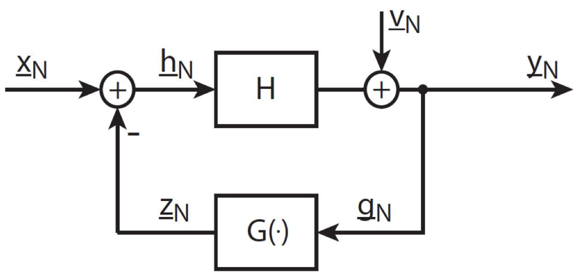
Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.
- Unless otherwise mentioned, all Python problems have to be solved using only Python standard libraries, `numpy` and `matplotlib`.

Problem 4.1 (0.5 points)



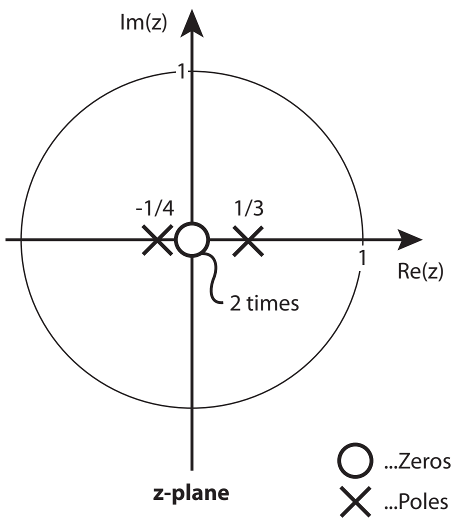


Figure 4.1.1: (a) Feedback system (left) (b) Pole-Zero plot of the system $H(z)$

Consider the feedback system shown in Figure 4.1.1 (a). The causal system $H(z)$ is given by its pole-zero diagram shown in Figure 4.1.1 (b). It has two poles and two zeros. The system G is given by the non-linear relationship

$$G(x) = \frac{x}{\sqrt{e^{-ax} + e^{bx}}}, \; a,b \in \mathbb{R}. \tag{4.1.1}$$

$\boxed{4.1.1}$ Calculate the gain of the system $H(z) = K\frac{\prod_{i=0}^{1} z-z_{0,i}}{\prod_{i=0}^{1} z-z_{p,i}}$, where K is a fixed constant, and $z_{0,i}$, $z_{p,i}$ are zeros and poles respectively.

$\boxed{4.1.2}$ Calculate the gain of the system $G(\cdot)$.

From now on we set $a = 1$ and $b = 1$.

$\boxed{4.1.3}$ Assuming $K = 1$ and the additive disturbance $v_n = 0\ \forall n$, is the feedback system stable? Justify your answer.

$\boxed{4.1.4}$ Assuming zero initial conditions for the systems H and G, derive an upper bound on $||\underline{y}_N||_2$ in terms of $\underline{x}_N$, $\underline{v}_N$ and the gains of the systems, where $\underline{y}_N$ is the length N vector of output samples.

Problem 4.2 (0.5 points)

$\boxed{4.2.1}$ Prove that

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \tag{4.2.1}$$

is a metric, where x_n and y_n are from the vector space of real-valued sequences

$$\mathbb{R}^{\mathbb{N}} = \{x = (x_n)_{n\in\mathbb{N}} | x_n \in \mathbb{R}\}. \tag{4.2.2}$$

$\boxed{4.2.2}$ Show that $d(x,y) = |\arctan(x) - \arctan(y)|$ is a metric on $\mathbb{R}$.

$\boxed{4.2.3}$ A norm is induced by an inner product iff it satisfies

$$\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2 = 2(\|\underline{x}\|^2 + \|\underline{y}\|^2). \tag{4.2.3}$$

Prove this identity and determine, if L_2 is a Hilbert space with inner product $\langle \underline{f}, \underline{g} \rangle = \int \bar{\underline{f}}(t)\underline{g}(t)dt$, where $\bar{\underline{f}}(t)$ denotes conjugate complex of $\underline{f}(t)$.

Problem 4.3 (0.5 points)

A pilot symbol s with signal power $|s|^2 = \sigma_s^2$ is transmitted over a channel with complex-valued channel attenuation h that introduces a frequency offset ε. The transmission is also affected by i.i.d. complex Gaussian noise with noise variance σ_ν^2, i.e., $\nu_k \sim \mathcal{CN}(0,\sigma_\nu^2)$. The received signal is expressed as

$$r_k = s\,h\,e^{j\varepsilon k} + \nu_k. \tag{4.3.1}$$

The frequency offset ε introduces a periodic change in phase. Your task is to estimate the frequency offset at the receiver. The complex-valued pilot symbol s is known at the receiver.

