Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.
- Unless otherwise mentioned, all Python problems have to be solved using only Python standard libraries, `numpy` and `matplotlib`.

Problem 3.1 (0.5 points)

Let X be the set of all sequences with l_2-norm equal to $\sqrt{2}$, i.e.,

$$\left\{ x \in X : \sqrt{\sum_{n=1}^{\infty} x_n^2} = \sqrt{2} \right\},$$

where $x_n \in \mathbb{R}$ denotes the n^{th} element of sequence x.
Consider

$$d = \sum_{n=1}^{\infty} x_n y_n,$$

with $\{x,y \in X\}$.

$\boxed{3.1.1}$ Do X and d constitute a metric space?

$\boxed{3.1.2}$ Do X and d constitute a pre-Hilbert space?

$\boxed{3.1.3}$ Calculate tight upper and lower bounds of d.

$\boxed{3.1.4}$ Consider $\{x,y \in X\}$ and the sequence y is given. Find the sequences x, for which the lower and upper bounds are achieved.

$\boxed{3.1.5}$ Calculate bounds for

$$\sum_{n=1}^{\infty} (x_n - y_n)^2 \pm \sum_{n=1}^{\infty} (x_n + y_n)^2, \quad \{x,y \in X\}.$$

Next, assume further the sequences $\{x,y \in X\}$ to be given as

$$x_n = \sin\left(\frac{\pi}{2}n\right)(u_n - u_{n-4}), \qquad n = 1,2,\ldots,\infty$$
$$y_n = a_1\delta_{n-1} + a_2\delta_{n-2} + a_3\delta_{n-3} \qquad n = 1,2,\ldots,\infty$$

with $\{a_1,a_2,a_3\} \in \mathbb{R}$.
The function u_n denotes the unit step function, i.e.,

$$u_n = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1, & \text{for} \quad n \geq 0 \end{cases} ,$$

and δ_n is the discrete unit sample function

$$\delta_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0, & \text{for} \quad n \neq 0 \end{cases} .$$

$\boxed{3.1.6}$ Calculate a_1, a_2, and a_3 such that d becomes zero. Determine the number of possible solutions.

Problem 3.2 (0.5 points)

The convergence or divergence of a series can, among other techniques, be checked by one of the following theorems:

(I) Comparison Test
Let $\sum a_k$ and $\sum b_k$ be series with a_k,$b_k \in \mathbb{R}^+$. Given that $a_k \leq b_k \ \forall k$, then

- $\sum a_k$ converges if $\sum b_k$ converges,
- $\sum b_k$ diverges if $\sum a_k$ diverges.

(II) Ratio Test (or d'Alembert's Ratio Test or Cauchy Ratio Test)
Let $\sum\limits_{k=1}^{\infty} a_k$ be a series where a_k is non-zero, when k is large. Then for $\rho = \lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k}$, it holds that

- the series converges if $\rho < 1$,
- the series diverges if $\rho > 1$,
- the test is inconclusive for $\rho = 1$.

(III) Root Test (or Cauchy Root Test or Cauchy's Radical Test)
Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with $a_k \in \mathbb{R}^+$. Then for $\rho = \lim\limits_{k\to\infty} (a_k)^{\frac{1}{k}}$, it holds that

- the series absolutely converges if $\rho < 1$,
- the series diverges if $\rho > 1$,
- the series converges if $\rho = 1$ and the limit approaches strictly from above,
- otherwise no statement about convergence or divergence can be made.

(IV) Integral Test
Let $f(x)$ be a monotonically decreasing, non-negative function with support $[N,\infty)$. Then,

- the series $\sum\limits_{k=N}^{\infty} f(k)$ converges to a real number iff $\int\limits_{N}^{\infty} f(x)dx$ is finite.

(V) p-series Test
Let a series be given by $\sum\limits_{n=1}^{\infty} \frac{1}{k^p}$. Then,

- the series diverges if $0 < p \leq 1$,
- the series converges if $p > 1$.

Apply the suitable theorem(s) from the list above to analyze the convergence or divergence of the following series.

$\boxed{3.2.1}$

$$S_1 = \sum_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$$

$\boxed{3.2.2}$

$$S_2 = \sum_{k=1}^{\infty} \frac{1}{k^{1/3} - 1}$$

$\boxed{3.2.3}$

$$S_3 = \sum_{k=1}^{\infty} \left[k^4 \sin^2 \left(\frac{3k}{2k^3 - 2k^2 + 5} \right) \right]^k$$

$\boxed{3.2.4}$

$$S_4 = \sum_{k=2}^{\infty} \frac{1}{k \log(k)}$$

$\boxed{3.2.5}$

$$S_5 = \sum_{k=1}^{\infty} \frac{(k-1)! + 4^{k+1}}{k!}$$

Problem 3.3 (0.5 points)

A common method for optimal resource allocation in mobile cellular networks is convex optimization. A convex function satisfies the following property:

$$f(\theta s + (1-\theta)t) \leq \theta f(s) + (1-\theta) f(t) \tag{3.3.1}$$

where $0 \leq \theta \leq 1$.

$\boxed{3.3.1}$ Assume $f : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary norm. Show that every norm on $\mathbb{R}^n$ is convex.

$\boxed{3.3.2}$ Show that the following expressions define norms on $\mathbb{R}^n$, with $\underline{x} = (x_1, x_2, \ldots, x_n)^T$:

- $\|\underline{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$,
- $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$,
- $\|\underline{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

Problem 3.4 (Python, 1.0 point)

Consider a linear time-variant filter with linear phase

$$H\left(e^{j\Omega}\right) = a_0 + \sum_{k=1}^{N} 2a_k \cos(k\Omega).$$

$\boxed{3.4.1}$ *Analytical part*: Determine the coefficients of this filter such that it optimally follows

$$H^{(d)}\left(e^{j\Omega}\right) = \begin{cases} 0.2 & ,|\Omega| < \Omega_1 \\ 1 & ,\Omega_1 \leq |\Omega| \leq \Omega_2 \\ 0.2 & ,\Omega_2 < |\Omega| < \pi \end{cases} \quad , \qquad \Omega_1 < \Omega_2 < \pi$$

in the Least Squares (LS) sense. This is equivalent to minimizing the metric

$$d_2\left(H^{(d)}\left(e^{j\Omega}\right), H\left(e^{j\Omega}\right)\right) = \int_{-\pi}^{\pi} \left|H^{(d)}\left(e^{j\Omega}\right) - H\left(e^{j\Omega}\right)\right|^2 d\Omega$$

Hint:

- *Apply the Parseval theorem to get an equivalent statement in the time domain.*
- *Write the desired transfer function as a linear combination of ideal low-pass filters.*
- *To minimize a convex function, you can differentiate it with respect to the variables and set the derivative equal to zero.*

$\boxed{3.4.2}$ *Python part*:

1. Implement a Python code to plot the frequency response of the filter $H\left(e^{j\Omega}\right)$, as obtained from your analytic calculations for arbitrary filter order. The limit frequencies are $\Omega_1 = \frac{\pi}{4}$ and $\Omega_2 = \frac{3\pi}{4}$.
2. Compare the filters for the orders $N = \{4,10,100\}$. What do you observe?