2022 IEEE 6th Information Technology and Mechatronics Engineering Conference (ITOEC) | 978-1-6654-3185-9/22/$31.00 ©2022 IEEE | DOI: 10.1109/ITOEC53115.2022.9734617

An Analytical Solution for Inverse Kinematics of 7-DOF Redundant Manipulators with Offsets at Elbow and Wrist

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***Abstract*—This paper proposes an analytical solution for inverse kinematics of the seven-degree-of-freedom (7-DOF) redundant manipulator with offsets at the elbow and the wrist which is modelled on Franka Emika manipulator. The modified Denavit-Hartenberg (D-H) model of the manipulator is established. Then according to the manipulator's configuration, the elbow offset structure is analyzed with geometric method. The wrist offset is solved by transforming coordinate, and the 7^{th} joint angle is given in advance. Based on the analysis of the elbow and wrist offsets, an inverse kinematic solution is derived by the fixed-arm-angle method. Moreover, due to the joint angle and workspace limits, the value range of the 7^{th} joint angle needed is analyzed and calculated. At the end of this paper, the joint motion of the manipulator is simulated in MATLAB, which verifies the correctness of the method for obtaining the inverse kinematic solution of the redundant manipulator with offsets at the elbow and the wrist.**

Keywords—7-DOF redundant manipulator; Elbow offset; Wrist offset; Inverse kinematics

I. INTRODUCTION

In three-dimensional space, in order to determine the position and orientation of the end-effector, the 6-DOF manipulator is required. Therefore, the 6-DOF manipulator can reach any given pose in its working space. However, due to the singularity of the structure and the existence of obstacles, the manipulator cannot complete some complex actions. The 7-DOF manipulator has a redundant degree of freedom, so compared with 6-DOF manipulator, it has higher flexibility, reliability and adaptability which can avoid the structural singularity and obstacle avoidance that often occur in 6-DOF manipulator. In some special requirements and environment, the 7-DOF manipulator can successfully complete complex tasks, so it is more and more applied to modern production and life.

Due to the redundancy of 7-DOF redundant manipulator, the same end-effector's pose corresponds to countless configurations in calculating inverse kinematics, so there are countless inverse kinematic solutions, which are difficult to solve. The general 7-DOF manipulator is S-R-S type. The rotation axes of the first three joints intersect at one point, and the latter three joints are the same. The middle joint connects the front and rear parts. This configuration is similar to the human arm. The first three joints can represent the shoulder joints, the fourth joint represents the elbow joint, and the last three joints can represent the wrist joints. So far, there has been a lot of research on the inverse kinematics of 7-DOF manipulator at home and abroad, mainly including numerical solutions [1], analytical method [2], iterative algorithms [3], and geometric parameter method [4]. In [2], an analytical method is used to obtain the inverse kinematic solution of the S-R-S 7-DOF manipulator, and the concept of arm angle is proposed. In [4], a geometric method for solving inverse kinematics is proposed, and the corresponding solution is obtained according to the given arm angle. These methods solve the inverse kinematics of the manipulator without offset. For the manipulator with offset, there is also a lot of research. In [5], the inverse kinematic solution of the manipulator with the wrist offset is obtained by using the "virtual joint", and the method of fixed-arm-angle is also used in the solution. In [6], an analytical solution for the inverse kinematics of manipulator with elbow offset is found, and the calculation methods for different elbow structures are analyzed. In [7], the modified gradient projection method is proposed for manipulator with wrist offset. In [8], for a redundant 7-DOF manipulator with elbow offset, a method based on joint parameterization is proposed to solve inverse kinematics. In [9], an analytical inverse kinematics solution of 7-DOF redundant manipulator with shoulder and wrist offsets is developed. It is based on a parametric method and uses different joints as redundant joints, and finally the suitable redundant joint is selected.

This paper takes Franka Emika manipulator as the model, which is 7-DOF manipulator with elbow and wrist offsets, and the method for inverse kinematics of the manipulator is proposed in this paper. In section II, we give the model of the manipulator, and establish coordinate system using modified D-H method to calculate the forward kinematics. In section III, the elbow offset structure is analyzed and for the wrist offset, the coordinate transformation method is introduced. Then the inverse kinematic solution of the manipulator is derived by the fixed-arm-angle method [2]. In section IV, the value range of the 7th joint angle is analyzed and calculated. In Section V, we perform kinematic simulation in MATLAB to verify the correctness of the method. In section IV, there is conclusion.

II. THE D-H MODEL OF THE MANIPULATOR

A. Manipulator model

This paper takes Franka Emika manipulator as the model. The model of the manipulator and the simplified model are as shown in Fig. 1. The rotation axes of the first three joints intersect at one point. The joint 4 is offset, the rotation axes of the joint 5 and the joint 6 intersect at one point, and the joint 7 is offset. The first three joints are called shoulder joints, the last three joints are called wrist joints, and joint 4 is called elbow joint. Use the modified D-H method to establish the coordinate system. For the state in Fig. 1, all joint angles are 0, and O is the origin of the base coordinate system, S is the origin of the first and second coordinate systems, C is the origin of the third coordinate system, E is the origin of the fourth coordinate system, and W_1 is the origin of the fifth coordinate system and the sixth coordinate system. W_2 is the origin of the seventh coordinate system, and F is the origin of the F coordinate system (i.e. the coordinate system of the end-effector) which is built at the end of the 7th joint.

