

# Non-linear sigma models for non-Hermitian random matrices in symmetry classes $\text{AI}^\dagger$ and $\text{AII}^\dagger$

Anish Kulkarni,<sup>1</sup> Kohei Kawabata,<sup>2</sup> and Shinsei Ryu<sup>1</sup>

<sup>1</sup>*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*

<sup>2</sup>*Institute for Solid State Physics, University of Tokyo, Kashiwa, Chiba 277-8581, Japan*

(Dated: November 1, 2024)

Symmetry of non-Hermitian matrices underpins many physical phenomena. In particular, chaotic open quantum systems exhibit universal bulk spectral correlations classified on the basis of time-reversal symmetry $^\dagger$  (TRS $^\dagger$ ), coinciding with those of non-Hermitian random matrices in the same symmetry class. Here, we analytically study the spectral correlations of non-Hermitian random matrices in the presence of TRS $^\dagger$  with signs +1 and -1, corresponding to symmetry classes  $\text{AI}^\dagger$  and  $\text{AII}^\dagger$ , respectively. Using the fermionic replica non-linear sigma model approach, we derive  $n$ -fold integral expressions for the  $n$ th moment of the one-point and two-point characteristic polynomials. Performing the replica limit  $n \rightarrow 0$ , we qualitatively reproduce the density of states and level-level correlations of non-Hermitian random matrices with TRS $^\dagger$ .

## CONTENTS

I. Introduction	1
II. NL $\sigma$ M for class $\text{AI}^\dagger$	2
A. Replica space matrix integral for characteristic polynomials	2
B. One-point characteristic polynomial	4
1. Density of states	5
C. Two-point characteristic polynomial	7
1. Cosine-sine decomposition	7
2. Two-point correlation function	9
III. NL $\sigma$ M for class $\text{AII}^\dagger$	10
A. Replica space matrix integral for characteristic polynomials	10
B. One-point characteristic polynomial	11
1. Density of states	12
C. Two-point characteristic polynomial	13
1. Two-point correlation function	14
IV. Discussions	15
Acknowledgments	16
A. Saddle-point integral	16
1. Class $\text{AI}^\dagger$	16
2. Class $\text{AII}^\dagger$	17
B. Two-point correlation functions of small non-Hermitian random matrices	18
References	18

## I. INTRODUCTION

Random matrix theory (RMT), originally developed by Wigner to describe the energy levels of heavy atomic nuclei [1, 2], has since evolved into a versatile tool for understanding complex systems [3]. Its applications span numerous subfields of physics, including quantum chaos, Anderson localization, quantum chromodynamics (QCD), statistical mechanics, quantum information, and quantum gravity. Beyond physics, RMT has found widespread use in disciplines such as number theory, biological systems, and data science, among others. At its core, RMT

studies the statistical properties of matrices with randomly chosen elements, providing insight into the behavior of large, interacting systems. In many cases, the eigenvalue distributions of these random matrices exhibit universal properties, meaning that they are largely independent of the specific details of the matrix ensemble. This universality is one of the most compelling features of RMT, making it applicable to diverse problems.

Non-Hermitian RMT extends the powerful framework of the traditional RMT to the study of matrices that lack Hermiticity [4]. While Hermitian matrices are central in quantum mechanics, non-Hermitian matrices emerge naturally in a wide array of physical systems where dissipation, gain, or non-conservative interactions play a role [5, 6]. These include open quantum systems, non-equilibrium statistical mechanics, disordered materials, and biological and sociological systems. The study of non-Hermitian random matrices began in earnest with the work of Ginibre [7], who introduced ensembles of non-Hermitian matrices. Unlike Hermitian matrices, eigenvalues of non-Hermitian matrices generally spread over the complex plane, leading to rich and intricate spectral structures. One of the most remarkable features of non-Hermitian RMT is the emergence of complex spectral distributions, such as Girko's circular law [8], which describes the eigenvalue distribution for large non-Hermitian matrices with independent, identically distributed entries. Other classic works on non-Hermitian RMT are found, for example, in Refs. [9–15]. The relevance of non-Hermitian RMT has grown considerably in recent decades, particularly with the recognition of non-Hermitian physics in various fields. For example, beyond the Hermitian regime, the physics of Anderson localization in non-Hermitian systems has attracted growing interest [16–32]. Additionally, topological phenomena intrinsic to non-Hermitian systems have been extensively explored, offering new insights into the behavior of these systems [33, 34].

One of the key impacts of non-Hermiticity is the enrichment of symmetry classification, expanding the traditional 10-fold classification [35–38] to a more complex 38-fold scheme [39, 40]. This expanded classification is crucial not only for non-Hermitian RMT but also for understanding chaotic behavior in open quantum systems [41–69]. Similar to their Hermitian counterparts, spectral properties of non-Hermitian random matrices encode information about the dynamics of the associated physical system, many of which are expected to be universal. They are insensitive to the details of the ensemble distribution and classified based on the symmetry of the matrices in the ensemble. For non-Hermitian matrices, time-reversal symmetry<sup>†</sup> (TRS<sup>†</sup>) is particularly relevant [48]. It is defined as  $\mathcal{T}H\mathcal{T}^{-1} = H^\dagger$ , where  $H$  is a non-Hermitian matrix and  $\mathcal{T}$  is an anti-unitary operator. Local correlations between bulk eigenvalues—away from symmetric lines, symmetric points, and edges of the spectrum—are classified based solely on the absence, presence, and sign of TRS<sup>†</sup>. Therefore, in this work, we consider symmetry classes AI<sup>†</sup> and AII<sup>†</sup> of non-Hermitian random matrices [40], defined to respect TRS<sup>†</sup> with signs  $\mathcal{T}^2 = +1$  and  $\mathcal{T}^2 = -1$ , respectively.

Non-linear  $\sigma$  models (NL $\sigma$ Ms) have long been employed to calculate correlation functions in disordered systems [70]. Subsequently, field-theoretic treatments using the replica trick have been successfully used, even for non-Hermitian matrices. In Ref. [71], this method was used to calculate the density of states for non-Hermitian random matrices in classes A, AI, and AII. In this work, we further develop replica space NL $\sigma$ Ms for classes AI<sup>†</sup> and AII<sup>†</sup>, which are relevant to the threefold universality classes of the bulk spectral correlations. Specifically, we consider the  $n$ th moments of the  $k$ -point characteristic polynomials  $Z_n^{(k)}(z_1, \dots, z_k)$ , defined in Eq. (3), for Gaussian non-Hermitian random matrices in classes AI<sup>†</sup> and AII<sup>†</sup>. We also study the density of states,  $R_1(z)$ , and the two-point correlation function,  $R_2(z_1, z_2)$ , of the complex spectrum, derived from these polynomials via the replica limit  $n \rightarrow 0$ .

Our main results are as follows. We derive general replica space matrix integrals for  $Z_n^{(k)}(z_1, \dots, z_k)$  in both symmetry classes AI<sup>†</sup> and AII<sup>†</sup>. We then study the cases  $k = 1$  and  $k = 2$  in detail. For non-Hermitian random matrices of size  $N$ , we obtain exact expressions for  $Z_n^{(1)}(z)$  with arbitrary  $N$  and for  $Z_n^{(2)}(z_1, z_2)$  in the limit  $N \rightarrow \infty$  in both symmetry classes. These expressions are in the form of  $n$ -fold or  $2n$ -fold integrals over replica space singular values or cosine-sine values. We confirm these exact results by comparing them with numerically computed ensemble averages of characteristic polynomials. Furthermore, we perform the replica limit  $n \rightarrow 0$  of these  $n$ -fold integrals to obtain closed form expressions for the spectral distributions  $R_1(z)$  and  $R_2(z_1, z_2)$  in the regimes  $|z| \ll \sqrt{N}$  and  $|z_1 - z_2| \ll \sqrt{N}$ , respectively. These expressions qualitatively capture salient features of the spectral distributions computed numerically.

## II. NL $\sigma$ M FOR CLASS AI<sup>†</sup>

### A. Replica space matrix integral for characteristic polynomials

Non-Hermitian matrices in class AI<sup>†</sup> are defined to respect TRS<sup>†</sup> with sign +1:

$$H^\dagger = \mathcal{T}H\mathcal{T}^{-1} = \bar{H}, \quad (1)$$

where we choose the symmetry operator  $\mathcal{T}$  to be complex conjugation  $\mathcal{K}$ , and  $\bar{H}$  denotes complex conjugate of  $H$ . This is equivalent to  $H = H^T$ , where  $H^T$  denotes transposition of  $H$ . We thus consider a Gaussian ensemble of

symmetric  $N \times N$  complex matrices  $H$ :

$$P(H) \propto \exp[-g^{-1} \text{tr}(H^\dagger H)], \quad (2)$$

where  $g$  parameterizes the width of the Gaussian. Our main objects of interest are the  $n$ th moments of  $k$ th-order characteristic polynomial of  $H$ , defined as

$$Z_n^{(k)}(z_1, \dots, z_k) = \left\langle \prod_{l=1}^k \det(z_l - H)^n \det(\bar{z}_l - H^\dagger)^n \right\rangle. \quad (3)$$

Following the replica trick, the  $k$ -point correlation functions of complex eigenvalues of  $H$  can be determined using  $Z_n^{(k)}(z_1, \dots, z_k)$  (see Refs. [71, 72] for details). For simplicity, we will first derive the replica space matrix integral for  $k = 1$ . A few modifications will give the auxiliary field matrix integral for general  $k$ .

The determinants are re-expressed as Grassmann integrals,

$$\begin{aligned} Z_n^{(1)}(z, \bar{z}) &= \int dH P(H) \int d\psi d\bar{\psi} \exp[-\bar{\psi}_a^i (z \delta^{ij} \delta^{ab} - H^{ij} \delta^{ab}) \psi_a^j] \\ &\quad \times \int d\chi d\bar{\chi} \exp[-\bar{\chi}_a^i (\bar{z} \delta^{ij} \delta^{ab} - (H^\dagger)^{ij} \delta^{ab}) \chi_a^j], \end{aligned} \quad (4)$$

where  $\psi_a^i, \bar{\psi}_a^i, \bar{\chi}_a^i, \chi_a^i$  are independent Grassmann variables. The indices  $i, j = 1, \dots, N$  are color/matrix indices, and  $a, b = 1, \dots, n$  are flavor/replica indices. Since  $H$  is symmetric, it is only coupled with the symmetric part of the fermion bilinears  $[\psi_a \bar{\psi}_a^T] = \frac{1}{2} (\psi_a \bar{\psi}_a^T - \bar{\psi}_a \psi_a^T)$ . Specifically, we have

$$\bar{\psi}_a^i H^{ij} \psi_a^j = -\text{tr}(H [\psi_a \bar{\psi}_a^T]). \quad (5)$$

The Gaussian integral over  $H$  gives a quartic action for the fermions,

$$Z_n^{(1)}(z, \bar{z}) = \int d\psi d\bar{\psi} d\chi d\bar{\chi} e^{S[\psi, \chi]}, \quad (6)$$

$$S[\psi, \chi] = \bar{z} \text{tr}([\chi_a \bar{\chi}_a^T]) + z \text{tr}([\psi_a \bar{\psi}_a^T]) + g \text{tr}([\chi_a \bar{\chi}_a^T] [\psi_b \bar{\psi}_b^T]). \quad (7)$$

The quartic term above is expressed in terms of color-space matrices. We rearrange it as follows to express it in terms of flavor/replica space matrices,

$$\text{tr}([\chi_a \bar{\chi}_a^T] [\psi_b \bar{\psi}_b^T]) = \frac{1}{2} ((\psi_b^T \chi_a)(\bar{\chi}_a^T \bar{\psi}_b) - (\bar{\psi}_b^T \chi_a)(\bar{\chi}_a^T \psi_b)). \quad (8)$$

