FREE BANACH LATTICES OVER PRE-ORDERED BANACH SPACES

MARCEL DE JEU AND XINGNI JIANG

ABSTRACT. We define the free Banach lattice over a pre-ordered Banach space in a category of Banach lattices of a given convexity type, and show its existence. The subsumption of a pre-ordering necessitates an approach that differs fundamentally from the known one for the free Banach lattice over a Banach space under a given convexity condition, which is a special case. The relation between the free vector lattice over a pre-ordered Banach space and the free Banach lattice of a given convexity type over it is made explicit. It is determined when precisely the free Banach lattice has a canonical realisation as a lattice of homogeneous continuous functions on the positive part of the unit ball of the dual space. For free p-convex Banach lattices with convexity constant 1 over pre-ordered Banach spaces, realisations as function lattices are obtained that generalise those for free Banach lattices of that type over Banach spaces.

A characterisation of p-convex Banach lattices in terms of vector lattice homomorphisms into L_p-spaces or into the real numbers is included.

1. INTRODUCTION AND OVERVIEW

The notion of the free Banach lattice over a set was introduced by de Pagter and Wickstead in 2015 [11]; it was followed by the definition of the free Banach lattice $\mathrm{FBL}[X]$ over a Banach space X by Avilés, Rodríguez, and Tradacete in [4]. The latter paper was the starting point of an active line of research.

In the present paper, we bring (pre-)ordering into the picture. For a pre-ordered Banach space X with positive wedge X^+ and a given convexity type $\mathcal{C}$ (a notion introduced in the paper), we define the free Banach lattice $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ over (X,X^+). When taking $\mathcal{C}=\emptyset$, our $\mathrm{FBL}^{\emptyset}[X,\{0\}]$ is $\mathrm{FBL}[X]$ from [4]; the free Banach lattices $\mathrm{FBL}^{\mathcal{D}}[X]$ under convexity conditions from [15] all occur as an $\mathrm{FBL}^{\mathcal{C}}[(X,\{0\}]$ in the present paper. When X is an ordered Banach space with a (not necessarily generating) closed positive cone X^+, there is a linear injection from X into $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ that is bipositive. Thus X sits inside $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$, and its ordering extends to a lattice ordering on the enveloping space. It is a non-trivial question, left for further research, to find conditions under which the image is closed; this is always the case if $X^+=\{0\}$ when the injection is even isometric. When the image is closed, X is embedded (though perhaps not necessarily isometrically) into the Banach lattice $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ as an ordered Banach space. The embedding of a Banach space X into $\mathrm{FBL}[X]$ makes a better

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understanding possible of the relation between Banach spaces and Banach lattices. The same can now be expected for that between ordered Banach spaces and Banach lattices.

Compared to earlier papers on free Banach lattices, our approach to the existence of the free Banach lattice is of a fundamentally different nature. We do not start from a function lattice, but instead from an algebraic free object which we know to exist, and proceed from there. This has not only the advantage that it works in all generality, whereas—as we shall argue—this cannot be expected for the function lattice method, but we also believe that it makes the picture more transparent. As will become clear, one can say quite a bit about the free Banach lattice without having a concrete function lattice realisation of it (which may not exist), or without even knowing that it actually exists.

This paper is organised as follows.

Section 2 contains the necessary preliminaries.

Section 3 contains preparatory material on convexity in Banach lattices. In its first part, using an idea from [15], a characterisation of p-convex Banach lattices is obtained in terms of vector lattice homomorphisms into L_p-spaces or the real numbers. This characterisation, which appears to be new, is instrumental to Section 7. It also allows one to give an alternate proof of the well-known fact that p-convex Banach lattices can always be renormed so that the p-convexity constant becomes 1. In its second part, the notion of a convexity type is introduced, generalising that of a convexity condition in [15]. If, for a Banach lattice, there is a norming family of contractive vector lattice homomorphisms into Banach lattices that are all of the same convexity type, then so is the domain lattice.

Section 4 introduces the free vector lattice $\mathrm{FVL}[(V,V^+)]$ over a pre-ordered vector space V with positive wedge V^+. It is shown that it exists. One can argue that, once this existence been established (a consequence of a basic principle in universal algebra), it is routine to verify that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ always exists.

In Section 5, the free Banach lattice $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ of a given convexity type over a pre-ordered Banach space (X,X^+) is defined. The definition is purely categorical, in terms of positive contractions that factor over contractive vector lattice homomorphisms, without any conditions on maps being injective or isometric, or on norms of operators being equal. The remainder of the section is a study of its properties, should it exist. It is not disappointing what can already be said a priori. Our definition of $\mathrm{FVL}^{\emptyset}[(X,\{0\})]$, for example, can be shown to be equivalent to the seemingly stronger one for $\mathrm{FBL}[X]$ in [4]. A formula for the norm can be derived, and it can be determined when $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ contains $\mathrm{FVL}[(V,V^+)]$. This is always the case when $X^+=\{0\}$. Consequently, the free Banach lattices in [4, 15] always contain a copy of the free vector lattice over the Banach space X. In the notation of these papers, the vector lattices of functions on the dual space that is generated by the δ_x is the free vector lattice over X. This seems to have gone unnoticed in the papers on free Banach lattices thus far, and one can regard it is an explanation why the approach in [4, 15], where one starts with a function lattice on the dual unit ball, actually works. We also give arguments why this will in all likelihood not work when $X^+ \neq \{0\}$. It can also be determined a priori when precisely there is a canonical realisation of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ as a function lattice on the positive part of the dual unit ball.

In Section 6, the existence of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is established in a manner as was already done in [10] for FBL$[X]$, and as is systematically exploited in [18]. The idea, which can already be found in the books by Dixmier and presumably also elsewhere, is to start with a free object in an algebraic context and proceed routinely along a standard path. There is no role for function lattices on the dual space, or even for the dual space itself.

In Section 7, it is shown that the free p-convex Banach lattices with p-convexity constant 1 over (X,X^+) can be realised as lattices of homogeneous continuous functions on the positive part of the dual unit ball. When $X^+ = \{0\}$, one retrieves the realisations in [15]. This section is clearly indebted to the ingenious techniques in [15], but in spite of the similarity the road to these realisations is very different. Whereas in [15] the proof of the existence of the free Banach lattice is essentially redone by showing that the proposed Banach function lattice has the correct universal property, we start from the free object itself, which we know to exist, and analyse it to the extent that it becomes a triviality that it has a realisation as a function lattice on the positive part of the dual unit ball. One could say that in the present paper these realisations are *derived*, and that in [4, 15] they are *found*.

A words on notation. Firstly, compared to many papers on free Banach lattices, our canonical choice of letters is reversed: X is a Banach space and E and F are Banach lattices or vector lattices. We believe that this reflects the notation in most of the existing textbooks on Banach lattices and vector lattices, and apologise for any inconvenience this may cause. Secondly, we have consistently used Greek letters for positive contractions and contractive vector lattice homomorphisms, rather than the usual operator letters such as T. This fits better into our categorical approach, where these are the morphisms we work with.

2. PRELIMINARIES

In this section, we collect the necessary notation, definitions, and conventions.

All vector spaces are over the real numbers. A vector lattice need not be Archimedean.

If V is a vector space, then a *wedge* in V is a non-empty subset V^+ such that $\alpha_1 x_1 + \alpha_2 x_2 \in V^+$ for all $\alpha_1, \alpha_2 \geq 0$ and $x_1, x_2 \in V^+$. A wedge V^+ is a *cone* when $V^+ \cap (-V^+) = \{0\}$. There is a well-known correspondence between preorderings on V and wedges in V, where the partial orderings correspond to the cones. A *pre-ordered vector space* is a pair (V, V^+), where V^+ is a wedge in the vector space V. If (V, V^+) and (W, W^+) are pre-ordered vector spaces, then a linear map $\varphi\colon V \to W$ is *positive* when $\varphi(V^+) \subseteq W^+$. When $V^+ = \{0\}$, these are just the linear maps.

A *pre-ordered Banach space* is a pair (X,X^+), where X^+ is a wedge in the Banach space X. It need not be closed.

If X is a Banach space, then X^* denotes its dual. When (X, X^+) is a pre-ordered Banach space, we write B_X for its unit ball, B_{X^*} for the unit ball of its dual, and $\mathrm{B}_{X^*_+}$ for the positive elements of B_{X^*}. If $X^+ = \{0\}$, then $\mathrm{B}_{X^*_+} = \mathrm{B}_{X^*}$.

We supply $\mathrm{B}_{X^*_+}$ with the relative weak*-topology, making it into a compact Hausdorff space, and let $\mathrm{C}^{\mathrm{h}}(\mathrm{B}_{X^*_+})$ denote the continuous, homogeneous, real-valued functions on it. For $x \in X$, we define $\delta_x \in \mathrm{C}^{\mathrm{h}}(\mathrm{B}_{X^*_+})$ by setting

$$\delta_x(x^*) := x^*(x) \tag{2.1}$$

for $x^* \in \mathrm{B}_{X^*_+}$. If $X^+ = \{0\}$, then δ_x is defined on B_{X^*}.

Contractions between Banach spaces are linear.

To conclude, we recall the following definition of free objects from [1, Definition 8.22].

Definition 2.1. Suppose that Cat_1 and Cat_2 are categories, and that $U\colon \mathsf{Cat}_2 \mapsto \mathsf{Cat}_1$ is a faithful functor.[1]. Take an object O_1 of Cat_1. A *free object over O_1 of* Cat_2 *with respect to U* is a pair $(j, \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$, where $\mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ is an object of Cat_2 and $j : O_1 \to U\left(\mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]\right)$ is a morphism of Cat_1, with the property that, for every object O_2 of Cat_2 and every morphism $\varphi : O_1 \to U(O_2)$ of Cat_1, there exists a unique morphism $\overline{\varphi} : \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1] \to O_2$ of Cat_2 such that the diagram

$$\begin{tikzcd} O_1 \arrow[r, "j"] \arrow[dr, "\varphi"'] & U\left(\mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]\right) \arrow[d, "U(\overline{\varphi})"] \\ & U(O_2) \end{tikzcd}$$

in Cat_1 is commutative.

A free object $(j, \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$ as in Definition 2.1 need not exist. However, if it exists, and if $(j', \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]')$ is another such pair, then a standard argument shows that the unique morphism $\overline{j'} : \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1] \to \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]'$ of Cat_2 such that $U(\overline{j'}) \circ j = j'$ is, in fact, an isomorphism with as its inverse the unique morphism $\overline{j} : \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]' \to \mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ of Cat_2 such that $U(\overline{j}) \circ j' = j$. In particular, a free object over O_1 of Cat_2 with respect to U, if it exists, is determined up to an isomorphism of Cat_2. In the case of existence we shall, therefore, sometimes simply speak of 'the' free object $\mathrm{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ over O_1 of Cat_2, the accompanying morphism j often being tacitly understood from the context.

In the current paper, we are concerned with two contexts for Definition 2.1, an algebraic and an analytic one. The results in the algebraic context are the stepping stone for those in the analytic context.

[1] Recall that U is *faithful* when the associated map $U\colon \mathrm{Hom}_{\mathsf{Cat}_2}(O_2, O_2') \to \mathrm{Hom}_{\mathsf{Cat}_1}(U(O_2), U(O_2'))$ is injective for all objects O_2, O_2' of Cat_2. This will always be the case in the present paper.

3. Convexity

In this section, we collect the material on convexity that will be needed in the sequel. For the positively homogeneous functional calculus for Banach lattices that is used in it, we refer to [16, Section 1.d]. It will be important that it is compatible with vector lattice homomorphisms; this follows from the uniqueness statement in [16, Theorem 1.d.1].

3.1. **p-convexity.** Section 7 is concerned with free p-convex Banach lattices. As a preparation for this, we recall the definitions and give characterisations of p-convex Banach lattices in terms of contractive vector lattice homomorphisms into L_p-spaces or $\mathbb{R}$.

