Sahand Communications in Mathematical Analysis (SCMA) Vol. 12 No. 1 (2018), 167-177

http://scma.maragheh.ac.ir

DOI: 10.22130/scma.2017.27822

Some Properties of Reproducing Kernel Banach and Hilbert Spaces

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ABSTRACT. This paper is devoted to the study of reproducing kernel Hilbert spaces. We focus on multipliers of reproducing kernel Banach and Hilbert spaces. In particular, we try to extend this concept and prove some related theorems. Moreover, we focus on reproducing kernels in vector-valued reproducing kernel Hilbert spaces. In particular, we extend reproducing kernels to relative reproducing kernels and prove some theorems in this subject.

1. Introduction

The study of positive definite kernels and the associated Hilbert spaces, plays a key role in both pure and applied mathematics. Let H be a Hilbert space of functions on some set X, which every point evaluation at points in X are continuous in the norm of H. It is known that such space H, can be represented by a positive definite kernel. This fact is first introduced by Aronszajn in his classic paper [2]. These Hilbert spaces are called reproducing kernel Hilbert spaces. Since distribution functions are well defined types of these kind of kernels, probability spaces are the best examples of practical reproducing kernel Hilbert spaces. More details about applications of reproducing kernel Hilbert spaces can be found in [3] and [5]. As an extension of reproducing kernels, relative reproducing kernels produced by Alpay in [1]. The kernels of relative reproducing kernel Hilbert spaces are characterized in terms of conditional negativity rather than definite positivity. The notion of conditional negativity plays a role in infinitely divisible distributions.

2010 *Mathematics Subject Classification.* 47B32; 47A70.

Key words and phrases. Reproducing kernel, Multipliers, Vector-valued spaces

Received: 22 October 2016, Accepted: 26 March 2017.

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On the applied side, relative kernels can be used to express electrical network models [1].

In other way, the study of reproducing kernel Banach spaces becomes important to use in machine learning. Learning a function from its finite samples is a fundamental science problem. The essence in achieving this is to choose an appropriate measurement of similarities between elements in the domain of the function. Motivated by multi-task machine learning with Banach spaces, notion of vector-valued reproducing kernel Banach spaces plays an important role. Basic properties of the spaces and the associated reproducing kernels are investigated.

In this paper, we try to extend some notions in reproducing kernel Banach and Hilbert spaces to vector-valued forms and find some useful applications. In Section 2, we express some fundamental definitions and theorems. In Section 3, we extend definitions and theorems to vector-valued spaces and verify relations between them.

2. Preliminaries

In following, we need some preliminaries of reproducing kernel Hilbert spacres, reproducing kernel Banach space and modules over C^* algebras. We review them in three subsections.

2.1. **Vector-Valued Reproducing Kernel Hilbert Spaces.** Given a set X and a normed vector space $\mathcal{Y}$, a map $K : X \times X \to \mathcal{L}(\mathcal{Y})$ is called a $\mathcal{Y}$-reproducing kernel if

$$\sum_{i,j=1}^{n} \langle K(x_i, x_j)y_j, y_i\rangle \geq 0, \tag{2.1}$$

for any $x_1, \ldots, x_n$ in X, $y_1, \ldots, y_n$ in $\mathcal{Y}$ and $n \geq 1$. Given $x \in X$, $K_x : \mathcal{Y} \to \mathcal{F}(X, \mathcal{Y})$ denotes the linear operator whose action on a vector $y \in \mathcal{Y}$ is the function $K_{xy} \in \mathcal{F}(X, \mathcal{Y})$ defined by

$$(K_{xy})(t) = K(t, x)y, \quad t \in X. \tag{2.2}$$

Given a $\mathcal{Y}$-reproducing kernel K, there is a unique Hilbert space $\mathcal{H}_K \subset \mathcal{F}(X, \mathcal{Y})$ satisfying

$$\begin{aligned} &K_x \in L(\mathcal{Y}, \mathcal{H}_K), \quad x \in X, \\ &f(x) = K_x^* f, \quad x \in X, f \in \mathcal{H}_K, \end{aligned} \tag{2.3}$$

where $K_x^* : \mathcal{H}_K \to \mathcal{Y}$ is the adjoint of K_x. The space $\mathcal{H}_K$ is called the reproducing kernel Hilbert space associated with K, the corresponding scalar product and norm are denoted by $\langle ., .\rangle_K$ and $\|.\|_K$, respectively.

