

# Goldstone Modes and the Coexistence Saddle: Spectral Bifurcation Analysis of Competing Ring Attractors Under Mean-Field Cross-Inhibition

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## Abstract

Persistent neural activity in working memory is often modeled as bump attractors on ring networks. When two such networks compete via cross-inhibition, the system must resolve which representation dominates – a winner-take-all (WTA) decision. We present a complete spectral bifurcation analysis of this transition in a coupled ring attractor model (two networks of  $N = 48$  neurons each, cosine connectivity within, mean-field cross-inhibition between). We identify four key results.

First, the coexistence fixed point – where both bumps self-sustain simultaneously – exists only below a critical cross-inhibition strength  $J_x^* \approx 0.36$ . Above this threshold, cross-inhibition is too strong for both representations to survive; the system admits only WTA solutions.

Second, the continuous rotational symmetry of each bump guarantees exactly two Goldstone modes (zero eigenvalues) that are protected against mean-field coupling. The first non-Goldstone eigenvalue – governing uniform amplitude competition – crosses zero at  $J_x^* \approx 0.3485$  via a pitchfork bifurcation, creating the coexistence saddle and two WTA stable states.

Third, the critical eigenvector is spatially uniform (DC mode), meaning the instability concerns total activity competition rather than spatial pattern rearrangement. This is a direct consequence of mean-field (spatially unstructured) cross-inhibition.

Fourth, large-scale stochastic simulations (128,000 trials across 256 parameter combinations) confirm the spectral predictions: swap errors emerge at the predicted  $J_x$  threshold, drive strength is secondary to cross-inhibition, and a non-monotonic valley at intermediate  $J_x$  identifies the functional operating point for working memory. We discuss implications for the behavioral cliff and argue that the brain operates in a valley regime where cross-inhibition and encoding drive are balanced, rather than near  $J_x^*$  itself.

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## 1. Introduction

### 1.1 Working Memory and Competing Representations

Persistent neural activity in prefrontal and parietal cortex underlies the short-term maintenance of information in working memory (Goldman-Rakic, 1995; Funahashi et al., 1989). Ring attractor models capture a key feature of this activity: spatially tuned neurons form a localized “bump” of elevated firing that persists through recurrent excitation even after the sensory stimulus is removed (Compte et al., 2000; Ben-Yishai et al., 1995; Amari, 1977). These bumps encode continuous variables such as spatial location or orientation, and their precision is set by a balance between recurrent drive and noise-induced diffusion along the ring (Wimmer et al., 2014; Burak and Fiete, 2012). Such ring attractor dynamics have been observed experimentally in the Drosophila head direction system (Kim et al., 2017) and are reviewed in the broader context of attractor and integrator networks by Khona and Fiete (2022).

When multiple items must be stored simultaneously, as in multi-item visual working memory tasks, the standard approach posits multiple bump networks coupled through cross-inhibition (Edin et al., 2009; Wei et al., 2012). The cross-inhibition creates a competition: if it is weak, both bumps coexist and the system maintains multiple items; if it is strong, one bump suppresses the other in a winner-take-all (WTA) decision. The transition between these regimes determines the capacity limit of the working memory circuit (Edin et al., 2009).

Despite the importance of this transition, its spectral structure – the full set of eigenvalues and eigenvectors of the system’s Jacobian – has not been characterized. Previous analyses have focused on one-dimensional (1D) mean-field reductions, projecting the high-dimensional dynamics onto a single dominance variable  $D = \bar{r}_A - \bar{r}_B$  and characterizing the resulting cusp catastrophe (Thom, 1972; Zeeman, 1977). While this captures the topology of the bifurcation, it discards the 96-dimensional dynamics that include rotational modes, drift modes, and the full stability structure of the coexistence state.

## 1.2 The Behavioral Cliff

Psychophysical experiments reveal a striking feature of working memory performance: below a critical stimulus strength, accuracy does not degrade gradually but collapses abruptly – a “behavioral cliff” (Bays et al., 2009; Zhang and Luck, 2008). In the mixture model framework, this manifests as a sharp increase in the probability of reporting a non-target item (swap errors) or of random guessing, even for small changes in signal-to-noise ratio.

The standard theoretical account attributes the cliff to noise-driven escape from a metastable state (Kramers, 1940; Hanggi et al., 1990): when the cue is weak, the barrier between the correct attractor and competing attractors is low, and stochastic fluctuations cause the system to fall to a wrong state. This yields a cusp catastrophe potential  $V(D) = D^4 + aD^2 + bD$ , where  $a$  is controlled by the circuit’s lateral inhibition and  $b$  by the cue strength. The cliff occurs at the cusp point where the barrier vanishes.

However, this account treats the cliff as a cue phenomenon – a consequence of weak sensory input. An alternative possibility, which we develop here, is that the cliff reflects a structural property of the circuit: the proximity of the effective cross-inhibition strength  $J_x$  to a critical value  $J_x^*$  where the coexistence state undergoes a spectral bifurcation.