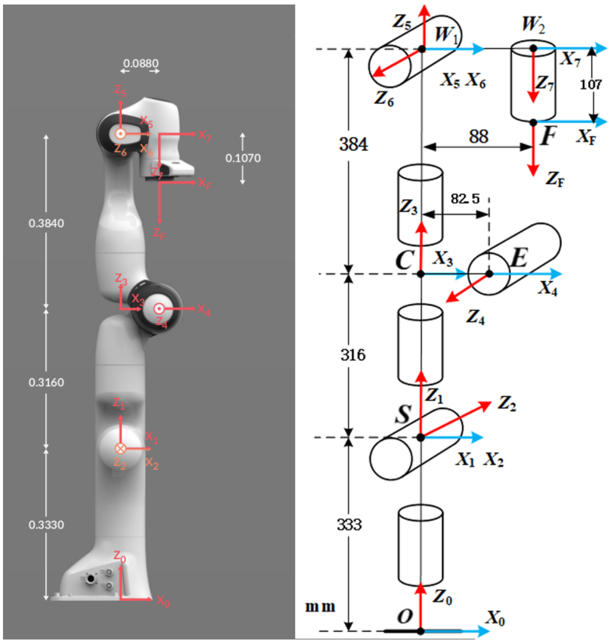

Fig. 1. The model of manipulator and its Simplified model.

B. Forward kinematics

The DH parameters of the 7-DOF manipulator are listed in Table I. The joint space limits are shown in Table II. From the Table I, we can know that a_3 and a_4 are elbow offsets, and a_6 is wrist offset.

TABLE I. THE D-H PARAMETERS OF MANIPULATOR

i	a_{i-1}(mm)	d_i(mm)	α_{i-1}(rad)	θ_i(rad)
1	0	333	0	θ_1
2	0	0	$-\pi/2$	θ_2
3	0	316	$\pi/2$	θ_3
4	82.5	0	$\pi/2$	θ_4
5	-82.5	384	$-\pi/2$	θ_5
6	0	0	$\pi/2$	θ_6
7	88	0	$\pi/2$	θ_7
F	0	107	0	0

TABLE II. JOINT SPACE LIMITS

	$q_{\min}$ /(rad)	$q_{\max}$ /(rad)
Joint 1	-2.8973	2.8973
Joint 2	-1.7628	1.7628
Joint 3	-2.8973	2.8973
Joint 4	-3.0718	-0.0698
Joint 5	-2.8973	2.8973
Joint 6	-0.0175	3.7525
Joint 7	-2.8973	2.8973

The forward kinematics equation can be calculated by using the homogeneous transformation matrix, and the parameters are given in Table I. The transformation matrix is given by

$$ {}^{i-1}T_i = \begin{bmatrix} {}^{i-1}R_i & {}^{i-1}P_i \\ 0..0 & 1 \end{bmatrix} \tag{1}$$

$$ {}^{i-1}R_i = \begin{bmatrix} C\theta_i & -S\theta_i & 0 \\ S\theta_i \bullet C\alpha_{i-1} & C\theta_i \bullet C\alpha_{i-1} & -S\alpha_{i-1} \\ S\theta_i \bullet S\alpha_{i-1} & C\theta_i \bullet S\alpha_{i-1} & C\alpha_{i-1} \end{bmatrix} \tag{2}$$

$$ {}^{i-1}P_i = \begin{bmatrix} a_{i-1} \\ -d_i \bullet S\alpha_{i-1} \\ d_i \bullet C\alpha_{i-1} \end{bmatrix} \tag{3}$$

$$ {}^{7}T_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_F \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$ {}^{0}T_F = {}^{0}T_1 \bullet {}^{1}T_2 \ldots \bullet {}^{6}T_7 \bullet {}^{7}T_F = \begin{bmatrix} {}^{0}R_F & {}^{0}P_F \\ 0 & 1 \end{bmatrix} \tag{5}$$

Where $C\theta_i$ and $S\theta_i$ represent $\cos\theta_i$ and $\sin\theta_i$. ${}^{i-1}T_i$ is the transformation matrix from $(i-1)^{th}$ coordinate system to i^{th} coordinate system. ${}^{i-1}R_i$ and ${}^{i-1}P_i$ give the rotation and position transformation of the i^{th} coordinate system relative to $(i-1)^{th}$ coordinate system. ${}^{7}T_F$ is the transformation matrix from 7th coordinate system to end-effector coordinate system. ${}^{0}T_F$ is the transformation matrix from base coordinate system to end-effector coordinate system. ${}^{0}R_F$ and ${}^{0}P_F$ can be used as the known parameters for solving inverse kinematics problems.

III. Inverse Kinematics Solution

A. The analysis of elbow offset structure

From Table I. we can know that the elbow joint has two offsets which is different from the general elbow offset shown in [8]. Therefore, a specific analysis of the elbow joint is needed to find a solution suitable for the elbow structure.