Now, we introduce replica space auxiliary fields to decouple these quartic terms into quadratic terms using the Hubbard-Stratonovich transformations. We use “Tr” for the trace in flavor space. We introduce flavor space matrices  $Q$  and  $R \in \mathbb{C}^{n \times n}$  to decouple, respectively, the first and second terms of Eq. (8),

$$\exp\left[-\frac{1}{2}g \text{Tr}((\bar{\psi}^T \chi)(\bar{\chi}^T \psi))\right] \propto \int dQ \exp\left[-\frac{1}{2}g^{-1} \text{Tr}(QQ^\dagger) - \frac{1}{2} \text{Tr}(\bar{\psi}^T \chi Q^\dagger) + \frac{1}{2} \text{Tr}(Q \bar{\chi}^T \psi)\right]. \quad (9)$$

Similarly, we have

$$\exp\left[\frac{1}{2}g \text{Tr}((\psi^T \chi)(\bar{\chi}^T \bar{\psi}))\right] \propto \int dR \exp\left[-\frac{1}{2}g^{-1} \text{Tr}(RR^\dagger) - \frac{1}{2} \text{Tr}(R^\dagger \psi^T \chi) - \frac{1}{2} \text{Tr}(\bar{\chi}^T \bar{\psi} R)\right]. \quad (10)$$

Schematically, the resulting action has the quadratic form  $S = -\frac{1}{2}\Theta^T M \Theta$  in terms of the Grassmann variables collectively denoted by  $\Theta$ . The matrix  $M$  is an antisymmetric matrix. The integral over the Grassmann fields can now be performed, which gives the Pfaffian of  $M$  and hence the following explicit form for the characteristic polynomial. Let us define the antisymmetric bilinear forms

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_n^y = \sigma^y \otimes I_n = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}. \quad (11)$$

Then, the characteristic polynomial is expressed as

$$Z_n^{(1)}(z, \bar{z}) = \int_{\mathcal{M}} d\mathcal{Q} e^{-g^{-1} \text{tr}(\mathcal{Q}^\dagger \mathcal{Q})} \det^{N/2} \begin{pmatrix} \bar{z} & \mathcal{Q} \\ -\mathcal{Q}^\dagger & z \end{pmatrix}, \quad (12)$$

with

$$\mathcal{M} = \{\mathcal{Q} \in \mathbb{C}^{2n \times 2n} \mid \Sigma_n^y \mathcal{Q} = \bar{\mathcal{Q}} \Sigma_n^y\}. \quad (13)$$

Now, we generalize this expression to the  $k$ -point characteristic polynomial. A straightforward extension of the above procedure gives the same expressions as above, with  $z$  and  $\Sigma_n^y$  replaced by matrices  $Z$  and  $\Sigma_{nk}^y$ , defined momentarily. Using a suitable (orthogonal) similarity transformation on  $Q$ ,  $Z$ , and  $\Sigma_{nk}^y$ , we can choose  $Z = \text{diag}(z_1, \dots, z_k) \otimes I_{2n}$  and  $\Sigma_{nk}^y = \sigma^y \otimes I_{nk}$  without loss of generality. The characteristic polynomial for the  $k$ -point function is then given as

$$Z_n^{(k)}(z_1, \dots, \bar{z}_1, \dots) = \int_{\mathcal{M}} d\mathcal{Q} e^{-g^{-1} \text{tr } \mathcal{Q}^\dagger \mathcal{Q}} \det^{N/2} \begin{pmatrix} \bar{Z} & \mathcal{Q} \\ -\mathcal{Q}^\dagger & Z \end{pmatrix}, \quad (14)$$

with

$$\mathcal{M} = \{\mathcal{Q} \in \mathbb{C}^{2nk \times 2nk} \mid \Sigma_{nk}^y \mathcal{Q} = \bar{\mathcal{Q}} \Sigma_{nk}^y\}. \quad (15)$$

In the subsequent sections, we will compute this integral for  $k = 1$  and  $k = 2$  under suitable limits.

## B. One-point characteristic polynomial

The integral expression in Eq. (12), playing a role of the partition function, can be further evaluated in various ways. Let us first discuss the behavior in the large- $N$  limit. In the limit  $N \rightarrow \infty$ , we can use the saddle-point approximation to calculate  $Z_n^{(1)}(z, \bar{z})$ . The saddle-point equation for the action in Eq. (12) is

$$\mathcal{Q}^\dagger \mathcal{Q} = \frac{gN}{2} - |z|^2. \quad (16)$$

For  $|z|^2 < gN/2$ , the solution is simply

$$\mathcal{Q} = \sqrt{\frac{gN}{2} - |z|^2} U, \quad (17)$$

where  $U$  is unitary matrix that satisfies  $\Sigma_n^y U = \bar{U} \Sigma_n^y$  (or equivalently,  $U \Sigma_n^y U^T = \Sigma_n^y$ ), i.e.,  $U \in \text{Sp}(n)$ . Now, we substitute this solution into Eq. (12) and obtain the dominant contribution at large  $N$ , leading to

$$\begin{aligned} Z_n^{(1)}(z, \bar{z}) &\simeq \int_{\text{Sp}(n)} dU e^{(g^{-1}|z|^2 - \frac{N}{2}) \text{tr } \mathbb{I}_{2n \times 2n}} \det^{N/2} \left( g \frac{N}{2} \mathbb{I}_{2n \times 2n} \right) \\ &= e^{2ng^{-1}|z|^2} \left[ e^{-nN} \left( \frac{gN}{2} \right)^{nN} \text{Vol}(\text{Sp}(n)) \right]. \end{aligned} \quad (18)$$

Note that the factors inside the square bracket drop out in the replica limit  $n \rightarrow 0$ . In the next subsection, we will use this expression to discuss the density of states.

Now, let us go back to Eq. (12) and evaluate it without the large- $N$  approximation. We consider the singular-value decomposition of  $\mathcal{Q}$ ,

$$\mathcal{Q} = U \Lambda V; \quad U, V \in \text{Sp}(n), \quad \Lambda = \text{diag}(\lambda_1^{\frac{1}{2}}, \dots, \lambda_n^{\frac{1}{2}}, \lambda_1^{\frac{1}{2}}, \dots, \lambda_n^{\frac{1}{2}}) \quad (\lambda_a \geq 0). \quad (19)$$

Accordingly, the measure is given as

$$d\mathcal{Q} = dU dV \Delta(\lambda)^4 \prod_{a=1}^n d\lambda_a \lambda_a, \quad \Delta(\lambda) = \prod_{a>b}^n |\lambda_a - \lambda_b|. \quad (20)$$

Here,  $dU$  and  $dV$  represent the Haar measure on  $\text{Sp}(n)$ , the integral over which is just an overall constant and is irrelevant in the replica limit. The only relevant degrees of freedom in the replica space are  $\lambda_a$ , and hence we have

$$Z_n^{(1)}(z, \bar{z}) \simeq \int_0^\infty \prod_{a=1}^n d\lambda_a e^{-2g^{-1}\lambda_a} (|z|^2 + \lambda_a)^N \lambda_a \Delta(\lambda)^4. \quad (21)$$

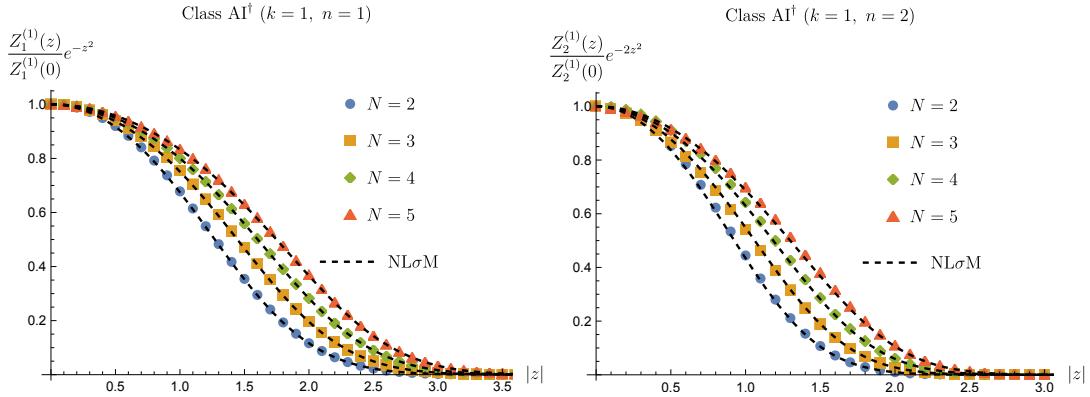


FIG. 1: Plot of  $\frac{Z_n^{(1)}(z)}{Z_n^{(1)}(0)} e^{-2ng^{-1}z^2}$  as a function of  $|z|$  for class  $\text{AI}^\dagger$ . For each value of  $N$ , we sample  $10^5$  non-Hermitian random matrices according to Eq. (2) with  $g = 2$ . The solid markers show the ensemble-averaged characteristic polynomials. The black dashed curves show the same quantity calculated by the  $\text{NL}\sigma\text{M}$  in Eq. (21).

We highlight that this is an exact expression for all moments of the one-point characteristic polynomial at arbitrary  $N$  and  $z$ , up to an overall constant. This can also be thought of as a partition function for replica space degrees of freedom  $\lambda_a$ .

In Fig. 1, we plot the analytical expression in Eq. (21) of the characteristic polynomial for various values of  $N$  and  $n$  as a function of  $|z|$ . We also compare the analytical result in Eq. (1) with numerical results where we numerically generate non-Hermitian random matrices from the Gaussian ensemble in Eq. (2) and calculate the characteristic polynomial. For all values of  $N$  and  $n$  we studied, we have a good agreement between the analytical and numerical results.

### 1. Density of states

The density of states,  $R_1(z)$ , is determined from the first-order characteristic polynomial by [71, 72]

$$\pi R_1(z) = \lim_{n \rightarrow 0} \frac{1}{n} \partial_z \partial_{\bar{z}} Z_n^{(1)}(z, \bar{z}). \quad (22)$$

To begin with, we extract the dominant contribution to  $R_1(z)$  at large  $N$ . We expect this contribution to be a uniform distribution on a disk centered at the spectral origin, which is verified from the  $\text{NL}\sigma\text{M}$  as follows. In the large  $N$  limit, using Eqs. (18) and (22), we obtain

$$\pi R_1(z) = \lim_{n \rightarrow 0} \frac{1}{n} \partial_{\bar{z}} \partial_z Z_n^{(1)}(z, \bar{z}) = \frac{2}{g} \quad \text{for } |z|^2 < \frac{gN}{2}. \quad (23)$$

Indeed, at large  $N$ , the density of states is a uniform distribution on a disc of radius  $\sqrt{gN/2}$  centered at the origin, which is consistent with Girko's circular law [8].