As a consequence of (2.3), we have

$$K(x,t) = K_x^* K_t, \quad x, t \in X,$$
$$\mathcal{H}_K = \overline{span}\{K_{xy} | x \in X, y \in Y\}.$$

Let $\mathcal{H}$ be a Hilbert space of functions defined on the set X. We say that it is a relative reproducing kernel Hilbert space if there exists a function $M_{x,y}$ from $X \times X$ into $\mathcal{H}$ such that

$$F(x) - F(y) = \langle F, M_{x,y} \rangle_{\mathcal{H}}, \quad \forall x, y \in X, \forall F \in \mathcal{H}.$$

Given a set X, we say that $\mathcal{H}$ is a reproducing kernel Hilbert space (abbreviation RKHS) on X over $\mathbb{F}$, provided that:

(i) $\mathcal{H}$ is a vector subspace of $\mathcal{F}(X, \mathbb{F})$,
(ii) $\mathcal{H}$ is endowed with an inner product $\langle ., . \rangle$ making it into a Hilbert space,
(iii) for every $y \in X$, the linear evaluation functional $E_y : \mathcal{H} \to \mathbb{F}$ defined by $E_y(f) = f(y)$ is bounded.

If $\mathcal{H}$ is a RKHS on X, then since every bounded linear functional is given by the inner product with a unique vector in $\mathcal{H}$, for every $y \in X$, there exists a unique vector $k_y \in \mathcal{H}$ such that

$$f(y) = \langle f, k_y \rangle, \quad \forall f \in \mathcal{H}.$$

The function k_y is called the reproducing kernel for the point y. The 2-variable function defined by $K(x,y) = k_y(x)$ is called the reproducing kernel for $\mathcal{H}$.
Note that we have,

$$K(x,y) = k_y(x) = \langle k_y, k_x \rangle,$$
$$\|E_y\|^2 = \|k_y\|^2 = \langle k_y, k_y \rangle = K(y,y).$$

Definition 2.1. Let $\mathcal{H}$ be a RKHS on X with kernel function K. A function $f : X \to \mathbb{C}$ is called a multiplier of $\mathcal{H}$ provided that $f\mathcal{H} = \{fh : h \in \mathcal{H}\} \subseteq \mathcal{H}$. Let $\mathcal{M}(\mathcal{H})$ or $\mathcal{M}(K)$ denote the set of multipliers of $\mathcal{H}$. More generally, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two RKHSs on X with reproducing kernels, K_1 and K_2, respectively. Then a function $f : X \to \mathbb{C}$ is called a multiplier of $\mathcal{H}_1$ into $\mathcal{H}_2$ if $f\mathcal{H}_1 \subseteq \mathcal{H}_2$. We denote by $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, the set of multipliers of $\mathcal{H}_1$ into $\mathcal{H}_2$. In this way, we define $\mathcal{M}(\mathcal{H}) := \mathcal{M}(\mathcal{H}, \mathcal{H})$.

Given a multiplier $f \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we define a linear map $\mathcal{M}_f : \mathcal{H}_1 \to \mathcal{H}_2$ by $\mathcal{M}_f(h) = fh$. Clearly, the set of multipliers $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ is a vector space and the set of multipliers $\mathcal{M}(\mathcal{H})$ is an algebra.