## 1.3 From Mean-Field Reduction to Full Spectral Analysis

The 1D reduction  $D = \bar{r}_A - \bar{r}_B$  captures the order parameter of the WTA transition but suppresses 95 of the 96 dynamical degrees of freedom. In particular, it cannot distinguish:

1. **Goldstone modes** – exactly-zero eigenvalues arising from the continuous rotational symmetry of each bump (Goldstone, 1961; Burak and Fiete, 2012). These modes govern bump drift and are protected by symmetry.
2. **Genuine instabilities** – eigenvalues that cross zero as parameters change, signaling structural reorganization of the attractor landscape.
3. **The character of the critical mode** – whether the instability that destroys coexistence is spatially uniform (DC), spatially patterned (cosine), or mixed.

Previous spectral approaches to ring networks have addressed non-Hermitian quasi-localization (Tanaka and Nelson, 2018) and the stability of persistent activity under short-term plasticity

(Seeholzer et al., 2019), but the full eigenvalue structure of the *coupled* system has not been resolved. We present the first complete eigenvalue decomposition of the coupled ring attractor Jacobian, resolving all  $2N = 96$  eigenvalues as a function of the cross-inhibition strength  $J_x$ . By cleanly separating Goldstone modes from genuine instabilities, we identify the precise location, character, and consequences of the coexistence-to-WTA pitchfork bifurcation.

## 1.4 Summary of Contributions

Our main results are:

1. **Existence threshold.** The coexistence fixed point exists only for  $J_x < J_x^{exist} \approx 0.36$ . At the commonly used value  $J_x = 0.5$ , coexistence is not a fixed point of the dynamics – both bumps cannot self-sustain under such strong cross-inhibition.
2. **Goldstone separation and pitchfork.** Two Goldstone modes (exactly-zero eigenvalues protected by rotational symmetry) persist at all  $J_x$  where coexistence exists. The first non-Goldstone eigenvalue  $\lambda_{dom}$  crosses zero at  $J_x^* \approx 0.3485$ , creating a pitchfork bifurcation where the symmetric coexistence state becomes a saddle point and two WTA attractors are born.
3. **DC critical mode.** The critical eigenvector is spatially uniform: a flat increase in network A’s activity coupled with a flat decrease in network B’s. This reflects the mean-field character of the cross-inhibition and means the instability is about total activity competition, not spatial pattern rearrangement.
4. **Stochastic phase diagram.** A 128,000-trial parameter sweep confirms the spectral predictions and reveals a non-monotonic valley at intermediate  $J_x$  ( $\approx 1.2$ – $1.6$ ) where swap error rates dip to 7–13% between two qualitatively different failure modes.

These results reframe the behavioral cliff as a  $J_x$ -space phenomenon and predict a functional operating regime (the valley) where the brain balances encoding drive against cross-inhibition for reliable working memory.

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## 2. Model

### 2.1 Single Ring Attractor

We consider a rate model with  $N = 48$  neurons uniformly distributed on a ring. Each neuron  $i$  has a preferred angle  $\theta_i = -\pi + 2\pi i/N$  and firing rate  $r_i(t)$  governed by:

$$\tau \frac{dr_i}{dt} = -r_i + \sigma(h_i)$$

where  $\tau = 10$  ms is the time constant and  $\sigma(h) = r_{max}/(1 + e^{-\beta(h-h_0)})$  is a sigmoidal activation function with parameters  $r_{max} = 1.0$ ,  $\beta = 5.0$ ,  $h_0 = 0.5$ . The total input to neuron  $i$  is:

$$h_i = \sum_{j=1}^N W_{ij} r_j + I_i^{ext}$$

where the within-network connectivity has cosine tuning:

$$W_{ij} = \frac{1}{N} (J_0 + J_1 \cos(\theta_i - \theta_j))$$

with  $J_0 = 1.0$  (uniform inhibition) and  $J_1 = 6.0$  (tuned excitation). This connectivity supports a family of bump solutions at any angular position, forming a ring attractor (Amari, 1977; Ben-Yishai et al., 1995).

## 2.2 Coupled System with Mean-Field Cross-Inhibition

We couple two identical ring networks A and B through mean-field cross-inhibition. The dynamics become:

$$\begin{aligned}\tau \frac{dr_i^A}{dt} &= -r_i^A + \sigma \left( \sum_j W_{ij} r_j^A + I_i^{cue} - J_{\times} \bar{r}^B \right) \\ \tau \frac{dr_i^B}{dt} &= -r_i^B + \sigma \left( \sum_j W_{ij} r_j^B - J_{\times} \bar{r}^A \right)\end{aligned}$$

where  $\bar{r}^X = \frac{1}{N} \sum_j r_j^X$  is the mean activity of network  $X$  and  $J_{\times} \geq 0$  is the cross-inhibition strength. The external cue input is a von Mises tuning curve applied to network A only:

$$I_i^{cue} = c \cdot \frac{e^{\kappa \cos(\theta_i - \theta_{stim})}}{I_0(\kappa)}$$

with concentration parameter  $\kappa = 2.0$  and cue gain  $c \geq 0$ .

The critical feature of mean-field cross-inhibition is that it depends only on the total activity  $\bar{r}^X$  of the opposing network, not on the spatial pattern of its bump. This has profound consequences for the symmetry structure of the system (Fig. 1).

**Figure 1.** Model schematic. Two ring networks (A, B) of  $N = 48$  neurons each, with cosine within-network connectivity ( $J_0 + J_1 \cos \Delta\theta$ ) and mean-field cross-inhibition ( $J_{\times} \bar{r}^X$ ). External cue input drives network A only. The cross-inhibition depends on mean activity, not bump position, preserving rotational symmetry.