To discuss the density of states near the edge of the spectrum, we use Eq. (21) and analyze its behavior at large  $N$  and in the limit  $n \rightarrow 0$ . For large  $N$ , we can further perform a saddle-point approximation. We set  $g = 2$  for simplicity and consider the factor  $I(\lambda) = e^{-\lambda_a} (|z|^2 + \lambda_a)^N$  in the integrand. For large  $N$ ,  $I(\lambda)$  approaches an un-normalized Gaussian. The saddle point obtained by solving  $\partial_\lambda \ln(I(\lambda)) = 0$  is  $\lambda_{\text{sp}} = N - |z|^2$ . We Taylor-expand the action around this saddle point,  $\ln I(\lambda) \sim -\lambda_{\text{sp}} + N \ln N - \frac{(\lambda - \lambda_{\text{sp}})^2}{2N}$ , and substitute this back into Eq. (21). Rescaling  $\lambda \rightarrow \lambda_{\text{sp}}\lambda$  and ignoring irrelevant overall factors, we obtain

$$Z_n^{(1)}(z, \bar{z}) \simeq e^{-n\lambda_{\text{sp}}} \lambda_{\text{sp}}^{2n^2} \int_0^\infty \prod_{a=1}^n d\lambda_a \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (\lambda_a - 1)^2\right) |\lambda_a| \Delta(\lambda)^4. \quad (24)$$

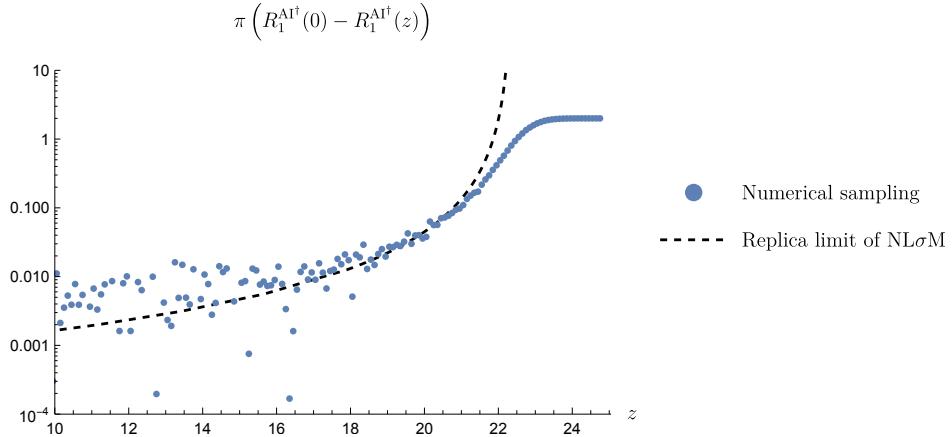


FIG. 2: Comparison between the analytical result in Eq. (29) and the density of states obtained from numerical calculations. The numerical results are obtained using  $2 \times 10^4$  realizations of  $10^3 \times 10^3$  non-Hermitian random matrices in class  $\text{AI}^\dagger$  sampled according to Eq. (2) with  $g = 1$ .

Now, we rewrite the contour of integration as a sum over two contours,

$$\begin{aligned} Z_n^{(1)}(z, \bar{z}) &\simeq e^{-n\lambda_{\text{sp}}^2} \lambda_{\text{sp}}^{2n^2} \sum_p (-1)^p \binom{n}{p} \int_{-\infty}^0 \prod_{a=1}^p dx_a \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (x_a - 1)^2\right) |x_a| \Delta(x)^4 \\ &\quad \times \int_{-\infty}^{\infty} \prod_{a=1}^{n-p} dy_a \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (y_a - 1)^2\right) |y_a| \Delta(y)^4. \end{aligned} \quad (25)$$

We have considered the regime where  $\lambda_{\text{sp}}^2/2N$  is large enough so that  $x$  and  $y$  accumulate near the maxima of the weight function, i.e., 0 and 1. As such, we can approximate  $\Delta(\lambda) \simeq \Delta(x)\Delta(y)$ , and the  $x$  and  $y$  variables become decoupled. In the  $y$ -integral, since the Gaussian is narrowly peaked at 1, we replace the factor of  $y_a$  in the integrand with 1. The remaining integral is then a Selberg integral that is evaluated exactly,

$$\int_{-\infty}^{\infty} \prod_{a=1}^{n-p} dy_a \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (y_a - 1)^2\right) \Delta(y)^4 \simeq \left(\frac{\lambda_{\text{sp}}}{\sqrt{N}}\right)^{-2(n-p)^2+(n-p)} \left(\frac{2}{\pi}\right)^{p/2} \prod_{a=1}^{n-p} \Gamma(1+2a). \quad (26)$$

Before proceeding to the  $x$ -integral, we should look at the coefficient of the  $p$ th term in this expansion and identify which terms survive in the replica limit  $n \rightarrow 0$ . The coefficient is  $\binom{n}{p} \prod_{a=1}^{n-p} \Gamma(1+2a)$ . At order  $n$ , the coefficient is 1 and  $n$  for  $p = 0$  and 1, respectively, and vanishes for all  $p \geq 2$ . We then have

$$\int_{-\infty}^0 dx_1 \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (x_1 - 1)^2\right) |x_1| \simeq \frac{N^2}{\lambda_{\text{sp}}^4} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N}\right) \quad \text{for } \frac{\lambda_{\text{sp}}^2}{2N} \gg 1. \quad (27)$$

Putting it all together, we have

$$Z_n^{(1)}(z, \bar{z}) \simeq 1 - n\lambda_{\text{sp}} + n \ln \lambda_{\text{sp}} - n \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\lambda_{\text{sp}}}{\sqrt{N}}\right)^{-7} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N}\right) + \mathcal{O}(n^2). \quad (28)$$

Finally, plugging this into Eq. (22) and keeping the leading-order terms lead to

$$\pi R_1(z) = 1 - \frac{1}{4u^2} - \sqrt{\frac{2}{\pi N}} \frac{e^{-2u^2}}{16u^4} - \frac{e^{-2u^2}}{16\sqrt{2\pi}u^5}, \quad (29)$$

where we introduce  $u = \sqrt{N} - |z|$  as the distance of  $z$  from the edge of the spectrum, and use  $\lambda_{\text{sp}} \simeq 2u\sqrt{N}$ . The  $g$  dependence can be added back using dimensional analysis.

In Fig. 2, we compare the analytical result in Eq. (29) and the density of states obtained from numerical calculations. We see that there is good agreement between the NL $\sigma$ M result and the numerical calculation up to  $|z| \simeq 21$ , even though Eq. (29) is strictly valid only for  $|z| \ll \sqrt{gN/2} \simeq 22.36$ .

### C. Two-point characteristic polynomial

We now discuss the second-order characteristic polynomial,  $Z_n^{(2)}$ , starting from Eq. (14). For large  $N$ , the integral over  $\mathcal{Q}$  can be approximated by an integral over the saddle-point manifold. We introduce the radius of the spectrum,  $r_s = \sqrt{gN/2}$ , and focus on the region in the bulk of the spectrum,  $|z_{1,2}| \ll r_s$ , where the saddle-point equation is simplified to

$$\mathcal{Q}^\dagger \mathcal{Q} = r_s^2. \quad (30)$$

The solution is  $\mathcal{Q} = r_s U$ , with a unitary matrix  $U$ . Moreover, owing to  $\mathcal{Q} \in \mathcal{M}$  [see Eq. (15)], we have

$$U \Sigma_{nk}^y U^T = \Sigma_{nk}^y \bar{U} U^T = \Sigma_{nk}^y. \quad (31)$$

Thus,  $U$  is symplectic and unitary, and thus belongs to the symplectic group  $\text{Sp}(2n)$ . We substitute the saddle-point solution back into the expression for  $Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2)$ , leading to

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &\simeq \int_{\text{Sp}(2n)} dU e^{-g^{-1}r_s^2 4n \det^{N/2}} \begin{pmatrix} \bar{Z} & r_s U \\ -r_s U^\dagger & Z \end{pmatrix} \\ &\simeq \int_{\text{Sp}(2n)} dU \exp[g^{-1} \text{tr}(U^\dagger \bar{Z} U Z)]. \end{aligned} \quad (32)$$

When we introduce  $z = (z_1 + z_2)/2$ ,  $\omega = z_1 - z_2$ , and  $s = \text{diag}(1, -1) \otimes \mathbb{I}_{2n}$ , we have  $Z = z\mathbb{I}_{4n} + \frac{1}{2}\omega s$  and hence

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &\simeq e^{g^{-1}4n|\omega|^2} \int_{\text{Sp}(2n)} dU \exp\left[\frac{|\omega|^2}{4g} \text{tr}(U^\dagger s U s)\right] \\ &\simeq e^{g^{-1}4n|\omega|^2} \int_{\text{Sp}(2n)/[\text{Sp}(n) \times \text{Sp}(n)]} dW \exp\left[\frac{|\omega|^2}{4g} \text{tr}(W s)\right]. \end{aligned} \quad (33)$$

Here, we have used the fact that for  $U \in \text{Sp}(2n)$ ,  $U^\dagger s U$  lies on the symplectic Grassmannian  $\text{Sp}(2n)/[\text{Sp}(n) \times \text{Sp}(n)]$ . Thus, the target space of this NL $\sigma$ M is the symplectic Grassmannian. In the following, we calculate this integral in the regime  $|\omega|^2 \gg g$ .

#### 1. Cosine-sine decomposition

Analogous to the singular-value decomposition used in Sec. II B, we find that a cosine-sine (CS) decomposition for the symplectic group is effective in calculating the second-order characteristic polynomial  $Z_n^{(2)}$ . The CS decomposition of a symplectic unitary matrix  $U \in \text{Sp}(2n)$  is described below [73]. We start with the following parametrization,

$$U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad (34)$$

with

$$\Theta = \text{diag}(\theta_1, \theta_1, \dots, \theta_n, \theta_n) \quad [\theta_i \in [0, \pi]], \quad u_{1,2}, v_{1,2} \in \text{Sp}(n). \quad (35)$$

Notice that each angle is repeated twice. One can regard each  $\theta_i$  as the commuting part of the  $2 \times 2$  matrix representation of a quaternion. While  $u_{1,2}$  and  $v_{1,2}$  are generic matrices in  $\text{Sp}(n)$ , they must be compensated by an  $\text{Sp}(1)^{\otimes n}$  phase factor inside the CS matrix. A parametrization for  $\text{Sp}(1) \cong S^3$  is  $e^{i\phi(\mathbf{m} \cdot \boldsymbol{\sigma})}$  with  $\phi \in [0, \pi]$  and  $\mathbf{m} \in \mathbb{R}^3$  satisfying  $|\mathbf{m}|^2 = 1$  (i.e.,  $\mathbf{m} \in S^2$ ). We thus obtain a new CS decomposition without redundancy by

$$U = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cos \Theta \exp(i\Phi \mathbf{M} \cdot \boldsymbol{\Sigma}) & \sin \Theta \\ -\sin \Theta & \cos \Theta \exp(-i\Phi \mathbf{M} \cdot \boldsymbol{\Sigma}) \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad (36)$$

with

$$u_{1,2} \in \text{Sp}(n)/\text{Sp}(1)^{\otimes n}, \quad v_{1,2} \in \text{Sp}(n); \quad (37)$$

$$\mathbf{M} \cdot \boldsymbol{\Sigma} = \text{diag}(\mathbf{m}_1 \cdot \boldsymbol{\sigma}, \dots, \mathbf{m}_n \cdot \boldsymbol{\sigma}), \quad \Phi = \text{diag}(\phi_1, \phi_1, \dots, \phi_n, \phi_n) \quad [\mathbf{m}_i \in S^2, \phi_i \in [0, \pi]]. \quad (38)$$

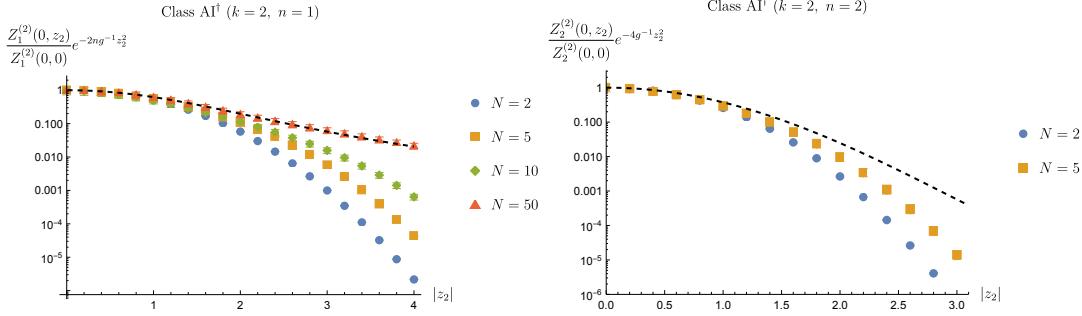


FIG. 3: Logarithmic plot of the two-point characteristic polynomial in class  $\text{AI}^\dagger$ . We plot the expression  $\frac{Z_n^{(2)}(0, z_2)}{Z_n^{(2)}(0, 0)} e^{-2ng^{-1}z_2^2}$  instead of  $Z_n^{(2)}(0, z_2)$  for better visualization. For each value of  $N = 2$  and  $N = 5$ , we collect  $10^6$  realizations of non-Hermitian random matrices, while for  $N = 10$  and  $N = 50$  we collect  $5 \times 10^6$  realizations. In all cases, we use  $g = 2$ . The solid markers show the numerically computed ensemble-averaged moments of characteristic polynomials. The black dashed curves show the same quantity calculated with the NL $\sigma$ M for large  $N$  in Eq. (43).