2.2. **Vector-Valued Reproducing Kernel Banach Spaces.** A semi-inner product on a Banach space V is a function from $V \times V$ to $\mathbb{C}$, denoted by $[\cdot, \cdot]_V$, such that for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$

(i) (linearity)$[\alpha f+\beta g,h]_V=\alpha\,[f,h]_V+\beta\,[g,h]_V$;
(ii) (positivity)$[f,f]_V>0$ for $f\neq 0$;
(iii) (conjugate homogeneity)$[f,\alpha g]_V=\overline{\alpha}\,[f,g]_V$;
(iv) (Cauchy-Schwartz inequality)$|[f,g]_V|\leq [f,f]_V^{1/2}\,[g,g]_V^{1/2}$.

A semi-inner product $[\cdot,\cdot]_V$ on V is said to be compatible if

$$[f,f]_V^{1/2}=\|f\|_V\,,\quad \forall f\in V,$$

where $\|\cdot\|_V$ denotes the norm on V. Every Banach space has a compatible semi-inner product. Let $[\cdot,\cdot]_V$ be a compatible semi-inner product on V. Then one can see by the Cauchy-Schwartz inequality that for each $f\in\mathcal{B}$, the linear functional f^* on V defined by

$$f^*(g):=[g,f]_V\,,\quad g\in V,$$

is bounded on V. In other words, f^* lies in the dual space $\mathcal{B}^*$ of $\mathcal{B}$. Moreover, we have

$$\|f^*\|_{V^*}=\|f\|_V\,.$$

We define the duality mapping $\mathcal{I}_V$ from V to V^* by setting

$$\mathcal{I}_V(f):=f^*,\quad f\in V.$$

Semi-inner product spaces have several interesting properties. For more details, we refer readers to [7, 8, 9, 10].

Let Λ be a Banach space. A space $\mathcal{B}$ is called a Banach space of Λ-valued functions on X if it consists of certain functions from X to Λ and the norm on $\mathcal{B}$ is compatible with point evaluations in the sense that

$$\|f\|_B=0\quad\Leftrightarrow\quad f(x)=0,\quad\forall x\in X.$$

We call $\mathcal{B}$ a Λ-valued RKBS on X if both $\mathcal{B}$ and Λ are uniform and $\mathcal{B}$ is a Banach space of functions from X to Λ such that for every $x\in X$, the point evaluation $\delta_x:\mathcal{B}\to\lambda$ defined by

$$\delta_x(f):=f(x),\quad f\in\mathcal{B},$$

is continuous from $\mathcal{B}$ to Λ. More interesting details can be found in [15].

Definition 2.2. A reproducing kernel Banach space (RKBS) on X is a reflexive Banach space of functions on X such that its topological dual B' is isometric to a Banach space of functions on X and the point evaluations are continuous linear functionals on both B and B'.

In this case, there is a kernel function $K:X\times X\to\mathbb{C}$ such that

$$[f,K(.,x)]_{\mathcal{B}}=f(x),\quad\forall f\in\mathcal{B}\quad\forall x\in X,$$

and $\mathcal{B}=\overline{span}\{K(.,x);\quad x\in X\}$.

Definition 2.3. Let X be a set and $\mathcal{B}$ be a uniformly convex and uniformly Frechet differentiable $RKBS$ on X. In this way, we call $\mathcal{B}$ a *s.i.p.* reproducing kernel Banach space (abbreviation *s.i.p.RKBS*).

Theorem 2.4 (Riesz representation Theorem [14])**.** *For each $g \in B'$, there exists a unique $h \in B$ such that $g = h^*$, i.e., $g(f) = [f, h]_B$, $f \in B$ and $\|g\|_{B'} = \|h\|_B$ where $[\cdot,\cdot]_B$ denotes the semi-inner product on B.*