When the end-effector's pose ${}^{0}T_F$ is given, the joints of the manipulator will rotate at a certain angle. The simplified model is as shown in Fig. 2. In Fig. 2, when the elbow joint rotates at a certain angle, C' will also move with it. And the point C' moves on a circle with the point E as the center and the length of the line CE as the radius. But we need to know that since θ_4 has a range of values, the trajectory of point C' is only a part of the circle. $\angle CEC'$ is the joint angle θ_4 after the elbow joint rotates. D is the intersection of the extension lines of line W_1C' and line SC. According to the geometric relationship in the elbow structure in Fig. 2, we can get

$$\| CD \| = \| CE \| \bullet \tan\left(\theta_4 / 2\right) \tag{6}$$

$$\| C'D \| = \| EC' \| \bullet \tan\left(\theta_4 / 2\right) \tag{7}$$

$$\| SD \| = \| SC \| + \| CD \| \tag{8}$$

$$\| W_1 D \| = \| W_1 C' \| + \| C'D \| \tag{9}$$

In ΔDSW_1, the equation about θ_4 can be obtained according to the law of cosines.

$$\cos\theta_4 = \frac{\| {}^{0}L_{SW_1} \|^2 - \| SD \|^2 - \| W_1 D \|^2}{2 \bullet \| SD \| \bullet \| W_1 D \|} \tag{10}$$

$${}^{0}L_{SW_1} = {}^{0}P_{W_1} - {}^{0}P_S \tag{11}$$

Where ${}^{0}P_{W_1}$ is the position vector of point W_1 in the base coordinate system, and ${}^{0}P_S$ is the position vector of point S in the base coordinate system. ${}^{0}P_S$ can be easily obtained and ${}^{0}P_{W_1}$ can be obtained by the method in the part B. Substituting (11) into (10). We can know that there is only one unknown in (10), and that is θ_4. So solving the equation (10) will get θ_4.

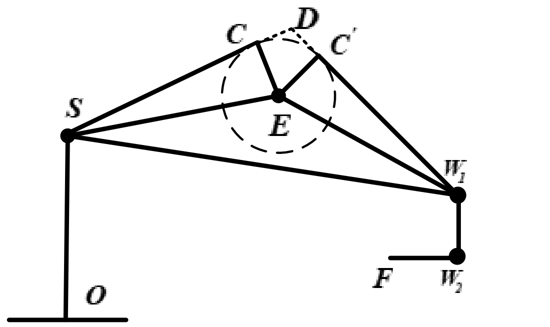

Fig. 2. Simplified model after the manipulator's joints rotate at an angle

B. The analysis of wrist joints

From the model of the manipulator, we can know that the rotation axis of joint 7 and the rotation axes of joint 5 and joint 6 do not intersect, which makes the latter three joints unable to form a spherical wrist joint. So, the fixed-arm-angle method cannot be used directly when solving the inverse kinematics.

We already know that W_2 is the origin of the 7th coordinate system, and W_1 is the origin of the 6th coordinate system. So the position of point W_1 is the same as the position of the 6th coordinate system, and the position of point W_2 is the same as the position of the 7th coordinate system. When the end-effector's pose ${}^{0}T_F$ is given, we can easily get ${}^{0}T_7$, so the point W_2 is fixed. If we can also get ${}^{0}T_6$, we will know the position of point W_1 relative to the base coordinate system. In this way, W_1 and W_2 are both fixed which means that the position of W_1 and the structure behind it are all fixed. Then we can use the position of W_1 and the structure in front of it to derive the inverse kinematic solution using fixed-arm-angle method. So, based on the above analysis, we can transform the coordinate system between the end-effector coordinate system and the 6th coordinate system to get the position of W_1.

The transformation matrix between the 6th coordinate system and the 7th coordinate system can be denoted by the homogeneous transformation matrix ${}^{6}T_7$. So, we have

$${}^{0}T_{W_2} = {}^{0}T_7 = {}^{0}T_F \bullet {}^{7}T_F^{-1} \tag{12}$$

$${}^{0}T_6 = {}^{0}T_{W_1} = {}^{0}T_{W_2} \bullet {}^{6}T_7^{-1} = \begin{bmatrix} {}^{0}R_6 & {}^{0}P_6 \\ 0 & 1 \end{bmatrix} \tag{13}$$

$${}^{0}P_{W_1} = {}^{0}P_6 \tag{14}$$

Where ${}^{7}T_F$ is known, ${}^{0}T_F$ is given, but the calculation of transformation matrix ${}^{6}T_7$ needs to know θ_7. Therefore, if we want to determine the position of point W_1, we need to give θ_7 in advance. After the position of W_1 is obtained, we can use the fixed-arm-angle method to obtain the inverse kinematics solution of the manipulator. In the following part, the detailed calculation method is given.

C. Inverse kinematics solution process

In the parts A and B, we analyze the elbow and wrist joints of the manipulator, and give the corresponding solutions to the elbow offset and wrist offset. In this part, the detailed calculation process of solving inverse kinematics is given.

We call the plane formed by the points S, E, and W_1 the arm plane. When $\theta_3 = 0$, the plane formed by points O, S, and W_1 is defined as the reference plane. Arm angle ψ is the angle between the reference plane and the arm plane. L_{SW} is the vector from S to W_1. L_{CE} is the vector from C to E. $L_{EC'}$ is the vector from E to C'. $L_{C'W_1}$ is the vector from C' to W_1. As shown in Fig. 3. The real movement of the manipulator can be regarded as reaching the target pose on the reference plane then rotating a certain angle around the axis SW_1.