First, let's assume that the pilot symbol s stays constant over time and that the channel attenuation h is static over time.

$\boxed{4.3.1}$ Calculate $y_k = r_k r_{k+1}^*$ as an observation for determining the frequency offset. Then, determine the constant signal term c and the noise term w_k that allow to express the observation in the form $y_k = ce^{-j\varepsilon} + w_k$, with $c \in \mathbb{R}$ constant.

$\boxed{4.3.2}$ Determine an expression for the frequency offset ε from the observation y_k.
Hint: You can neglect the noise.

$\boxed{4.3.3}$ Calculate the signal-to-noise ratio (SNR) of the observation y_k

$$\gamma = \frac{\mathrm{E}\{|c|^2\}}{\mathrm{E}\{|w_k|^2\}}. \tag{4.3.2}$$

What is the SNR in particular if the noise variance σ_ν^2 is small compared to the signal power?

Next, we assume to receive M signals

$$r_k^{(m)} = s_m h_m e^{j\varepsilon k} + \nu_k^{(m)}, \quad m = 1,2,...M, \tag{4.3.3}$$

at the same time, where the symbols s_m are statistically independent. The complex-valued channel attenuation h_m is different for each of the M signals. We compute the M observations $y_k^{(m)}$ and combine them as

$$\sum_{m=1}^{M} y_k^{(m)}. \tag{4.3.4}$$

$\boxed{4.3.4}$ Calculate the SNR after combining.

Now we combine the M observations $y_k^{(m)}$ using weighting factors g_m, i.e.,

$$\sum_{m=1}^{M} g_m y_k^{(m)}. \tag{4.3.5}$$

$\boxed{4.3.5}$ Determine the weighting factors g_m such that the SNR after combining is maximized.

Hint: Try to rewrite the expression for the SNR as $\frac{\left|\sum_{m=1}^{M} p_m q_m\right|^2}{\sigma_\nu^2 \sum_{m=1}^{M} |p_m|^2}$ *and use the Cauchy-Schwarz inequality to find the maximum.*

$\boxed{4.3.6}$ Determine the SNR after combining with optimal weights.

Problem 4.4 (Python, 1.0 point)

In this exercise, we evaluate the frequency offset problem from Problem 4.3 by simulations. The received signal is expressed as

$$r_k = h\, e^{j\varepsilon k} + \nu_k, \tag{4.4.1}$$

i.e., the all-one sequence is used for the pilot symbols. The channel attenuation h is assumed constant for a simulation run.
Your task is to determine and plot the average estimation error $|\hat{\varepsilon}-\varepsilon|^2$ over 2000 simulation runs for an SNR of the received symbol r_k ranging from -20 dB to 20 dB. Consider a frequency offset $\varepsilon = \pi/4$. For each simulation run, use `numpy.random.randn()` to generate the complex valued channel(s) and noise, and normalize them properly. Use `numpy.angle()` to evaluate the angle of a complex number.

$\boxed{4.4.1}$ **Single channel**: compute the average frequency offset $\hat{\varepsilon}$ based on one observation $y_k = r_k r_{k+1}^*$.

$\boxed{4.4.2}$ **Single channel, multiple observations**: consider that we receive 5 symbols r_k, $k = 1, \ldots, 5$, from a single channel. This results in 4 instantaneous observations y_k, $k = 1, \ldots, 4$. Compute the frequency offset $\hat{\varepsilon}$ by evaluating the angle of $\sum_{k=1}^{4} y_k$. *Hint: The channel h stays constant for one simulation run, but each received symbol r_k is affected by an independent noise realization.*

$\boxed{4.4.3}$ **Multiple channels**: consider four independent channels $\{h_1, h_2, h_3, h_4\}$ simultaneously. At each time instant k, four observations $(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, y_k^{(4)})$ are available. Compute the frequency offset $\hat{\varepsilon}$ by evaluating the angle of $\sum_{m=1}^{4} y_k^{(m)}$.

$\boxed{4.4.4}$ **Multiple channels, SNR-optimal combining**: repeat the four-channel experiment. Employ the weighting factors derived in Task 4.3.5 and compute the frequency offset by evaluating the angle of $\sum_{m=1}^{4} g_m y_k^{(m)}$.

$\boxed{4.4.5}$ Plot the average estimation error $|\hat{\varepsilon} - \varepsilon|^2$ over SNR and analyze the performance of the different cases.