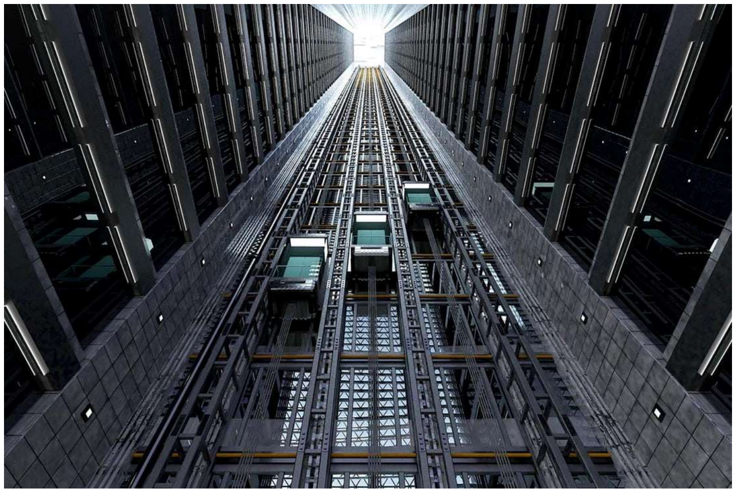
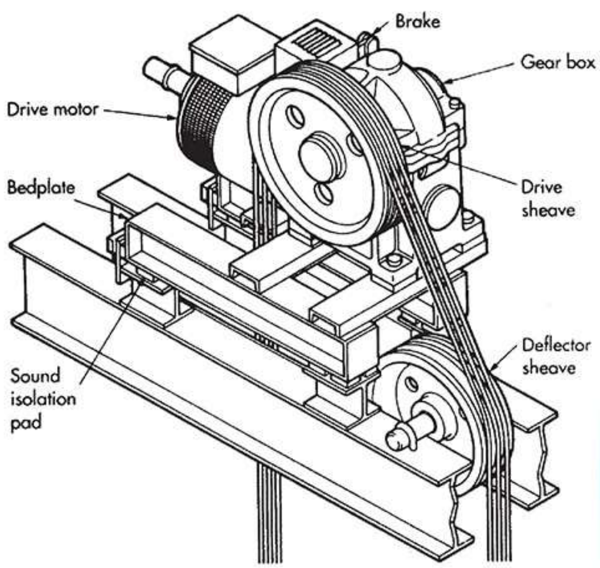
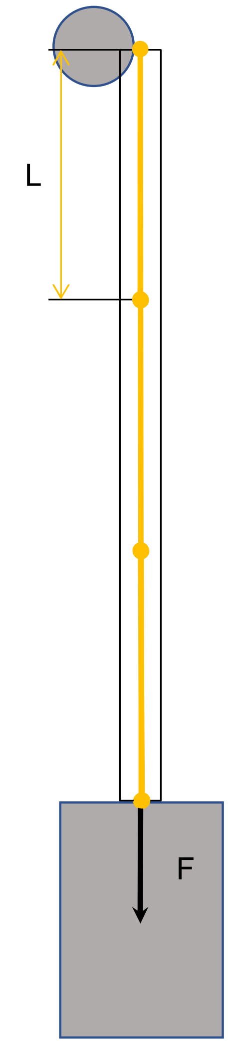
Today’s Session Summary

- Elevator problem
- Rod Element
- Truss Structures

Elevator Problem



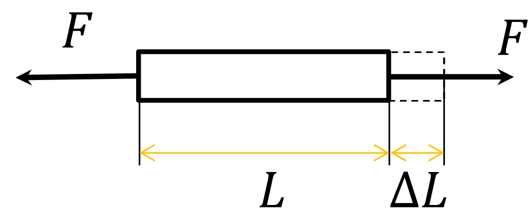




Rod stiffness

Known data:

- Length (L[m])
- Area (A[m2])
- Longitudinal Elasticity (Young's Modulus E[Pa])



Stress $\frac{F}{A} = \sigma$

Strain $\frac{\Delta L}{L} = \varepsilon$

Hooke's Law
(Elastic regime) $\sigma = E\varepsilon$

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

$$F = \frac{AE}{L}\Delta L$$

Rod Stiffness $F = k\ \Delta L$

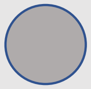
Elevator Problem

$$\frac{F}{\sigma_{adm}} = A = \frac{5000\text{kg } 9{,}81\text{m/s}^2}{250MPa} = 0{,}0001962m^2$$

