Compute the determinants

1) a) $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1\cdot 4 - 3\cdot 2 = -2$

b) $\det\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = 0\cdot 0 - (-3)\cdot 3 = 9$

c) $\det\begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix} = 1\cdot 6 - 3\cdot 0 = 6$

+ −

d) $\det\begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 0 \\ 3 & 5 & 0 \end{pmatrix} = \begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 3 & 6 & 0 & 3 & 6 \\ 3 & 5 & 0 & 3 & 5 \end{array} =$

$1\cdot 6\cdot 0 + 2\cdot 0\cdot 3 + 1\cdot 3\cdot 5 - 1\cdot 6\cdot 3 - 1\cdot 0\cdot 5 - 2\cdot 3\cdot 0$

$= 15 - 18 = -3$

e) $\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & -2 & 0 & 0 \end{array} =$

$1\cdot 6\cdot(-2) + 0\cdot 0\cdot 0 + 0\cdot 0\cdot 0 - 0\cdot 6\cdot 0 - 1\cdot 0\cdot 0 - 0\cdot 0\cdot$

$= -12$

f) $\det\begin{pmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 3 & 5 & 4 \end{pmatrix} = \begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 3 & 6 & 0 & 3 & 6 \\ 3 & 5 & 4 & 3 & 5 \end{array} =$

$1\cdot 6\cdot 4 + 0\cdot 0\cdot 3 + 0\cdot 3\cdot 5 - 0\cdot 6\cdot 3 - 1\cdot 0\cdot 5 - 0\cdot 3\cdot 4$

$= 24$

g) $\det\begin{pmatrix} 1 & -2 & -1 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{array}{ccc|cc} 1 & -2 & -1 & 1 & -2 \\ 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 4 & 0 & 0 \end{array} =$

$1\cdot 6\cdot 4 + (-2)\cdot 0\cdot 0 + (-1)\cdot 0\cdot 0 - (-1)\cdot 6\cdot 0 - 1\cdot 0\cdot 0$

$- (-2)\cdot 0\cdot 4 = 24$

1)

Compute the determinants

h)

$$\det\begin{pmatrix} 1 & -2 & -1 \\ 0 & 6 & 0 \\ 1 & 4 & 1 \end{pmatrix} = \begin{matrix} 1 & -2 & -1 \\ 0 & 6 & 0 \\ 1 & 4 & 1 \end{matrix} \left| \begin{matrix} 1 & -2 \\ 0 & 6 \\ 1 & 4 \end{matrix} \right. =$$

$$1\cdot 6\cdot 1 + (-2)\cdot 0\cdot 1 + (-1)\cdot 0\cdot 4 - (-1)\cdot 6\cdot 1 - 1\cdot 0\cdot 4 -$$

$$(-2)\cdot 0\cdot 1 = 6 + 6 = 12$$

i) No more we have to use the determinant formula:

$$\det(X) = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} |X_{ij}|$$

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \quad A_{[13]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \quad A_{[23]} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A_{[33]} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad A_{[43]} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\det(A) = (-1)^{1+3}\cdot 0\cdot \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + (-1)^{2+3}\cdot 0\cdot \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$+ (-1)^{3+3}\cdot 0\cdot \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} + (-1)^{4+3}\cdot 0\cdot \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \underline{\underline{0}}$$

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l)

$$A = \begin{pmatrix} 3 & 9 & 3 & -10 \\ 3 & 0 & 3 & 0 \\ 3 & 7 & 3 & -1 \\ 3 & 11 & 3 & -1 \end{pmatrix} \quad A_{[21]} = \begin{bmatrix} 9 & 3 & -10 \\ 7 & 3 & -1 \\ 11 & 3 & -1 \end{bmatrix} \quad A_{[23]} = \begin{bmatrix} 3 & 9 & -10 \\ 3 & 7 & -1 \\ 3 & 11 & -1 \end{bmatrix}$$

$$A_{[22]} = \begin{bmatrix} 3 & 3 & -10 \\ 3 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \quad A_{[24]} = \begin{bmatrix} 3 & 9 & 3 \\ 3 & 7 & 3 \\ 3 & 11 & 3 \end{bmatrix}$$

$$\det(A) = (-1)^{2+1}\cdot 3\cdot \begin{vmatrix} 9 & 3 & -10 \\ 7 & 3 & -1 \\ 11 & 3 & -1 \end{vmatrix} + (-1)^{2+2}\cdot 0\cdot \begin{vmatrix} 3 & 3 & -10 \\ 3 & 3 & -1 \\ 3 & 3 & -1 \end{vmatrix}$$

