Methods 2 - 5

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Linear Algebra: Vectors and Matrices – Gill, chapter 3

- **Vectors**
  - Vector algebra
  - Inner, cross, outer products
  - Transpose
  - Norms
- **Matrices**
  - Symmetric
  - Diagonal
  - Identity, J, zero, triangular
  - Multiplication
  - Transposition

Elementary Formal Properties of Vector Algebra

$\rightarrowtail$	Commutative Property	$\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$
$\rightarrowtail$	Additive Associative Property	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
$\rightarrowtail$	Vector Distributive Property	$s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$
$\rightarrowtail$	Scalar Distributive Property	$(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$
$\rightarrowtail$	Zero Property	$\mathbf{u} + \mathbf{0} = \mathbf{u} \Longleftrightarrow \mathbf{u} - \mathbf{u} = \mathbf{0}$
$\rightarrowtail$	Zero Multiplicative Property	$\mathbf{0}\mathbf{u} = \mathbf{0}$
$\rightarrowtail$	Unit Rule	$\mathbf{1}\mathbf{u} = \mathbf{u}$

Inner product ("dot" product)

$$\mathbf{u} \cdot \mathbf{v} = \left[u_1 v_1 + u_2 v_2 + \cdots u_k v_k\right] = \sum_{i=1}^{k} u_i v_i.$$

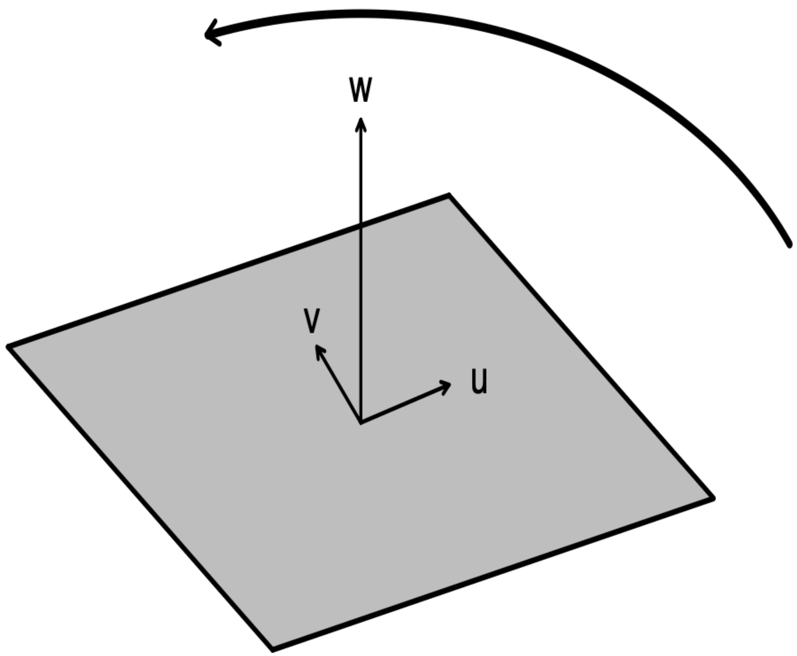
Inner Product Formal Properties of Vector Algebra

- $\rightarrowtail$ Commutative Property $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\rightarrowtail$ Associative Property $s(\mathbf{u} \cdot \mathbf{v}) = (s\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (s\mathbf{v})$
- $\rightarrowtail$ Distributive Property $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $\rightarrowtail$ Zero Property $\mathbf{u} \cdot \mathbf{0} = 0$
- $\rightarrowtail$ Unit Rule $\mathbf{1u} = \mathbf{u}$
- $\rightarrowtail$ Unit Rule $\mathbf{1u} = \sum_{i=1}^{k} \mathbf{u}_i$, for $\mathbf{u}$ of length k

Cross product

$$\mathbf{u} \times \mathbf{v} = [u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1]$$