Now, we proceed to derive the Haar measure in terms of these new variables. Let  $\Lambda$  be the CS matrix. Using the invariance properties of the Haar measure, it is sufficient to consider a neighborhood of  $u_{1,2} = v_{1,2} = \mathbb{I}_{2n}$ , leading to

$$U^\dagger dU = \begin{pmatrix} dv_1 & 0 \\ 0 & dv_2 \end{pmatrix} + \Lambda^\dagger \begin{pmatrix} du_1 & 0 \\ 0 & du_2 \end{pmatrix} \Lambda + \Lambda^\dagger d\Lambda. \quad (39)$$

The calculations follow quite similarly to the unitary case from here on. The contribution from  $\Lambda^\dagger d\Lambda$  is simply an extension of the measure for  $\text{Sp}(2)$  matrices given in Ref. [74]. The second term is the non-trivial part, which by comparison to the unitary case contributes  $\Delta(\cos^2 \Theta)^4$ . Finally, the Haar measure is obtained as

$$dU^{\text{Haar}} \propto du_1^{\text{Haar}} du_2^{\text{Haar}} dv_1^{\text{Haar}} dv_2^{\text{Haar}} \Delta(\cos^2 \Theta)^4 \prod_i \sin^3(2\theta_i) \sin^2(\phi_i) d\theta_i d\phi_i d^2 \mathbf{m}_i. \quad (40)$$

Now, we recall the saddle-point integral in Eq. (33). The integrand in terms of the new variables is

$$\exp\left(\frac{\omega^2}{4g} \text{tr}(UsU^\dagger s)\right) = \exp\left(\frac{\omega^2}{2g} \text{tr}(\cos 2\Theta)\right). \quad (41)$$

Thus, ignoring irrelevant overall constants, we have

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &\simeq e^{g^{-1}4n|z|^2} \int_0^\pi \prod_i d\theta_i \sin^3(2\theta_i) \exp\left(\frac{\omega^2}{2g} \text{tr} \cos 2\Theta\right) \Delta(\cos^2 \Theta)^4 \\ &= e^{4n|z|^2} \int_{-1}^1 \prod_i d\lambda_i (1 - \lambda_i^2) e^{\omega^2 \sum_i \lambda_i} \Delta(\lambda)^4, \end{aligned} \quad (42)$$

where we set  $g = 1$  for simplicity and introduce  $\lambda_i = \cos 2\theta_i$ , satisfying  $\text{tr} \cos 2\Theta = 2 \sum_i \lambda_i$ . To make a comparison with Selberg integrals, we make the shift  $\lambda \rightarrow 1 - \lambda$  and then rescale  $\lambda \rightarrow 2\lambda$ , resulting in

$$Z_n(z_1, \bar{z}_1, z_2, \bar{z}_2) \simeq e^{4n|z|^2} e^{n\omega^2} \int_0^1 \prod_i d\lambda_i \lambda_i (1 - \lambda_i) e^{-2\omega^2 \sum_i \lambda_i} \Delta(\lambda)^4. \quad (43)$$

In Fig. 3, we plot the analytical expression in Eq. (43) of the second-order characteristic polynomial for various values of  $N$  and  $n$  as a function of  $|z_2 - z_1|$ . We also compare the analytical result in Eq. (43) with numerical results where we numerically generate non-Hermitian random matrices from the Gaussian ensemble in Eq. (2) and calculate the characteristic polynomial. Here, it should be noted that Eq. (43) is valid only for large  $N$ . While the numerics for small  $N$  deviates from Eq. (43) for large  $z_2$ , it is consistent with Eq. (43) for large enough  $N$  and for small enough  $z_2$ .

## 2. Two-point correlation function

We now discuss the two-point correlation function. The two-point correlation function is obtained from the second-order characteristic polynomial  $Z_n^{(2)}$  by [71, 72]

$$\pi^2 R_2(z_1, z_2) = \lim_{n \rightarrow 0} \frac{1}{n^2} \partial_{z_1} \partial_{\bar{z}_1} \partial_{z_2} \partial_{\bar{z}_2} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2), \quad (44)$$

where  $Z_n^{(2)}$  is given by Eq. (14) with  $k = 2$ . To evaluate  $R_2$ , we go back to Eq. (43) and analyze its behavior in the limit  $n \rightarrow 0$ . It is difficult to evaluate this integral exactly on the domain  $[0, 1]^n$ . However, we aim to evaluate it only up to order  $n^2$ . For this purpose, we can expand the domain to  $[0, \infty]^n$  and then systematically add or subtract corrections at increasing orders in  $n$ . Only finitely many terms in this series survive at order  $n^2$ . To perform this procedure, we begin with

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &\simeq e^{4n|z|^2} e^{n\frac{t}{2}} \int_0^\infty \prod_i d\lambda_i |\lambda_i(1 - \lambda_i)|(1 - \Theta(\lambda_i - 1)) e^{-t \sum_i \lambda_i} \Delta(\lambda)^4 \\ &\simeq e^{4n|z|^2} e^{n\frac{t}{2}} \sum_{p=0}^n \binom{n}{p} (-)^p \int_1^\infty \prod_{i=1}^p dx_i |x_i(1 - x_i)| e^{-t \sum_i x_i} \int_0^\infty \prod_{i=1}^{n-p} dy_i |y_i(1 - y_i)| e^{-t \sum_i y_i} \Delta(x, y)^4 \end{aligned} \quad (45)$$

with  $t = 2\omega^2$ . For large  $t$ , the  $x$  and  $y$  variables accumulate near the peak of the weight function  $|\lambda(1 - \lambda)|e^{-t\lambda}$  within their respective domain. To the leading order in  $1/t$ , these are simply located at  $x = 1$  and  $y = 0$ , respectively, and hence we can make the approximation  $\Delta(x, y) \simeq \Delta(x)\Delta(y)$ . Let us focus on the  $y$ -integral. Since the  $y$  variables accumulate close to the origin, we can approximate  $|y_i(1 - y_i)| \simeq y_i$ . Thus, we have

$$\begin{aligned} \int_0^\infty \prod_{i=1}^l dy_i y_i e^{-t \sum_i y_i} \Delta(y)^4 &= \left(\frac{1}{t}\right)^{2l^2} \Gamma(3)^{-l} \prod_{k=1}^l \Gamma(2k)\Gamma(1+2k) \\ &\simeq \left(\frac{1}{t}\right)^{2l^2} \Gamma(3)^p \prod_{k=l+1}^n (\Gamma(2k)\Gamma(1+2k))^{-1} \end{aligned} \quad (46)$$

with  $n - p = l$ . After taking the  $\binom{n}{p}$  factor into account, only  $p = 0, 1$  terms survive at order  $n^2$ . On the other hand, the  $x$ -integral for  $p = 0$  is trivially 1. For  $p = 1$ , it is

$$\int_1^\infty dx_1 x_1(x_1 - 1) e^{-tx_1} = \frac{e^{-t}(t+2)}{t^3}. \quad (47)$$

Thus, we have

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &\simeq e^{4n|z|^2} e^{n\frac{t}{2}} \left( \frac{1}{t^{2n^2}} - \frac{4n^2 e^{-t}(t+2)}{t^5} \right) \\ &\simeq e^{4n|z|^2} \left( 1 + \frac{nt}{2} + \frac{n^2 t^2}{8} - 2n^2 \ln t - \frac{4n^2 e^{-t}}{t^4} \right). \end{aligned} \quad (48)$$

Then, the two-point correlation function is obtained from Eq. (44), to leading order in  $1/|\omega|$ , as

$$\pi^2 R_2^{\text{AI}^\dagger}(\omega) \simeq 4 \left( 1 - \frac{e^{-2\omega^2}}{\omega^4} \right). \quad (49)$$

A similar saddle-point integral was also calculated in Ref. [75] (see Appendix A for details). However, the relevant parameter  $\mu$  is assumed to be real in Ref. [75], while it should be imaginary in our calculations. For comparison, we also analytically present the two-point correlation function  $R_2(z_1, z_2)$  for  $2 \times 2$  non-Hermitian random matrices in class  $\text{AI}^\dagger$  in Appendix B. We recall that the regime of validity for this result is  $\sqrt{g} \ll |z_1 - z_2|$  and  $|z_1|, |z_2| \ll \sqrt{gN/2}$ . While it is difficult to access this regime numerically, we present a comparison of this result with numerical calculations in Fig. 4. However, the derivation of Eq. (43) is valid even when the separation  $|z_1 - z_2|$  is small. Taking Eq. (43) as a starting point, future work may provide important insights into the correlations of nearby eigenvalues in class  $\text{AI}^\dagger$ .

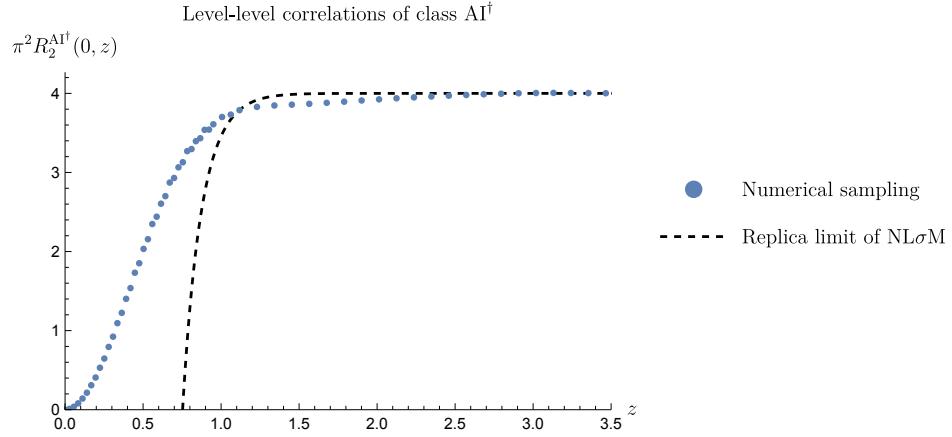


FIG. 4: Comparison between the analytical result in Eq. (49) and the two-point correlation function obtained from numerical calculations. The numerical results are obtained by  $2 \times 10^4$  realizations of  $10^3 \times 10^3$  non-Hermitian random matrices in class  $\text{AI}^\dagger$  sampled according to Eq. (2) with  $g = 1$ .

