Big O Notation

Analyzing the runtime of your algorithm

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What is Big O Notation?

Runtime

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You're not allowed to try it out and time it.

Space complexity

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Finding the *space complexity* works just the same as analyzing the runtime.

Definition

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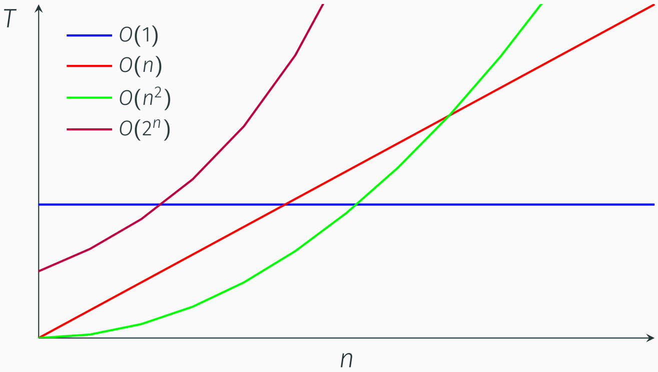
For double the size of the input, will your algorithm take twice as long? Four times as long? More?

Example



Definition (continued)

For an algorithm that runs in time $O(n)$, its runtime is a constant times the size of the input.

$$c \cdot n$$

where c is a real number.

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For an algorithm that runs in time $O(n^2)$, its runtime is a constant times the square of the size of the input.

$$c \cdot n^2$$

A simple `for` loop

For loop: example

Finding the maximum number in a list of n random numbers.

- initialize $M = 0$
- compare each number in the list to the number max. If it is bigger, it becomes the new M

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- Total cost $c_1 + n \cdot c_2$

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No, because it does not grow as fast as the cost of iterating over the loop.

We only care about the fastest-growing term. Therefore we drop the smaller terms in our cost analysis.

Drop the constants

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Recap: What will happen when an algorithm with $O(n)$ runtime gets twice as much input?

Answer: the runtime will double.

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What happens if we double the lengths of both lists?

We get $(2n)^2 = 4n^2$ comparisons: if the input is doubled, the runtime is multiplied by 4.

Repeatedly dividing by 3

Logarithm (1/3)

We start with a big number n and we keep dividing it by 3 until it is smaller than 3.

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If $3^{k-1} \leq n \leq 3^k$, we make k divisions, where $k = {}^3\log(n)$.

Logarithm (2/3)

Logarithm to which base?

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$$^{a}\log(x) = \frac{^{b}\log(x)}{^{b}\log(a)}$$

Does not matter, since $^{b}\log(a)$ is just a constant.

Conclusion: repeatedly dividing by 3 has runtime $O(\log n)$.

Logarithm (3/3)

What happens if we multiply the number n by 5, and run it through the same algorithm of dividing by 3?

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Answer: we get $\log(5n) = \log(5) + \log(n)$ divisions.

The knapsack problem

Knapsack: example

You have n items, each with their own weight $w_1, \ldots, w_n$. Since you're flying, the contents of your suitcase cannot be heavier than X. This means you have to leave some items at home. How do you fill up your suitcase as heavy as possible?

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There are 2^n possibilities of how to fill up your suitcase. You have to consider each of them, so your runtime is $O(2^n)$.

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Because the knapsack problem is NP-complete, you can't do better if you want to be sure to get the best solution. But maybe you'll get an acceptable outcome with an *approximation algorithm*.

Advice

Practice

- Solve a ton of exercises.

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- Then solve some more.

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- Then solve some more.
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- Math is great and so are you!
- Ask questions! Ask anyone, anytime, anywhere.