Methods 2 - 7

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Linear Algebra: Vectors and Matrices – Gill, chapter 4

- **Matrices**
    - Eigen-analysis

Elementary scalar calculus – Gill, chapter 5

- **Derivatives**
    - Definition
    - Notation
    - Rules

Eigenvalues and eigenvectors

A useful and theoretically important feature of a given square matrix is the set of **eigenvalues** associated with this matrix. Every $p \times p$ matrix $\mathbf{X}$ has p scalar values, $\lambda_i, i = 1, \ldots, p$, such that

$$\mathbf{X}\mathbf{h}_i = \lambda_i \mathbf{h}_i$$

for some corresponding vector $\mathbf{h}_i$. In this decomposition, λ_i is called an eigenvalue of $\mathbf{X}$ and h_i is called an **eigenvector** of $\mathbf{X}$. These eigenvalues show important structural features of the matrix. Confusingly, these are also called the **characteristic roots** and **characteristic vectors** of $\mathbf{X}$, and the process is also called **spectral decomposition**.

Properties of Eigenvalues for a Nonsingular $(n \times n)$ Matrix

$\rightarrowtail$	Inverse Property	If λ_i is an eigenvalue of $\mathbf{X}$, then $\frac{1}{\lambda_i}$ is an eigenvalue of $\mathbf{X}^{-1}$
$\rightarrowtail$	Transpose Property	$\mathbf{X}$ and $\mathbf{X}'$ have the same eigenvalues
$\rightarrowtail$	Identity Matrix	For $\mathbf{I}$, $\sum \lambda_i = n$
$\rightarrowtail$	Exponentiation	If λ_i is an eigenvalue of $\mathbf{X}$, thenλ_i^k is an eigenvalue of $\mathbf{X}^k$ and k a positive integer

Properties of the Quadratic, $\mathbf{y}$ Non-Null

<u>Non-Negative Definite:</u>

$\rightarrowtail$	positive definite	$\mathbf{y}'\mathbf{X}\mathbf{y} > 0$
$\rightarrowtail$	positive semidefinite	$\mathbf{y}'\mathbf{X}\mathbf{y} \geq 0$

<u>Non-Positive Definite:</u>

$\rightarrowtail$	negative definite	$\mathbf{y}'\mathbf{X}\mathbf{y} < 0$
$\rightarrowtail$	negative semidefinite	$\mathbf{y}'\mathbf{X}\mathbf{y} \leq 0$

We can also say that $\mathbf{X}$ is **indefinite** if it is neither nonnegative definite nor nonpositive definite. The big result is worth stating with emphasis:

A positive definite matrix is always nonsingular.

Properties of Limits, $\exists \lim_{x \to X} f(x), \lim_{x \to X} g(x)$**, Constant** k

$\rightarrowtail$ Addition and Subtraction

$$\lim_{x \to X} [f(x) + g(x)] = \lim_{x \to X} f(x) + \lim_{x \to X} g(x)$$

$$\lim_{x \to X} [f(x) - g(x)] = \lim_{x \to X} f(x) - \lim_{x \to X} g(x)$$

$\rightarrowtail$ Multiplication

$$\lim_{x \to X} [f(x)g(x)] = \lim_{x \to X} f(x) \lim_{x \to X} g(x)$$

$\rightarrowtail$ Scalar Multiplication

$$\lim_{x \to X} [kg(x)] = k \lim_{x \to X} g(x)$$

$\rightarrowtail$ Division $\left(\lim_{x \to X} g(x) \neq 0\right)$

$$\lim_{x \to X} [f(x)/g(x)] = \frac{\lim_{x \to X} f(x)}{\lim_{x \to X} g(x)}$$

$\rightarrowtail$ Constants

$$\lim_{x \to X} k = k$$

$\rightarrowtail$ Natural Exponent

$$\lim_{x \to \infty} \left[1 + \frac{k}{x}\right]^x = e^k$$

Summary of Derivative Theory

$\rightarrowtail$	Existence	$f'(x)$ at x exists iff $f(x)$ is continuous at x, and there is no point where the right-hand derivative and the left-hand derivative are different
$\rightarrowtail$	Definition	$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
$\rightarrowtail$	Tangent Line	$f'(x)$ is the slope of the line tangent to $f()$ at x; this is the limit of the enclosed secant lines

Derivative rules

Constant Rule: $\dfrac{d}{dx}(c) = 0$, where c is a constant

Power Rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

Chain Rule: $(f(g(x))' = f'(g(x))g'(x)$

$$\frac{d}{dx}(a^x) = a^x \ln(a) \qquad \frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Where I messed up

- **Video 4a, 7:00**

  Square 2-by-2 matrix should read:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

  (I reversed the indices of the off-diagonal elements.)

- **Video 5c, 1:35**

  Factor rule is $\left(kf(x)\right)' = kf'(x)$. k does **not** multiply x, it multiples f.