## 2.3 Jacobian of the Coupled System

The steady-state condition  $F(\mathbf{r}^*) = 0$  defines the fixed points, where  $F_i^A = -r_i^A + \sigma(h_i^A)$  and similarly for B. The Jacobian  $\mathbf{J} = \partial F / \partial \mathbf{r}$  evaluated at a fixed point  $\mathbf{r}^*$  has a  $2 \times 2$  block structure:

$$\mathbf{J} = \begin{pmatrix} -\mathbf{I} + \mathbf{D}_A \mathbf{W} & \mathbf{D}_A \mathbf{C} \\ \mathbf{D}_B \mathbf{C} & -\mathbf{I} + \mathbf{D}_B \mathbf{W} \end{pmatrix}$$

where  $\mathbf{D}_X = \text{diag}(\sigma'(h_i^X))$  is the diagonal matrix of sigmoid derivatives at the fixed point, and  $\mathbf{C} = -\frac{J_{\times}}{N} \mathbf{1} \mathbf{1}^T$  is the rank-1 mean-field coupling matrix. The full Jacobian is  $2N \times 2N = 96 \times 96$ .

The block structure reveals that the cross-coupling enters only through the rank-1 matrix  $\mathbf{C}$ . This low-rank perturbation to the block-diagonal within-network dynamics is what makes the spectral analysis tractable: the cross-inhibition can shift at most one eigenvalue per symmetry sector.

## 2.4 Symmetries

The coupled system possesses two symmetries at zero cue ( $c = 0$ ):

**Continuous rotational symmetry.** The mean-field coupling  $J_{\times}\bar{r}^X$  is invariant under any rotation of the bump profile: if  $r_i^X$  is a fixed point, so is  $r_{i+k}^X$  for any shift  $k$ . This gives a continuous family of fixed points parametrized by bump position, and by Goldstone's theorem (Goldstone, 1961), each such continuous symmetry produces an eigenvalue that is exactly zero. With two independent bumps, there are two Goldstone modes.

**Discrete exchange symmetry.** At  $c = 0$ , the system is invariant under  $A \leftrightarrow B$ . The coexistence fixed point (where both bumps are present with  $\bar{r}^A = \bar{r}^B$ ) respects this symmetry; the WTA states ( $\bar{r}^A \gg \bar{r}^B$  or vice versa) break it. The transition between these is governed by a pitchfork bifurcation.

The nonzero cue  $c > 0$  breaks the exchange symmetry (favoring network A) and deforms the pitchfork into an imperfect bifurcation with hysteresis.

## 3. Results

### 3.1 Existence of the Coexistence Fixed Point

**3.1.1 Numerical Method** We locate fixed points using a two-phase approach. In Phase 1 (simulation), we establish both bumps using strong external drives ( $I_{ext} = 5.0$ ), then remove the drives and simulate the coupled system for  $5 \times 10^4$  time steps ( $\Delta t = 0.1, \tau = 10$ ) under the target cross-inhibition  $J_{\times}$ . In Phase 2 (Newton), we polish the resulting state using Newton's method (scipy.optimize.fsolve) with the analytical Jacobian, achieving residuals  $\|F(\mathbf{r}^*)\| < 10^{-10}$ .

We verify convergence by checking: (i) the residual  $\max_i |F_i(\mathbf{r}^*)| < 10^{-6}$ ; (ii) both bumps are active ( $\max_i r_i^A > 0.3$  and  $\max_i r_i^B > 0.3$ ); and (iii) the solution is not a WTA state ( $|D| = |\bar{r}^A - \bar{r}^B| < 0.1$ ).

**3.1.2 Critical Existence Threshold** **Figure 2.** Coexistence existence boundary. Peak firing rate of each network as a function of cross-inhibition strength  $J_{\times}$ . Below  $J_{\times}^{exist} \approx 0.358$ , both networks sustain bumps at matched amplitudes (coexistence). Above this threshold, one network collapses to baseline (WTA). The pitchfork bifurcation at  $J_{\times}^* \approx 0.349$  (orange dashed) and existence boundary at  $J_{\times}^{exist} \approx 0.358$  (purple dashed) delimit the narrow instability window  $\Delta J_{\times} \approx 0.01$ .

We scan  $J_{\times}$  from 0 to 0.5 (Fig. 2). Below  $J_{\times} \approx 0.36$ , Newton converges to a genuine coexistence fixed point with residual  $< 10^{-10}$ . Above this threshold, one bump suppresses the other during the simulation phase; Newton converges only to WTA solutions. At  $J_{\times} = 0.50$  – a commonly used parameter value – coexistence does not exist as a fixed point of the deterministic dynamics.

The critical existence threshold lies between  $J_{\times} = 0.355$  (both bumps survive,  $\max r^A = 0.88, \max r^B = 0.88$ ) and  $J_{\times} = 0.360$  (one bump collapses,  $\max r^A = 0.97, \max r^B = 0.04$ ). The transition is sharp: a 1.4% increase in  $J_{\times}$  converts stable coexistence into complete dominance.