Let $M \geq 1$. For $1 \leq p < \infty$, a Banach lattice E is said to be *p-convex with p-convexity constant at most M* when

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

for $x_1, \ldots, x_n \in E$. The smallest such M is called the p-convexity constant $M^{(p)}(E)$ of E. All L_p-spaces are p-convex with p-convexity constant 1. All Banach lattices are 1-convex with 1-convexity constant 1.

It is said to be *∞-convex with ∞-convexity constant at most M* when

$$\left\| \bigvee_{i=1}^{n} |x_i| \right\| \leq M \bigvee_{i=1}^{n} \|x_i\|$$

for $x_1, \ldots, x_n \in E$. The smallest such M is the ∞-convexity constant $M^{(\infty)}(E)$ of E.

Parts of the proofs of the following results are inspired by ideas in the proof of [15, Theorem 6.1].

Proposition 3.1. *Let E be a Banach lattice. Then, for $x \in E$,*

$$\|x\| = \max_{\varphi} \|\varphi(x)\|,$$

where φ runs over all contractive vector lattice homomorphisms $\varphi \colon E \to L_1(\mu)$ for probability measures μ.

Proof. Take $x^* \in E^*$ with $\|x^*\| \leq 1$ such that $x^*(|x|) = \|x\|$. We let $N_{x^*} := \{x \in F : x^*(|x|)\}$ be its null-ideal, and define F to be the completion of $E/N(x^*)$ in the norm $\|x + N_{x^*}\| := x^*(|x|)$. As disjoint elements in a quotient have disjoint lifts, F is an AL-space, so that it can be identified with a concrete L_1-space $L_1(\nu)$ on a set Ω. If we let $\psi \colon E \to \mathrm{L}_1(\nu)$ be the resulting contractive vector lattice homomorphism, then $\|x\| = \|\psi(x)\|_{\mathrm{L}_1(\nu)}$. To obtain a probability measure μ, we restrict ν to $S := \{\omega \in \Omega : [\psi(x)](\omega) \neq 0\}$ and let μ be this restriction, divided by $|\psi(x|$. Then $\chi : \mathrm{L}_1(\nu) \to \mathrm{L}_1(\mu)$, defined by setting $\chi(f) := |\psi(x)| \, f\restriction_S$ for $f \in \mathrm{L}_1(\nu)$, is a contractive vector lattice homomorphism such that $\|(\chi \circ \psi)(x)\|_{\mathrm{L}_1(\mu)} = \|\psi(x)\|_{\mathrm{L}_1(\nu)} = \|x\|$. This result follows. $\square$

Theorem 3.2. *Let $1 < p < \infty$ and let $M \geq 1$. The following are equivalent for a Banach lattice E:*

1. *E is p-convex with p-convexity constant at most M;*
2. *for $x \in E$,*

$$\|x\| \leq M \sup_{\varphi} \|\varphi(x)\|,$$

where φ runs over all contractive vector lattice homomorphisms $\varphi\colon E \to L_p(\mu)$ for probability measures μ.

Proof. We start with the easy proof that part (2) implies part (1). Take $x_1, \ldots, x_n \in E$. Using that the codomains are L_p-spaces in the third step, we then have

$$\begin{aligned}
\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| &\leq M \sup_{\varphi} \left\| \varphi\left(\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right) \right\| \\
&= M \sup_{\varphi} \left\| \left(\sum_{i=1}^n |\varphi(x_i)|^p \right)^{1/p} \right\| \\
&= M \sup_{\varphi} \left(\sum_{i=1}^n \|\varphi(x_i)\|^p \right)^{1/p} \\
&=\leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}.
\end{aligned}$$

We now prove that part (1) implies part (2). Take $x \in E$. Proposition 3.1 furnishes a probability measure μ and a contractive vector lattice homomorphism $\varphi\colon E \to \mathrm{L}_1(\mu)$ such that $\|x\|_E = \|\varphi(x)\|_{\mathrm{L}_1(\mu)}$. Take $x_1, \ldots, x_n \in E$. Then

$$\begin{aligned}
\left\| \left(\sum_{i=1}^n |\varphi(x_i)|^p \right)^{1/p} \right\|_{\mathrm{L}_1(\mu)} &= \left\| \varphi\left(\left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right) \right\|_{\mathrm{L}_1(\mu)} \\
&\leq \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_E \\
&\leq M \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p}
\end{aligned}$$

The Maurey-Nikishin factorisation theorem (see [2, Theorem 7.1.2]) now yields a probability measure ν, a vector lattice homomorphism $S\colon E \to \mathrm{L}_p(\nu)$ with $\|S\| \leq M$, and an isometric embedding $j\colon \mathrm{L}_1(\nu) \to \mathrm{L}_1(\mu)$ such that the diagram

$$
\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "S"'] & \mathrm{L}_1(\mu) \\
\mathrm{L}_p(\nu) \arrow[r, hook, "i"] & \mathrm{L}_1(\nu) \arrow[u, hook, "j"']
\end{tikzcd} \tag{3.1}
$$

is commutative. Here i is the inclusion map which—this is the point—is contractive as ν is a probability measure. We then have

$$\|x\|_E = \|\varphi(x)\|_{\mathrm{L}_1(\mu)} = \|(j \circ i \circ S)(x)\|_{\mathrm{L}_1(\mu)} = \|(i \circ S)(x)\|_{\mathrm{L}_1(\nu)} \leq \|S(x)\|_{\mathrm{L}_p(\nu)}.$$

On writing $S = M \cdot (S/M)$ we see that the inequality in part (2) holds for x. □

Corollary 3.3. *Let $1 \leq p < \infty$. The following are equivalent for a Banach lattice E:*

1. *E is p-convex with p-convexity constant 1;*
2. *for $x \in E$,*
$$\|x\| = \sup_{\varphi} \|\varphi(x)\|,$$
where φ runs over all contractive vector lattice homomorphisms $\varphi\colon E \to L_p(\mu)$ for probability measures μ.

We complement this with the following.

Proposition 3.4. *The following are equivalent for a Banach lattice E:*

1. *E is ∞-convex with ∞-convexity constant 1;*
2. *for $x \in E$,*
$$\|x\| = \max_{\varphi} \|\varphi(x)\|,$$
where φ runs over all contractive vector lattice homomorphisms $\varphi\colon E \to \mathbb{R}$.

Proof. It follows from the Kakutani representation theorem (see [16, Theorem 1.b.6], for example) that the ∞-convex Banach lattices with ∞-convexity constant 1 are, up to an isometric vector lattice homomorphism, precisely the Banach sublattices of $\mathrm{C}(K)$ for some compact Hausdorff space K. On considering the point evaluations in such a functional representation, we see that part (1) implies part (2). For the converse, one realises E canonically as a Banach lattice of continuous functions on the space of the real-valued contractive vector lattice homomorphisms, supplied with the relative weak*-topology, to see that it is a Banach sublattice of a $\mathrm{C}(K)$-space. □

Remark 3.5. For $1 \leq p < \infty$, we shall see in Remark 3.8 how Corollary 3.3 provides an alternate proof of the fact that a p-convex Banach lattice has an equivalent norm in which it is p-convex with p-convexity constant 1.

3.2. **Convexity types.** In this section, we introduce the notion of a convexity type for Banach lattices. This generalises the notion of a $\mathcal{D}$-convexity condition in [15]. As we shall see, if there exists a norming family of contractive vector lattice homomorphisms from a Banach lattice into codomains which are all of the same convexity type, then the domain is also of this type.

The definition of a convexity type uses the positively homogeneous functional calculus. For $n \in \mathbb{N}$, we let $\mathcal{H}_n$ denote the set of all continuous positively homogeneous real-valued functions on $\mathbb{R}^n$. We set

$$\mathcal{H}_n^{\mathrm{i}} := \{h \in \mathcal{H}_n : h(t_1, \ldots, t_n) \leq h(s_1, \ldots, s_n) \text{ when } 0 \leq t_i \leq s_i \text{ for } i = 1, \ldots, n\}$$

for its elements that are increasing on the positive cone $\mathbb{R}_+^n$ of $\mathbb{R}^n$.

A *convexity implication* $\mathcal{I}$ is a triple $(\mathcal{I}_{\mathrm{a}}, \mathcal{I}_{\mathrm{c}}; n)$, where $n \in \mathbb{N}$, $\mathcal{I}_{\mathrm{a}} \subseteq \mathcal{H}_n$, and $\mathcal{I}_{\mathrm{c}} \subseteq \mathcal{H}_n \times \mathcal{H}_n^{\mathrm{i}}$. We allow that $\mathcal{I}_{\mathrm{a}} = \emptyset$, but require that $\mathcal{I}_{\mathrm{c}} \neq \emptyset$. A Banach lattice E is said to *satisfy* $\mathcal{I}$ if

$$\|c_1(x_1, \ldots, x_n\| \leq c_2(\|x_1\|, \ldots, \|x_n\|) \tag{3.2}$$

for every $(c_1, c_2) \in \mathcal{I}_{\mathrm{c}}$ whenever $x_1, \cdots, x_n \in E$ are such that $a(x_1, \ldots, x_n) \leq 0$ for all $a \in \mathcal{I}_{\mathrm{a}}$. When $\mathcal{I}_a = \emptyset$, we require that (3.2) hold for all $x_1, \cdots, x_n \in E$. [2] A *convexity type* $\mathcal{C}$ is a (possibly) empty set of convexity implications. A Banach lattice is said to be *$\mathcal{C}$-convex* if it satisfies all convexity implications in $\mathcal{C}$. Every Banach lattice is $\emptyset$-convex.

For $a \in \mathcal{H}_n$, the condition that $a(x_1, \ldots, x_n) = 0$ is equivalent to requiring that $a(x_1, \ldots, x_n) \leq 0$ and $(-a)(x_1, \ldots, x_n) \leq 0$. Thus also antecedents in terms of equalities are covered by the formalism. Likewise, if c_1 is also increasing on $\mathbb{R}_+^n$, then including both (c_1, c_2) and (c_2, c_1) in $\mathcal{I}_{\mathrm{c}}$ makes it possible to include certain equalities in the consequent.

Take $n \in \mathbb{N}$ and define $a \in \mathcal{H}_n^{\mathrm{i}}$ by setting $a(t_1, \ldots, t_n) := \sum_{i=1}^n |t_i - |t_i||$. Then $a(x_1, \ldots, x_n) \leq 0$ if and only if $x_1, \ldots, x_n \in E^+$. As a variation, set $a'(t_1, \ldots, t_n) := \sum_{i,j=1}^n |t_1| \wedge |t_2|$. Then $a'(x_1, \ldots, x_n) \leq 0$ if and only if the x_i are pairwise disjoint.

After these two observations, it is clear how every convexity condition $\mathcal{D}$ as defined in [15, Section 3] leads to a corresponding convexity type $\mathcal{C}$ as above, such that the $\mathcal{D}$-convex Banach lattices as defined in [15] are precisely the $\mathcal{C}$-convex Banach lattices as defined above. The present set-up is, however, more general and it is for this reason that we have chosen a different terminology ('type' versus 'condition') and a different letter. This avoids a possible misunderstanding when using results from [15].

Example 3.6. Let $M \geq 1$. Take $1 \leq p < \infty$. For $n \geq 1$, we set $c_p^n(t_1, \ldots, t_n) := \left(\sum_{i=1}^n |t_i|^p\right)^{1/p}$. Let $\mathcal{C}_p^M := \left\{\emptyset, \{(c_p^n, Mc_p^n; n) : n = 1, 2, \ldots\}\right\}$. Then the $\mathcal{C}_p^M$-convex Banach lattices are the p-convex Banach lattices with p-convexity constant at most M.

[2] Thus E satisfies $\mathcal{I}$ if an implication is valid for it, the (possibly vacuously fulfilled) antecedent of which is expressed in terms of $\mathcal{I}_a$ and its elements a, and the consequent of which is expressed in terms of $\mathcal{I}_{\mathrm{c}}$ and its elements (c_1, c_2). This motivates the notation.