Known data:
- $L = 40m$

Material
- $E = 200GPa$

$$0{,}0001962m^2 = \frac{\pi D^2}{4}$$

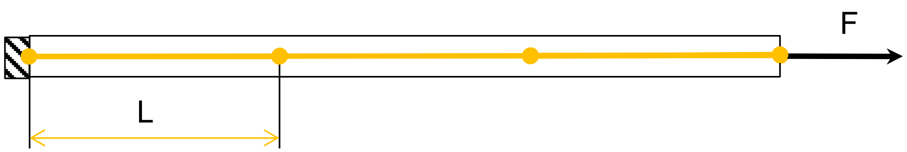


Yield Stress
- $\sigma_y = 500MPa$
- $\sigma_{adm} = 250MPa$

- Necessary area
- Extension

$$\frac{\text{LF}}{\text{AE}} = \Delta\text{L} = \frac{40\text{m } 5000\text{kg } 9{,}81\text{m/s}^2}{0.0001962m^2 200\text{GPa}}$$

Rod element



$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} u^I{}_1 \\ u^I{}_2 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix}$$

$$[K_e]\{D_e\} = \{R_e\}$$

$$u^I{}_1 = 1m; u^I{}_2 = 0$$
$$u^I{}_1 = 0; u^I{}_2 = 1m$$

Unitary displacements

$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ 1m \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} k^I{}_{12} \\ k^I{}_{22} \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} = \frac{A_e E_e}{L_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Stiffness Matrix:

$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} 1m \\ 0 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} k^I{}_{11} \\ k^I{}_{21} \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} = \frac{A_e E_e}{L_e} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$[K_e] = \begin{bmatrix} X^I & -X^I \\ -X^I & X^I \end{bmatrix}; X^I = \frac{A_e E_e}{L_e}$$

Rod element - Assembly

Local and Global Dof
$$\begin{cases} u_1 = {u^{I}}_{1} \\ u_2 = {u^{I}}_{2} = {u^{II}}_{1} \\ u_3 = {u^{II}}_{2} \end{cases}$$

$$\begin{bmatrix} X^I & -X^I \\ -X^I & X^I \end{bmatrix} \begin{Bmatrix} {u^I}_1 \\ {u^I}_2 \end{Bmatrix} = \begin{Bmatrix} {F^I}_1 \\ {F^I}_2 \end{Bmatrix} \rightarrow \begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} {F^I}_1 \\ {F^I}_2 \\ 0 \end{Bmatrix}$$

$$\underbrace{\begin{bmatrix} X^{II} & -X^{II} \\ -X^{II} & X^{II} \end{bmatrix}}_{[K_e]} \underbrace{\begin{Bmatrix} {u^{II}}_1 \\ {u^{II}}_2 \end{Bmatrix}}_{\{D_e\}} = \underbrace{\begin{Bmatrix} {F^{II}}_1 \\ {F^{II}}_2 \end{Bmatrix}}_{\{R_e\}} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ {F^{II}}_1 \\ {F^{II}}_2 \end{Bmatrix}$$

$$\left(\begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} {F^I}_1 \\ {F^I}_2 + {F^{II}}_1 \\ {F^{II}}_2 \end{Bmatrix} \rightarrow \begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I + X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Rod element - Assembly

Local and Global Dof $\begin{cases} u_1 = {u^I}_1 \\ u_2 = {u^I}_2 = {u^{II}}_1 \\ u_3 = {u^{II}}_2 \end{cases}$

$$\underbrace{\begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I + X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}_{\{D\}} = \underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}}_{\{R\}} \quad ; \quad \textit{Boundary conditions} \qquad \begin{cases} u_1 = 0 \\ F_2 = 0 \end{cases}$$