### III. $\text{NL}\sigma\text{M}$ FOR CLASS $\text{AII}^\dagger$

#### A. Replica space matrix integral for characteristic polynomials

Non-Hermitian matrices in class  $\text{AII}^\dagger$  respect  $\text{TRS}^\dagger$  with sign  $\mathcal{T}^2 = -1$ . We choose the symmetry operator to be  $\mathcal{T} = \Sigma^y \mathcal{K}$ , with complex conjugation  $\mathcal{K}$  and  $\Sigma^y = \sigma^y \otimes \mathbb{I}_N$ . Thus, the symmetry condition reads

$$H^\dagger = \mathcal{T} H \mathcal{T}^{-1}, \quad \text{i.e.,} \quad \Sigma^y H^T \Sigma^y = H. \quad (50)$$

We consider a Gaussian ensemble in Eq. (2). As before, the general characteristic polynomial for  $k$ -point functions is defined as in Eq. (3). We first focus on the  $k = 1$  case and then generalize to  $k > 1$ . After rewriting the determinants in terms of Grassmann variables, the replica characteristic polynomial for the one-point function is given by

$$Z_n^{(1)}(z, \bar{z}) = \int dH P(H) \int d\psi d\bar{\psi} d\chi d\bar{\chi} \exp[-\bar{\psi}_a^i (z \delta^{ij} \delta^{ab} - H^{ij} \delta^{ab}) \psi_a^j - \bar{\chi}_a^i (\bar{z} \delta^{ij} \delta^{ab} - (H^\dagger)^{ij} \delta^{ab}) \chi_a^j], \quad (51)$$

where the indices  $i, j$  now run from 1 to  $2N$ , and  $a, b$  run from 1 to  $n$ . Now, we account for the symmetry of  $H$  and distill the relevant fermionic degrees of freedom by noting

$$\bar{\psi}_a^i H^{ij} \psi_a^j = -\text{tr}(H \psi_a \bar{\psi}_a^T) = +\text{tr}(H \Sigma^y \bar{\psi}_a \psi_a^T \Sigma^y) = -\text{tr}(H [\psi_a \bar{\psi}_a^T]), \quad (52)$$

where we define the modified symmetric part of the fermion bilinear as

$$[\psi_a \bar{\psi}_a^T] = \frac{1}{2} (\psi_a \bar{\psi}_a^T + (\Sigma^y \psi_a \bar{\psi}_a^T \Sigma^y)^T) = \frac{1}{2} (\psi_a \bar{\psi}_a^T - \Sigma^y \bar{\psi}_a \psi_a^T \Sigma^y). \quad (53)$$

Indeed, the modified symmetric part respects  $\text{TRS}^\dagger$  in class  $\text{AII}^\dagger$ :  $\Sigma^y [\psi_a \bar{\psi}_a^T]^T \Sigma^y = [\psi_a \bar{\psi}_a^T]$ . In perfect analogy to class  $\text{AI}^\dagger$  (see Sec. II A), we integrate out the  $H$  variables and are left with the following integral,

$$Z_n^{(1)}(z, \bar{z}) = \int d\psi d\bar{\psi} d\chi d\bar{\chi} \exp[g \text{tr}([\chi_a \bar{\chi}_a^T] [\psi_b \bar{\psi}_b^T]) + \bar{z} \text{tr}([\chi_a \bar{\chi}_a^T]) + z \text{tr}([\psi_a \bar{\psi}_a^T])]. \quad (54)$$

We expand the four-fermion term as

$$g \text{tr}([\chi_a \bar{\chi}_a^T] [\psi_b \bar{\psi}_b^T]) = \frac{1}{2} g [-(\bar{\psi}_b^T \chi_a)(\bar{\chi}_a^T \psi_b) + (\psi_b^T \Sigma^y \chi_a)(\bar{\chi}_a^T \Sigma^y \bar{\psi}_b)]. \quad (55)$$

We introduce flavor-space matrices  $Q$  and  $R \in \mathbb{C}^{n \times n}$  to decouple, respectively, the first and second terms in Eq. (55),

$$\exp\left[-\frac{1}{2} g \text{Tr}((\bar{\psi}^T \chi)(\bar{\chi}^T \psi))\right] \propto \int dQ \exp\left[-\frac{1}{2} g^{-1} \text{Tr}(QQ^\dagger) - \frac{1}{2} \text{Tr}(\bar{\psi}^T \chi Q^\dagger) + \frac{1}{2} \text{Tr}(Q \bar{\chi}^T \psi)\right]. \quad (56)$$

Similarly, we have

$$\exp\left[\frac{1}{2}g \operatorname{Tr}\left((\psi^T \Sigma^y \chi)(\bar{\chi}^T \Sigma^y \bar{\psi})\right)\right] \propto \int dR \exp\left[-\frac{1}{2}g^{-1} \operatorname{Tr}(RR^\dagger) - \frac{1}{2} \operatorname{Tr}(R^\dagger \psi^T \Sigma^y \chi) - \frac{1}{2} \operatorname{Tr}(\bar{\chi}^T \Sigma^y \bar{\psi} R)\right]. \quad (57)$$

Following similar steps as before, we express the characteristic polynomial as

$$Z_n^{(1)}(z, \bar{z}) \simeq \int_{\mathbb{R}^{2n \times 2n}} d\mathcal{Q} e^{-g^{-1} \operatorname{tr} \mathcal{Q}^T \mathcal{Q}} \det^N \begin{pmatrix} -z & \mathcal{Q} \\ \mathcal{Q}^T & \bar{z} \end{pmatrix}. \quad (58)$$

Now, we generalize this expression to the  $k$ -point characteristic polynomial. The procedure is largely identical to the one above, resulting in

$$Z_n^{(k)}(z_1, \dots, z_k, \bar{z}_1, \dots, \bar{z}_k) = \int_{\mathbb{R}^{2nk \times 2nk}} d\mathcal{Q} e^{-g^{-1} \operatorname{tr} \mathcal{Q}^T \mathcal{Q}} \det^N \begin{pmatrix} -Z & \mathcal{Q} \\ \mathcal{Q}^T & \bar{Z} \end{pmatrix}, \quad (59)$$

with

$$Z = \operatorname{diag}(z_1, \dots, z_k) \otimes I_{2n}. \quad (60)$$

In the subsequent sections, we will compute this integral for  $k = 1$  and  $k = 2$  under suitable limits.

## B. One-point characteristic polynomial

We study the first-order characteristic polynomial  $Z_n^{(1)}(z, \bar{z})$  in more detail and use it to derive the density of states for class AII $^\dagger$ . First, we extract only the dominant contribution for large  $N$ . In this limit, the integral in Eq. (58) can be approximated by integrating over the saddle point. The saddle-point equation is

$$\mathcal{Q}^T \mathcal{Q} = gN - |z|^2. \quad (61)$$

For  $|z| < \sqrt{gN}$ , it is solved by  $\mathcal{Q} = \sqrt{gN - |z|^2} O$ , with  $O \in \mathrm{O}(2n)$ . Dropping the overall factors irrelevant in the replica limit, we get  $Z_n^{(1)}(z, \bar{z}) \simeq e^{2ng^{-1}|z|^2}$ , which gives  $\pi R_1(z) = 2g^{-1}$  from Eq. (22). This is consistent with numerical calculations and Girko's circular law [8].

Now, let us study the characteristic polynomial  $Z_n^{(1)}(z, \bar{z})$  in more detail without any approximations. Below, we assume  $g = 1$  for simplicity. Notice that the integrand in Eq. (58) remains invariant if we multiply  $\mathcal{Q}$  by an orthogonal matrix on the left or on the right. Thus, we can reduce the matrix integral to an integral over singular values of  $\mathcal{Q}$ . Let the singular-value decomposition of  $\mathcal{Q} \in \mathbb{R}^{2n \times 2n}$  be

$$\mathcal{Q} = U \Lambda V; \quad U, V \in \mathrm{O}(2n), \quad \Lambda = \operatorname{diag}(\lambda_1^{\frac{1}{2}}, \dots, \lambda_{2n}^{\frac{1}{2}}) \quad (\lambda_a \geq 0). \quad (62)$$

The integration measure on  $\mathcal{Q}$  is transformed into the following measure on  $U$ ,  $V$ , and  $\Lambda$ ,

$$d\mathcal{Q} = dU dV \Delta(\lambda) \prod_{a=1}^{2n} d\lambda_a \lambda_a^{-\frac{1}{2}}, \quad \Delta(\lambda) = \prod_{a>b}^n |\lambda_a - \lambda_b|. \quad (63)$$

Since the integrand is independent of  $U$  and  $V$ , the integrals over  $U$  and  $V$  simply give the volume of  $\mathrm{O}(2n)$ . This is an irrelevant overall factor in the replica limit and hence ignored. The remaining integral is

$$Z_n^{(1)}(z, \bar{z}) \simeq \int_0^\infty \prod_{a=1}^{2n} d\lambda_a I(\lambda_a) \lambda_a^{-\frac{1}{2}} \Delta(\lambda), \quad I(\lambda) = e^{-\lambda} (\lambda + |z|^2)^N. \quad (64)$$

In Fig. 5, we plot the analytical expression in Eq. (64) of the first-order characteristic polynomial for various values of  $N$  and  $n$  as a function of  $|z|$ . We also compare the analytical result in Eq. (64) with numerical results where we numerically generate non-Hermitian random matrices from the Gaussian ensemble in Eq. (2) and calculate the characteristic polynomial. For all values of  $N$  and  $n$  we studied, the analytical and numerical results are consistent.

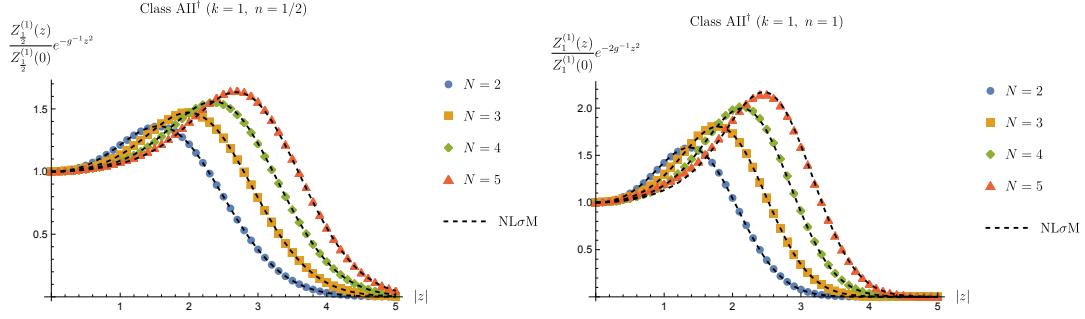


FIG. 5: Plot of  $\frac{Z_n^{(1)}(z)}{Z_n^{(1)}(0)} e^{-2ng^{-1}z^2}$  as a function of  $|z|$  for class  $\text{AII}^\dagger$ . For each value of  $N$ , we sample  $10^5$  realizations of non-Hermitian random matrices according to Eq. (2) with  $g = 2$ . The solid markers show the ensemble-averaged characteristic polynomials. The black dashed curves show the same quantity calculated by Eq. (64) obtained from the  $\text{NL}\sigma\text{M}$ .