Fig. 3.2. The Right-Hand Rule Illustrated



Cross Product Formal Properties of Vector Algebra

$\rightarrowtail$	Commutative Property	$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
$\rightarrowtail$	Associative Property	$s(\mathbf{u} \times \mathbf{v}) = (s\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (s\mathbf{v})$
$\rightarrowtail$	Distributive Property	$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
$\rightarrowtail$	Zero Property	$\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
$\rightarrowtail$	Self-Orthogonality	$\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Vector transpose

$$\underset{k\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}}' = \underset{1\times k}{[u_1, u_2, \ldots, u_k]}, \qquad \underset{1\times k}{[u_1, u_2, \ldots, u_k]}' = \underset{k\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}}.$$

Outer product

★ **Example 3.9: Outer Product Calculation**. Once again using the simple numerical forms, we now calculate the outer product instead of the cross product:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3, 3, 3] = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}.$$

And to show that order matters, consider:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} [1, 2, 3] = \begin{bmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 3 & 6 & 9 \end{bmatrix}.$$

Vector norms

$$\|\mathbf{v}\| = \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)^{\frac{1}{2}} = \left(\mathbf{v}' \cdot \mathbf{v}\right)^{\frac{1}{2}}$$

Properties of the Standard Vector Norm

$\rightarrowtail$ Vector Norm $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$

$\rightarrowtail$ Difference Norm $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$

$\rightarrowtail$ Multiplication Norm $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$

p-norms

Actually, the norm used above is the most commonly used form of a **p-norm**:

$$\|\mathbf{v}\|_p = (|v_1|^p + |v_2|^p + \cdots + |v_n|^p)^{\frac{1}{p}}, \ p \geq 0,$$

where $p = 2$ so far. Other important cases include $p = 1$ and $p = \infty$:

$$\|\mathbf{v}\|_\infty = \max_{1 \leq i \leq n} |x_i|,$$

Properties of Vector Norms, Length-n

- $\rightarrowtail$ Triangle Inequality $\quad \|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\|+\|\mathbf{w}\|$
- $\rightarrowtail$ Hölder's Inequality $\quad$ for $\frac{1}{p}+\frac{1}{q}=1$, $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|_p \|\mathbf{w}\|_q$
- $\rightarrowtail$ Cauchy-Schwarz Ineq. $\quad |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|_2 \|\mathbf{w}\|_2$
- $\rightarrowtail$ Cosine Rule $\quad \cos(\theta)=\frac{\mathbf{v}\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$
- $\rightarrowtail$ Vector Distance $\quad d(\mathbf{v}, \mathbf{w})=\|\mathbf{v}-\mathbf{w}\|$
- $\rightarrowtail$ Scalar Property $\quad \|s\mathbf{v}\|=|s|\|\mathbf{v}\|$

Matrices

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1(p-1)} & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2(p-1)} & x_{2p} \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & \cdots & x_{(n-1)(p-1)} & x_{(n-1)p} \\ x_{n1} & x_{n2} & \cdots & \cdots & x_{n(p-1)} & x_{np} \end{bmatrix}$$

Symmetric and skew-symmetric matrices

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 5 & 6 \\ 3 & 5 & 1 & 7 \\ 4 & 6 & 7 & 8 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

Diagonal matrices

$$\mathbf{X} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & d_{n-1} & 0 \\ 0 & 0 & 0 & 0 & d_n \end{bmatrix}$$

Identity, J, and zero matrices

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lower and upper triangular matrices

$$\mathbf{X}_{LT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 6 & 7 & 8 \end{bmatrix}, \quad \mathbf{X}_{UT} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 8 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Properties of (Conformable) Matrix Manipulation

$\rightarrowtail$	Commutative Property	$\mathbf{X}+\mathbf{Y}=\mathbf{Y}+\mathbf{X}$
$\rightarrowtail$	Additive Associative Property	$(\mathbf{X}+\mathbf{Y})+\mathbf{Z}=\mathbf{X}+(\mathbf{Y}+\mathbf{Z})$
$\rightarrowtail$	Matrix Distributive Property	$s(\mathbf{X}+\mathbf{Y})=s\mathbf{X}+s\mathbf{Y}$
$\rightarrowtail$	Scalar Distributive Property	$(s+t)\mathbf{X}=s\mathbf{X}+t\mathbf{X}$
$\rightarrowtail$	Zero Property	$\mathbf{X}+\mathbf{0}=\mathbf{X}$ and $\mathbf{X}-\mathbf{X}=\mathbf{0}$