$$\begin{matrix} [-X^I \quad 0] \\ \underbrace{\begin{bmatrix} X^I + X^{II} & -X^{II} \\ -X^{II} & X^{II} \end{bmatrix}}_{[K_r]} \end{matrix} \underbrace{\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}}_{\{D_r\}} = \underbrace{\begin{matrix} \{F_1\} \\ \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix} \end{matrix}}_{\{R_r\}} \quad \therefore$$

$$[-X^I \quad 0] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{AE}{L} [-1 \quad 0] \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \{F_1\}$$

undetermined

$$\begin{bmatrix} X^I + X^{II} & -X^{II} \\ -X^{II} & X^{II} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix}$$

determined

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{X} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{LF_3}{AE} \\ \dfrac{2LF_3}{AE} \end{Bmatrix}$$

$$\frac{AE}{L} [-1 \quad 0] \begin{Bmatrix} \dfrac{LF_3}{AE} \\ \dfrac{2LF_3}{AE} \end{Bmatrix} = -\{F_3\}$$

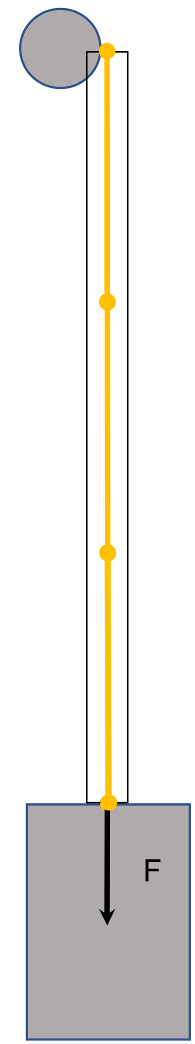
Rod wire - Structure

$$\frac{A_eE_e}{L_e}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}F_1\\ F_2\\ F_3\\ F_4\end{Bmatrix}$$

$$F^I{}_1=F^I{}_2=\frac{qL_e}{2}\rightarrow\frac{A_eE_e}{L_e}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}R\\ qL\\ qL\\ 0.5qL+F\end{Bmatrix}$$

$$\begin{Bmatrix}u_2\\ u_3\\ u_4\end{Bmatrix}=\frac{L_e}{A_eE_e}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{bmatrix}\begin{Bmatrix}qL_e\\ qL_e\\ 0.5qL_e+F\end{Bmatrix}=\frac{L_e}{A_eE_e}\begin{Bmatrix}2.5qL_e+F\\ 4qL_e+F\\ 4.5qL_e+F\end{Bmatrix}$$

$$\{R\}=\frac{A_eE_e}{L_e}\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\frac{L_e}{A_eE_e}\begin{Bmatrix}2.5qL_e+F\\ 4qL_e+F\\ 4.5qL_e+F\end{Bmatrix}=-2.5qL_e-F$$



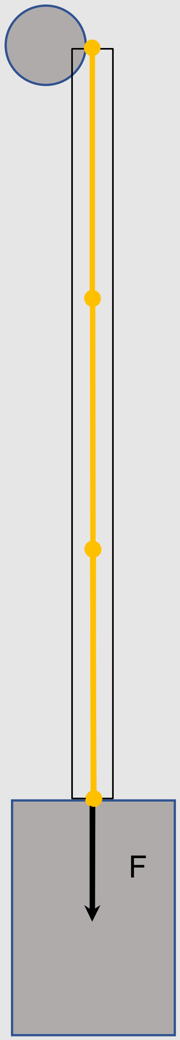
Elevator Problem

Data:
- $L = 200m$
- $F = 4000kg\ 9.81m/s^2$

Material
- $E = 200GPa$

Yield Stress
- $\sigma_y = 500MPa$
- $\sigma_{adm} = 250MPa$

Requests:
- Necessary area
- Consider wire weight
- Extension



UNIVERSIDAD TORCUATO DI TELLA

Today’s Session Summary

- Truss Structures

Transformations – Bar elements in plane structures

$$V_{12} = (x_2 - x_1, y_2 - y_1)$$

$$\frac{V_{12}}{|V_{12}|} = \{l, m\}; \underbrace{\begin{bmatrix} u_1' \\ v_1' \end{bmatrix}}_{\{d'\}} = \underbrace{\begin{bmatrix} l & m \\ -m & l \end{bmatrix}}_{[T]} \underbrace{\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}}_{\{d\}}; [T][T]^T = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{bmatrix} l & -m \\ m & l \end{bmatrix}; [T]^T = [T]^{-1}$$