Definition 2.5 (The adjoint operator in a semi-inner product space)**.** Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are two s.i.p. Banach spaces. For an operator $T : \mathcal{B}_1 \to \mathcal{B}_2$, the adjoint operator $T^* : D(T^*) \to \mathcal{B}_1^c$ is defined by $T^*g^* = g^*T$, where $\mathcal{B}_1^c$ is the space of all continuous functionals on $\mathcal{B}_1$ and

$$D(T^*) = \{g^* \in \mathcal{B}_2^* : g^*T \quad \text{is continuous on} \quad \mathcal{B}_1\}.$$

Definition 2.6. A normed vector space V of functions on X satisfies the Norm Consistency Property if for every Cauchy sequence $\{f_n : n \in \mathbb{N}\}$ in V,

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \in X \quad \Rightarrow \quad \lim_{n\to\infty} \|f_n\|_V = 0.$$

Suppose X be a set and $\mathcal{B}$ be a s.i.p. RKBS on X with K as its kernel. Let

$$\mathcal{B}^\sharp = span\{K(x,.); \quad x \in X\}.$$

We can define a new norm as follows

$$\|g\|_{\mathcal{B}^\sharp} = \sup_{f\in\mathcal{B}, f\neq 0} \frac{|[f,g]_{\mathcal{B}}|}{\|f\|_{\mathcal{B}}}, \quad g \in \mathcal{B}^\sharp.$$

Theorem 2.7 ([4])**.** *The norm $\|\cdot\|_{\mathcal{B}^\sharp}$ is well-defined and point evaluation functionals are continuous on $\mathcal{B}^\sharp$ if and only if point evaluation functionals are continuous on $\mathcal{B}$.*

2.3. **Modules Over C^* Algebras.** Let B be a C^* algebra and W be a right B-module. We denote the module action of B on W by $(x, b) \to x.b$ $(x \in W, b \in B)$. We assume that any module treated below has a vector space structure over the complex numbers $\mathbb{C}$ compatible with that of B in the sense that $\lambda(x.b) = (\lambda x).b = x.(\lambda b)$ $(x \in W, b \in B, \lambda \in \mathbb{C})$. Let Z be a right B-module. A mapping $T : W \to Z$ is called a module map if T satisfies $T(x.b) = (Tx).b$ $(x \in W, b \in B)$.

Definition 2.8. A right B-module W is called a Banach B-module if W is a Banach space with the norm $\|\cdot\|_w$ satisfying

$$\|X.b\|_w \leq \|x\|_w \|b\|, \quad x \in W, b \in B.$$

Definition 2.9. A right B-module X is said to be a pre-Hilbert B-module if X is equipped with a conjugate bilinear map $\langle\cdot,\cdot\rangle : X \times X \to B$ (called a B-valued inner product on X) satisfying the following conditions:

- $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ only if $x = 0$;
- $\langle x, y\rangle = \langle y, x\rangle^*$;
- $\langle x.b, y\rangle = \langle x, y\rangle\, b, \quad (x, y \in X, b \in B)$.

Definition 2.10. A pre-Hilbert B-module X is called a Hilbert B-module if X is complete with respect to the norm

$$\|x\| = \|\langle x, x\rangle\|^{1/2}, \quad x \in X.$$

Hilbert B-modules are Banach B-modules. The C^*-algebra B is also a Hilbert B-module with $\langle a, b\rangle = b^*a$ $(a, b \in B)$.

Definition 2.11. Let S be a set, W and W' be Banach B-modules and $M(W, W')$ be the set of bounded linear module maps of W into W'. A kernel $K : S \times S \to M(W, W')$ is said to be positive definite (PD) if for any $f \in F(S, W)$,

$$\sum_{s,t} (K(s,t)f(s))\,(f(t)) \geq 0.$$

Let $\phi(S, W)$ be the set of W-valued mappings on S. Let $K : S \times S \to M(W, W')$ be a kernel and Y be a B-submodule of $\phi(S, W')$ forming a Hilbert B-module.