For the choice $c_\infty^n(t_1,\ldots,t_n) := \sum_{i=1}^n |t_i|$ and $\mathcal{C}_\infty^M := \big\{\emptyset, \{(c_\infty^n, Mc_\infty^n; n) : n = 1,2,\ldots\}\big\}$ one obtains the ∞-convex Banach lattices with ∞-convexity constant at most M.

In [15], the existence of free $\mathcal{D}$-convex Banach lattices was established. For this, a study of the positively homogeneous functional calculus and its continuity properties was needed for suitable, not necessarily uniformly complete, vector lattices, to show that the $\mathcal{D}$-convexity passes from a certain normed vector lattice to its completion in the final step of the existence proof. In our approach, this is not necessary. The $\mathcal{C}$-convexity of the free $\mathcal{C}$-convex Banach lattice will be immediate from the following result, the proof of which uses only the compatibility of the homogeneous functional calculus with vector lattice homomorphisms. It will now become clear why the condition above that c_2 be increasing on $\mathbb{R}_+^n$ is necessary.

Lemma 3.7. *Let E be a Banach lattice and let $\mathcal{C}$ be a convexity type. Suppose that, for all $x \in E$,*

$$\|x\| = \sup_\varphi \|\varphi(x)\|$$

where φ runs over a class of contractive vector lattice homomorphisms $\varphi\colon E \to E_\varphi$ into Banach lattices E_φ. If all E_φ are $\mathcal{C}$-convex, then E is $\mathcal{C}$-convex.

Proof. If $\mathcal{C} = \emptyset$, there is nothing to prove, so we suppose that $\mathcal{C}$ is non-empty. Take a convexity implication $\mathcal{I} = (\mathcal{I}_\mathrm{a}, \mathcal{I}_\mathrm{c}; n)$ in $\mathcal{C}$. Suppose that $x_1, \cdots, x_n \in E$ are such that $a(x_1,\ldots,x_n) \le 0$ for all $a \in \mathcal{I}_\mathrm{a}$; this is vacuously fulfilled if $\mathcal{I}_\mathrm{a} = \emptyset$. Take a $\varphi\colon E \to E_\varphi$ as in the statement. Then $a(\varphi(x_1),\ldots,\varphi(x_n)) = \varphi(a(x_1,\ldots,x_n)) \le 0$ for all $a \in \mathcal{I}_\mathrm{a}$. Since E_φ satisfies $\mathcal{I}$, we have

$$\|c_1(\varphi(x_1),\ldots,\varphi(x_n)\| \le c_2(\|\varphi(x_1)\|,\ldots,\|\varphi(x_n)\|)$$

for all $(c_1,c_2) \in \mathcal{I}_\mathrm{c}$. Using the fact that c_2 is increasing on $\mathbb{R}_+^n$ in the final step, we then see that, for all $(c_1,c_2) \in \mathcal{I}_c$,

$$\begin{aligned}
\|c_1(x_1,\ldots,x_n\| &= \sup_\varphi \|\varphi\left(c_1(x_1,\ldots,x_n)\right)\| \\
&= \sup_\varphi \|c_1(\varphi(x_1),\ldots,\varphi(x_n))\| \\
&\le \sup_\varphi c_2(\|\varphi(x_1)\|,\ldots,\|\varphi(x_n)\|) \\
&\le c_2(\|x_1\|,\ldots,\|x_2\|).
\end{aligned}$$

Hence E satisfies $\mathcal{I}$, as required. □

Remark 3.8. Let $1 \le p < \infty$, and let E be a p-convex Banach lattice. The definition of p-convexity and Theorem 3.2 imply that there exists an $M \ge 1$ such that $\|x\| \le M \sup_\varphi \|\varphi(x)\|$, where φ runs over all contractive vector lattice homomorphisms $\varphi\colon E \to \mathrm{L}_p(\mu)$ for probability measures μ. For $x \in E$, set $\|x\|' := \sup_\varphi \|\varphi(x)\|$. Since $\|x\|' \le \|x\| \le M\|x\|'$, $\|\cdot\|$ is a complete lattice norm on X that is equivalent to $\|\cdot\|$. As all codomains of the φ are p-convex with p-convexity constant 1, Lemma 3.7 shows that the same is true for E in the norm $\|\cdot\|'$. We have thus retrieved a part of [16, Proposition 1..8].

The zero lattice is always $\mathcal{C}$-convex. It can happen that this is the only $\mathcal{C}$-convex Banach lattice. As an example where this is the case, take $n=1$, define $c_1(t_1)=2t_1$ and $c_2(t_1)=t_1$, and set $\mathcal{C}=\big\{(\emptyset,\{(c_1,c_2)\};1)\big\}$. There is a convenient criterion to avoid such a collapse of Sections 5 and 6 to the study of the zero lattice. For this, we note that a Banach sublattice of a $\mathcal{C}$-convex Banach lattice is also $\mathcal{C}$-convex. On considering one-dimensional Banach sublattices, it now follows that the existence of non-zero $\mathcal{C}$-convex Banach lattices is equivalent to $\mathbb{R}$ being $\mathcal{C}$-convex.

We assume in the sequel that $\mathbb{R}$ is $\mathcal{C}$-convex.

For each convexity condition $\mathcal{D}$ in [15], $\mathbb{R}$ is $\mathcal{D}$-convex, hence also $\mathcal{C}$-convex in the corresponding reformulation of $\mathcal{D}$ as a convexity type $\mathcal{C}$.[3].

4. Free vector lattices over pre-ordered vector spaces

The existence of free Banach lattices over pre-ordered spaces in Section 6 is a routinely verified consequence of the existence of free vector lattices over pre-ordered vector spaces. The latter will be taken care of in this section.

We apply Definition 2.1 in the following context. For Cat_1, we take the pre-ordered vector spaces as objects and the positive linear maps as morphisms. For Cat_2, we take the vector lattices as objects and the vector lattice homomorphisms as morphisms. We define the functor U from Cat_1 to Cat_2 by sending a vector lattice E to the pre-ordered vector space (E,E^+) and viewing a vector lattice homomorphism as a positive linear map. Simplifying the notation and terminology a little, this leads to the following definition.

Definition 4.1. A *free vector lattice over the pre-ordered vector space* (V,V^+) is a pair $(j,\mathrm{FVL}[(V,V^+)])$, where $\mathrm{FVL}[(V,V^+)]$ is a vector lattice and $j\colon V\to \mathrm{FVL}[(V,V^+)]$ is a positive linear map with the property that, for every positive linear map $\varphi\colon V\to F$ into a vector lattice F, there exists a unique vector lattice homomorphism $\overline{\varphi}\colon \mathrm{FVL}[(V,V^+)]\to F$ such that the diagram

$$
\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FVL}[(V,V^+)] \arrow[d, "\overline{\varphi}"] \\
& F
\end{tikzcd}
\tag{4.1}
$$

commutes.

Remark 4.2.

1. When $V^+=\{0\}$, every linear map $\varphi\colon V\to F$ is positive, so that $(j,\mathrm{FVL}[(V,\{0\})])$ is what is generally called the free vector lattice over the vector space V.

[3]For this we need to check, in the notation of [15, p. 15], that, under certain conditions on $x_1,\dots,x_n\in\mathbb{R}^n_+$, $|g(x_1,\dots,x_n)|\leq M(g)g(|x_1|,\dots,|x_n|)$. Since $M(g)\geq 1$ and g is positive on $\mathbb{R}^n_+$, this is even true without any conditions

2. If $\mathrm{FVL}[(V, V^+)]$ exists, then it is clear that its vector sublattice that is generated by $j(V)$ is also a free vector lattice over (V, V^+). Hence the inclusion map from this vector sublattice into $\mathrm{FVL}[(V, V^+)]$ is a (surjective) vector lattice isomorphism. Thus $j(V)$ generates $\mathrm{FVL}[(V, V^+)]$.
3. If $\mathrm{FVL}[(V, V^+)]$ exists, then it is clear that the map j in (4.1) is injective if and only if the positive linear maps $\varphi\colon V \to F$ into vector lattices F separate the points of V. When $V^+ = \{0\}$, the linear maps into $\mathbb{R}$ already do this, so that the free vector lattice over a vector space V will contain V as a subspace. This is not always the case. When $V^+ = V$, the only positive linear map φ in (4.1) is the zero map. Hence $\mathrm{FVL}[(V, V)] = \{0\}$ and j is the zero map.

Theorem 4.3. *Let (V, V^+) be a pre-ordered vector space. There exists a free vector lattice $(j, \mathrm{FVL}[(V, V^+)])$ over (V, V^+). It is generated by $j(V)$ as a vector lattice.*

Proof. There exists a free vector lattice $(j, \mathrm{FVL}[(V, \{0\})])$ over $(V, \{0\})$. This is a part of [10, Theorem 6.2]. Alternatively, one can infer this existence from the well-known fact that the free vector lattice over a set exists (see [5–7, 19]). Indeed, the free vector lattice over a basis of V is then as needed. [4].

Take the order ideal I of $\mathrm{FVL}[(V, \{0\})]$ that is generated by $\{j(v)^+ - j(v) : v \in V^+\}$. Let $q\colon \mathrm{FVL}[(V, \{0\})] \to \mathrm{FVL}[(V, \{0\})]/I$ denote the quotient map. We claim that $(q \circ j', \mathrm{FVL}[(V, \{0\})]/I)$ is a free vector lattice over the pre-ordered vector space (V, V^+).

Take $v \in V^+$. Then $[(q \circ j)(v)]^+ = [q(j(v))]^+ = q(j(v)^+) = q(j(v)) = (q \circ j)(v)$. Hence $q \circ j$ is positive, as required.

Let $E \to F$ be a positive linear map into a vector lattice F. There exists a unique vector lattice homomorphism $\varphi\colon \mathrm{FVL}[(V, \{0\})] \to F$ such that $\overline{\varphi} \circ j = \varphi$. Take $v \in V^+$. Then, using the positivity of φ in the final step, we have

$$\begin{aligned}
\overline{\varphi}\big(j(v)^+ - j(v)\big) &= \overline{\varphi}\big(j(v)^+\big) - \overline{\varphi}(j(v)) \\
&= [\overline{\varphi}(j(v))]^+ - \varphi(v) \\
&= \varphi(v)^+ - \varphi(v) \\
&= 0.
\end{aligned}$$

Hence $I \subseteq \ker \varphi$, so that there exists a unique vector lattice homomorphism $\overline{\overline{\varphi}}\colon \mathrm{FVL}[(V, \{0\})]/I \to F$ such that $\overline{\overline{\varphi}} \circ q = \overline{\varphi}$. Since the diagram

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FVL}[(V, \{0\})] \arrow[r, "q"] \arrow[d, "\overline{\varphi}"] & \mathrm{FVL}[(V, \{0\})]/I \arrow[dl, "\overline{\overline{\varphi}}"] \\
& F &
\end{tikzcd}$$

[4] In both approaches, the existence ultimately relies on the existence theorem for free objects in equational classes over sets; see [14, Corollary to Theorem 2.10], for example. We refer to [10] for a detailed and self-contained exposition in the context of vector lattices and vector lattice algebras.

is commutative, we have $\overline{\overline{\varphi}} \circ (q \circ j) = \varphi$. To show uniqueness, suppose that $\psi \colon \mathrm{FVL}[(V, \{0\})]/I \to F$ is a vector lattice homomorphism such that $\psi \circ (q \circ j) = \varphi$. Then $(\psi \circ q) \circ j = \varphi$, so $\psi \circ q = \overline{\varphi}$ and then $\psi = \overline{\overline{\varphi}}$. $\square$

The following separation result is elementary, and in analogous forms obviously valid in many other contexts. We still include its proof to put later results in a similar vein into perspective.