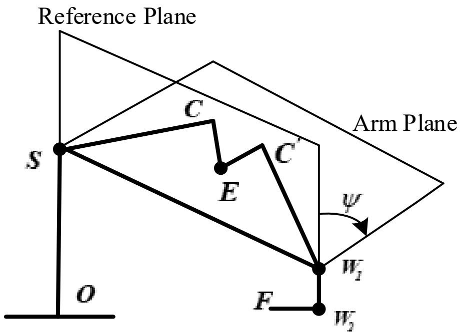

Fig. 3. Definition of arm plane, reference plane and arm angle

$$ {}^0R_\psi = \mathrm{I}_3 + \sin\psi \bullet \left[{}^0u_{sw} \times \right] + (1-\cos\psi) \bullet \left[{}^0u_{sw} \times \right]^2 \quad (15)$$

$$ {}^0R_k = {}^0R_\psi \bullet {}^0R_k^0 \quad (k = 1, 2..7) \quad (16)$$

Where $I_3 \in R^{3\times 3}$ is the identity matrix, ${}^0u_{sw} \in R^3$ is the unit vector of ${}^0L_{SW_1}$, and $\left[{}^0u_{sw} \times \right]$ denotes the skew-symmetric matrix of the vector ${}^0u_{sw}$. ${}^0R_k^0$ is the rotation matrix on the reference plane, ${}^0R_\psi$ is the rotation matrix rotating around the axis SW_1.

In the part B, using (12), (13), (14), we can get the position of point W_1 relative to the base coordinate system which is ${}^0P_{W_1}$.

1) Elbow joint

When analyzing the elbow joint, we know that C' will move on the circle with E as the center. And according to the part A, the elbow joint θ_4 is available.

2) The joint angle θ_1^0, θ_2^0, θ_3^0 in the reference plane

The transformation matrix ${}^3R_4$ can be calculated using θ_4. In addition to calculating the vector ${}^0L_{SW_1}$ using (11), there is another calculation method:

$$ {}^0L_{SW_1} = {}^0R_3^0 \bullet {}^3L_{SW_1} \quad (17)$$

$$ {}^3L_{SW_1} = {}^3L_{SC} + {}^3L_{CE} + {}^3R_4 \bullet {}^4L_{EC'} + {}^3R_4 \bullet {}^4L_{C'W_1} \quad (18)$$

$$ {}^0R_3^0 = R(\theta_1^0) \bullet R(\theta_2^0) \bullet \mathrm{R}(\theta_3^0) \quad (19)$$

θ_1^0, θ_2^0, θ_3^0 are the joint angles in the reference plane, and $\theta_3^0 = 0$. R_3^0 is the transformation matrix in the reference plane. R_3^0 is given by

$$ {}^0R_3^0 = \begin{bmatrix} \cos\theta_1^0\cos\theta_2^0 & -\sin\theta_1^0 & \cos\theta_1^0\sin\theta_2^0 \\ \sin\theta_1^0\cos\theta_2^0 & \cos\theta_1^0 & \sin\theta_1^0\sin\theta_2^0 \\ -\sin\theta_2^0 & o & \cos\theta_2^0 \end{bmatrix} \quad (20)$$

Substituting (11), (18) into (17), we have

$$ (-{}^3L_{SW_1}(1))\sin\theta_2^0 + {}^3L_{SW_1}(3)\cos\theta_2^0 = {}^0L_{SW_1}(3) \quad (21)$$

Solving the equation, we will get θ_2^0.

Because the Y coordinate of the vector ${}^3L_{SW_1}$ is 0, so we have

$$ \sin\theta_1^0 = \frac{{}^0L_{SW_1}(2)}{{}^3L_{SW_1}(1)\cos\theta_2^0 + {}^3L_{SW_1}(3)\sin\theta_2^0} \quad (22)$$

$$ \cos\theta_1^0 = \frac{{}^0L_{SW_1}(1)}{{}^3L_{SW_1}(1)\cos\theta_2^0 + {}^3L_{SW_1}(3)\sin\theta_2^0} \quad (23)$$

So far, θ_1^0, θ_2^0, θ_3^0 are all known. Substituting them into (20) to get ${}^0R_3^0$.

3) Shoulder joints

According to (16), we have

$$ {}^0R_3 = {}^0R_\psi \bullet {}^0R_3^0 \quad (24)$$

Substituting ${}^0R_3^0$ and ${}^0R_\psi$ into (24), we have

$$ {}^0R_3 = As \bullet \sin\psi + Bs \bullet \cos\psi + Cs \quad (25)$$

Where As, Bs, Cs are constant matrices given by

$$ As = \left[{}^0u_{sw} \times \right] {}^0R_3^0 $$

$$ Bs = -\left[{}^0u_{sw} \times \right]^2 {}^0R_3^0 $$

$$ Cs = \left[{}^0u_{sw} \, {}^0u_{sw}^{\ T} \right] {}^0R_3^0 $$

The rotation matrix ${}^0R_3$ is given by

$$ {}^0R_3 = \begin{bmatrix} * & * & \cos\theta_1\sin\theta_2 \\ * & * & \sin\theta_1\sin\theta_2 \\ -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 & \cos\theta_2 \end{bmatrix} \quad (26)$$

Where the elements denoted by * are omitted here. Combining some elements of this matrix, we can get the relations between the arm angle and the shoulder joint angles.