**3.1.3 Diagnostic: Fixed Point vs. Slow Manifold** At  $J_x = 0.50$ , a time-resolved diagnostic reveals that the system does not converge: residuals remain at  $\sim 10^{-3}$  and the dominance variable  $D$  drifts monotonically toward  $\pm 0.27$ . At  $J_x = 0.35$ , residuals converge exponentially to machine precision ( $\sim 10^{-15}$ ). This confirms that the coexistence state is a genuine fixed point below threshold and does not exist (even as a slow manifold) above it.

## 3.2 Goldstone Modes and the Protected Symmetry

**3.2.1 Origin of the Goldstone Modes** The mean-field cross-coupling  $J_x \bar{r}^B$  is a function of the mean activity  $\bar{r}^B = \frac{1}{N} \sum_j r_j^B$  only. Any rotation of the bump profile  $r_j^B \rightarrow r_{j+k}^B$  preserves the mean, hence  $\partial F / \partial \theta_{bump} = 0$  identically. The derivative of the bump profile with respect to its angular position is therefore a null vector of the Jacobian:  $\mathbf{J} \cdot \frac{\partial \mathbf{r}^*}{\partial \theta} = 0$ . Each network contributes one such null vector, yielding two Goldstone modes with eigenvalue exactly zero.

This is the neural circuit analog of the Goldstone theorem in quantum field theory (Goldstone, 1961): a spontaneously broken continuous symmetry produces a massless (zero-energy) excitation. In our context, the “mass” is the eigenvalue, and “massless” means neutrally stable – perturbations along the Goldstone direction neither grow nor decay.

**3.2.2 Numerical Identification** We classify eigenvalues into Goldstone candidates ( $|\lambda| < 10^{-3}$ ) and genuine modes ( $|\lambda| \geq 10^{-3}$ ). For each eigenvector  $\mathbf{v}$ , we compute projections onto six basis directions:

- $\mathbf{d}_{dom}$ : symmetric dominance (cosine envelope,  $A \uparrow B \downarrow$ )
- $\mathbf{d}_{drift,+}$ : co-directional drift (sine envelope, both shift same way)
- $\mathbf{d}_{drift,-}$ : anti-directional drift (sine envelope, shift opposite ways)
- $\mathbf{d}_{uni}$ : uniform/DC ( $A \uparrow B \downarrow$  flat)
- $\mathbf{d}_{gold,A}$ : rotation of bump A (sine envelope, A only)
- $\mathbf{d}_{gold,B}$ : rotation of bump B (sine envelope, B only)

The Goldstone eigenvectors project strongly onto  $\mathbf{d}_{gold,A}$  and  $\mathbf{d}_{gold,B}$ , confirming their rotational character (Fig. 4, left panel).

**3.2.3 Goldstone Count Across  $J_x$**  Across the entire range  $J_x \in [0, 0.36]$  where coexistence exists, we find exactly two Goldstone modes. Their eigenvalues remain at  $|\lambda| \sim 10^{-8}$  to  $10^{-11}$  (machine precision for our iterative solver), and they are never lifted by increasing cross-inhibition. This confirms the symmetry protection: mean-field coupling cannot break rotational invariance.

**Figure 4.** Eigenvector comparison at  $J_x = 0.34$  (near the pitchfork). Left: Goldstone mode ( $\lambda \approx -5.8 \times 10^{-11}$ , effectively zero), showing sinusoidal spatial structure in Network A – this is the rotational mode that slides the bump around the ring. Right: Critical dominance mode ( $\lambda_{dom} = -0.026$ , DC/uniform), showing spatially flat anti-phase activation (Network A uniformly up, Network B uniformly down). The two modes are qualitatively distinct: the Goldstone mode encodes *where* the bump sits; the critical mode encodes *which network wins*.

## 3.3 The Pitchfork Bifurcation

**3.3.1 The First Non-Goldstone Eigenvalue** After removing the two Goldstone modes, we track the dominant genuine eigenvalue  $\lambda_{dom}$  as a function of  $J_x$  (Fig. 3). Key findings:

- At  $J_x = 0$ :  $\lambda_{dom} = -0.572$  (strongly stable). Without cross-inhibition, the coexistence state is deeply attractive.
- $\lambda_{dom}$  increases monotonically with  $J_x$ , crossing zero at  $J_x^* \approx 0.3485$ .
- Above  $J_x^*$ :  $\lambda_{dom} > 0$  (saddle). The coexistence state acquires one unstable direction.
- At  $J_x = 0.356$ :  $\lambda_{dom} = +0.025$ , and coexistence ceases to exist shortly after at  $J_x \approx 0.358$ .

The crossing at  $J_x^*$  is a pitchfork bifurcation: the symmetric coexistence state ( $D = 0$ ) loses stability, and two WTA states ( $D > 0$  and  $D < 0$ ) emerge as the new stable attractors. The  $A \leftrightarrow B$  exchange symmetry is spontaneously broken.

**3.3.2 Character of the Critical Eigenvector** At  $J_x^*$ , the critical eigenvector has the following projections:

Direction	$ \langle \mathbf{v}_1, \mathbf{d} \rangle $
Uniform (DC)	0.43
Dominance (cosine)	0.34
Anti-drift (sine)	0.34
Co-drift (sine)	0.00

The largest projection is onto the uniform/DC direction (Fig. 4, right panel): a spatially flat increase in all of network A’s neurons coupled with a spatially flat decrease in all of network B’s neurons. This means the instability is about which network has more total activity, not about the spatial pattern of either bump.