Proposition 4.4. *Let (V, V^+) be a pre-ordered vector spaces. Take $\mathfrak{f} \in \mathrm{FVL}[(V, V^+)]$. If $\overline{\varphi}(\mathfrak{f}) = 0$ for all positive linear maps $\varphi \colon V \to F$ into a vector lattice F, then $\mathfrak{f} = 0$.*

Proof. Set $I := \bigcap_{\varphi} \ker \overline{\varphi}$, where φ runs over all positive linear maps $\varphi \colon V \to F$ into a vector lattice F. For each such φ, we have $I \subseteq \ker \varphi$, so there is a unique vector lattice homomorphism $\overline{\overline{\varphi}} \colon \mathrm{FVL}[(V, V^+)]/I \to F$ making the diagram

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FVL}[(V, V^+)] \arrow[r, "q"] \arrow[d, "\overline{\varphi}"] & \mathrm{FVL}[(V, V^+)]/I \arrow[dl, "\overline{\overline{\varphi}}"] \\ & F & \end{tikzcd} \tag{4.2}$$

commutative. Hence $(q \circ j, \mathrm{FVL}[(V, V^+)]/I)$ is also a free vector lattice over (V, V^+). Then the unique vector lattice homomorphism $\psi \colon \mathrm{FVL}[(V, V^+)] \to \mathrm{FVL}[(V, V^+)]/I$ such that $\psi \circ j = q \circ j$ is an isomorphism. We conclude that q is injective. $\square$

5. Free $\mathcal{C}$-convex Banach lattices over pre-ordered Banach spaces: definition and a priori properties

In this section, we define the free $\mathcal{C}$-convex Banach lattice $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ over a pre-ordered Banach space (X, X^+). We shall see in Section 6 that it always exists. A number of its properties can, however, be determined a priori. We include these here, rather than after the existence has been established, thus emphasizing that they are an immediate consequence of the categorical definition and basic facts about the duals of Banach spaces and Banach lattices. It will become clear that an existing, seemingly stronger, definition in the literature for $\mathrm{FBL}^{\mathcal{C}}[X, \{0\}]$ is equivalent to ours, but that its natural analogue for the general case where X^+ need not be trivial appears not to be suitable to work with; see Remark 5.12.

Fix a convexity type $\mathcal{C}$. In the general context of Definition 2.1, we take for Cat_1 the pre-ordered Banach spaces as objects and the positive contractions as morphisms. Note that, under our standard assumption that $\mathbb{R}$ be $\mathcal{C}$-convex, $\mathrm{B}_{X^*_+}$ consists of morphisms in Cat_1 for every pre-ordered Banach space (X, X^+). For Cat_2, we take the $\mathcal{C}$-convex Banach lattices as objects and the contractive vector lattice homomorphisms as morphisms. The functor U is defined by sending a Banach lattice E to the pre-ordered Banach space (E, E^+) and viewing a contractive vector lattice homomorphism as a positive contraction. This leads to the following definition.

Definition 5.1. Let $\mathcal{C}$ be a convexity type. A *free $\mathcal{C}$-convex Banach lattice over the pre-ordered Banach* (X,X^+) is a pair $(j,\mathrm{FBL}^{\mathcal{C}}[(X,X^+)])$, where $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is a $\mathcal{C}$-convex Banach lattice and $j\colon X\to\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is a positive contraction with the property that, for every positive contraction $\varphi\colon X\to F$ into a $\mathcal{C}$-convex Banach lattice F, there exists a unique contractive vector lattice homomorphism $\overline{\varphi}\colon\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]\to F$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \arrow[d, "\overline{\varphi}"] \\ & F \end{tikzcd} \tag{5.1}$$

commutes.

Remark 5.2. When $T\colon X\to F$ is an arbitrary bounded linear operator, scaling yields that there is a unique vector lattice homomorphism $\overline{T}$ making the diagram commutative. Then $\|\overline{T}\|=\|T\|$ by Lemma 5.8, below.

If $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists, then it is uniquely determined up to an isometric vector lattice isomorphism. Indeed, for any two free objects, there are mutually inverse contractive vector lattice isomorphisms between them. These contractions must then, in fact, be isometries.

If $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists, then, as for the free vector lattice over a pre-ordered vector space, general principles imply that $j(X)$ generates $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ as a Banach lattice. An argument as that in the proof of Proposition 4.4 shows that the family of all $\overline{\varphi}$ obtained from (5.1) separate the points of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$.

The following simple observation is pivotal to the results in Section 7.

Lemma 5.3. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Let F and F' be $\mathcal{C}$-convex Banach lattices, let $\varphi\colon X\to F$ be a positive contraction, and let $\psi\colon F\to F'$ be a contractive vector lattice homomorphism. Then $\overline{\psi\circ\varphi}=\psi\circ\overline{\varphi}$.*

Proof. We have $(\psi\circ\overline{\varphi})\circ j=\psi\circ\varphi$. As $\psi\circ\overline{\varphi}$ is a contractive vector lattice homomorphism, it must be $\overline{\psi\circ\varphi}$. □

Under our standing assumption that $\mathbb{R}$ be $\mathcal{C}$-convex it is still possible that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is trivial. This is obviously the case when $X^+=X$. The general criterion is as follows. In it, and in other results, we prefer to say that $\mathrm{B}_{X_+^*}$ is nontrivial or that it separates the points of X, rather than saying that X_+^* has this property. The reason is that the first set consists of morphisms and the second does not when $X\neq\{0\}$.

Lemma 5.4. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Then $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]\neq\{0\}$ if and only if $\mathrm{B}_{X_+^*}\neq\{0\}$.*

Proof. Clearly, if $\mathrm{B}_{X_+^*}\neq\{0\}$, then $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]\neq\{0\}$, since otherwise the factoring of an arbitrary x^* in $\mathrm{B}_{X_+^*}$ would imply that it is 0. Conversely, if

$\mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \neq \{0\}$, then its generating linear subspace $j(X)$ is not trivial. Pick $x \in X$ such that $j(x) \neq 0$ and then a positive contractive x^* in the dual of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \neq$ such that $x^*(j(x)) \neq 0$. Then $x^* \circ j$ is a non-zero element of $\mathrm{B}_{X^*_+}$. ☐

A similar argument, again using that the positive contractive functionals on $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ separate its points, yields the following.

Lemma 5.5. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Take $x \in X$. Then $j(x) = 0$ if and only if $x^*(x) = 0$ for all $x^* \in \mathrm{B}_{X^*_+}$. Consequently, j is injective if and only if $\mathrm{B}_{X^*_+}$ separates the points of X.*

Lemma 5.6. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Then $\|j(x)\| = \max_{x^* \in \mathrm{B}_{X^*_+}} x^*(x)$ for $x \in X^+$.*

Proof. Take $x^* \in \mathrm{B}_{X^*_+}$. Then $x^*(x) = |x^*(x)| = |(\overline{x^*} \circ j)(x)| \leq \|j(x)\|$. Hence $\|j(x)\| \geq \sup_{x^* \in \mathrm{B}_{X^*_+}} x^*(x)$. For the reverse inequality, take a positive contractive x^* in the dual of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ such that $(x^* \circ j)(x) = x^*(j(x)) = \|j(x)\|$, which is possible as $j(x)$ is positive. This implies the reverse inequality, and also that the supremum is a maximum. ☐

As a consequence of Lemmas 5.5 and 5.6 we have the following.

Corollary 5.7. *Let X be a Banach lattice, and let $\mathcal{C}$-type be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Then j is injective on X and isometric on X^+.*

Lemma 5.8. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. If $\varphi : X \to F$ is a positive contraction into a $\mathcal{C}$-convex Banach lattice, then $\|\varphi\| = \|\overline{\varphi}\|$.*

Proof. We may suppose that $\varphi \neq 0$. As $\varphi = \overline{\varphi} \circ j$, we have $\|\varphi\| \leq \|\overline{\varphi}\|$. The reverse inequality follows from the observation that $\overline{\varphi}$ is even the unique vector lattice homomorphism that makes (5.1) commutative. To see this, we note that any such vector lattice homomorphism $\overline{\varphi}$ is uniquely determined on $j(X)$ and then also on the vector sublattice generated by $j(X)$. As this is dense and $\overline{\varphi}$ is automatically continuous, $\overline{\varphi}$ is unique.[5] Now $\varphi/\|\varphi\|$ is also a positive contraction, so it has an accompanying contraction $\overline{(\varphi/\|\varphi\|)}$ such that $\varphi/\|\varphi\| = \overline{(\varphi/\|\varphi\|)} \circ j$, which is also the unique vector lattice homomorphism $\psi : \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \to F$ such that $\psi \circ j = \varphi/\|\varphi\|$. As $\overline{\varphi}/\|\varphi\|$ obviously has this property, it must be the contraction $\overline{(\varphi/\|\varphi\|)}$. Thus $\|\overline{\varphi}/\|\varphi\|\| \leq 1$ and it follows that $\|\overline{\varphi}\| \leq \|\varphi\|$, as desired. ☐

Lemma 5.9. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Then $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \neq \{0\}$ if and only if $\|j\| = 1$.*

[5] This argument is also used in[15, Corollary 3.5].

Proof. Suppose that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \neq \{0\}$. By Lemma 5.4, there exists a non-zero $x^* \in \mathrm{B}_{X_+^*}$. Since $\|x^*\| = \|\overline{x^*} \circ j\| \leq \|\overline{x^*}\|\|j\| = \|x^*\|\|j\|$ we have $\|j\| \geq 1$. Hence $\|j\| = 1$. The converse statement is trivial. □

Lemma 5.10. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. If $\mathrm{B}_{X_+^*}$ is norming for X, then j is isometric. This is the case when $X^+ = \{0\}$.*

Proof. Let $x \in X$. As $\mathrm{B}_{X_+^*}$ is norming, there exists an $x^* \in \mathrm{B}_{X_+^*}$ such that $\|x\| = |x^*(x)|$. Then $\|x\| = |x^*(x)| = |\overline{x^*}(j(x))| \leq \|\overline{x^*}\|\|j(x)\| \leq \|j(x)\|$. As j is contractive by definition, it is isometric. Clearly, when $X^+ = \{0\}$, $\mathrm{B}_{X_+^*} = \mathrm{B}_{X^*}$ is norming for X.

□

We refrain from claiming any originality for the following.

Lemma 5.11. *Let (X,X^+) be a pre-ordered Banach space, where X^+ is a closed cone.*

1. *For $x \in X$, $x \in X^+$ if and only if $x^*(x) \geq 0$ for all $x^* \in \mathrm{B}_{X_+^*}$*
2. *X_+^* separates the points of X.*
3. *j is bipositive.*

Proof. The separation theorem [8, Theorem 3.9] shows that, for $x \notin X^+$, there exists an $x^* \in \mathrm{B}_{X_+^*}$ such that $x^*(x) < 0$. This implies the first part, which is still true for closed wedges. For cones, the second part then follows easily. For the third, suppose that $j(x) \geq 0$. Then $x^*(x) = \overline{x^*}(j(x)) \geq 0$ all $x^* \in \mathrm{B}_{X_+^*}$. Hence $x \geq 0$. □

Remark 5.12. Let X be a Banach space. As all Banach lattices are $\emptyset$-convex, a free $\emptyset$-convex Banach lattice over $(X,\{0\})$ is a pair $(j,\mathrm{FBL}[X])$, where $\mathrm{FBL}[X]$ is a Banach lattice and $j\colon X \to \mathrm{FBL}[X]$ is a contraction with the property that, for every contraction $\varphi\colon X \to F$ into a Banach lattice F, there exists a unique contractive vector lattice homomorphism $\overline{\varphi}\colon \mathrm{FBL}[X] \to F$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FBL}[X] \arrow[d, "\overline{\varphi}"] \\ & F \end{tikzcd} \tag{5.2}$$

commutes. This is almost the definition/description of the free Banach lattice over the Banach space X in the sense of [4], but there it is also required that $\|\overline{\varphi}\| = \|\varphi\|$ and that j be isometric. Lemmas 5.8 and 5.10, however, show that both are automatic for our free objects. Thus the specialisation of our general categorical definition to the case where $\mathcal{C} = \emptyset$ and $X^+ = \{0\}$ and the definition in [4] are in agreement after all.