$$\cos\theta_2 = As_{33}\sin\psi + Bs_{33}\cos\psi + Cs_{33} \tag{27}$$

$$\sin\theta_1 = \frac{As_{23}\sin\psi + Bs_{23}\cos\psi + Cs_{23}}{\sin\theta_2} \tag{28}$$

$$\cos\theta_1 = \frac{As_{13}\sin\psi + Bs_{13}\cos\psi + Cs_{13}}{\sin\theta_2} \tag{29}$$

$$\sin\theta_3 = \frac{As_{32}\sin\psi + Bs_{32}\cos\psi + Cs_{32}}{\sin\theta_2} \tag{30}$$

$$\cos\theta_3 = \frac{As_{31}\sin\psi + Bs_{31}\cos\psi + Cs_{31}}{-\sin\theta_2} \tag{31}$$

Where As_{ij}, Bs_{ij} and Cs_{ij} are the (i, j) elements of the As, Bs and Cs.

4) Wrist joints

We have

$$^0R_7 = {}^0R_3 \bullet {}^3R_4 \bullet {}^4R_7 \tag{32}$$

After transformation, we can get

$$^4R_7 = ({}^3R_4)^T \bullet ({}^0R_3)^T \bullet {}^0R_7 \tag{33}$$

Substituting (25) into (33), we have

$$^4R_7 = Aw \bullet \sin\psi + Bw \bullet \cos\psi + Cw \tag{34}$$

Where Aw, Bw and Cw are constant matrices given by

$$Aw = ({}^3R_4)^T \bullet As^T \bullet {}^0R_7$$
$$Bw = ({}^3R_4)^T \bullet Bs^T \bullet {}^0R_7$$
$$Cw = ({}^3R_4)^T \bullet Cs^T \bullet {}^0R_7$$

The rotation matrix $^4R_7$ is given by

$$^4R_7 = \begin{bmatrix} * & * & \cos\theta_5\sin\theta_6 \\ \sin\theta_6\cos\theta_7 & -\sin\theta_6\sin\theta_7 & -\cos\theta_6 \\ * & * & -\sin\theta_5\sin\theta_6 \end{bmatrix} \tag{35}$$

Thus, the relations between the arm angle and the wrist joint angles are derived as

$$\cos\theta_6 = -\left(Aw_{23}\sin\psi + Bw_{23}\cos\psi + Cw_{23}\right) \tag{36}$$

$$\sin\theta_5 = \frac{Aw_{33}\sin\psi + Bw_{33}\cos\psi + Cw_{33}}{-\sin\theta_6} \tag{37}$$

$$\cos\theta_5 = \frac{Aw_{13}\sin\psi + Bw_{13}\cos\psi + Cw_{13}}{\sin\theta_6} \tag{38}$$

$$\sin\theta_7 = \frac{Aw_{22}\sin\psi + Bw_{22}\cos\psi + Cw_{22}}{-\sin\theta_6} \tag{39}$$

$$\cos\theta_7 = \frac{Aw_{21}\sin\psi + Bw_{21}\cos\psi + Cw_{21}}{\sin\theta_6} \tag{40}$$

Given the arm angle, the wrist joint angles can be calculated from these equations.

5) The calculation of arm angle ψ

We analyzed that using the fixed-arm-angle method to calculate the inverse kinematics solution, in addition to the given end-effector's pose ^0T_F, the joint angle θ_7 also needs to be given in advance. According to (39) and (40), we know that the arm angle ψ has a corresponding relationship with joint angle θ_7. When the joint angle θ_7 is given, the arm angle ψ is also determined. Using (39), we have

$$A\sin\psi + B\cos\psi + C = 0 \tag{41}$$

Where A, B, and C are given by

$$A = Aw_{22} + Aw_{21}\tan\theta_7$$
$$B = Bw_{22} + Bw_{21}\tan\theta_7$$
$$C = Cw_{22} + Cw_{21}\tan\theta_7$$

Using the method of trigonometric identity transformation to process the equation, we have

$$\psi = \operatorname{atan2}(-\mathrm{C}, \pm\sqrt{A^2 + B^2 - C^2}) - \operatorname{atan2}(\mathrm{B}, \mathrm{A}) \tag{42}$$

ψ can be calculated. Then substitute ψ into the equations of all the joint angles.

IV. The Value Range of θ_7

The given θ_7 directly affects the calculation result. Using different θ_7, we will get a different set of joint angles. However, the joint angle θ_7 cannot be any value in the joint limit which is [-2.8973, 2.8973] (rad). There is another limit on θ_7. In this section, the calculation process of value range will be given in detail.

Using (11), (12), (13) and (14), we can get the length of L_{SW_1}. The joint angle θ_7 is used as a parameter in the calculation, so the value range of θ_7 is related to the length of L_{SW_1}. However, during the movement of the manipulator, the length of L_{SW_1} is limited. so the value of θ_7 is also limited. If we want to get the value range of θ_7, we first need to calculate the length range of L_{SW_1}.