### 1. Density of states

As in the case of class  $\text{AI}^\dagger$ , we consider the large- $N$  limit and approximate  $I(\lambda)$  with an un-normalized Gaussian:  $I(\lambda) \simeq N^N e^{-\lambda_{\text{sp}}} \exp\left(-\frac{(\lambda - \lambda_{\text{sp}})^2}{2N}\right)$ . Here,  $\lambda_{\text{sp}} = N - |z|^2$  is the solution to the saddle-point equation  $\partial_\lambda \ln I(\lambda) = 0$ . Rescaling  $\lambda \rightarrow \lambda_{\text{sp}}\lambda$ , we obtain

$$Z_n^{(1)}(z, \bar{z}) \simeq e^{-2n\lambda_{\text{sp}}\lambda_{\text{sp}}^{2n^2}} \int_0^\infty \prod_{a=1}^{2n} \frac{d\lambda_a}{\sqrt{|\lambda_a|}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (\lambda_a - 1)^2\right) |\Delta(\lambda)|. \quad (65)$$

We rewrite the range of integration as  $\int_0^\infty d\lambda = \int_{-\infty}^\infty d\lambda - \int_{-\infty}^0 d\lambda$ . We consider the regime where  $\frac{\lambda_{\text{sp}}^2}{2N}$  is large enough so that  $x$  and  $y$  accumulate near the maxima of the weight function, i.e., 0 and 1, respectively. As such, we can approximate  $\Delta(\lambda) \simeq \Delta(x)\Delta(y)$ , and hence the  $x$  and  $y$  variables become decoupled, as follows:

$$\begin{aligned} Z_n^{(1)}(z, \bar{z}) &\simeq e^{-2n\lambda_{\text{sp}}\lambda_{\text{sp}}^{2n^2}} \sum_p (-1)^p \binom{2n}{p} \int_{-\infty}^0 \prod_{a=1}^p \frac{dx_a}{\sqrt{|x_a|}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (x_a - 1)^2\right) |\Delta(x)| \\ &\quad \times \int_{-\infty}^\infty \prod_{a=1}^{2n-p} \frac{dy_a}{\sqrt{|y_a|}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (y_a - 1)^2\right) |\Delta(y)| \end{aligned} \quad (66)$$

Furthermore, in the  $y$ -integral, since the Gaussian is narrowly peaked at 1, we replace the factor of  $|y_a|^{-\frac{1}{2}}$  in the integrand with 1. The remaining integral is a Selberg integral, evaluated exactly as

$$\int_{-\infty}^\infty \prod_{a=1}^{2n-p} dy_a \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (y_a - 1)^2\right) |\Delta(y)| \simeq \left(\frac{\lambda_{\text{sp}}}{\sqrt{N}}\right)^{-\frac{(2n-p+1)(2n-p)}{2}} 2^{\frac{3}{2}(2n-p)} \prod_{a=1}^{2n-p} \Gamma\left(1 + \frac{1}{2}a\right). \quad (67)$$

We should check the coefficient of the  $p$ th term in this expansion and identify which terms survive in the limit  $n \rightarrow 0$ . The coefficient is  $\binom{2n}{p} \prod_{a=1}^{2n-p} \Gamma\left(1 + \frac{1}{2}a\right)$ . At order  $n$ , the coefficients are 1,  $2n$ , and  $-n/\sqrt{\pi}$  for  $p = 0, 1$ , and 2, respectively, and zero for all  $p \geq 3$ . On the other hand, there is no  $x$ -integral for  $p = 0$ . For  $p = 1$ , it is

$$\int_{-\infty}^0 \frac{dx_1}{\sqrt{|x_1|}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} (x_1 - 1)^2\right) \simeq \frac{\sqrt{\pi N}}{\lambda_{\text{sp}}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N}\right) \quad \text{for } \frac{\lambda_{\text{sp}}^2}{2N} \gg 1. \quad (68)$$

For  $p = 2$ , it is

$$\int_{-\infty}^0 \frac{dx_1}{\sqrt{|x_1|}} \int_{-\infty}^0 \frac{dx_2}{\sqrt{|x_2|}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N} [(x_1 - 1)^2 + (x_2 - 1)^2]\right) |x_1 - x_2| \simeq \frac{2N^2}{\lambda_{\text{sp}}^4} \exp\left(-\frac{\lambda_{\text{sp}}^2}{N}\right) \quad \text{for } \frac{\lambda_{\text{sp}}^2}{2N} \gg 1. \quad (69)$$

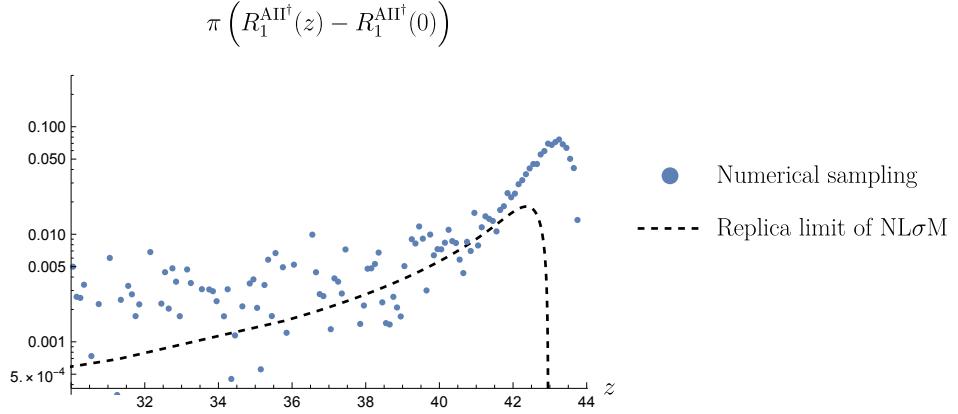


FIG. 6: Comparison between the analytical result in Eq. (71) and the density of states obtained from numerical calculations. The numerical results are obtained by  $10^4$  realizations of  $(2 \times 10^3) \times (2 \times 10^3)$  non-Hermitian random matrices in class  $\text{AII}^\dagger$  sampled according to Eq. (2) with  $g = 2$ .

Putting it all together, we have

$$Z_n^{(1)}(z, \bar{z}) \simeq 1 - 2n\lambda_{\text{sp}} - n \ln \lambda_{\text{sp}} - \frac{n}{\lambda_{\text{sp}}} \sqrt{\frac{\pi N}{2}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{2N}\right) - \frac{n}{4\lambda_{\text{sp}}^5} \sqrt{\frac{N^5}{\pi}} \exp\left(-\frac{\lambda_{\text{sp}}^2}{N}\right) + \mathcal{O}(n^2). \quad (70)$$

Plugging this into Eq. (22) and keeping the leading-order terms, we obtain

$$\pi R_1(z) = 2 + \frac{1}{4u^2} - \sqrt{2\pi} e^{-2u^2} \left(u + \frac{1}{4u} + \frac{1}{8u^3}\right) - \frac{e^{-4u^2}}{8\sqrt{\pi}u^3} + \mathcal{O}\left(\frac{1}{u^5}\right), \quad (71)$$

where we introduce  $u = \sqrt{N} - |z|$  as the distance of  $z$  from the edge of the spectrum and use  $\lambda_{\text{sp}} \simeq 2u\sqrt{N}$ .

In Fig. 6, we compare the analytical result in Eq. (71) and the density of states obtained from numerical calculations. We notice that the positive term  $+1/4u^2$  in Eq. (71) is unique to class  $\text{AII}^\dagger$  and does not appear for class A or  $\text{AI}^\dagger$ . As seen in Fig. 6, this term allows us to qualitatively reproduce the observed increase in the density of states near the edge of the spectrum. The agreement is not precise since the above result is strictly valid only for  $|z| \ll \sqrt{gN} \simeq 44.72$ .

### C. Two-point characteristic polynomial

We recall the replica space matrix integral for the second-order characteristic polynomial in Eq. (59). For large  $N$ , we can approximate the  $\mathcal{Q}$ -integral by an integral over the saddle-point manifold of  $\mathcal{Q}$ . Unlike our discussion in Sec. III B for the density of states,  $Z$  and  $\mathcal{Q}$  do not commute, which makes the saddle-point equation more involved. However, in the regime  $z_{1,2} \ll \sqrt{gN}$ , it is simplified to

$$\mathcal{Q}^T \mathcal{Q} = gN, \quad (72)$$

which is solved as  $\mathcal{Q} = \sqrt{gN}O$  with an orthogonal matrix  $O \in \text{O}(4n)$ . The integral over the saddle-point manifold is

$$\begin{aligned} Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) &= \int_{\text{O}(4n)} dO e^{-4nN} \det^N \begin{pmatrix} -Z & \sqrt{gN}O \\ \sqrt{gN}O^T & \bar{Z} \end{pmatrix} \\ &\simeq \int_{\text{O}(4n)} dO \exp[g^{-1} \text{tr}(O^T Z O \bar{Z})]. \end{aligned} \quad (73)$$

Here, we introduce  $z = (z_1 + z_2)/2$ ,  $\omega = (z_1 - z_2)$ , and  $s = \text{diag}(1, -1) \otimes \mathbb{I}_{2n}$ , and then have  $Z = z\mathbb{I}_{4n} + \frac{1}{2}\omega s$ , leading to

$$Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \simeq e^{g^{-1}4n|z|^2} \int_{\text{O}(4n)} dO \exp\left[g^{-1} \text{tr}\left(\frac{|\omega|^2}{4} O^T s O s\right)\right] \quad (74)$$

When we define  $W = O^T s O$ , this change of variables maps the integration manifold to the orthogonal Grassmannian,

$$Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \simeq e^{g^{-1} 4n|z|^2} \int_{\mathrm{O}(4n)/[\mathrm{O}(2n) \times \mathrm{O}(2n)]} dW \exp \left[ g^{-1} \mathrm{tr} \left( \frac{|\omega|^2}{4} W s \right) \right]. \quad (75)$$

We further simplify the above integrals by using the CS decomposition of the orthogonal group  $\mathrm{O}(4n)$ ,

$$O = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} \cos \Theta & \sin \Theta \\ -\sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad (76)$$

with

$$\Theta = \mathrm{diag}(\theta_1, \dots, \theta_{2n}), \quad \theta_i \in [0, \pi), \quad u_{1,2}, v_{1,2} \in \mathrm{O}(2n). \quad (77)$$

The integrand is only a function of  $\Theta$ , not dependent on  $u$  and  $v$ ,

$$\exp \left( \frac{\omega^2}{4} \mathrm{tr}(OsO^T s) \right) = \exp \left( \frac{\omega^2}{2} \cos 2\Theta \right). \quad (78)$$

Now, we express the Haar measure on  $\mathrm{O}(4n)$  in terms of  $u, v$ , and  $\Theta$ . Due to the invariance properties of the Haar measure, we can assume that  $u_{1,2}$  and  $v_{1,2}$  are in the neighborhood of identity. Thus, the Haar measure in terms of  $u, v$ , and  $\Theta$  is given as

$$\bigwedge_{4n \geq i > j \geq 1} (O^T dO)_{ij} = \bigwedge_{k > l} (dv_1)_{kl} (dv_2)_{kl} \bigwedge_m d\theta_m \bigwedge_{i > j} (du_1)_{ij} \wedge (du_2)_{ij} \Delta(\cos 2\Theta). \quad (79)$$

For  $u$  and  $v$ , the measure is just the Haar measure on  $\mathrm{O}(2n)$ , which only contributes an irrelevant overall factor. The relevant part is the measure on  $\Theta$ . Thus, we finally have

$$Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \simeq e^{g^{-1} 4n|z|^2} \int_0^\pi d\Theta |\Delta(\cos 2\Theta)| \exp \left( \frac{\omega^2}{2} \sum_i \cos 2\theta_i \right). \quad (80)$$

Introducing  $\lambda_i = \sin(\theta_i)^2$  and  $t = \omega^2$ , we further have

$$Z_n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \simeq e^{g^{-1} 4n|z|^2} e^{nt} I_{2n}(t), \quad (81)$$

$$I_m(t) = \int_0^1 \prod_{i=1}^m d\lambda_i (\lambda_i(1-\lambda_i))^{-\frac{1}{2}} e^{-t \sum_i \lambda_i} |\Delta(\lambda)|. \quad (82)$$

### 1. Two-point correlation function

As before, we extend the range of integration from  $[0, 1]$  to  $[0, \infty)$  and deform the integration contour for each  $\lambda_i$  as  $\int_0^1 d\lambda \rightarrow \int_0^\infty d\lambda - \int_1^\infty d\lambda$ . Now, we make several approximations. We observe that in each branch of the contour, the dominant contribution comes from the endpoints 0 and 1. At these endpoints,  $(\lambda_i(1-\lambda_i))^{-\frac{1}{2}}$  can be approximated as  $\lambda^{-\frac{1}{2}}$  and  $(1-\lambda)^{-\frac{1}{2}}$ . We can also factorize the Vandermonde determinant such that variables on different contours are decoupled. We thus have

$$I_m(t) = \sum_{p=0}^m \binom{m}{p} (-1)^p \int_0^\infty \prod_{i=1}^{m-p} dy_i |y_i|^{-\frac{1}{2}} e^{-t \sum_i y_i} |\Delta(y)| \times e^{-pt} \int_0^\infty \prod_{i=1}^p dx_i |x_i|^{-\frac{1}{2}} e^{-t \sum_i x_i} |\Delta(x)|. \quad (83)$$