The fact that j is automatically an isometry when $X^+ = \{0\}$ is a consequence of the fact that $\mathrm{B}_{X_+^*}$ is then norming for X. Since, in the general situation of Definition 5.1, where it is possible that $X^+ \neq \{0\}$, $\mathrm{B}_{X_+^*}$ cannot be expected to be norming for X, a definition analogous to that in [4], including the requirement that j be an isometry, would not, or at least not obviously, lead to a theory in which $(j, \mathrm{FBL}^{\mathcal{C}}[(X,X^+)])$ exists.

The norm on $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is determined by its universal property in an explicit way.

Lemma 5.13. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ exists. Then, for $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$,*

$$\|\mathfrak{f}\| = \max_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|,$$

where φ runs over the positive contractions $\varphi : X \to F_\varphi$ into $\mathcal{C}$-convex Banach lattices F_φ. In particular, for $x \in X$, we have

$$\|j(x)\| = \max_{\varphi} \|\varphi(x)\|,$$

where φ runs over the positive contractions $\varphi : X \to F_\varphi$ into $\mathcal{C}$-convex Banach lattices F_φ.

Proof. Take $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ and a positive contraction $\varphi : \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \to F_\varphi$ into a $\mathcal{C}$-convex Banach lattice F_φ. Then $\|\overline{\varphi}(\mathfrak{f})\| \leq \|\overline{\varphi}\|\|\mathfrak{f}\| \leq \|\mathfrak{f}\|$. Hence $\|\mathfrak{f}\| \geq \sup_\varphi \|\overline{\varphi}(\mathfrak{f})\|$. For $\varphi = j$, $\overline{\varphi}$ is the identity map. This gives the reverse inequality, and also that the supremum is a maximum. The expression for $\|j(x\|$ is then clear. $\square$

Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type. What is the relation between the free vector lattice $\mathrm{FVL}[(X,X^+)]$ and the free Banach lattice $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$? To make this question precise, we take—if it exists—a free $\mathcal{C}$-convex Banach lattice $(j, \mathrm{FBL}^{\mathcal{C}}[(X,X^+)])$ over (X,X^+) and a free vector lattice $(j_{\mathrm{po}}, \mathrm{FVL}[(X,X^+)])$ over the pre-ordered vector space (X,X^+). Here the subscript in the latter indicates that are working with positive linear maps between pre-ordered vector spaces. There is a unique vector lattice homomorphism $\overline{j} : \mathrm{FVL}[(X,X^+)] \to \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "j_{\mathrm{po}}"] \arrow[dr, "j"'] & \mathrm{FVL}[(X,X^+)] \arrow[d, "\overline{j}"] \\ & \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \end{tikzcd} \tag{5.3}$$

is commutative. Hence $\overline{j}\big(\mathrm{FVL}[(X,X^+)]\big)$ coincides with the dense vector sublattice of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ that is generated by $j(X)$. Thus, when $\overline{j}$ is injective, $\mathrm{FVL}[(X,X^+)]$ is embedded in $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ as a dense vector sublattice. We shall now determine when this is the case. Take a positive contraction $\varphi : X \to F_\varphi$ into a $\mathcal{C}$-convex Banach lattice F. There exist a unique vector contractive lattice homomorphism $\overline{\varphi} : \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \to F$ such that $\overline{\varphi} \circ j = \varphi$, and a unique vector lattice homomorphism $\overline{\varphi_{\mathrm{po}}} : \mathrm{FVL}[(X,X^+)] \to F$ such that $\overline{\varphi_{\mathrm{po}}} \circ j_{\mathrm{po}} = \varphi_{\mathrm{po}}$, where φ_{po} indicates that we view φ as a positive linear map between pre-ordered vector spaces. For $x \in X$, we have

$$(\overline{\varphi} \circ \overline{j})(j_{\mathrm{po}}(x)) = \overline{\varphi}(j(x)) = \varphi(x) = \overline{\varphi_{\mathrm{po}}}(j_{\mathrm{po}}(x)). \tag{5.4}$$

Since they agree on a generating subset of $\mathrm{FVL}[(X,X^+)]$, we have $\overline{\varphi} \circ \overline{j} = \overline{\varphi_{\mathrm{po}}}$. As the $\overline{\varphi}$ as thus obtained separate the points of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$, we now see that, for $\mathfrak{f} \in \mathrm{FVL}[(X,X^+)]$, $\overline{j}(\mathfrak{f}) = 0$ if and only if $\overline{\varphi_{\mathrm{po}}}(\mathfrak{f}) = 0$ for all positive contractions φ from X into a $\mathcal{C}$-convex Banach lattice, seen as positive linear maps with the vector space X as domain. Therefore, $\overline{j}$ *in* (5.3) *embeds* $\mathrm{FVL}[(X,X^+)]$ *into* $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ *as a (dense) vector sublattice if and only if the* $\overline{\varphi_{\mathrm{po}}}$ *thus obtained separate the points of* $\mathrm{FVL}[(V,V^+)]$. In that case, if we let E denote the vector sublattice of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ that is generated by $j(V)$, and view j as a map with codomain E, then (j,E) is a free vector lattice over the pre-ordered vector space (X,X^+). Conversely, if this is the case, then the unique positive linear map $\overline{j}$ that makes (5.3) commutative is an isomorphism, so $\overline{j}$ is injective.

We know from Proposition 4.4 that allowing φ to be a positive linear map from X into an arbitrary vector lattice gives a separating family of morphisms $\overline{\varphi}$ on $\mathrm{FVL}[(V,V^+)]$, but there is no reason why this should be the case for the smaller family that is obtained by allowing only positive contractions into Banach lattices. Hence the dense vector sublattice of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ that is generated by $j(X)$ will, in general, be a proper quotient of $\mathrm{FVL}[(V,V^+)]$ via the map $\overline{j}$ in (5.3).

When $X^+ = \{0\}$, however, $\overline{j}$ *is* always injective. Then the positive contractive $\varphi : X \to \mathbb{R}$ already yield a separating family of vector lattice homomorphisms $\overline{\varphi}_{\mathrm{po}}$ for $\mathrm{FVL}[X,\{0\}]$, as a consequence of the following result from [9].

Theorem 5.14. *Let V be a vector space. Take a separating vector space L^* of linear functionals on V. Then the $\overline{l^*}$ for $l^* \in L^*$ separate the points of* $\mathrm{FVL}[V,\{0\}]$.

For a Banach space X, X^* is such a separating vector space, so the $\overline{x^*_{\mathrm{po}}}$ for $x^* \in \mathrm{B}_{X^*}$ are a separating family for $\mathrm{FVL}[(X,\{0\})]$.

We summarise the above as follows. As said, the existence condition is always fulfilled.

Theorem 5.15. *Let X be a Banach space, and let $\mathcal{C}$ be a convexity type such that the free $\mathcal{C}$-convex Banach lattice $(j,\mathrm{FBL}^{\mathcal{C}}[(X,X^+)])$ over X exists. Let E be the dense vector sublattice of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ that is generated by $j(X)$, and view j as a positive linear map from X into E. Then the following are equivalent:*

1. *(j,E) is a free vector lattice over (X,X^+);*
2. *The family of vector lattice homomorphism $\overline{\varphi}_{\mathrm{po}} : \mathrm{FVL}[(X,X^+)] \to F_\varphi$, obtained as the unique vector lattice homomorphism such that $\overline{\varphi}_{\mathrm{po}} \circ j = \varphi$ as φ runs over the positive contractions $\varphi : X \to F_\varphi$ into a $\mathcal{C}$-convex Banach lattice F_φ, separate the points of* $\mathrm{FVL}[(X,X^+)]$.

When $X^+ = \{0\}$, this is the case.

Remark 5.16. Suppose that V is a vector space and that L^* is a separating vector space of linear functionals on V. For $x \in V$, define $\delta'_x : L^* \to \mathbb{R}$ by setting $\delta'_x(l^*) := l^*(x)$ for $l^* \in L^*$, and let E' be the function lattice on L^* that is generated by the δ'_x for $x \in V$. With $\delta' : V \to E'$ denoting the linear map thus obtained, Theorem 5.14 shows that (δ', E') is a free vector lattice over $(V, \{0\})$.

This applies, in particular, to a Banach space X and its dual X^*. Since restriction of homogeneous functions on X^* to B_{X^*} yields an isomorphic function lattice, we can define δ_x for $x \in X$ as in (2.1) (for $X^+ = \{0\}$), let E be the function lattice on B_{X^*} that is generated by the δ_x, and let $\delta : X \to E$ denote the linear map thus obtained. Then (δ, E) is a free vector lattice over $(X, 0)$.

This fact seems to have escaped notice in the papers on free Banach lattices so far. In [17], for example, the lattice linear calculus is used to proved that (δ, E) has the pertinent universal property with respect to Archimedean vector lattices as codomains (see [17, p. 24]). As is now clear, the Archimedean property can be omitted, and the lattice linear calculus is not needed.

The proof of Theorem 5.14 in [9] is surprisingly easy, using elegant arguments due to Bleier in [7]. It has the existence of $\mathrm{FVL}[V, \{0\}]$, known from universal algebra, as a starting point. In the general case, we do know from Theorem 4.3 that $\mathrm{FVL}[(V, V^+)]$ exists, but the method of proof in [9] does not appear to generalise to give any conjectured generalisation of Theorem 5.14 at all. Consequently, to the minds of the authors, a conjecture that the δ_x from (2.1) always generate $\mathrm{FVL}[(X, X^+)]$ is suspect. This also makes it doubtful whether the methods in [4] and [15] to construct free Banach lattices can be successful when $X^+ \neq \{0\}$. These start from—what we now know to be—a canonical realisation of $\mathrm{FVL}[(X, \emptyset)]$ on X^*, establish 'by hand' its universal property in the pertinent context (which we now know to be a consequence of it being a free vector lattice over X), use this to supply it with a suitable norm, and then complete the space to obtain the sought free object. When the δ_x on $\mathrm{B}_{X^*_+}$ do not generated $\mathrm{FVL}[(X, X^+)]$, this method is bound to run aground in the very beginning. As we shall see in the next section, however, everything works smoothly as long one does not attempt to start from a function lattice on $\mathrm{B}_{X^*_+}$.

Another matter is the question whether $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ can be realised as a function lattice on some set. As $\mathbb{R}$ is $\mathcal{C}$-convex, there is a canonical candidate for such a set, namely X^*_+. For $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$, define $\widehat{\mathfrak{f}} \colon \to \mathbb{R}$ by setting

$$\widehat{\mathfrak{f}} := \overline{x^*}(\mathfrak{f}) \tag{5.5}$$

for $x^* \in \mathrm{B}_{X^*_+}$. The map $\mathfrak{f} \mapsto \widehat{\mathfrak{f}}$ is then a vector lattice homomorphism from $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ into the vector lattice of all homogeneous functions on $\mathrm{B}_{X^*_+}$. Lemma 5.13 makes clear that convergence in $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ implies uniform convergence in $\mathrm{C^h}(\mathrm{B}_{X^*_+})$. As the dense vector sublattice of $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ that is generated by $j(X)$ clearly maps into $\mathrm{C^h}(\mathrm{B}_{X^*_+})$, this is true for $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ itself.