A. *The maximum length of* L_{SW_1}

When the length of L_{SW_1} is maximum, the movement of the manipulator is shown in Fig. 4. In this state, points S, E, W_1 are collinear. We have

$$\left\|L_{SW_1}\right\|_{\max} = \| SE \| + \| EW_1 \| \tag{43}$$

$$\| SE^2 \| = \| SC \|^2 + \| CE \|^2 \tag{44}$$

$$\| EW_1 \|^2 = \| C'E \|^2 + \| C'W_1 \|^2 \tag{45}$$

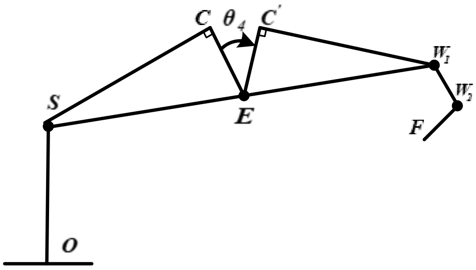

Fig. 4. Simplified model of the manipulator when the length of L_{SW_1} is maximum

B. The minimum length of L_{SW_1}

When the length of L_{SW_1} is minimum, the movement of the manipulator is shown in Fig. 5. In this state, $\|\angle CEC'\|$ is maximum (this is, $\|\theta_4\|$). We have

$$\angle SEW_1 = \pi - \angle CES - \angle MEW_1 \tag{46}$$

Where $\angle CES$ and $\angle MEW$ can be easily obtained.

In ΔESW_1, using the law of cosines, we have

$$\cos(\angle SEW_1) = \frac{\|SE\|^2 + \|EW_1\|^2 - \|SW_1\|_{\min}^2}{2\|SE\| \times \|EW_1\|} \tag{47}$$

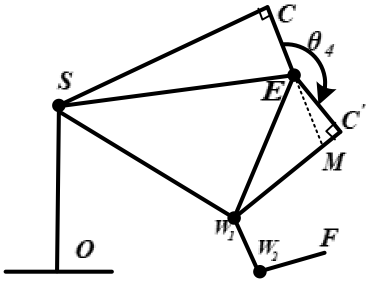

Fig. 5. Simplified model of the manipulator when the length of L_{SW_1} is minimum

When the end-effector's pose ${}^0T_F$ is given by

$$^0T_F = \begin{bmatrix} m_1 & n_1 & * & c_1 \\ m_2 & n_2 & * & c_2 \\ m_3 & n_3 & * & c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

we can get the expression of ${}^0L_{SW_1}$ which contains the parameter θ_7. According to the length range of L_{SW_1}, we have

$$\left\|{}^0L_{SW_1}\right\|^2 = A\cos\theta_7 + B\sin\theta_7 + C \tag{49}$$

$$\left\|L_{SW_1}\right\|_{\min} \le \left\|{}^0L_{SW_1}\right\| \le \left\|L_{SW_1}\right\|_{\max} \tag{50}$$

Where A, B, and C are given by

$$A = -2a_6\left(c_1 \bullet m_1 + c_2 \bullet m_2 + (c_3 - d_1) \bullet m_3\right)$$

$$B = 2a_6\left(c_1 \bullet n_1 + c_2 \bullet n_2 + (c_3 - d_1) \bullet n_3\right)$$

$$C = c_1^2 + c_2^2 + (c_3 - d_1)^2 + a_6^2$$

We can get the value range of θ_7 by using (50). Using any value within this range will get the corresponding solution for inverse kinematics, and there will be no solution beyond this range.

V. Simulations

In this section, we will perform kinematics simulation to verify the correctness of the proposed method for solving inverse kinematics. We will verify this in two ways. First, we give a specific pose in the workspace of the manipulator, and test whether the manipulator can reach the pose by solving inverse kinematics. Then, given a specific trajectory, test whether the manipulator can move according to the trajectory by solving inverse kinematics.

A. Simulation for a specific point

The specified pose of the manipulator is given by

$$^0T_F = \begin{bmatrix} -0.2811 & 0.4729 & -0.8351 & 349.7275 \\ 0.2559 & 0.8017 & -0.5401 & 490.6813 \\ -0.9249 & -0.3655 & 0.1044 & 299.1303 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \tag{51}$$

Given $\theta_7 = 30^\circ$. We can get a set of joint angles

[$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$]=[30, 59.9999, 29.9999, -60, 30, 30, 30] ($^\circ$)

Substituting the joint angles into (5). We have

$$^0T_F = \begin{bmatrix} -0.2811 & 0.4729 & -0.8351 & 349.7275 \\ 0.2559 & 0.8017 & -0.5401 & 490.6813 \\ -0.9249 & -0.3655 & 0.1044 & 299.1303 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \tag{52}$$

Given $\theta_7 = 60^\circ$. We can get a set of joint angles

[$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$]=[10.5839, 83.2376, 75.3846, -63.9206, 9.9610, 39.4243, 60] ($^\circ$)

Substituting the joint angles into (5). We have

$$^0T_F = \begin{bmatrix} -0.2811 & 0.4729 & -0.8351 & 349.7275 \\ 0.2559 & 0.8017 & -0.5401 & 490.6813 \\ -0.9249 & -0.3655 & 0.1044 & 299.1303 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \tag{53}$$

Comparing (50), (51) and (49), we can know that the method of solving inverse kinematics is correct. For different θ_7, we will get different solutions. When the value range of the given θ_7 is [0.5, 1.3] (rad), different motion poses of the manipulator are obtained, as shown in Fig. 6