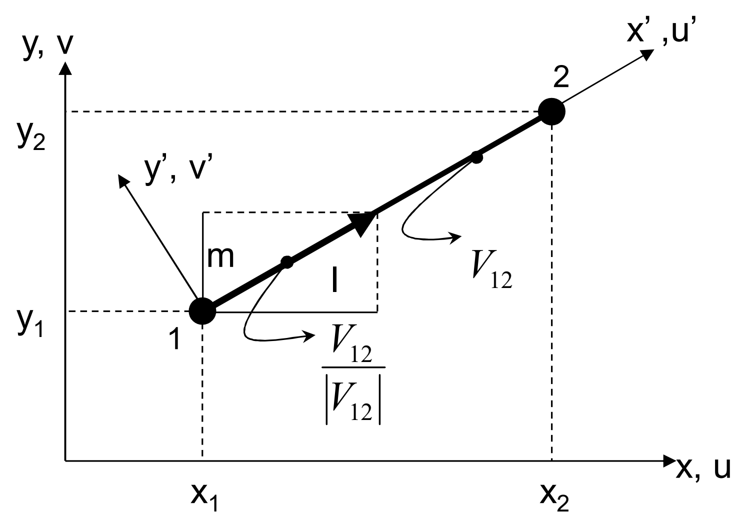
$$l = \cos(\beta)$$
$$m = \sin(\beta)$$

$$\{d'\} = [T]\{d\} \rightarrow \underbrace{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}}_{\{d'\}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}}_{[T]} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}}_{\{d\}}$$

$$[k']\{d'\} = \{r'\} \rightarrow [k'][T]\{d\} = \{r'\}$$

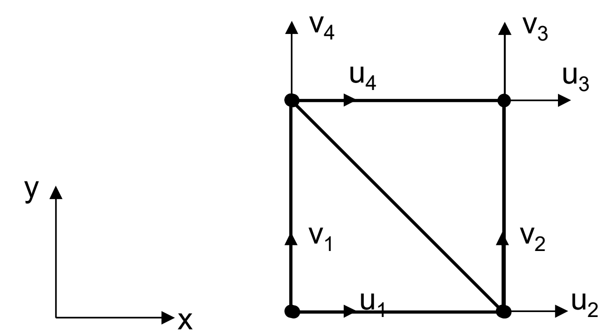
$$[T]^T[k'][T]\{d\} = [T]^T\{r'\}\, and \{r'\} = [T]\{r\}$$

$$[T]^T[k'][T]\{d\} = \{r\} \rightarrow [k] = [T]^T[k'][T]$$



Structure - Assembly

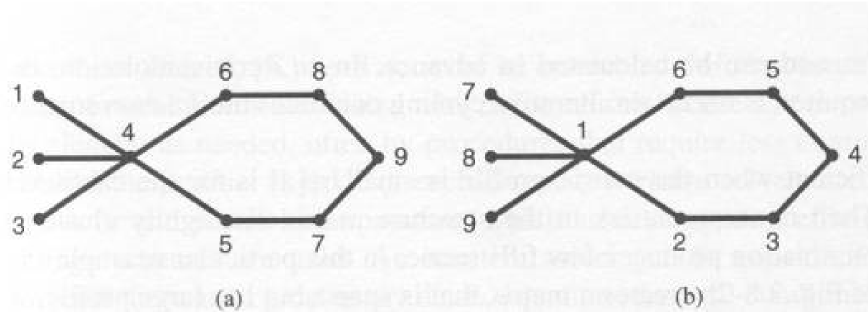
$$\begin{array}{ccc}
\begin{bmatrix}
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & -
\end{bmatrix}
&
\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix}
\;\leftrightarrow\;
\begin{matrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{matrix}
\\
\begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} &
\end{array}$$