This is a direct and falsifiable prediction of mean-field coupling. If cross-inhibition were spatially structured (depending on the relative positions of the two bumps), the critical eigenvector would acquire spatial structure (cosine or higher Fourier modes). The DC character is specific to coupling that “sees” only total activity.

**3.3.3 The Narrow Existence Window** The coexistence saddle – genuinely unstable, not merely Goldstone-neutral – exists only in the interval  $J_x \in [0.3485, 0.358]$ , a width of  $\Delta J_x \approx 0.01$ . Below  $J_x^*$ , coexistence is a stable node. Above  $J_x^{exist} \approx 0.358$ , it ceases to exist entirely.

This razor-thin window has two implications:

1. **Structural precision.** The bifurcation is sharp: a 3% change in  $J_x$  (from 0.348 to 0.358) takes the system from stable coexistence through saddle instability to complete collapse. The system is tuned near a critical point.
2. **Heterogeneity prediction.** In biological circuits with heterogeneous connectivity, the sharp boundary should be smeared into a broader regime where saddle-like dynamics persist (see Discussion).

**Figure 3.** Pitchfork bifurcation diagram. Dominance  $D = \bar{r}^A - \bar{r}^B$  vs. cross-inhibition strength  $J_x$ . The coexistence branch ( $D = 0$ ) is stable (solid) for  $J_x < J_x^*$  and becomes a saddle (dashed red) above the pitchfork at  $J_x^* \approx 0.349$ . Two WTA branches ( $D > 0$  and  $D < 0$ , blue) emerge at  $J_x^{exist} \approx 0.358$  as stable attractors. The subcritical structure creates the narrow existence window  $\Delta J_x \approx 0.01$ .

### 3.4 The Coexistence Saddle Under Cue

Newton continuation in cue gain  $c$  from 0 to 0.5 at  $J_x = 0.35$  (within the saddle window; see Fig. 3) reveals:

- The coexistence branch maintains both bumps across the full cue range, with  $D$  growing slowly as the cue favors network A.
- All 51/56 tracked solutions are unstable (saddle points), with  $n_{positive} = 1$  for  $c < 0.15$  and  $n_{positive} = 2$  for  $c > 0.15$  (a second eigenvalue crossing).
- The critical eigenvector's projection onto cosine and sine directions is  $|\cos| \approx |\sin| \approx 0.47$  – a mixed dominance-drift mode at 45 degrees.

The WTA branch (tracked simultaneously) is stable across all cue values, confirming that the saddle's unstable manifold connects to the two WTA basins.

### 3.5 Stochastic Phase Diagram of Swap Errors

**3.5.1 Parameter Sweep** To bridge the deterministic bifurcation analysis with behavioral predictions, we performed a large-scale stochastic simulation across the  $(J_x, c)$  parameter space. At each of 256 grid points (16 values of  $J_x$  from 0.05 to 8.0, 16 values of drive strength from 1.0 to 8.0), we ran 500 stochastic trials ( $N_{total} = 128,000$ ). Two ring networks encoded items separated by  $\pi/2$  radians, with independent Gaussian noise ( $\sigma = 0.1$ ) added to each neuron during a 500-step maintenance period. Swap errors were classified as decoded responses within 0.3 rad of the non-target item's location.

**3.5.2 Onset of Swap Errors** **Figure 5.** Stochastic phase diagram in  $(J_x, c)$  space (128,000 trials across 256 parameter combinations). Color indicates swap error rate. Near-vertical isocontours show that swap probability depends primarily on  $J_x$ , not drive strength. A non-monotonic valley at  $J_x \approx 1.2\text{--}1.6$  with strong drive identifies the functional operating regime where WTA dynamics and encoding strength are balanced.

Swap errors emerge at  $J_x \approx 0.25$ , consistent with the spectral prediction of the pitchfork bifurcation at  $J_x^* \approx 0.3485$ . The stochastic onset is lower than the deterministic bifurcation because noise-mediated escape from the metastable coexistence well occurs when the barrier height  $\Delta V \sim \sigma^2$ , which corresponds to a  $J_x$  slightly below the eigenvalue crossing. This is precisely the Kramers mechanism.

Between  $J_x \approx 0.25$  and 0.5, swap rates increase from 5% to approximately 45%. Above  $J_x \approx 1.0$ , swap rates plateau near 50% – the system has become a noise-driven bistable switch with no memory of the initial encoding.

**3.5.3 Drive Strength Is Secondary** The phase diagram shows near-vertical isocontours of swap rate (Fig. 5): swap error probability depends primarily on  $J_x$  and only weakly on drive strength. This is a direct prediction of the spectral analysis: the critical eigenvector is spatially uniform (DC mode), governing total activity competition rather than spatial encoding. Stronger drive does not protect against the dominance instability because the instability is orthogonal to the encoding direction.

This has a counterintuitive implication: increasing stimulus strength – the commonly proposed intervention for working memory failures – targets the wrong degree of freedom. The cliff is a  $J_x$  phenomenon, not a cue phenomenon.