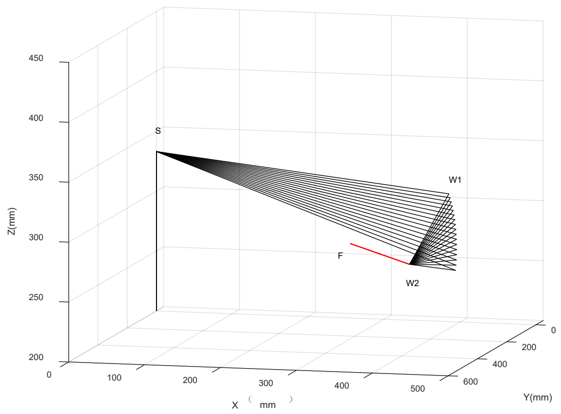

Fig. 6. The manipulator posture obtained when the value range of is [0.5, 1.3]

B. Simulation for a specific trajectory

Given a trajectory for the end position of the manipulator.

$$
\begin{aligned}
p_x &= 349.7275mm + 10 * \sin t \\
p_y &= 490.6813mm + 10 * \cos t \\
p_z &= 299.1303mm + 10 * t \\
t &\in [0,7], \Delta t = 0.1s
\end{aligned} \tag{54}
$$

Each time point corresponds to a position of end-effector, and for each position point, the inverse kinematics is solved by the method in this paper. The motion trajectory of the manipulator is obtained, as shown in Fig. 7, Fig. 8, Fig. 9, and Fig. 10. The change of all joint angles during movement is as shown in Fig. 11. The overall motion posture of the manipulator is as shown in Fig. 12. The position error of the manipulator is as shown in Fig. 13. When we change the posture of the manipulator, the obtained posture tracking curve is as shown in Fig. 14. The posture error is as shown in Fig. 15.