Definition 2.12. A kernel K is called a reproducing kernel (RK) of Y if K satisfies the following conditions:

- for any $s \in S$ and $w \in W$, $K_s w \in Y$ and the set $\{K_s w : s \in S, w \in W\}$ spans a dense B-submodule of Y;
- for any $s \in S$, $w \in W$ and $y \in Y$,

  $$(y(s))\,(w) = \langle K_s w, y\rangle\,.$$

  In this case, Y is called a reproducing kernel Hilbert B-module of K.

In the sequel, let Γ be a unital $*$-semigroup, that is, Γ is a semigroup with the unit e and the involution $*$ such that $\xi^{**} = \xi$ and $(\xi\eta)^* = \eta^*\xi^*(\xi, \eta \in \Gamma)$. We denote by $\mathcal{U}(X, Y)$ the set of operators $T \in B(X, Y)$, (X, Y are two B-module Hilbert spaces), for which there exists an operator $T^* \in B(Y, X)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for every $x \in X, y \in Y$. It is easy to see that $\mathcal{U}(X, Y) \subseteq M(X, Y)$. It is shown that $\mathcal{U}(X)$ is a unital C^*-algebra.

Definition 2.13. A mapping $\pi : \Gamma \to \mathcal{U}(X)$ is called a representation of T on X if for every $\xi, \eta \in \Gamma, \pi(\xi)^* = \pi(\xi^*), \pi(\xi\eta) = \pi(\xi)\pi(\eta)$ and $\pi(e) = 1_X$ (denotes the identity mapping on X).

Theorem 2.14. *Let Φ be a completely positive linear map from a unital C^*-algebra X into another Y. Then there is a Hilbert B-module Z, an* *element z in Z and a $*$-representation π_Φ of X into $\mathcal{U}(X)$ such that $\Phi(x) = \langle \pi_\Phi(x)z, z\rangle$ for all x in X.*

Proof. For a complete proof, we refer readers to [13]. □

3. Main Theorems

The definition of RKHS for vector valued functions parallels the one in the scalar, with the main difference that a reproducing kernel can be expressed as a matrix valued. Gaussian process methods for modeling vector-valued functions is a good example of application of vector valued reproducing kernel Hilbert spaces. Gaussian process is defined as a collection of random variables, such that any finite number of them follows a joint Gaussian distribution. In the single output case, the random variables are associated to a single process f evaluated at different values of x while in the multiple output case, the random variables are associated to different processes $\{f_d\}_{d=1}^{D}$, evaluated at different values of x. In this article, we just focus on the mathematical concepts and refer readers for interesting details to [6, 11, 12, 16].

Motivated by the need of processing vector-valued data, we extend the notion of the relative reproducing kernel Hilbert space to vector-valued spaces.

Definition 3.1. Let X be an arbitrary set, $\mathcal{Y}$ a normed vector space and $\mathcal{H}$ a vector-valued reproducing kernel Hilbert space with K as its reproducing kernel satisfying in relations (2.1) and (2.2). We say that it is a vector-valued relative reproducing kernel Hilbert space (abbreviation RRKHS) if there exists a function $M_{x,y}$ such that

$$\begin{aligned} &M_{x,y} \in \mathcal{L}(\mathcal{Y}, \mathcal{H}), \quad \forall x, y \in X; \\ &f(y) - f(x) = M^*_{x,y} f, \quad \forall x, y \in X, \forall f \in \mathcal{H}, \end{aligned} \tag{3.1}$$

where $M^*_{x,y} : \mathcal{H} \to \mathcal{Y}$ is the adjoint of $M_{x,y}$.

A vector-valued relative reproducing kernel Hilbert space equipped with a relative reproducing kernel $M_{x,y}$ is denoted by $\mathcal{H}_M$.

Obviously, we expect basic properties of reproducing kernel Hilbert spaces in this kind of spaces. So, it is easy to check a basic theorem.