**3.5.4 The Non-Monotonic Valley** At  $J_x \approx 1.2\text{--}1.6$  with moderate to strong drive, the phase diagram reveals a non-monotonic feature: swap rates dip to 7–13% between two distinct failure modes:

1. **Near-critical swaps** ( $J_x \approx 0.3\text{--}0.5$ ): Near the pitchfork, barriers are small and noise escapes freely. Swap rate  $\rightarrow 50\%$ .
2. **Overpowering swaps** ( $J_x > 2.0$ , weak drive): Strong cross-inhibition overwhelms feedforward encoding. One network suppresses the other regardless of initial drive.
3. **The valley** ( $J_x \approx 1.2\text{--}1.6$ , strong drive): Cross-inhibition is strong enough for clear WTA, but encoding drive is also strong enough to “lock in” the correct network during stimulus presentation. The correct representation is captured by a deep attractor basin before maintenance-period competition begins.

The valley represents the predicted functional operating point for a healthy working memory circuit: cross-inhibition strong enough to resolve competition, drive strong enough to bias competition correctly. The circuit need not be tuned precisely to  $J_x^*$  but rather to the valley regime where WTA dynamics and encoding strength are balanced.

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## 4. Discussion

### 4.1 The Coexistence Threshold as a Structural Constraint

The finding that coexistence does not exist at the commonly used  $J_x = 0.5$  is a structural constraint on models of multi-item working memory. If the brain maintains multiple items simultaneously – as behavioral data strongly suggest (Bays et al., 2009; Ma et al., 2014) – then the effective cross-inhibition must be below threshold. This constrains the balance between lateral inhibition and recurrent excitation: the circuit cannot be in the WTA regime and store multiple items.

Our threshold  $J_x^{exist} \approx 0.36$  depends on the specific parameters  $(J_0, J_1, \beta, h_0)$ . Different parameter regimes will yield different thresholds. The key result is qualitative: there is always a finite critical cross-inhibition beyond which coexistence is structurally impossible.

### 4.2 Heterogeneity Transforms the Bifurcation Type

In our symmetric model, the coexistence saddle exists only in a narrow window ( $\Delta J_x \approx 0.01$ ). However, biological circuits have heterogeneous connectivity, non-uniform firing thresholds, and spatially structured inhibition (Kilpatrick et al., 2013). We tested the effect of connectivity heterogeneity by adding symmetric Gaussian noise to the within-network weight matrices ( $W \rightarrow W + \sigma \xi$ , where  $\xi_{ij} \sim \mathcal{N}(0, 1/N)$ ) and repeating the eigenvalue analysis across six noise levels ( $\sigma \in \{0, 0.05, 0.1, 0.2, 0.3, 0.5\}$ , five random seeds each, 30 values of  $J_x$  per condition).

The result refutes the intuitive prediction that heterogeneity would widen the instability window. Instead, heterogeneity *destroys* it. At  $\sigma = 0.05$ , two of three trials lost the instability entirely (the dominance eigenvalue  $\lambda_{dom}$  never crossed zero), while one trial showed a wider window ( $\Delta J_x \approx 0.04$ ). At  $\sigma \geq 0.10$ , no trial exhibited a positive  $\lambda_{dom}$  at any  $J_x$  – the sharp pitchfork bifurcation had vanished completely.

The mechanism is the breaking of exact A↔B exchange symmetry. The pitchfork bifurcation at  $J_x^*$  requires that the two networks be related by an exact symmetry operation: if  $(r_A^*, r_B^*)$  is a fixed point,

then  $(r_B^*, r_A^*)$  must also be one, and the bifurcation occurs when the symmetric fixed point ( $D = 0$ ) loses stability to the antisymmetric perturbation ( $D \neq 0$ ). Heterogeneity in the weight matrices breaks this exchange symmetry, because the two networks no longer have identical connectivity. In the language of the cusp potential  $V(D) = D^4 + aD^2 + bD$ , heterogeneity introduces a nonzero  $b$  even at zero cue – the potential is always tilted, and there is no sharp symmetry-restoring point where  $b = 0$  exactly.

This converts the pitchfork into an *imperfect bifurcation* (Strogatz, 2015). Instead of a sharp zero-crossing of  $\lambda_{dom}$ , the system shows a smooth crossover: one network is always slightly favored, and the dominance eigenvalue approaches zero asymptotically without crossing it. The “window” does not widen – it dissolves, because the phase transition changes type from a sharp symmetry-breaking event to a smooth preference gradient.

Two consequences follow. First, the Goldstone modes, which are exactly zero in the symmetric model, become “soft modes” with small but nonzero eigenvalues at  $\sigma > 0$  (Kilpatrick et al., 2013; Poll et al., 2015). Bumps become pinned to preferred locations rather than freely rotating. Second, and more importantly, the razor-thin window ( $\Delta J_x \approx 0.01$ ) is a symmetry artifact of the clean model, not a biological constraint. Real circuits operate in a regime of smooth crossover where no parameter precision is required.

This result strengthens the valley interpretation (Section 3.5.4). The non-monotonic valley at  $J_x \approx 1.2\text{--}1.6$  exists in the stochastic simulations regardless of whether the underlying deterministic bifurcation is a sharp pitchfork or a smooth crossover. What matters for behavior is the *landscape* – the barrier heights and basin depths – not the exact location of a mathematical bifurcation point. Heterogeneity smears the transition without eliminating the functional operating regime.

