Guidelines

- Write your name and matriculation number on each sheet of paper.
- Only clearly readable exercise-elaborations are evaluated.
- Results have to be provided together with an evident way of calculation.
- Keep textual answers short and concise. Lengthy or vague statements won't gain points.
- Unless otherwise mentioned, all Python problems have to be solved using only Python standard libraries, `numpy` and `matplotlib`.

Problem 5.1 (0.5 points)

Consider the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$ in Figure 5.1.1.

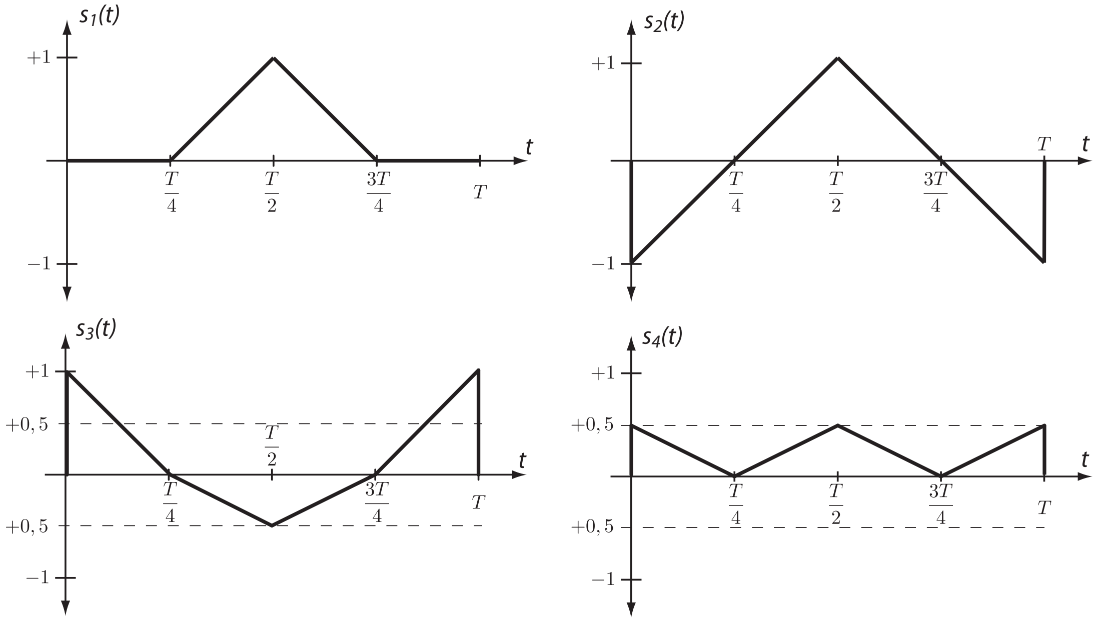

Figure 5.1.1: Signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$.

$\boxed{5.1.1}$ Evaluate $\|s_1(t)\|_2^2$, $\|s_2(t)\|_2^2$, $\|s_3(t)\|_2^2$ and $\|s_4(t)\|_2^2$.
Hint: Try to avoid unnecessary calculations, split the signals in smaller parts, reuse results.

$\boxed{5.1.2}$ Derive an orthonormal basis, $\phi_k(t)$, k = 1,2,3,4, for the space spanned by $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$. Use the Gram-Schmidt orthogonalization method, start with $s_1(t)$ and continue with $s_2(t)$ and $s_3(t)$. Sketch the evaluated basis functions. Which dimension has the signal space?
Hint: Again, avoid unnecessary calculations. What does it mean for the orthonormaliza-

tion if a signal can be represented by a linear combination of others?

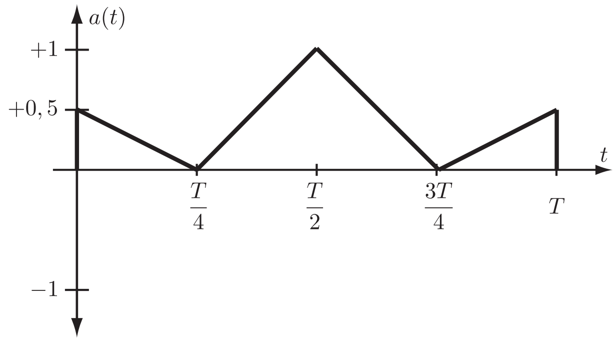

Figure 5.1.2: Signal $a(t)$.