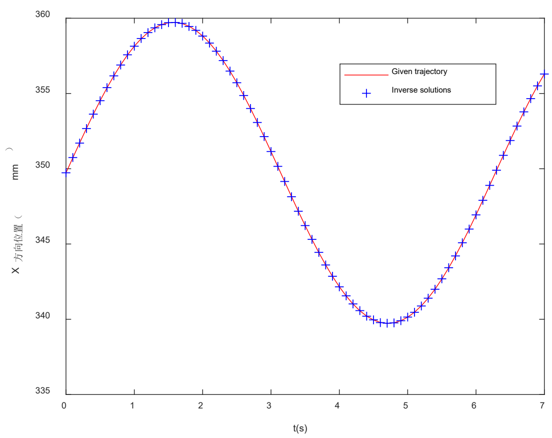

Fig. 7. Trajectory in X direction

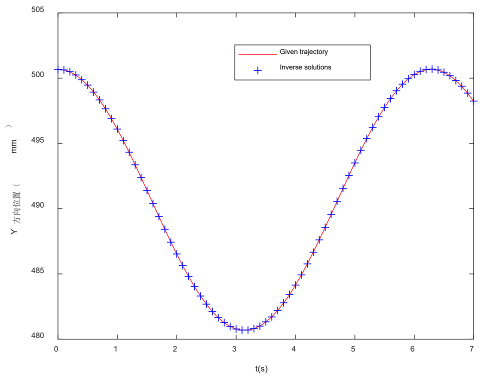

Fig. 8. Trajectory in Y direction

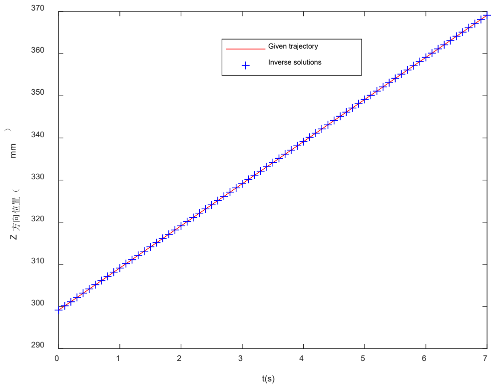

Fig. 9. Trajectory in Z direction

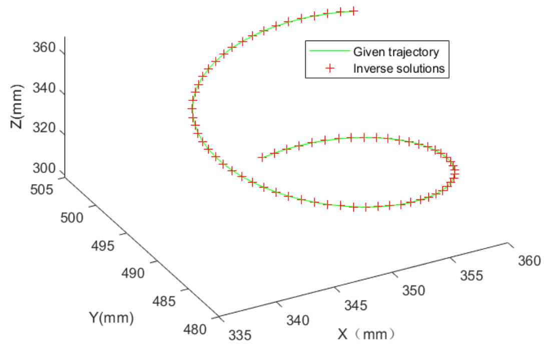

Fig. 10. Trajectory in X-Y-Z space

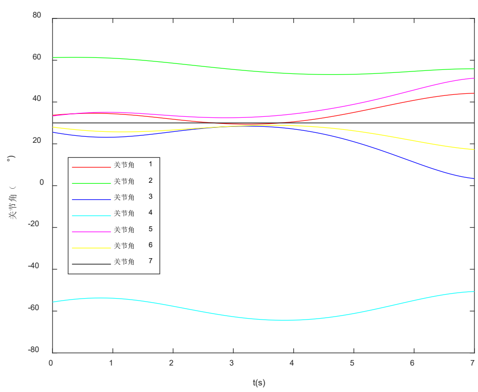

Fig. 11. Curves of joint angles

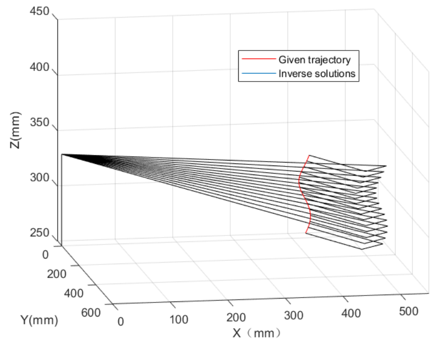

Fig. 12. The overall posture of the manipulator

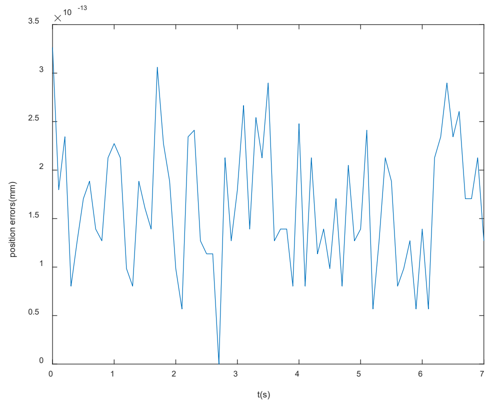

Fig. 13. The space position errors between the expected trajectory and the actual trajectory of the manipulator

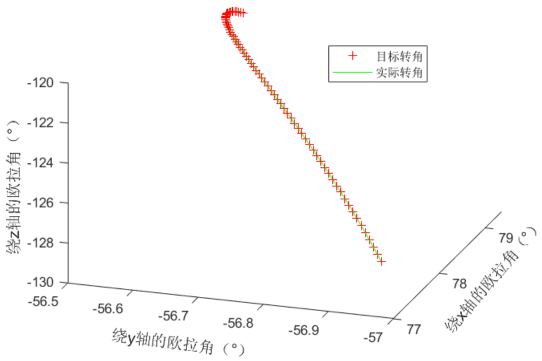

Fig. 14. Posture tracking

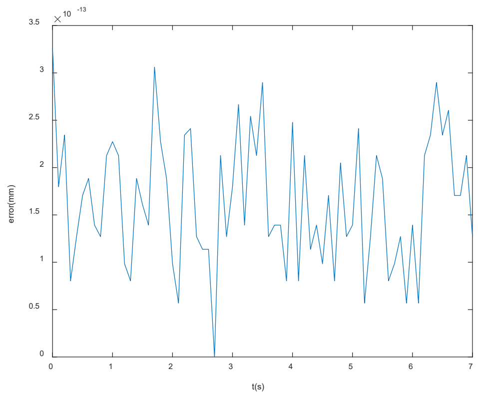

Fig. 15. The Posture errors between the given and actual posture

From Fig. 7-15, we can know that using the proposed method for inverse kinematics, the end-effector of the manipulator can move well in accordance with the given trajectory.

VI. CONCLUSIONS

This paper proposes a method to solve the inverse kinematics solution for a 7-DOF redundant manipulator with elbow offset and wrist offset. In this method, the specific elbow structure is analyzed, and the elbow joint angle is obtained by using geometric relations. For the wrist joints, we deal with wrist offset by transforming coordinate system. This method transforms the biased structure into an unbiased structure to solve the inverse kinematics. The actual motion posture of the manipulator is determined by θ_7, and each θ_7 corresponds to a motion posture of the manipulator. The θ_7 need to be given in advance in this method, and the value range can be calculated by geometric method. In the end, the correctness of the method is verified by MATLAB simulation.

REFERENCES

[1] K. Kreutzdelgado, M. Long, and H. Seraji, "kinematic analysis of 7-DOF manipulators," International Journal of Robotics Research, vol. 11, no. 5, pp. 469-481, 1992.

[2] M. Shimizu, H. Kakuya, and W. K. Yoon, "Analytical Inverse Kinematic Computation for 7-DOF Redundan Manipulators With Joint Limits and Its Application to Redundancy Resolution," IEEE Transactions on Robotics, vol. 24, no. 5, pp. 1131-1142, 2008.

[3] P. H. Chang, "a closed-form solution for inverse kinematics of robot manipulators with redundancy," IEEE Journal of Robotics and Automation, vol. 3, no. 5, pp. 393-403, 1987.

[4] B. C. Chen, G. Z. Cao, W. B. Li. An analytical solution of inverse kinematics for a 7- DOF redundant manipulator, 15th International Conference on Ubiquitous Robots, pp. 523-527., 2018.

[5] C. Yu, M. Jin, and H. Liu, "An analytical solution for inverse kinematic of 7-DOF redundant manipulators with offset-wrist," IEEE International Conference on Mechatronics and Automation, pp. 92- 97, 2012.

[6] G. K. Singh and J. Claassens, "An analytical solution for the inverse kinematics of a redundant 7DoF Manipulator with link offsets," IEEE/RSJ International Conference on Intelligent Robots and Systems, pp. 2976-2982, 2010.

[7] L. Jiang, X. Huo, Y. Liu, and H. Liu, "An integrated inverse kinematic approach for the 7-DOF humanoid arm with offset wrist," IEEE International Conference on Robotics and Biomimetics (ROBIO), pp. 2737-2742, 2013.

[8] Z. Chen, Z. Zeng, G. Shu, and Q. Chen, "Kinematic solution and singularity analysis for 7-DOF redundant manipulators with offsets at the elbow," IEEE Industrial Cyber-Physical Systems (ICPS), pp. 422-427, 2018.