pi(x)$ as its stationary distribution.
- It is in general not possible to find such a kernel explicitly. A convenient solution is the **Metropolis–Hastings** algorithm.
- Assume that for some x, y

$$\pi(x)q(x, y) > \pi(y)q(y, x)$$

  Then the chain evolves from x to y more often than from y to x.
- To correct this, the frequency of evolutions from x to y is reduced by introducing an **acceptance probability** $\alpha(x, y) < 1$.

Statistical Methods
of Data Analysis

W. Waltenberger

Introduction

Prior distributions

Binomially Distributed
Data

Poisson Distributed Data

Normally Distributed
Data

Exponentially Distributed
Data

Markov Chain Monte
Carlo

Markov Chain Monte Carlo

- From the detailed balance condition we get

$$\alpha(x,y) = \min\left[1, \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)}\right]$$

- $\alpha(x,y)$ can be calculated **without knowledge of the normalization constant of** $\pi(x)$!

The Metropolis–Hastings Algorithm

- Generate a starting value x_0.
- For i=1,...,N:
  - Generate a proposal value y from $q(x_{i-1}, y)$ and u from $\mathrm{Un}(0,1)$.
  - If $u \leq \alpha(x_{i-1}, y)$, set $x_i = y$, otherwise $x_i = x_{i-1}$.
- Discard the first n links of the chain (,,burn-in").

Markov Chain Monte Carlo

- The M–H algorithm can be used to draw from densities with unknown normalization, such as posteriori densities.
- If $q(x, y) = q(y, x)$, then the acceptance probability is

$$\alpha(x, y) = \min\left[1, \frac{\pi(y)}{\pi(x)}\right]$$

- If $\pi(y) > \pi(x)$, then take the proposed step; otherwise take the step with probability $\pi(y)/\pi(x)$. This is the basis of **optimization by "simulated annealing"'**.
- If $q(x, y) = g(y - x)$, then the chain is a **random walk**, as $y = x + z$, z being the noise term with a distribution $g(z)$. If $g(x)$ is symmetric, the probability of acceptance reduces again as above.

- In case $q(x, y) = p(y)$ does not depend on x, the algorithm is also called **Independence Sampler**, and $p(y)$ is called the **Proposal Density**. The acceptance probability is

$$\alpha(x, y) = \min \left[1, \frac{\pi(y)p(x)}{\pi(x)p(y)} \right]$$

Markov Chain Monte Carlo

Literature

1. S. Chib and E. Greenberg, Understanding the Metropolis–Hastings Algorithm. The American Statistician 49/4, 1995.

2. L. Tierney, Markov Chains for Exploring Posterior Distributions. The Annals of Statistics 22/4, 1994.

3. W.K. Hastings, Monte Carlo Sampling Methods Using Markov Chains and Their Applications. Biometrika 57/1, 1970.

4. O. Cencic and R. Frühwirth, Data reconciliation with non-normal distributions I: linear constraints. Submitted to Computers & Chemical Engineering.