Matrix multiplication

$$
\begin{aligned}
\mathbf{XY} &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\
&= \begin{bmatrix} (x_{11}\ x_{12}) \cdot (y_{11}\ y_{21}) & (x_{11}\ x_{12}) \cdot (y_{12}\ y_{22}) \\ (x_{21}\ x_{22}) \cdot (y_{11}\ y_{21}) & (x_{21}\ x_{22}) \cdot (y_{12}\ y_{22}) \end{bmatrix} \\
&= \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}.
\end{aligned}
$$

Matrix multiplication

$$\underset{(k\times n)}{\mathbf{X}}\underset{(n\times p)}{\mathbf{Y}} = \begin{bmatrix} \sum_{i=1}^{n} x_{1i}y_{i1} & \sum_{i=1}^{n} x_{1i}y_{i2} & \cdots & \sum_{i=1}^{n} x_{1i}y_{ip} \\ \sum_{i=1}^{n} x_{2i}y_{i1} & \sum_{i=1}^{n} x_{2i}y_{i2} & \cdots & \sum_{i=1}^{n} x_{2i}y_{ip} \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^{n} x_{ki}y_{i1} & \cdots & \cdots & \sum_{i=1}^{n} x_{ki}y_{ip} \end{bmatrix}$$

Matrix multiplication

Starting with the matrices

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix},$$

calculate

$$\mathbf{XY} = \begin{bmatrix} (1 \;\; 2) \cdot (-2 \;\; 0) & (1 \;\; 2) \cdot (2 \;\; 1) \\ (3 \;\; 4) \cdot (-2 \;\; 0) & (3 \;\; 4) \cdot (2 \;\; 1) \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-2) + (2)(0) & (1)(2) + (2)(1) \\ (3)(-2) + (4)(0) & (3)(2) + (4)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -6 & 10 \end{bmatrix}.$$

Properties of (Conformable) Matrix Multiplication

- $\rightarrowtail$ Associative Property $(\mathbf{XY})\mathbf{Z} = \mathbf{X}(\mathbf{YZ})$

- $\rightarrowtail$ Additive Distributive Property $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$

- $\rightarrowtail$ Scalar Distributive Property
$$\begin{aligned} s\mathbf{XY} &= (\mathbf{X}s)\mathbf{Y} \\ &= \mathbf{X}(s\mathbf{Y}) = \mathbf{XY}s \end{aligned}$$

- $\rightarrowtail$ Zero Property $\mathbf{X0} = \mathbf{0}$

Matrix multiplication

$$\mathbf{XY} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$\mathbf{YX} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

This is a very simple example, but the implications are obvious. Even in cases where pre-multiplication and post-multiplication are possible, these are different operations and matrix multiplication is not commutative.

Matrix transposition

$$\mathbf{X}' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of Matrix Transposition

$\rightarrowtail$	Invertibility	$(\mathbf{X}')' = \mathbf{X}$
$\rightarrowtail$	Additive Property	$(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$
$\rightarrowtail$	Multiplicative Property	$(\mathbf{X}\mathbf{Y})' = \mathbf{Y}'\mathbf{X}'$
$\rightarrowtail$	General Multiplicative Property	$(\mathbf{X}_1\mathbf{X}_2 \ldots \mathbf{X}_{n-1}\mathbf{X}_n)' = \mathbf{X}_n'\mathbf{X}_{n-1}' \ldots \mathbf{X}_2'\mathbf{X}_1'$
$\rightarrowtail$	Symmetric Matrix	$\mathbf{X}' = \mathbf{X}$
$\rightarrowtail$	Skew-Symmetric Matrix	$\mathbf{X} = -\mathbf{X}'$