Theorem 3.2. *The function $M_{x,y}$ in (3.1) is unique and satisfies*

$$M_{x_1,x_2}(t) + M_{x_2,x_3}(t) = M_{x_1,x_3}(t), \quad \forall x_1, x_2, x_3, t \in X. \tag{3.2}$$

Proof. The uniqueness of the function M_{x_1,x_2} follows from Riesz representation theorem. This uniqueness and the following equality imply (3.2):

$$f(x_1) - f(x_3) = f(x_1) - f(x_2) + f(x_2) - f(x_3).$$

□

Similarly, the following definition tries to extend this concept to Λ-valued Banach spaces. It is important, because of its application in machine learning, specially.

Definition 3.3. Let $\mathcal{B}$ be a Banach space of Λ-valued functions on X. We call $\mathcal{B}$ a Λ-valued relative reproducing kernel Banach space (abbreviation ΛRRKBS) on X if both $\mathcal{B}$ and Λ are uniform and $\mathcal{B}$ is a Banach space of functions from X to Λ such that for every $x \in X$, the function $\zeta_{x,y} : \mathcal{B} \to \lambda$ defined by

$$\zeta_{x,y}(f) := f(y) - f(x), \quad \forall f \in \mathcal{B}, \tag{3.3}$$

is continuous from $\mathcal{B}$ to Λ.

Next theorem verifies transitivity property in such Banach spaces.

Theorem 3.4. *Equation (3.2) holds in ΛRRKBS but the function is not necessarily unique in these spaces.*

Proof. It is easy to check the equation but we have some problems for uniqueness. Obviously the Riesz representation theorem does not hold in general Banach spaces and we need to add some conditions to establish this theorem. So in general, the above uniqueness does not hold. $\square$

Under particular conditions, Riesz representation theorem holds on Banach spaces. It helps us to prove uniqueness of the previews theorem.

Theorem 3.5. *Let $\mathcal{B}$ be a RRKBS on a locally compact Hausdorf Banach space Λ. Then the function defined in (3.3) is unique.*

Proof. In this case, a version of Riesz representation theorem holds and it implies the uniqueness. see Theorem 1 in [3]. $\square$

Since we construct a reproducing kernel Hilbert spaces and a relative reproducing kernel Hilbert spaces of a specific Hilbert space of operators, inclusion problem can be attractive.

Theorem 3.6. *Let $\mathcal{H}$ be a Hilbert space and there exist functions K and M with reproducing and relative reproducing property, respectively, then*

$$\mathcal{H}_M \subseteq \mathcal{H}_K \subseteq \mathcal{H}.$$

Proof. Obviously if a function belongs to $\mathcal{H}_M$, it is a linear combination of two functions of $\mathcal{H}_K$. Since $\mathcal{H}_K$ is a closed space so $\mathcal{H}_M \subseteq \mathcal{H}_K$. $\square$

As a representation tries to describe properties of a space in an other space, we start with the following theorem to attend such properties in Banach spaces.

Theorem 3.7. *Let X be a set and $\mathcal{H}$ be a reproducing kernel Hilbert space on X. Then the function $\pi_{\mathcal{H}} : \mathcal{M}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ with $\pi_{\mathcal{H}}(f, h) = \mathcal{M}_f(h) = fh$ is a representation.*

Proof. It is easy to check the conditions of a representation. □

Similarly, we can extend the notion of a multiplier to Banach spaces.

Definition 3.8. Suppose X be a set and $\mathcal{B}$ be a s.i.p. RKBS on X. A function $f : X \to \mathbb{C}$ is called a multiplier of $\mathcal{B}$ provided $f\mathcal{B} = \{fg : g \in \mathcal{B}\} \subseteq \mathcal{B}$. We denoted by $\mathcal{M}(\mathcal{B})$ the set of multipliers of $\mathcal{B}$.

Suppose $\mathcal{M}_{\mathcal{B}}$ be the set of multipliers of a s.i.p. RKBS. It is endowed with a semi inner product inherited of $\mathcal{B}$. So, it can be embedded in an inner product space. We denote by $\mathcal{H}_{\mathcal{MB}}$ a Hilbert space spanned by $\mathcal{M}_{\mathcal{B}}$.