Critical slowing down provides partial confirmation: convergence time after small perturbations increases as  $J_x$  approaches  $J_x^*$  in the clean model ( $\sigma = 0$ ), consistent with the expected  $\tau \sim 1/|\lambda_{dom}|$  scaling near the bifurcation. The data are noisy but directionally consistent with critical slowing down (Fig. 6).

**Figure 6.** Heterogeneity destroys the instability window. Dominance eigenvalue  $\lambda_{dom}$  as a function of  $J_x$  at six levels of connectivity heterogeneity ( $\sigma \in \{0, 0.05, 0.1, 0.2, 0.3, 0.5\}$ , five random seeds each). At  $\sigma = 0$  (clean model), the pitchfork zero-crossing is sharp. At  $\sigma = 0.05$ , two of three trials lose the instability entirely. At  $\sigma \geq 0.10$ , no trial exhibits a positive  $\lambda_{dom}$  at any  $J_x$  – the imperfect bifurcation has no zero-crossing. Right panel: convergence time data, consistent with critical slowing down near  $J_x^*$  in the clean model.

### 4.3 The Behavioral Cliff as a $J_x$ Phenomenon

The traditional view attributes the behavioral cliff to weak cues: below a threshold cue strength, the sensory signal cannot stabilize the correct bump, and noise-driven escape to a competing attractor becomes rapid (Kramers escape). Our analysis suggests a reinterpretation.

The cliff reflects the system’s proximity to critical  $J_x^*$ . The cue does not create the competition –  $J_x$  does. The cue provides stabilization against the dominance instability driven by  $J_x$ . When cue-mediated stabilization fails (cue too weak relative to  $|J_x - J_x^*|$ ), the system transitions to WTA.

This reinterpretation makes two testable predictions:

1. **Manipulating cross-inhibition should shift the cliff.** Pharmacological modulation of GABAergic inhibition, or inter-hemispheric TMS suppression, should change the effective  $J_x$ .

Stronger cross-inhibition should move the cliff to higher cue values; weaker cross-inhibition should make it disappear.

2. **Individual differences may reflect  $J_{\times}$  variation.** Subject-to-subject variability in cliff location may arise from variation in effective cross-inhibition strength (neuromodulation, connectivity differences) rather than variation in sensory sensitivity.

#### 4.4 Goldstone Protection and Functional Significance

The two Goldstone modes have a functional interpretation. They protect bump positions (the stored memory content) from being disrupted by the dominance competition. The system can resolve “which network wins” without disturbing “where each bump sits.” This separation of concerns – amplitude competition in the dominance subspace, position preservation in the Goldstone subspace – may be a design principle for neural circuits that must make decisions while maintaining stored information.

The Goldstone protection is specific to mean-field coupling. Spatially structured inhibition (e.g., lateral inhibition that depends on the angular distance between bumps) would break the rotational symmetry and couple dominance dynamics to positional dynamics. Whether biological cross-inhibition is closer to mean-field or structured is an empirical question with spectral consequences.

#### 4.5 Connection to the Cusp Catastrophe

The 1D cusp potential  $V(D) = D^4 + aD^2 + bD$  is the projection of the 96-dimensional dynamics onto the critical eigenvector. The spectral analysis validates this projection by identifying the critical direction explicitly: it is the uniform/DC mode, and the order parameter  $D = \bar{r}^A - \bar{r}^B$  is its natural coordinate. The cusp coefficient  $a$  is controlled by  $J_{\times}$  (with  $a = 0$  at  $J_{\times}^*$ ); the symmetry-breaking coefficient  $b$  is controlled by cue gain  $c$ .

However, the full spectral analysis adds what the 1D reduction cannot provide: the Goldstone modes (which require zero-mode regularization in a Kramers calculation), the stability of non-critical directions (which set the high-dimensional prefactor), and the quantitative location of  $J_{\times}^*$  (which the 1D reduction treats as a free parameter).

#### 4.6 Connection to Stochastic Attractor Models

Our deterministic spectral analysis and the stochastic attractor models of Penny (2024) provide complementary descriptions of working memory dynamics. Penny modeled maintenance as a stochastic differential equation  $dx = \beta g(x)dt + \sigma dw$ , where swap errors arise when “memory traces diffuse away from their initial state and are captured by the attractors of other items.” Our cusp potential  $V(D) = D^4 + aD^2 + bD$  provides the deterministic landscape on which Penny’s stochastic dynamics unfold. The Kramers escape rate  $\tau^{-1} \sim \exp(-\Delta V/\sigma^2)$  bridges the two: barrier height (set by the spectral bifurcation at  $J_{\times}^*$ ) determines swap error probability under stochastic dynamics. Penny’s model fits behavioral data from 90 subjects; our analysis provides the spectral characterization of the underlying attractor landscape.

A prediction emerges from the synthesis: swap error rate should increase continuously with maintenance delay (accumulated diffusion), but the rate of increase should exhibit a sharp change near  $J_{\times}^*$  (barrier collapse). The cusp sets the landscape; noise navigates it.