$\boxed{5.1.3}$ Now consider the signal $a(t)$ in Figure 5.1.2. Express the signal in terms of the derived basis and calculate $\|a - s_3\|_2^2$.

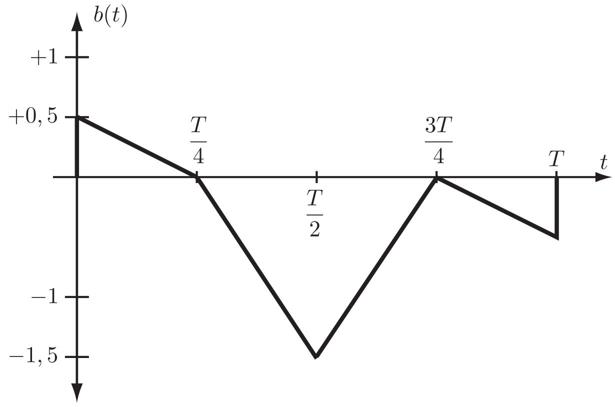

Figure 5.1.3: Signal $b(t)$.

$\boxed{5.1.4}$ Finally consider the signal $b(t)$ in Figure 5.1.3. Find the best approximation $\hat{b}(t)$ in terms of the L_2 norm of this signal in the space spanned by the basis derived in Task 5.1.2.

$\boxed{5.1.5}$ Sketch $\hat{b}(t)$.

Problem 5.2 (0.5 points)

In this exercise we consider linear least squares regression and derive some important statistical properties of the least squares estimator[1].
Consider the general linear measurement model of the form

$$\underline{x} = A\underline{c} + \underline{e}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & & & \vdots \\ a_{p1} & a_{12} & \dots & a_{pk} \end{bmatrix} \tag{5.2.1}$$

where $\underline{x} \in \mathbb{C}^{p\times 1}$ is the measured data, $\underline{c} \in \mathbb{C}^{k\times 1}$ are unobservable parameters ($k \leq p$), $A \in \mathbb{C}^{p\times k}$ is the input data of the measurement system and $\underline{e}$ is the measurement error (the input data is selected such that $\text{rank}(A) = k$). We want to apply the linear regression model

$$\hat{\underline{x}} = A\hat{\underline{c}} \tag{5.2.2}$$

to this measurement model.

$\boxed{5.2.1}$ Apply an ordinary least squares estimation to determine an expression for the parameters $\hat{\underline{c}}$.

$\boxed{5.2.2}$ An estimator is said to be unbiased, if the expected value of the estimate is equal to the true value, i.e., $\text{E}\{\hat{\underline{c}}\} = \underline{c}$. Assume that each component of $\underline{e}$ is a zero-mean, i.i.d. variable with variance σ_e^2 and show that the estimator obtained in Task 5.2.1 is unbiased.

$\boxed{5.2.3}$ Using the results above, calculate the covariance of the estimator, i.e.,

$$\text{Cov}(\hat{\underline{c}}) = \text{E}\Big\{(\hat{\underline{c}} - \text{E}\{\hat{\underline{c}}\})(\hat{\underline{c}} - \text{E}\{\hat{\underline{c}}\})^{\text{H}}\Big\}. \tag{5.2.3}$$

Next, we want to show that our linear least squares estimator is the best linear unbiased estimator (BLUE), in the sense that it minimizes the mean squared error $\text{E}\{\|\hat{\underline{c}} - \underline{c}\|^2\}$, for the case that $\text{E}\{\underline{e}\} = \underline{0}$, $\text{E}\big\{\underline{e}\,\underline{e}^{\text{H}}\big\} = \sigma_e^2 I$. The mean squared error (MSE) can be written as

$$\text{E}\big\{\|\hat{\underline{c}} - \underline{c}\|^2\big\} = \text{tr}(\text{Cov}(\hat{\underline{c}})) + \|\text{Bias}(\hat{\underline{c}},\underline{c})\|^2, \tag{5.2.4}$$

where $\text{Bias}(\hat{\underline{c}},\underline{c}) = \text{E}\{\hat{\underline{c}}\} - \underline{c}$. As we consider unbiased estimators, the bias term above is zero. Hence, we can focus on the covariance matrix of the estimator.