$$+ (-1)^{2+3}\cdot 3\cdot \begin{vmatrix} 3 & 9 & -10 \\ 3 & 7 & -1 \\ 3 & 11 & -1 \end{vmatrix} + (-1)^{2+4}\cdot 0\cdot \begin{vmatrix} 3 & 9 & 3 \\ 3 & 7 & 3 \\ 3 & 11 & 3 \end{vmatrix}$$

$$= -3\cdot \begin{vmatrix} 9 & 3 & -10 \\ 7 & 3 & -1 \\ 11 & 3 & -1 \end{vmatrix} - 3\cdot \begin{vmatrix} 3 & 9 & -10 \\ 3 & 7 & -1 \\ 3 & 11 & -1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 9 & -10 \\ 3 & 7 & -1 \\ 3 & 11 & -1 \end{vmatrix} - 3\cdot \begin{vmatrix} 3 & 9 & -10 \\ 3 & 7 & -1 \\ 3 & 11 & -1 \end{vmatrix}$$

3) Finding an orthogonal vector

a) $\vec{v} = (1\ 2\ 3) \qquad \vec{w} = (1,0,3)$

$$\vec{x} = \vec{v} \times \vec{w} = \begin{matrix} 1 & 1 \\ 2 & 0 \\ 3 & 3 \end{matrix} = \begin{bmatrix} 2\cdot 3 - 3\cdot 0 \\ 1\cdot 3 - 3\cdot 1 \\ 1\cdot 0 - 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

b)

$$\vec{x} = \vec{v} \times \vec{w} = \begin{matrix} v & w \\ 5 & 1 \\ 3 & 0 \\ 7 & 0 \end{matrix} = \begin{bmatrix} 3\cdot 0 - 7\cdot 0 \\ -(5\cdot 0 - 7\cdot 1) \\ 5\cdot 0 - 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ -3 \end{bmatrix}$$

6) Matrix Inversion

For 2x2 matrices we can utilize

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \qquad \mathbf{X}^{-1} = \det(\mathbf{X})^{-1} \begin{bmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{bmatrix}$$

a) $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2 \qquad A^{-1} = (-2)^{-1} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$= -\frac{1}{2} \cdot \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

b) $\det\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} = 0 + 9 = 9 \qquad A^{-1} = 9^{-1} \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$

$$= \frac{1}{9} \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$$

c) $\det\begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix} = 6 - 0 = 6 \qquad A^{-1} = 6^{-1} \begin{bmatrix} 6 & 0 \\ -3 & 1 \end{bmatrix}$

$$= \frac{1}{6} \begin{bmatrix} 6 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

For larger matrices use Gauss-Jordan

6) d) $\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 3 & 5 & 0 & 0 & 0 & 1 \end{array}\right)$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$ $R_3 - 3 \cdot R_1$

$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$ $R_2 - 3R_1$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & -5 & 0 & 2 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$ $R_1 + 2 \cdot R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{5}{3} & 2 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$ $R_1 - \frac{5}{3} \cdot R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{5}{3} & 2 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -3 & -3 & 1 & 0 \end{array}\right]$ $R_2 \leftrightarrow R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{5}{3} & 2 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & -3 & 1 & 0 \end{array}\right]$ $R_2 - R_3$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{5}{3} & 2 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array}\right]$ $\frac{R_3}{-3}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{5}{3} & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array}\right]$ $R_2 \cdot (-1)$

6)

e)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \quad \frac{R_2}{6} \;\&\; \frac{R_3}{-2}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

f)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & -1 \\ 3 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \quad R_2 - R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & -1 \\ 0 & 5 & 4 & -3 & 0 & 1 \end{array}\right] \quad R_3 - 3 \cdot R_1$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & -3 & 1 & 0 \\ 0 & 5 & 4 & -3 & 0 & 1 \end{array}\right] \quad R_2 + R_3$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & -3 & 1 & 0 \\ 0 & 0 & 4 & -\frac{1}{2} & -\frac{5}{6} & 1 \end{array}\right] \quad R_3 - \frac{5}{6} \cdot R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{5}{24} & \frac{1}{4} \end{array}\right] \quad \frac{R_2}{6} \;\&\; \frac{R_3}{4}$$