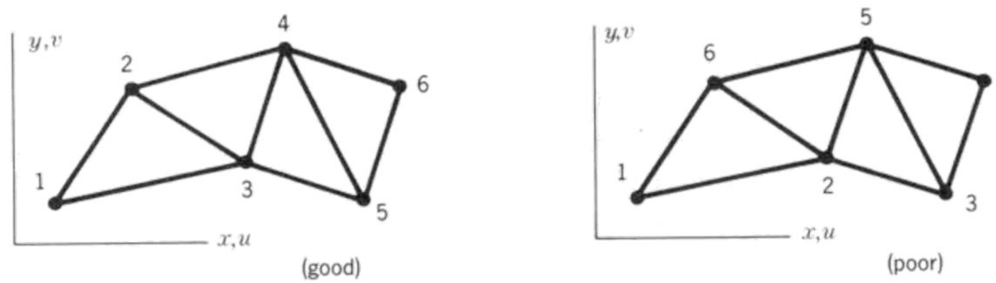


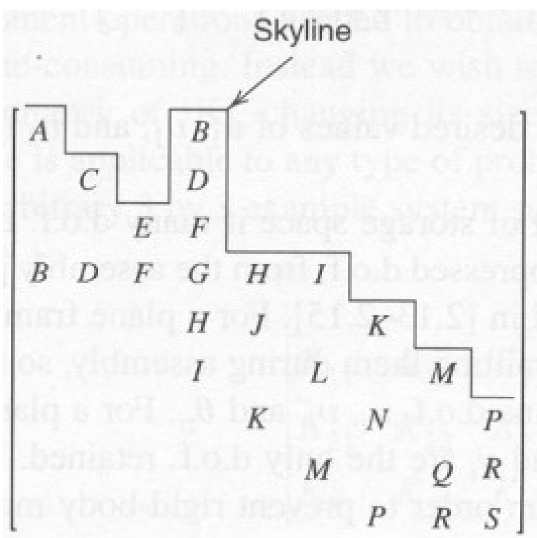
Local or elemental dof vs Global or structural dof

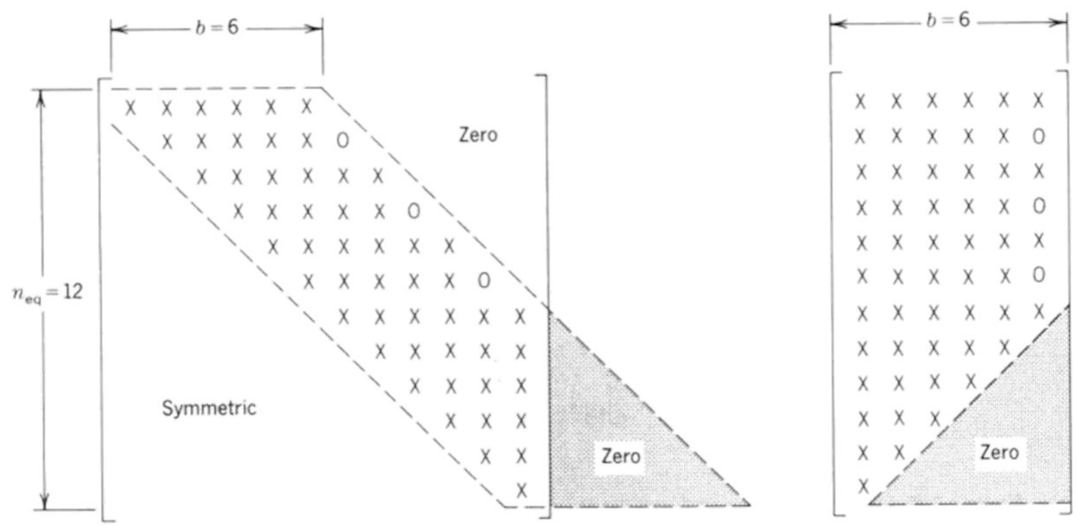
Stiffness Matrix Conectivity

- Storage
- Dof numbering
- Solve









Structure - Assembly

$$M = \begin{bmatrix} A & 0 & 0 & B & 0 & 0 & 0 & 0 \\ 0 & C & 0 & D & 0 & 0 & 0 & 0 \\ 0 & 0 & E & F & 0 & 0 & 0 & 0 \\ B & D & F & G & H & I & 0 & 0 \\ 0 & 0 & 0 & H & J & 0 & K & 0 \\ 0 & 0 & 0 & I & 0 & L & 0 & M \\ 0 & 0 & 0 & 0 & K & 0 & N & R \\ 0 & 0 & 0 & 0 & 0 & M & R & S \end{bmatrix}$$