Suppose we have another linear unbiased estimator of the parameter vector $\underline{c}$ which we write as

$$\tilde{\underline{c}} = \hat{\underline{c}} + B\underline{x} = L\underline{x}. \tag{5.2.5}$$

[1][Section 3.5], Moon., T.K. and Stirling, W.C.: *Mathematical Methods and Algorithms for Signal Processing*, Prentice Hall, 2000

$\boxed{5.2.4}$ Which condition does B have to satisfy for the estimator to be unbiased?

$\boxed{5.2.5}$ Calculate the covariance matrix of $\underline{\tilde{c}}$.

$\boxed{5.2.6}$ Express L in terms of A and B.

$\boxed{5.2.7}$ Show that $\text{Cov}(\underline{\tilde{c}})$ exceeds $\text{Cov}(\underline{\hat{c}})$ by a positive semi-definite matrix, and hence its trace is larger (implying a larger MSE).

$\boxed{5.2.8}$ Explain, in your own words, why our linear least squares estimator is the best linear unbiased estimator (BLUE).

Problem 5.3 (0.5 points)

Laguerre polynomials, which arise in quantum mechanics and are named after Edmond Laguerre (1834–1886), are solutions of the Laguerre differential equation:

$$xL_n^{''}(x) + (1-x)L_n^{'} + nL_n(x) = 0$$

.

$\boxed{5.3.1}$ Prove that the Laguerre differential equation is solved by the Laguerre polynomials given in their closed form as:

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k \tag{5.3.1}$$

where $n = 0,1,2,3,...$

$\boxed{5.3.2}$ Given the weighted inner product

$$\langle p, q \rangle_w = \int_0^\infty p(x)q(x)w(x)dx$$

and the weighting function $w(x) = e^{-x}$, derive an orthonormal set of polynomials from the standard monomials $\{1,x,x^2,x^3\}$. Compare your resulting polynomials with the polynomials generated using (5.3.1). After solving this task what can you state about Laguerre polynomials?

$\boxed{5.3.3}$ Now using the first four Laguerre polynomials from Task 5.3.2 calculate the polynomial approximation $\tilde{f}(x)$ of the function $f(x) = e^{-2x}$ by determining the coefficients c_i in

$$\tilde{f}(x) = \sum_{n=0}^{k} c_n L_n(x)$$

in the LS sense.

$\boxed{5.3.4}$ Using Python, plot $f(x)$ and the approximations $\tilde{f}(x)$ from the previous task for $k = 0,1,2,3$ on the interval $x \in [0,5]$. What do you observe?

Problem 5.4 (0.5 points)

Given the set of data points x and corresponding measurements y

$$x = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}; \quad y = \{45, 37, 28, 14, 13, 3, 1, 4, 2, 9\}.$$

$\boxed{5.4.1}$ Determine the quadratic approximation $y = ax^2 + bx + c$, which best fits the data in the LS sense. Write the measurements in the matrix-vector notation:

$$\underline{y} = A\underline{c} + \underline{e}$$

where matrix A is a function of $\underline{x}$ and $\underline{e}$ is the measurement error. Compute the LS solution $\underline{c}_{LS}$ in Python using the derived expression.

$\boxed{5.4.2}$ The least squares error can be written as $\underline{e} = B\underline{y}$. Determine an analytical expression for matrix B. What can you say about this matrix - what properties does it have? State a matrix and a vector to which the error $\underline{e}$ is orthogonal. Verify the orthogonality in Python.

$\boxed{5.4.3}$ Depict the data in a scatter plot.

$\boxed{5.4.4}$ Assume that the first- and the last data points are the most accurate ones, and both have a weight w. Formulate the corresponding weighting matrix W and derive the quadratic approximation $\underline{c}_{wLS}$ based on weighted LS.

$\boxed{5.4.5}$ Plot different approximation curves for following values of $w = \{0.1, 1, 10, 100\}$ and calculate the weighted Mean Squared Error (MSE_w) for each. Which of those is the best fit?
Hint: Weighted MSE is defined as

$$MSE_w := \frac{1}{N} \frac{\sum_{i=0}^{N} w_i \cdot \left(\underline{y_i} - \underline{y}_{wLS_i}\right)^2}{\sum_{i=0}^{N} w_i}$$