Proposition 5.17. *Let X be a Banach space, and let $\mathcal{C}$ be a convexity type such that the free $\mathcal{C}$-convex Banach lattice $(j, \mathrm{FBL}^{\mathcal{C}}[(X, X^+)])$ over X exists. The following are equivalent:*

1. *The map $\mathfrak{f} \mapsto \widehat{\mathfrak{f}}$ in* (5.5) *gives a realisation of* $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ *as a vector sublattice of $C^{\mathrm{h}}(\mathrm{B}_{X^*_+})$.*
2. *The contractive vector lattice homomorphisms $\overline{\varphi}\colon \mathrm{FBL}^{\mathcal{C}}[(X, X^+)] \to F$ separate the points of* $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ *as φ runs over all positive contractions $\varphi\colon X \to F_\varphi$ into finite dimensional Banach lattices F_φ.*

Proof. If $\mathfrak{f} \mapsto \widehat{\mathfrak{f}}$ is injective, then the $\overline{\varphi}$ for one-dimensional Banach lattice already suffice to separate the points of $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$. Conversely, suppose that the $\overline{\varphi}\colon X \to F_\varphi$ for positive contractions φ into finite dimensional Banach lattices F_φ separate the points of $\mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$, and that $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X, X^+)]$ is such that $\widehat{\mathfrak{f}} = 0$. Take such a φ. As F_φ is isomorphic as a vector lattice to $\mathbb{R}^n$ for some n, there are—after this identification—$x_1^*, \ldots, x_n^* \in \mathrm{B}_{X^*_+}$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\lambda_1 x_1^*, \cdots, \lambda_n x_n^*$ are the coordinate components of φ. By the uniqueness of the factoring morphism, the coordinate components of $\overline{\varphi}$ are $\lambda_1 \overline{x_1^*}, \cdots, \lambda_n \overline{x_n^*}$. Hence $\overline{\varphi}(\mathfrak{f}) = 0$. As φ was arbitrary, we have $\mathfrak{f} = 0$. □

For the free Banach lattices $\mathrm{FBL}^{(\infty)}[(X, X^+)]$ and $\mathrm{FBL}^{(p)}[(X, X^+)]$ in Section 7, the condition in part (2) of Proposition 5.17 is satisfied; see (7.2) and Theorem 7.7. It is thus clear that they can be realised as a vector sublattice of $\mathrm{C}^{\mathrm{h}}(\mathrm{B}_{X^*_+})$. The formulas for the norms in the images are just the transported formulas for the norms in the free Banach lattice.

6. Free $\mathcal{C}$-convex Banach lattices over pre-ordered Banach spaces: existence

We shall now show that the free $\mathcal{C}$-convex Banach lattice over a pre-ordered Banach space always exists. This generalises [15, Theorem 3.3] in two ways. At the technical level, our convexity types are more general than the convexity conditions in [15]. Subsuming a pre-ordering, however, is a fundamental change.

With the existence of the free vector lattice over a pre-ordered vector space as a starting point, we follow the method that already appears in [10, pp.103-106] and which is systematically exploited in [18]. The first step is the introduction of a (semi)norm on $\mathrm{FVL}[(X, X^+)]$ in terms of its universal property. When $X^+ = \{0\}$, so that the δ_x generated $\mathrm{FVL}[X, \{0\}]$, this is, in fact, also what is done in [4] and [15], but apparently without the authors being aware of the fact that they were working with $\mathrm{FVL}[X, \{0\}]$. As argued in Remark 5.16, it seems doubtful that a similar approach, where one starts from a vector lattice on $\mathrm{B}_{X^*_+}$, will work when $X^+ \neq \{0\}$.

We believe that even when $X^+ = \{0\}$ the construction below is easier. The latter is perhaps best illustrated at the very end, when it needs to be verified that the Banach lattice as constructed is $\mathcal{C}$-convex. A non-trivial preparatory technical effort is needed for this in [15], but in the proof below it is an immediate consequence of Lemma 3.7.

As in [10] and [18], the idea is to first take a free object in an algebraic category, and then exploit its universal property to arrive at the desired object in an analytic category. This concept occurs already as early as 1964 in the construction of the enveloping C*-algebra of an involutive Banach algebra A with an isometric involution and an approximate identity; see [12, Proposition 2.7.1] (or [13, Proposition 2.7.1]). Knowing that C*-algebras are subalgebras of some $B(H)$, what is done is the following: take the free involutive algebra over A (which is just A itself), introduce a semi-norm on it in terms of its universal property with respect to C*-algebras seen as involutive algebras, quotient out the kernel, and complete the quotient to obtain the so-called enveloping C*-algebra of A. This enveloping C*-algebra is, in fact, the free C*-algebra over A in the sense of Definition 2.1. We shall follow the exact same path.

We shall have use for the following preparatory lemma about norming classes of operators. It can very likely already be found elsewhere, but we are not aware of a concrete reference. In view of its pivotal role in the present paper, we include the easy proof.

Lemma 6.1. *Let X be a normed space. Suppose that, for all x in a dense subset of X,*

$$\|x\| = \sup_{\varphi} \|\varphi(x)\| \tag{6.1}$$

where φ runs over a class of contractions $\varphi: X \to X_\varphi$ into normed spaces X_φ. Then (6.1) *holds for all $x \in X$.*

Proof. Take $x \in X$. It is clear that $\|x\| \geq \sup_\varphi \|\varphi(x)\|$. For the reverse inequality, take $\varepsilon > 0$ and then an x_0 with $\|x - x_0\| < \varepsilon/3$ such that $\|x_0\| = \sup_\varphi \|\varphi(x_0)\|$. There exists a φ_0 in the class such that $\|x_0\| \leq \|\varphi_0(x_0)\| + \varepsilon/3$. Then

$$\begin{aligned}
\|x\| &\leq \|x - x_0\| + \|x_0\| \\
&\leq \varepsilon/3 + \|\varphi_0(x_0)\| + \varepsilon/3 \\
&\leq \|\varphi_0(x_0 - x)\| + \|\varphi_0(x)\| + 2\varepsilon/3 \\
&\leq \|\varphi_0(x)\| + \varepsilon \\
&\leq \sup_\varphi \|\varphi(x)\| + \varepsilon.
\end{aligned}$$

As ε was arbitrary, we are done. □

Starting with the existence proof, let (X, X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type. We start by viewing (X, X^+) as a pre-ordered vector space and take the free vector lattice $(j, \mathrm{FVL}[(X, X^+)])$ over it, which we know to exist from Theorem 4.3. If $\varphi: X \to F$ is a positive contraction into a

$\mathcal{C}$-convex Banach lattice F, then there exists a unique vector lattice homomorphism $\overline{\varphi}\colon \mathrm{FVL}[(X,X^+)] \to F$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FVL}[(X,X^+)] \arrow[d, "\overline{\varphi}"] \\ & F \end{tikzcd} \tag{6.2}$$

commutes.[6] Take $\xi \in \mathrm{FVL}[(X,X^+)]$. Motivated by Lemma 5.13, we set

$$\rho(\xi) := \sup_{\varphi} \|\overline{\varphi}(\xi)\| \tag{6.3}$$

where φ runs over the positive contractions $\varphi\colon X \to F_\varphi$ into $\mathcal{C}$-convex Banach lattices F_φ. Evidently, $\rho(j(x)) = \sup_\varphi \|\varphi(x)\| \le \|x\|$ for $x \in X$. Since the subset of $\mathrm{FVL}[(X,X^+)]$ where ρ is finite is easily seen to be a vector sublattice, and since it contains the generating subset $j(X)$, ρ is finite everywhere. It is a lattice semi-norm. The kernel $\ker\rho(f)$ is an order ideal of $\mathrm{FVL}[(X,X^+)]$. Let $q\colon \mathrm{FVL}[(X,X^+)] \to \mathrm{FVL}[(X,X^+)]/\ker\rho$ denote the quotient map and introduce a lattice norm $\|\cdot\|$ on $\mathrm{FVL}[(X,X^+)]/\ker\rho$ by setting $\|q(\xi)\| := \rho(\xi)$ for $\xi \in \mathrm{FVL}[(X,X^+)]$. Let $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ be the completion of $(\mathrm{FVL}[(X,X^+)]/\ker\rho, \|\cdot\|)$ and view q as a map from $\mathrm{FVL}[(X,X^+)]$ into $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$. We claim that, as the notation suggests, $(q\circ j, \mathrm{FBL}^{\mathcal{C}}[(X,X^+)])$ is a free $\mathcal{C}$-convex Banach lattice over (X,X^+). There are a number of things to be checked. First of all, $q\circ j$ is positive since q and j are; it is contractive as $\|(q\circ j)(x)\| = \rho(j(x)) = \sup_\varphi \|\varphi(x)\| \le \|x\|$ for $x \in X$. Next, take a positive contraction $\varphi\colon X \to F$ into a $\mathcal{C}$-convex Banach lattice F. It follows from (6.3) that $\ker\rho \subseteq \ker\overline{\varphi}$, so there exists a unique vector lattice homomorphism $\overline{\overline{\varphi}}\colon \mathrm{FVL}[(X,X^+)]/\ker\rho \to F$ such that $\overline{\overline{\varphi}} \circ q = \overline{\varphi}$. For $\xi \in \mathrm{FVL}[(X,X^+)]$, we have $\|\overline{\overline{\varphi}}(q(\xi))\| = \|\overline{\varphi}(\xi)\| \le \rho(\xi) = \|q(\xi)\|$. Hence $\overline{\overline{\varphi}}$ is a contractive vector lattice homomorphism. It extends to a contractive vector lattice homomorphism $\overline{\overline{\overline{\varphi}}}\colon \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \to F$. It is a consequence of the construction that $\overline{\overline{\overline{\varphi}}} \circ (q\circ j) = \varphi$. Furthermore, since $j(X)$ generates $\mathrm{FVL}[(X,X^+)]$ as a vector lattice, $(q\circ j)(X)$ generates $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ as a Banach lattice. Hence $\overline{\overline{\overline{\varphi}}}$ is the only contractive vector lattice homomorphism (by automatic continuity: even the only vector lattice homomorphism) such that $\overline{\overline{\overline{\varphi}}} \circ (q\circ j) = \varphi$.

The construction is summarised in the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FVL}[(X,X^+)] \arrow[r, "q"] \arrow[d, "\overline{\varphi}"] & \mathrm{FVL}[(X,X^+)]/\ker\rho_a \arrow[r, hook] \arrow[dl, "\overline{\overline{\varphi}}"] & \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \arrow[dll, "\overline{\overline{\overline{\varphi}}}"] \\ & F & & \end{tikzcd} \tag{6.4}$$

[6]In Section 5, this was denoted by $\overline{\varphi}_{\mathrm{po}}$, but here there is no possibility of confusion.

It remains to verify that $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is $\mathcal{C}$-convex. For this, take $\xi \in \mathrm{FVL}[(X,X^+)]$. Then

$$\|q(\xi)\| = \rho(\xi) = \sup_{\varphi} \|\overline{\varphi}(\xi)\| = \sup_{\varphi} \left\|\overline{\overline{\varphi}}(q(\xi))\right\| = \sup_{\varphi} \left\|\overline{\overline{\overline{\varphi}}}(q(\xi))\right\|.$$

Since $q(\mathrm{FVL}[(X,X^+)])$ is dense in $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$, Lemma 6.1 yields that

$$\|\mathfrak{f}\| = \sup_{\varphi} \left\|\overline{\overline{\overline{\varphi}}}(\mathfrak{f})\right\|$$

for $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$. By Lemma 3.7, $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ is $\mathcal{C}$-convex since all codomains are.

Remark 6.2. The above construction also works when the zero lattice is the only $\mathcal{C}$-convex Banach lattice, in which case the zero outcome is already clear from the start. Under our standing assumption that $\mathbb{R}$ is $\mathcal{C}$-convex, it is clear from the a priori results in Section 5 that there is a role to be played by $\mathrm{B}_{X_+^*}$ for the *properties* of $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$, but for its *existence* as established above there is none.

After brushing up our notation to match that in Definition 5.1, and collecting additional a priori information from Section 5, we have thus established the following.