Theorem 3.9. *Let X be a set and $\mathcal{B}$ be a reproducing kernel Banach space on X. Then function $\pi_{\mathcal{B}} : \mathcal{M}(\mathcal{B}) \times \mathcal{B} \to \mathcal{H}_{\mathcal{MB}}$ with $\pi_{\mathcal{B}}(f, g) = \mathcal{M}_f(g) = fg$ is a representation.*

Proof. It is not hard to verify conditions of π to be a representation. □

Theorem 3.10. *Let $\mathcal{B}_i$, $i = 1, 2$ be a s.i.p. RKBS's on X with reproducing kernels, $K_i(x, y) = k_y^i(x)$, $i = 1, 2$. If $f \in \mathcal{M}(\mathcal{B}_1, \mathcal{B}_2)$, then for every $y \in X$, $M_f^*(k_y^2) = \overline{f(y)}k_y^1$.*

Proof. For any $h \in \mathcal{B}_1$, we have

$$\left[h, \overline{f(y)}k_y^1\right]_1 = f(y)h(y) = \left[\mathcal{M}_f(h), k_y^2\right]_2 = \left[h, \mathcal{M}_f^*(k_y^2)\right],$$

and hence, $\overline{f(y)}k_y^1 = \mathcal{M}_f^*(k_y^2)$. □

Theorem 3.11. *Suppose $\mathcal{B}$ and $\mathcal{B}^\sharp$ defined as above. then $\mathcal{M}_{\mathcal{B}} \cong \mathcal{M}_{\mathcal{B}^\sharp}$*

Proof. Since $B \cong B^\sharp$ as Banach spaces and all of the multipliers come from a specific set $\mathcal{H}$, so $\mathcal{M}_{\mathcal{B}} \cong \mathcal{M}_{\mathcal{B}^\sharp}$. □

As a corollary of Theorem 3.11, we have the following statement.

Corollary 3.12. *The space $\mathcal{B}_0 = \{g \in \mathcal{B}^\sharp; \quad \|g\|_{\mathcal{B}^\sharp} = 1\}$ is a subspace of $\mathcal{B}$ as a s.i.p. RKBS.*

Similar to the previews definitions, we can also define a relative reproducing property for Hilbert B-modules.

Definition 3.13. A kernel M is called a relative reproducing kernel (abbreviation RRK) of Y if M satisfies the following conditions:

- for any $s, t \in S$ and $w \in W$, $K_{st}w \in Y$ and the set $\{M_{st}w : s, t \in S, w \in W\}$ spans a dense B-submodule of Y;

- for any $s, t \in S$, $w \in W$ and $y \in Y$,

$$(y(s) - y(t))\,(w) = \langle M_{st} w, y \rangle \, .$$

In this case, Y is called a relative reproducing kernel Hilbert B-module of K.

Let X be a reproducing kernel Hilbert B-module of functions over a set. Multipliers of X can be defined similarly. Like the previews notations, the set of all multipliers of X is denoted by $\mathcal{M}(X)$.

One of the essential distinctions between Hilbert C^*-modules and Hilbert spaces is the existence of non-self-dual modules (i.e. such that $X^* \neq X$). In other words, there is no Riesz representation theorem for Hilbert C^*-modules and not all operators have an adjoint. So, a relative reproducing kernel in the previews definition is not unique.

Theorem 3.14. *Let X be a reproducing kernel Hilbert B-module of functions. Then, the function Φ defined in Theorem 2.14 is a multiplier for X.*

Proof. For every $f \in X$ and for a Hilbert B-module Y we have

$$\Phi f = \langle \pi y, y \rangle \, f \in X \quad \Rightarrow \quad \Phi \in \mathcal{M}(X).$$

□

Acknowledgment. A part of this research was carried out while the first author was visiting the university of Iowa. The author is grateful to the members and staff of the university of Iowa. The authors also thanks the referees for their suggestions.

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