## 4.7 Selection Versus Representation Failure

Neural recordings from monkey prefrontal cortex during multi-item working memory reveal that swap errors can arise from misselection of correctly remembered items rather than representation failure (Alleman et al., 2024). Both representations persist in the population, but the readout process selects the wrong item.

This distinction maps onto our phase diagram. The non-monotonic valley at  $J_x \approx 1.2\text{--}1.6$  corresponds to the selection-failure regime: both bump representations coexist but WTA competition during readout can select the wrong network. At higher  $J_x (> 2.0)$ , one bump is suppressed during maintenance – representation failure. Our model predicts both mechanisms in different parameter regimes, with  $J_x$  governing the boundary.

The Alleman et al. finding that swap errors in healthy subjects arise from misselection suggests the brain operates in or near the valley, where cross-inhibition is strong enough for reliable WTA but not so strong as to destroy representations. A testable prediction follows: conditions that increase effective cross-inhibition (distractor-rich environments, high cognitive load) should shift swap errors from selection-type to representation-type.

## 4.8 Universality of the Saddle-Point Bifurcation

The pitchfork bifurcation we identify – attractors extinguished after merging with saddle points at high cross-inhibition – has structural analogs in decision-making circuits. Roach, Churchland, and Engel (2023) showed that in circuits with choice-selective inhibition, working memory attractors are extinguished after merging with saddle points as ipsispecific inhibition increases. Disjoint neural groups with within-group excitation and across-group inhibition exhibit group WTA dynamics, and the coexistence-to-WTA transition occurs via saddle-point annihilation (Roach et al., 2023; Wong and Wang, 2006; Machens et al., 2005).

This structural isomorphism suggests the spectral separatrix is not specific to working memory but describes the generic bifurcation of any neural circuit with competing stable states. Decision-making, attention, and working memory all involve population competition, and the mathematical structure – pitchfork at critical coupling, Goldstone protection of positional degrees of freedom, DC instability under mean-field coupling – should appear across domains. The spectral analysis presented here provides a template for characterizing these transitions.

## 4.9 Limitations

- (i) The model uses rate neurons, not spiking neurons; the noise structure differs qualitatively.
- (ii) Mean-field cross-inhibition is a simplification; realistic inhibitory interneuron pools have spatial and temporal structure, as demonstrated by Roach et al. (2023), where ipsispecific versus contraspecific inhibition creates qualitatively different attractor landscapes.
- (iii)  $N = 48$  is moderate; the Goldstone mode identification becomes cleaner at larger  $N$ .
- (iv) The stochastic phase diagram uses additive Gaussian noise; biologically realistic noise is multiplicative and state-dependent.
- (v) Our model conflates maintenance and selection into a single dynamical process; the Alleman et al. (2024) finding that swap errors arise at the selection stage suggests that a two-stage model (coexistence during maintenance, WTA competition at readout) may be more biologically appropriate.
- (vi) The mapping from our neural space (96 dimensions) to behavioral feature space (1D circular, as in Penny, 2024) requires assumptions about decoding that have not been derived from first principles.
- (vii) The sharp pitchfork bifurcation at

$J_x^*$  is a symmetry artifact: even modest connectivity heterogeneity ( $\sigma \geq 0.10$ ) destroys the instability window entirely, converting it to a smooth crossover. The deterministic bifurcation analysis characterizes the symmetric limit; biological relevance depends on the stochastic landscape (barrier heights, basin depths) rather than the exact bifurcation structure.

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## 5. Conclusion

We have presented the first complete spectral bifurcation analysis of competing ring attractors under mean-field cross-inhibition. Four results stand:

1. **Existence threshold.** Coexistence has a sharp existence threshold at  $J_x^{exist} \approx 0.36$ , below which it is a genuine fixed point and above which it does not exist.
2. **Goldstone separation and pitchfork.** Goldstone modes are symmetry-protected and separate cleanly from genuine instabilities. The first non-Goldstone eigenvalue crosses zero at  $J_x^* \approx 0.3485$  in a pitchfork bifurcation, creating the coexistence saddle.
3. **DC critical mode.** The critical eigenvector is spatially uniform, reflecting the mean-field character of cross-inhibition and predicting that the WTA instability concerns total activity competition rather than spatial pattern rearrangement.
4. **Stochastic phase diagram.** Large-scale stochastic simulations (128,000 trials) confirm the spectral prediction: swap errors emerge at  $J_x \approx 0.25$ , drive strength is secondary to cross-inhibition, and a non-monotonic valley at  $J_x \approx 1.2\text{--}1.6$  identifies the predicted functional operating point.

Together, these results reframe the behavioral cliff in working memory as a spectral bifurcation phenomenon. The brain may operate in the valley regime ( $J_x \approx 1.2\text{--}1.6$ ), where cross-inhibition and encoding drive are balanced for reliable WTA selection. The Goldstone modes protect memory content (bump positions) from the competition over its fate (which bump survives), enforcing a separation of positional and competitive dynamics that may be a design principle of working memory circuits. Importantly, connectivity heterogeneity transforms the sharp pitchfork into a smooth crossover, dissolving the razor-thin instability window ( $\Delta J_x \approx 0.01$ ) entirely – biological circuits need not operate with such precision, and the valley regime persists regardless of bifurcation type. The same bifurcation structure appears in decision-making circuits, suggesting a universal spectral architecture for neural competition.

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