Theorem 6.3. *Let (X,X^+) be a pre-ordered Banach space, and let $\mathcal{C}$ be a convexity type such that $\mathbb{R}$ is $\mathcal{C}$-convex. Then there exist a $\mathcal{C}$-convex Banach lattice and a positive contraction $j\colon X \to \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$ with the property that, for every positive contraction $\varphi\colon X \to F$ into a $\mathcal{C}$-convex Banach lattice F, there exists a unique contractive vector lattice homomorphism $\overline{\varphi}\colon \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \to F$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "\varphi"'] & \mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \arrow[d, "\overline{\varphi}"] \\ & F \end{tikzcd} \tag{6.5}$$

commutes. It is generated by $j(X)$. Furthermore:

1. *For $\mathfrak{f} \in \mathrm{FBL}^{\mathcal{C}}[(X,X^+)]$, we have*

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|, \tag{6.6}$$

where φ runs over the positive contractions $\varphi\colon X \to F_\varphi$ into $\mathcal{C}$-convex Banach lattices F_φ. In particular, for $x \in X$, we have

$$\|j(x)\| = \sup_{\varphi} \|\varphi(x)\|, \tag{6.7}$$

where φ runs over the positive contractions $\varphi\colon X \to F_\varphi$ into $\mathcal{C}$-convex Banach lattices F_φ.

For $x \in X^+$, we have

$$\|j(x)\| = \max_{x^* \in \mathrm{B}_{X_+^*}} x^*(x).$$

2. $\|\varphi\| = \|\overline{\varphi}\|$.
3. *When X^+ is a closed cone, j is a bipositive injection.*
4. *If $\mathrm{B}_{X_+^*}$ is norming for X, then j is isometric. This is the case when $X^+ = 0$.*
5. *If X is a Banach lattice, then j is injective on X and isometric on X^+.*
6. *For $x \in X$, $j(x) = 0$ if and only if $x^*(x) = 0$ for all $x^* \in \mathrm{B}_{X_+^*}$.*
7. *The map j is injective if and only if $\mathrm{B}_{X_+^*}$ separates the points of X.*
8. *The following are equivalent:*
   (a) $\mathrm{FBL}^{\mathcal{C}}[(X,X^+)] \neq \{0\}$*;*
   (b) $\mathrm{B}_{X_+^*} \neq \{0\}$*;*
   (c) $\|j\| = 1$.
   When X^+ is a closed cone, these are also equivalent to:
   (d) $X \neq \{0\}$.

We refer to Theorem 5.15 for the precise criterion when $j(X)$ generates $\mathrm{FVL}[(X,X^+)]$, and recall that this is always the case when $X^+ = \{0\}$.

7. Realisation of the free p-convex Banach lattice with p-convexity constant 1 over a pre-ordered Banach space as a function lattice

We recall from Example 3.6 that (quite obviously) there is a convexity types $\mathcal{C}_p$ for $1 \leq p \leq \infty$ such that the $\mathcal{C}_p$-convex Banach lattices are the p-convex Banach lattices with p-convexity constant 1. Hence we know from Theorem 6.3 that, for every $1 \leq p \leq \infty$, the free p-convex Banach lattice with p-convexity constant 1 over a pre-ordered Banach (X,X^+) space exists. We shall denote it by $\mathrm{FBL}^{(p)}[(X,X^+)]$. The free Banach lattices $\mathrm{FBL}^{(p)}[X]$ as in [15] are then our $\mathrm{FBL}^{(p)}[X,\{0\}]$. In this section, we shall give a concrete realisation of $\mathrm{FBL}^{(p)}[(X,X^+)]$ as a Banach lattice of functions. For $\mathrm{FBL}^{(p)}[X,\{0\}]$ such a realisation is known: for $p = 1$ this can be found in [4] and for $1 < p \leq \infty$ in [15]. When looking at the pertinent realisations as lattices of functions on B_{X^*}, there is a natural Ansatz for the general case: write down the obvious analogues of lattices of functions, but then on $\mathrm{B}_{X_+^*}$, rather than on B_{X^*}. At least when $X^+ = \{0\}$ we know that this gives the right answer. As we shall see, this is indeed correct.

As with existence, our proofs different from those in [4] and [15], where the existence of $\mathrm{FBL}^{(p)}[X,\{0\}]$ is re-proven by showing that the pertinent function lattice has the correct universal property. Instead, we shall analyse the structure of $\mathrm{FBL}^{(p)}[(X,X^+)]$, which we *know* to exist, to the point where it is immediate, almost as an afterthought, that it can be realised as a function lattice. To illustrate the difference further, and at the risk of then also annoying the reader further, we remark that, as in Section 5, this section can be read under the hypothesis that $\mathrm{FBL}^{(p)}[(X,X^+)]$ exist. Theorems 7.2 and 7.8 then show what it must look like, should it exist.

Our proof for $1 \leq p < \infty$ profits from the ingenuity of some of the arguments in the proof of [15, Theorem 6.1].

Remark 7.1. As in [15, Example 3.6], if $T : X \to F$ is a bounded linear operator into a p-convex Banach lattice with p-convexity constant $M^{(p)}(F)$, then there exists a unique vector lattice homomorphism $\overline{T} : \mathrm{FBL}^{\mathcal{C}}[(X, X^+)] \to F$ such that $T \circ j = \varphi$. Moreover $\|\overline{T}\| \leq M^{(p)}(F)\|T\|$. This follows from the values of the constants in the renorming in Remark 3.8.

7.1. **The case** $p = \infty$**.** This is the easy case. Take $\mathfrak{f} \in \mathrm{FBL}^{(\infty)}[(X, X^+)]$. Then

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|, \tag{7.1}$$

where φ runs over all positive contractions $\varphi : X \to F_\varphi$ into ∞-convex Banach lattice with ∞-convexity constant 1. Take such a φ. By Proposition 3.4, there exists a contractive vector lattice homomorphism $\psi : F_\varphi \to \mathbb{R}$ such that $\|\overline{\varphi}(\mathfrak{f})\| = |\psi(\overline{\varphi}(\mathfrak{f}))|$. By Lemma 5.3, $\overline{\psi \circ \varphi} = \psi \circ \overline{\varphi}$. Hence $\|\overline{\varphi}(\mathfrak{f})\| = |\overline{\psi \circ \varphi}(\mathfrak{f})|$. It is now clear that

$$\|\mathfrak{f}\| = \sup_{x^* \in \mathrm{B}_{X_+^*}} |\overline{x^*}(\mathfrak{f})|. \tag{7.2}$$

In view of Proposition 5.17, (7.2) implies that we have a canonical realisation of $\mathrm{FBL}^{(\infty)}[(X, X^+)]$ as a vector sublattice of $\mathrm{C^h}(\mathrm{B}_{X_+^*})$.

For $\mathfrak{f} \in \mathrm{FBL}^{(\infty)}[(X, X^+)]$, define $\widehat{\mathfrak{f}} : \mathrm{B}_{X_+^*} \to \mathbb{R}$ by setting

$$\widehat{\mathfrak{f}}(x^*) = \overline{x^*}(\mathfrak{f}).$$

Using Proposition 5.17, we thus have the following.

Theorem 7.2. *Let* (X, X^+) *be a pre-ordered Banach space. Supply* $C^{\mathrm{h}}(\mathrm{B}_{X_+^*})$ *with the maximum norm, and let* E *be the closed vector sublattice of the normed vector lattice* $C^{\mathrm{h}}(\mathrm{B}_{X_+^*})$ *that is generated by the* δ_x *for* $x \in X$*. Define* $j : X \to E$ *by setting* $j(x) := \delta_x$*. Then* (j, E) *is a free* ∞*-convex Banach lattice with* ∞*-convexity constant 1* (*i.e., a free AM-space*) *over* (X, X^+).

We recall from Theorem 5.15 that, when $X^+ = \{0\}$, the δ_x generate a free vector lattice over $(X, \{0\})$.

Remark 7.3. Of course, $\mathrm{C^h}(\mathrm{B}_{X_+^*})$ is a Banach lattice in the maximum norm. We have, nevertheless, formulated the result without this to illustrate the fundamental difference between the present approach and that in [4] and [15]. Being an isometric image of $\mathrm{FBL}^{(\infty)}[(X, X^+)]$, we *know* E to be complete. In the approach as in [4] and [15] for $X^+ = \{0\}$, however, it is essential that everything takes place in what is known to be a complete function lattice. Without this, there is no guarantee that the completion that is taken in the construction yields a function lattice again. In the present approach, this is immaterial.

7.2. **The case $1 \leq p < \infty$.** We now turn to the remaining case $1 \leq p < \infty$, which is more demanding.[7] Our aim is to show that, for $\mathfrak{f} \in \mathrm{FBL}^{(p)}[(X, X^+)]$,

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|, \tag{7.3}$$

where φ runs over the positive contractions from X into ℓ_p^n for $n = 1, 2, \ldots$. Then Proposition 5.17 will take care of the rest.

The first step towards (7.3) is to note that, for $\mathfrak{f} \in \mathrm{FBL}^{(p)}[(X, X^+)]$,

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\| \tag{7.4}$$

where φ runs over the positive contractions from X into $\mathrm{L}_p(\mu)$-spaces for probability measures μ. This is an immediate consequence of Corollary 3.3 and Lemma 5.3.

We now start the next step, which is the passage from (7.4) to (7.3).

For $1 \leq p \leq \infty$, we let q denote its conjugate exponent. For $g \in L_q(\mu)$, we write g^* for the corresponding element of $L_p(\mu)^*$. The following preparatory observation from [15] is at the same time trivial and crucial.

Lemma 7.4. *Let $1 \leq p \leq \infty$, and let $f_1, \ldots, f_k \in L_p(\mu)$ for some (not necessarily probability) measure μ on a set Ω. Suppose that S is a measurable subset of Ω with the property that there exists $1 \leq i' \leq k$ such that $\bigvee_{i=i}^k f_i(\omega) = f_{i'}(\omega)$ for all $\omega \in S$. If $g \in L_q(\mu)$ is positive on S and zero outside S, then*

$$g^*\left(\bigvee_{i=i}^k f_i\right) = \bigvee_{i=1}^k g^*(f_i).$$

Similarly for the infimum.

We shall also use the following, which was essentially proved in [15].

Lemma 7.5. *Let $1 \leq p \leq \infty$. Suppose that $\Gamma = \{g_1, \ldots, g_k\} \subset L_q(\mu)$ for some (not necessarily probability) measure μ is such that $g_i g_j = 0$ for $1 \leq i < j \leq k$ and that $\|g_i\|_q \leq 1$ for $i = 1, \ldots, k$. For $f \in L_p(\mu)$, set*

$$\psi_\Gamma(f) := (g_1^*(f), \ldots, g_k^*(f)) \in \ell_p^k.$$

Then $\psi_\Gamma : L_p(\mu) \to \ell_p^k$ is a contraction. If all g_i are positive, then ψ_Γ is positive.

Proof. Using that the unit ball of ℓ_q^k is norming for ℓ_p^k in the second equality, we have

$$\begin{aligned}\|\psi_\Gamma(f)\| &= \left(\sum_{i=1}^k |g_i^*(f)|^p\right)^{1/p} \\ &= \max_{(a_1,\ldots,a_k)\in \mathrm{B}_{\ell_q^k}} \sum_{i=1}^k a_i g_i^*(f)\end{aligned}$$

[7] We also include the case $p = 1$. This is excluded in [15, Section 6], perhaps because [4] then already provides the answer with an easier proof, but the same method works.

$$= \max_{(a_1,\dots,a_k)\in \mathrm{B}_{\ell_q^k}} \left(\sum_{i=1}^{k} a_i g_i \right)^{*} (f).$$

The disjointness of the g_i is easily seen to imply that $\|\sum_{i=1}^{k} a_i g_i\|_q \leq 1$ whenever $(a_1,\dots,a_k) \in \mathrm{B}_{\ell_q^k}$. An application of the Hölder inequality concludes the proof. □

We now come to the proof proper for $1 \leq p < \infty$. We use χ_A for the characteristic function of a set A.