Bandwidth

$$M = \begin{bmatrix} A & 0 & 0 & B \\ C & 0 & D & 0 \\ E & F & 0 & 0 \\ G & H & I & 0 \\ J & 0 & K & 0 \\ L & 0 & M & \otimes \\ N & R & \otimes & \otimes \\ S & \otimes & \otimes & \otimes \end{bmatrix}$$

Skyline

$$M = [A \quad C \quad E \quad B \quad D \quad F \quad G \quad H \quad J \quad I \quad 0 \quad L \quad K \quad \ldots]$$
$$v = [1 \quad 2 \quad 3 \quad 7 \quad 9 \quad 12 \quad 15 \quad 18]$$

Sparse

$$M = [A \quad B \quad C \quad D \quad E \quad F \quad G \quad H \quad I \quad J \quad K \quad L \quad M \quad \ldots]$$
$$c = [1 \quad 4 \quad 2 \quad 4 \quad 3 \quad 4 \quad 1 \quad \ldots]$$
$$r = [1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 12 \quad 15 \quad 18]$$

Storage: Full $O(n^2)$ vs sparse $O(n)$

Truss Structure – Test case

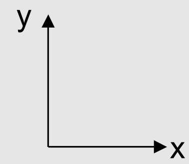
For the following structures made of steel (E=200GPa, A=50mm2):

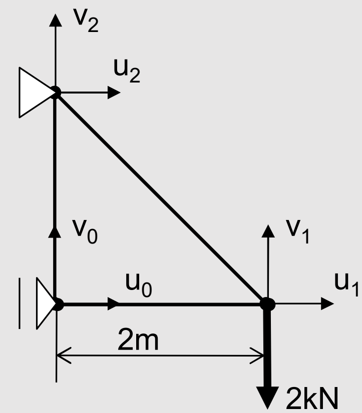
Determine:

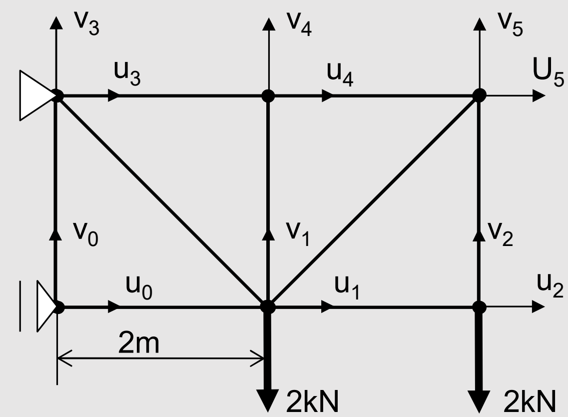
1. Nodal Displacements
2. Reaction forces (Validate!)
3. Element stresses (Validate!)

Analyze what changes when:

1. Fixed support is added to node 5.
2. Element connecting node 1 to with node 5 is removed.
3. Loads P are doubled.
4. Loads P are halved.

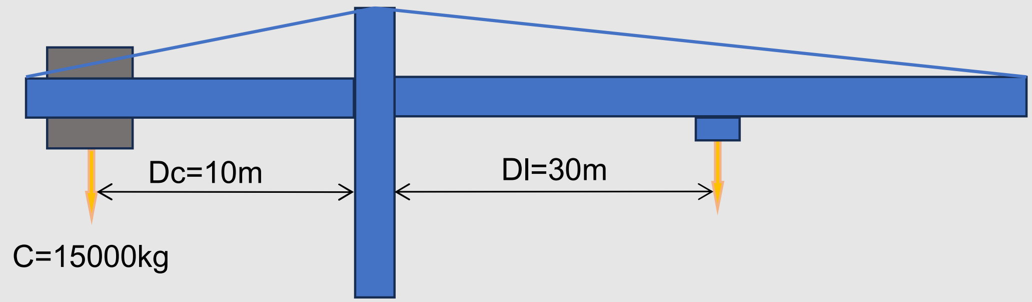






Practice

Truss Structure



End Session 14