The  $y$ -integral is a Selberg integral evaluated as

$$\int_0^\infty \prod_{i=1}^{m-p} dy_i |y_i|^{-\frac{1}{2}} e^{-t \sum_i y_i} |\Delta(y)| = t^{-(m-p)^2/2} \frac{1}{\Gamma(\frac{3}{2})^{m-p}} \prod_{k=1}^{m-p} \Gamma \left( \frac{k}{2} \right) \Gamma \left( 1 + \frac{k}{2} \right). \quad (84)$$

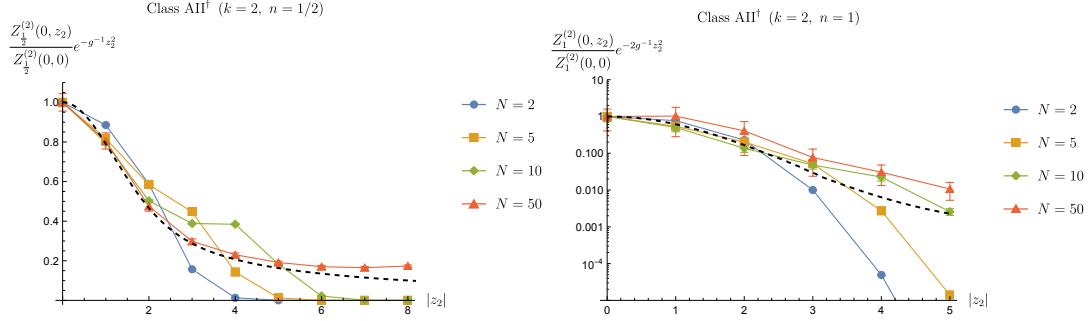


FIG. 7: Plots of the two-point characteristic polynomial in class  $\text{AII}^\dagger$ . We plot the expression  $\frac{Z_n^{(2)}(0, z)}{Z_n^{(2)}(0, 0)} e^{-2g^{-1}z^2}$  instead of  $Z_n^{(2)}(0, z)$  for better visualization. For each value of  $N$ , we sample  $2 \times 10^6$  realizations of non-Hermitian random matrices in class  $\text{AII}^\dagger$ . In all cases, we use  $g = 2$ . The solid markers show the ensemble-averaged characteristic polynomials. The black dashed curves show the same quantity calculated by the NL $\sigma$ M for large  $N$  in Eq. (81).

The combinatorial factor to be expanded in small  $n$  is then

$$F_n^p = \frac{\Gamma(m+1)}{\Gamma(p+1)\Gamma(m-p+1)} \prod_{k=m-p+1}^m \Gamma\left(\frac{k}{2}\right)^{-1} \Gamma\left(1 + \frac{k}{2}\right)^{-1}. \quad (85)$$

At order  $n^2$ , the non-zero terms are  $p = 0, 1, 2$ . The relevant  $x$ -integrals for  $p = 1$  and  $2$  are respectively

$$\int_0^\infty dx_1 x_1^{-\frac{1}{2}} e^{-tx_1} = \sqrt{\frac{\pi}{t}}, \quad (86)$$

$$\int_0^\infty dx_1 dx_2 (x_1 x_2)^{-\frac{1}{2}} e^{-t(x_1+x_2)} |x_1 - x_2| = \frac{2}{t^2}. \quad (87)$$

Thus, to leading order in  $t$ , we have

$$I_m(t) \simeq 1 - \frac{m^2}{2} \ln t - \frac{m^2 \pi}{4} \frac{e^{-t}}{t} + \frac{m^2}{16} \frac{e^{-2t}}{t^4} + \mathcal{O}(m^3). \quad (88)$$

Finally, using Eq. (44), we obtain the two-point correlation function to leading order in  $|\omega|^2$ :

$$\pi^2 R_2(\omega) \simeq 4 - \pi |\omega|^2 e^{-|\omega|^2}. \quad (89)$$

This is compatible with the saddle-point integral in Ref. [75] (see Appendix A for details). For comparison, we also analytically present the two-point correlation function  $R_2(z_1, z_2)$  for  $4 \times 4$  non-Hermitian random matrices in class  $\text{AII}^\dagger$  in Appendix B. In Fig. 8, we compare the analytical result in Eq. (89) and the numerically generated two-point correlation function. As in the case of class  $\text{AI}^\dagger$ , the range of the validity of Eq. (89) may hinder a direct comparison with the numerical results. However, once again we note that Eq. (81) is valid even when the separation  $|z_1 - z_2|$  is small. Therefore, it can be helpful in elucidating the correlations of nearby eigenvalues in class  $\text{AII}^\dagger$  in future work.

#### IV. DISCUSSIONS

In this work, we investigated the spectral properties of Gaussian non-Hermitian random matrices in symmetry classes  $\text{AI}^\dagger$  and  $\text{AII}^\dagger$ . Using the fermionic replica NL $\sigma$ M, we computed the one-point and two-point characteristic polynomials. We compared our results with finite- $N$  numerical calculations and found good agreements. The method developed in this work can be applied to other symmetry classes of non-Hermitian RMT [72]. Taking the replica limit, we also calculated the density of states and the two-point correlation functions. There remains room for improvement on these calculations. For example, in the standard case of the Gaussian unitary ensemble in Hermitian RMT, the replica limit can be taken more systematically once we know the recursion relation satisfied by the replica

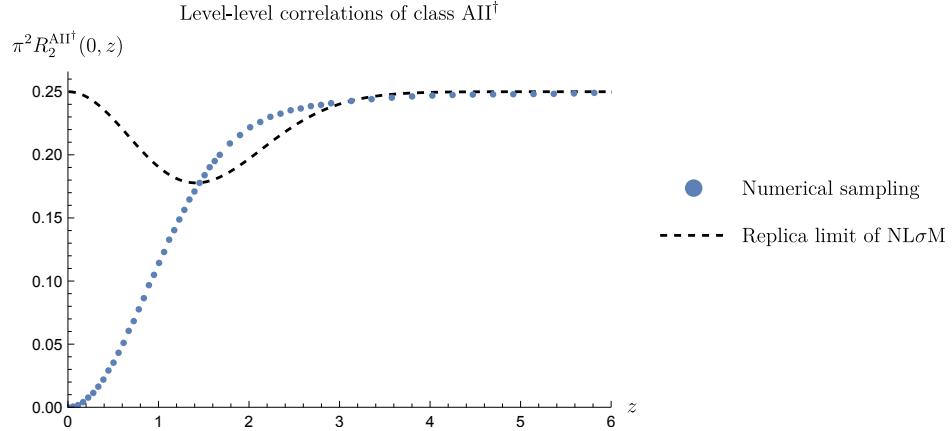


FIG. 8: Comparison between the analytical result in Eq. (89) and the two-point correlation function obtained from numerical calculations. The numerical results are obtained by  $10^4$  realizations of  $(2 \times 10^3) \times (2 \times 10^3)$  non-Hermitian random matrices in class  $\text{AII}^\dagger$  sampled according to Eq. (2) with  $g = 2$ .

partition functions with different replica indices, and also the partition functions with negative replica indices, which can be obtained from bosonic replica  $\text{NL}\sigma\text{M}$  [75–79]. Additionally, it should be useful to develop a supersymmetric  $\text{NL}\sigma\text{M}$  approach. Finally, it is interesting to explore applications of our results to physical systems, such as Lindblad superoperators.

*Note added.*—A part of this work was presented in Ref. [80]. When we were finalizing the draft, we learned about the related work [81].

## ACKNOWLEDGMENTS

We thank Ze Chen, Giorgio Cipolloni, and Jacobus Verbaarschot for helpful discussion. K.K. is supported by MEXT KAKENHI Grant-in-Aid for Transformative Research Areas A “Extreme Universe” No. 24H00945. S.R. is supported by Simons Investigator Grant from the Simons Foundation (Grant No. 566116). We gratefully acknowledge support from the Simons Center for Geometry and Physics, Stony Brook University at which some of the research for this work was performed.

## Appendix A: Saddle-point integral

### 1. Class $\text{AI}^\dagger$

We calculate the saddle-point integral using the results from Ref. [75], which computed the following sigma model matrix integral:

$$Y_n(\mu) = \int_{S(2n)/[S(n) \times S(n)]} \mathcal{D}W \exp \left[ -\frac{i\mu\alpha}{4} \text{Tr}(sW) \right]. \quad (\text{A1})$$

The group  $S(n)$  can be  $O(n)$ ,  $U(n)$ , or  $Sp(n)$ , corresponding to  $\alpha = 2$ ,  $\alpha = 2$ , or  $\alpha = 1$ , respectively. When we define  $W = U^\dagger sU$  and substitute  $\mu = 2ig^{-1}|\omega|^2$ , this becomes almost the same matrix integral in Eq. (33). We note, however, that  $\mu$  is assumed to be real in Ref. [75], while it should be imaginary in our calculations. Assuming that the measure is identical, we have the following expression for  $Z_n$  up to  $z$ -independent factors that reduce to 1 in the replica limit,

$$Z_n(z_1, \bar{z}_1, z_2, \bar{z}_2) = e^{4g^{-1}n|z|^2} Y_n(ig^{-1}|\omega|^2). \quad (\text{A2})$$

We rewrite  $R_2$  in terms of  $Y_n$  and  $\partial_\omega, \partial_z$ , and so on, and assume  $g = 1$  for simplicity, leading to

$$\begin{aligned}\pi^2 R_2(z_1, z_2) &= \lim_{n \rightarrow 0} \frac{1}{n^2} (\partial_\omega^2 - \partial_{2z}^2)(\partial_{\bar{\omega}}^2 - \partial_{2\bar{z}}^2) e^{4n|z|^2} Y_n(i|\omega|^2) \\ &= 2 + \lim_{n \rightarrow 0} \frac{1}{n^2} \partial_\omega^2 \partial_{\bar{\omega}}^2 Y_n(i|\omega|^2).\end{aligned}\quad (\text{A3})$$

Hence, we only need to keep terms of order  $n^2$  in  $Y_n$ . For the symplectic case, at large  $\mu$ , we use the  $\beta = 1$  result from Ref. [75]:

$$Y_{n \rightarrow 0}(\mu) = 1 - i\mu n + n^2 \left( -\frac{\mu^2}{2} - 2 \ln \mu + \frac{e^{2i\mu}}{4\mu^4} \right). \quad (\text{A4})$$