Let $\mathfrak{f}$ be an element of the vector sublattice of $\mathrm{FBL}^{(p)}[(X,X^+)]$ that is generated by $j(X)$. Varying on [15], we use (see [3, Exercise 4.1.8]) that there exist m,n and $x_{kl} \in X$ for $1 \leq k \leq m$ and $1 \leq l \leq n$ such that

$$\mathfrak{f} = \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} j(x_{kl}).$$

Let $\varphi : X \to \mathrm{L}_p(\mu)$ be a positive contraction for a probability measure μ on a set Ω. Then

$$\overline{\varphi}(\mathfrak{f}) = \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} \varphi(x_{kl}).$$

For $k' = 1,\dots,m$, set

$$\widetilde{A}_{k'} := \left\{ \omega \in \Omega : \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} [\varphi(x_{kl})](\omega) = \bigwedge_{l=1}^{n} [\varphi(x_{k'l})](\omega) \right\}.$$

Then $\Omega = \bigcup_{k'=1}^{m} \widetilde{A}_{k'}$. For $k' = 1,\dots,m$, set $A_{k'} = \widetilde{A}_{k'} \setminus \bigcup_{k<k'} \widetilde{A}_k$, so that $\Omega = \bigcup_{k'=1}^{m} A_k'$ as a disjoint union and

$$A_{k'} \subseteq \left\{ \omega \in \Omega : \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} [\varphi(x_{kl})](\omega) = \bigwedge_{l=1}^{n} [\varphi(x_{k'l})](\omega) \right\}.$$

Proceeding similarly for $\bigwedge_{l=1}^{n} \varphi(x_{k'l})$ on $A_{k'}$, we obtain a disjoint union $A_{k'} = \bigcup_{l'=1}^{n} B_{k'l'}$ such that

$$B_{k'l'} \subseteq \left\{ \omega \in \Omega : \bigwedge_{l=1}^{n} [\varphi(x_{k'l})](\omega) = [\varphi(x_{k'l'})](\omega) \right\}.$$

Finally, set

$$B_{k'l'}^{+} = \left\{ \omega \in B_{k'l'} : \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} [\varphi(x_{kl})](\omega) \geq 0 \right\}$$

and

$$B_{k'l'}^{-} = \left\{ \omega \in B_{k'l'} : \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} [\varphi(x_{kl})](\omega) < 0 \right\},$$

so that Ω is the disjoint union of the $B^+_{k'l'}$ and $B^-_{k'l'}$. For all k' and l', choose a positive $g^+_{k'l'} \in \mathrm{L}_q(\mu)$ with $\|g^+_{k'l'}\| \leq 1$ which is zero outside $B^+_{k'l'}$ and such that

$$\left(g^+_{k'l'}\right)^* (\overline{\varphi}(\mathfrak{f})) = \left\|\chi_{B^+_{k'l'}} \overline{\varphi}(\mathfrak{f})\right\|_{\mathrm{L}_p(\mu)},$$

and a positive $g^-_{k'l'} \in \mathrm{L}_q(\mu)$ with $\|g^-_{k'l'}\| \leq 1$ which is zero outside $B^-_{k'l'}$ and such that

$$\left|\left(g^-_{k'l'}\right)^* (\overline{\varphi}(\mathfrak{f}))\right| = \left\|\chi_{B^-_{k'l'}} \overline{\varphi}(\mathfrak{f})\right\|_{\mathrm{L}_p(\mu)}.$$

From the disjointness of the union, we have

$$\begin{aligned} \|\overline{\varphi}(\mathfrak{f})\|^p_{\mathrm{L}_p(\mu)} &= \sum_{k'=1}^{m}\sum_{l'=1}^{n} \left(\left\|\chi_{B^+_{k'l'}} \overline{\varphi}(\mathfrak{f})\right\|^p_{\mathrm{L}_p(\mu)} + \left\|\chi_{B^-_{k'l'}} \overline{\varphi}(\mathfrak{f})\right\|^p_{\mathrm{L}_p(\mu)} \right) \\ &= \sum_{k'=1}^{m}\sum_{l'=1}^{n} \left(\left|\left(g^+_{k'l'}\right)^* (\overline{\varphi}(\mathfrak{f}))\right|^p + \left|\left(g^-_{k'l'}\right)^* (\overline{\varphi}(\mathfrak{f}))\right|^p \right). \end{aligned} \tag{7.5}$$

The salient point of all this is that, by a double application of Lemma 7.4 in the first equality,

$$\begin{aligned} \left(g^\pm_{k'l'}\right)^* (\overline{\varphi}(\mathfrak{f})) &= \bigvee_{k=1}^{m}\bigwedge_{l=1}^{n} \left(g^\pm_{k'l'}\right)^* [\varphi(x_{kl})] \\ &= \bigvee_{k=1}^{m}\bigwedge_{l=1}^{n} \left(\left(g^\pm_{k'l'}\right)^* \circ \varphi\right)(x_{kl}) \\ &= \bigvee_{k=1}^{m}\bigwedge_{l=1}^{n} \overline{\left(\left(g^\pm_{k'l'}\right)^* \circ \varphi\right)}(j(x_{kl})) \\ &= \overline{\left(\left(g^\pm_{k'l'}\right)^* \circ \varphi\right)}\left(\bigvee_{k=1}^{m}\bigwedge_{l=1}^{n} j(x_{kl})\right) \\ &= \overline{\left(\left(g^\pm_{k'l'}\right)^* \circ \varphi\right)}(\mathfrak{f}). \end{aligned}$$

Thus (7.5) gives that

$$\|\overline{\varphi}(\mathfrak{f})\|^p_{\mathrm{L}_p(\mu)} = \sum_{k'=1}^{m}\sum_{l'=1}^{n} \left(\left|\overline{\left(\left(g^+_{k'l'}\right)^* \circ \varphi\right)}(\mathfrak{f})\right|^p + \left|\overline{\left(\left(g^-_{k'l'}\right)^* \circ \varphi\right)}(\mathfrak{f})\right|^p \right). \tag{7.6}$$

We shall now interpret this. In view of Lemma 7.5, the set

$$\Gamma := \left\{\left(g^\pm_{k'l'}\right)^* : 1 \leq k' \leq m, 1 \leq l' \leq n\right\}$$

yields a positive contraction ψ_Γ from $\mathrm{L}_p(\mu)$ into ℓ_p^{2mn}. Then $\psi_\Gamma \circ \varphi : X \to \ell_p^{2mn}$ is a positive contraction, the coordinate components of which are the $\left(g^\pm_{k'l'}\right)^* \circ \varphi$. By the uniqueness of the factor morphism, the coordinate components of $\overline{\psi_\Gamma \circ \varphi}$ are the $\overline{\left(g^\pm_{k'l'}\right)^* \circ \varphi}$. Thus (7.6) states that $\|\overline{\varphi}(\mathfrak{f})\|^p_{\mathrm{L}_p(\mu)} = \|\overline{(\psi_\Gamma \circ \varphi)}(\mathfrak{f})\|^p_{\ell_p^{2mn}}$.

Combining this with (7.4), we have established the following.[8]

Proposition 7.6. *For $\mathfrak{f} = \bigvee_{k=1}^{m} \bigwedge_{l=1}^{n} j(x_{kl})$ in $\mathrm{FBL}^{(p)}[(X, X^+)]$, its norm is given by*

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|,$$

where φ runs over the positive contractions from X into ℓ_p^{2mn}.

Now the work has been done. Evidently, if $\mathfrak{f}$ is any element of the vector sublattice of $\mathrm{FBL}^{(p)}[(X, X^+)]$ that is generated by $j(X)$, then

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|$$

as φ runs over the positive contractions from X into ℓ_p^n for $n \geq 1$. On invoking Lemma 6.1, we have the following.

Theorem 7.7. *Let (X, X^+) be a pre-ordered Banach space. For $\mathfrak{f} \in \mathrm{FBL}^{(p)}[(X, X^+)]$, its norm is given by*

$$\|\mathfrak{f}\| = \sup_{\varphi} \|\overline{\varphi}(\mathfrak{f})\|,$$

where φ runs over the positive contractions from X into ℓ_p^n for $n \geq 1$.

As for $p = \infty$, we shall now finally transport $\mathrm{FBL}^{(p)}[(X, X^+)]$ to $\mathrm{B}_{X_+^*}$.

A positive contraction $\varphi \colon X \to \ell_p^n$ corresponds to a subset $\{x_1^*, \ldots, \ldots x_n^*\} \subset \mathrm{B}_{X_+^*}$ such that

$$\sup_{x \in \mathrm{B}_X} \sum_{i=1}^{n} |x_i^*(x)|^p \leq 1,$$

in which case $\overline{\varphi}$ is easily seen to be given by $\{\overline{x_1^*}, \ldots, \overline{x_n^*}\}$, as a consequence of the uniqueness of a factor morphism.

Hence

$$\|\mathfrak{f}\| = \sup \left\{ \left(\sum_{i=1}^{n} \left| \overline{x_i^*}(\mathfrak{f}) \right|^p \right)^{1/p} : n \geq 1, x_1^*, \ldots, x_n^* \in \mathrm{B}_{X_+^*} \text{ s.t. } \sup_{x \in \mathrm{B}_X} \sum_{i=1}^{n} |x_i^*(x)|^p \leq 1 \right\}. \tag{7.7}$$

For $f \in \mathrm{C}^{\mathrm{h}}(\mathrm{B}_{X_+^*})$, define $\|f\|_p \in [0, \infty]$ by setting

$$\|f\|_p = \sup \left\{ \left(\sum_{i=1}^{n} \left| f(x_i^*) \right|^p \right)^{1/p} : n \geq 1, x_1^*, \ldots, x_n^* \in \mathrm{B}_{X_+^*} \text{ s.t. } \sup_{x \in \mathrm{B}_X} \sum_{i=1}^{n} |x_i^*(x)|^p \leq 1 \right\}.$$

Set

$$\mathrm{C}_p^{\mathrm{h}}(\mathrm{B}_{X_+^*}) := \left\{ f \in \mathrm{C}^{\mathrm{h}}(\mathrm{B}_{X_+^*}) : \|f\|_p < \infty \right\}.$$

This is a vector lattice, and we supply it with the lattice norm $\|\cdot\|_p$.

For $\mathfrak{f} \in \mathrm{FBL}^{(p)}[(X, X^+)]$, define $\widehat{\mathfrak{f}} \colon \mathrm{B}_{X_+^*} \to \mathbb{R}$ by setting $\widehat{\mathfrak{f}}(x^*) := \overline{x^*}(\mathfrak{f})$.

[8] Naturally, when working with elements of the form $\bigvee_{k=1}^{m} j(x_j) - \bigvee_{l=1}^{n} j(y_l)$, analogously to [15], the norm is obtained for positive contractions into ℓ_p^{2mn}.

Then Proposition 5.17 yields the following.

Theorem 7.8. *Let (X, X^+) be a pre-ordered Banach space. Let E be the closed sublattice of the normed vector lattice $\mathrm{C}_p^{\mathrm{h}}(\mathrm{B}_{X_+^*})$ that is generated by the δ_x for $x \in X$. Define $j\colon X \to E$ by setting $j(x) := \delta_x$. Then (j, E) is a free p-convex Banach lattice with p-convexity constant 1 over (X, X^+).*

We recall from Theorem 5.15 that, when $X^+ = \{0\}$, the δ_x generate a free vector lattice over $(X, \{0\})$.

Remark 7.9. It is routine to verify that $\mathrm{C}_p^{\mathrm{h}}(\mathrm{B}_{X_+^*})$ is a p-convex Banach lattice with p-convexity constant 1. As for the case $p = \infty$, it is not necessary to know that this space is complete to realise $\mathrm{FBL}^{(p)}[(X, X^+)]$ as a Banach lattice of homogeneous continuous functions on $\mathrm{B}_{X_+^*}$.

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(Marcel de Jeu) Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands, and, Department of Mathematics and Applied Mathematics, University of Pretoria, Corner of Lynnwood Road and Roper Street, Hatfield 0083, Pretoria, South Africa

Email address, Marcel de Jeu: mdejeu@math.leidenuniv.nl

(Xingni Jiang) College of Mathematics, Sichuan University, No. 24, South Section, First Ring Road, Chengdu, P.R. China

Email address, Xingni Jiang: x.jiang@scu.edu.cn