In our calculations, we instead need  $\mu = i\nu$ ,  $\nu \in \mathbb{R}_{>0}$ . The sign of the  $e^{2i\mu}/\mu^4$  term is ambiguous for  $\mu = i\nu$ . We here choose the sign based on a physical expectation and replace  $\ln \mu \rightarrow \ln |\mu|$ . Then, we have

$$Y_{n \rightarrow 0}(i\nu) = 1 + \nu n + n^2 \left( \frac{\nu^2}{2} - 2 \ln \nu - \frac{e^{-2\nu}}{4\nu^4} \right) \quad (\text{A5})$$

for  $\nu \in \mathbb{R}_{>0}$ . Using the above expression for  $Y_n$ , the term  $z^2 \partial_\omega^2 + \bar{z}^2 \partial_{\bar{\omega}}^2$  should vanish in the  $n \rightarrow 0$  limit. So we are left with

$$\begin{aligned}\pi^2 R_2(z_1, z_2) &= 2 + \partial_\omega^2 \partial_{\bar{\omega}}^2 \left( \frac{\nu^2}{2} - 2 \ln \nu - \frac{e^{-2\nu}}{4\nu^4} \right) \\ &= 4 - \frac{4e^{-2|\omega|^2}}{|\omega|^4} + \mathcal{O} \left( \frac{e^{-|\omega|^2}}{|\omega|^2} \right) + \mathcal{O} \left( \frac{e^{-2|\omega|^2}}{|\omega|^4} \right),\end{aligned}\quad (\text{A6})$$

where we introduce  $\omega = z_1 - z_2$  and  $\nu = g^{-1}\omega\bar{\omega}$ . While we here assume  $g = 1$  for simplicity, the  $g$  dependence is recovered as

$$\pi^2 R_2(z_1, z_2) = 4g^{-2} \left( 1 - \frac{e^{-2g^{-1}|\omega|^2}}{g^{-2}|\omega|^4} + \mathcal{O} \left( \frac{e^{-g^{-1}|\omega|^2}}{g^{-1}|\omega|^2} \right) + \mathcal{O} \left( \frac{e^{-2g^{-1}|\omega|^2}}{g^{-2}|\omega|^4} \right) \right). \quad (\text{A7})$$

## 2. Class AII<sup>†</sup>

For the orthogonal case, at large  $\mu$ , we use the  $\beta = 4$  result from Ref. [75].

$$Y_{2n \rightarrow 0}(\mu) = 1 - 2i\mu n + 4n^2 \left( -\frac{\mu^2}{2} - \frac{1}{2} \ln \mu + \Gamma^2(3/2) \frac{e^{2i\mu}}{2\mu} + \frac{e^{4i\mu}}{2^8 \mu^4} \right) \quad (\text{A8})$$

for  $\mu \in \mathbb{R}$ . As Eq. (A3), we rewrite the two-point correlation function in terms of  $z$  and  $\omega$  derivatives and assume  $g = 1$  for simplicity, leading to

$$\pi^2 R_2(z_1, z_2) = 2 + \lim_{n \rightarrow 0} \frac{1}{n^2} \partial_\omega^2 \partial_{\bar{\omega}}^2 Y_{2n}(i|\omega|^2/2). \quad (\text{A9})$$

Thus, we have

$$\begin{aligned}\pi^2 R_2(z_1, z_2) &= 2 + 4\partial_\omega^2 \partial_{\bar{\omega}}^2 \left( \frac{|\omega|^4}{8} - \frac{1}{2} \ln \left( \frac{i|\omega|^2}{2} \right) - \frac{\pi}{4} \frac{e^{-|\omega|^2}}{|\omega|^2} + \frac{e^{-2|\omega|^2}}{2^4 |\omega|^8} \right) \\ &= 4 - \pi|\omega|^2 e^{-|\omega|^2} + e^{-2|\omega|^2} \mathcal{O} \left( \frac{1}{|\omega|^6} \right).\end{aligned}\quad (\text{A10})$$

Putting the  $g$  dependence back by dimensional analysis, we have

$$\pi^2 R_2(z_1, z_2) = g^{-2} \left( 4 - \pi g^{-1} |\omega|^2 e^{-g^{-1}|\omega|^2} + e^{-2g^{-1}|\omega|^2} \mathcal{O} \left( \frac{1}{g^{-3} |\omega|^6} \right) \right). \quad (\text{A11})$$

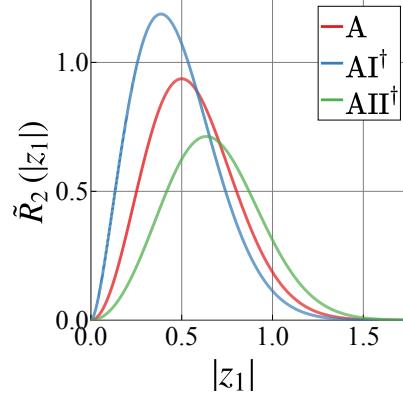


FIG. 9: Two-point correlation functions  $R_2(z_1, z_2)$  of  $2 \times 2$  ( $4 \times 4$ ) non-Hermitian random matrices in classes A and  $\text{AI}^\dagger$  ( $\text{AII}^\dagger$ ). The vertical axis  $\tilde{R}_2(|z_1|)$  is defined by  $R_2(z_1, z_2) := \delta(z_1 + z_2) \tilde{R}_2(|z_1|)$ . The normalization constants are chosen as  $C_2 = C_3 = C_5 = 1$ .

#### Appendix B: Two-point correlation functions of small non-Hermitian random matrices

We calculate the two-point correlation function of  $2 \times 2$  ( $4 \times 4$ ) non-Hermitian random matrices in classes A and  $\text{AI}^\dagger$  (class  $\text{AII}^\dagger$ ). In this case, non-Hermitian random matrices host only an opposite-sign pair of complex eigenvalues. The two-point correlation function reads

$$R_2(z_1, z_2) = \delta(z_1 + z_2) \langle \text{tr} \delta(z_1 - H) \rangle. \quad (\text{B1})$$

Here,  $\langle \text{tr} \delta(z_1 - z) \rangle$  depends only on  $|z_1|$  and is given as  $2 \langle \delta(z_1 - z) \rangle$  with an eigenvalue  $z \in \mathbb{C}$  of  $H$ . Hence, we have

$$R_2(z_1, z_2) = \frac{\delta(z_1 + z_2) \langle \delta(|z_1| - |z|) \rangle}{\pi |z_1|}. \quad (\text{B2})$$

On the other hand, the level-spacing distributions are defined as

$$p_s(s) := \langle \delta(s - 2|z|) \rangle. \quad (\text{B3})$$

Consequently, we have (Fig. 9)

$$R_2(z_1, z_2) = \frac{2\delta(z_1 + z_2) p_s(2|z_1|)}{\pi |z_1|} = \frac{2}{\pi} \delta(z_1 + z_2) \begin{cases} 16C_2^4 |z_1|^2 K_0(4C_2^2 |z_1|^2) & (\text{class } \text{AI}^\dagger); \\ 16C_3^4 |z_1|^2 e^{-4C_3^2 |z_1|^2} & (\text{class A}); \\ (16C_5^4 |z_1|^2 / 3) (1 + 4C_5^2 |z_1|^2) e^{-4C_5^2 |z_1|^2} & (\text{class } \text{AII}^\dagger), \end{cases} \quad (\text{B4})$$

with the modified Bessel function of the second kind,  $K_0(x)$ . Here, we use the analytical results of  $p_s(s)$  in Ref. [48] with arbitrary positive constants  $C_2, C_3, C_5 \geq 0$ . For  $|z_1|, |z_2| \ll 1$ , we have

$$R_2(z_1, z_2) \simeq \frac{2}{\pi} \delta(z_1 + z_2) \begin{cases} -16C_2^4 |z_1|^2 \ln(4C_2^2 |z_1|^2) & (\text{class } \text{AI}^\dagger); \\ 16C_3^4 |z_1|^2 & (\text{class A}); \\ 16C_5^4 |z_1|^2 / 3 & (\text{class } \text{AII}^\dagger). \end{cases} \quad (\text{B5})$$

- [1] E. P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, *Math. Proc. Cambridge Philos. Soc.* **47**, 790 (1951).
- [2] E. P. Wigner, On the Distribution of the Roots of Certain Symmetric Matrices, *Ann. Math.* **67**, 325 (1958).
- [3] G. Akemann, J. Baik, and P. Di Francesco, eds., *The Oxford Handbook of Random Matrix Theory*, Oxford Handbooks in Mathematics (Oxford University Press, Oxford, 2015).
- [4] S.-S. Byun and P. J. Forrester, *Progress on the Study of the Ginibre Ensembles* (Springer, 2024).

- [5] V. V. Konotop, J. Yang, and D. A. Zezyulin, Nonlinear waves in  $\mathcal{PT}$ -symmetric systems, *Rev. Mod. Phys.* **88**, 035002 (2016).
- [6] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Non-Hermitian physics and PT symmetry, *Nat. Phys.* **14**, 11 (2018).
- [7] J. Ginibre, Statistical Ensembles of Complex, Quaternion, and Real Matrices, *J. Math. Phys.* **6**, 440 (1965).
- [8] V. L. Girko, Circular Law, *Theory Probab. Appl.* **29**, 694 (1985).
- [9] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Grassmann integration in stochastic quantum physics: The case of compound-nucleus scattering, *Phys. Rep.* **129**, 367 (1985).
- [10] V. V. Sokolov and V. G. Zelevinsky, On a statistical theory of overlapping resonances, *Phys. Lett. B* **202**, 10 (1988).
- [11] V. V. Sokolov and V. G. Zelevinsky, Dynamics and statistics of unstable quantum states, *Nucl. Phys. A* **504**, 562 (1989).
- [12] F. Haake, F. Izrailev, N. Lehmann, D. Saher, and H.-J. Sommers, Statistics of complex levels of random matrices for decaying systems, *Z. Physik B* **88**, 359 (1992).
- [13] M. A. Stephanov, Random Matrix Model of QCD at Finite Density and the Nature of the Quenched Limit, *Phys. Rev. Lett.* **76**, 4472 (1996).
- [14] Y. V. Fyodorov, B. A. Khoruzhenko, and H.-J. Sommers, Almost-Hermitian random matrices: eigenvalue density in the complex plane, *Phys. Lett. A* **226**, 46 (1997).
- [15] J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. M. Verbaarschot, and A. Zhitnitsky, QCD-like theories at finite baryon density, *Nucl. Phys. B* **582**, 477 (2000).
- [16] N. Hatano and D. R. Nelson, Localization Transitions in Non-Hermitian Quantum Mechanics, *Phys. Rev. Lett.* **77**, 570 (1996).
- [17] K. B. Efetov, Directed Quantum Chaos, *Phys. Rev. Lett.* **79**, 491 (1997).
- [18] J. Feinberg and A. Zee, Non-hermitian random matrix theory: Method of hermitian reduction, *Nucl. Phys. B* **504**, 579 (1997).
- [19] K. B. Efetov, Quantum disordered systems with a direction, *Phys. Rev. B* **56**, 9630 (1997).
- [20] N. Hatano and D. R. Nelson, Vortex pinning and non-Hermitian quantum mechanics, *Phys. Rev. B* **56**, 8651 (1997).
- [21] P. W. Brouwer, P. G. Silvestrov, and C. W. J. Beenakker, Theory of directed localization in one dimension, *Phys. Rev. B* **56**, R4333(R) (1997).
- [22] J. Feinberg and A. Zee, Non-Hermitian localization and delocalization, *Phys. Rev. E* **59**, 6433 (1999).
- [23] S. Longhi, Topological Phase Transition in non-Hermitian Quasicrystals, *Phys. Rev. Lett.* **122**, 237601 (2019).
- [24] Q.-B. Zeng, Y.-B. Yang, and Y. Xu, Topological phases in non-Hermitian Aubry-André-Harper models, *Phys. Rev. B* **101**, 020201 (2020).
- [25] A. F. Tzortzakis, K. G. Makris, and E. N. Economou, Non-Hermitian disorder in two-dimensional optical lattices, *Phys. Rev. B* **101**, 014202 (2020).
- [26] Y. Huang and B. I. Shklovskii, Anderson transition in three-dimensional systems with non-Hermitian disorder, *Phys. Rev. B* **101**, 014204 (2020).
- [27] K. Kawabata and S. Ryu, Nonunitary Scaling Theory of Non-Hermitian Localization, *Phys. Rev. Lett.* **126**, 166801 (2021).
- [28] J. Claes and T. L. Hughes, Skin effect and winding number in disordered non-Hermitian systems, *Phys. Rev. B* **103**, L140201 (2021).
- [29] X. Luo, T. Ohtsuki, and R. Shindou, Universality Classes of the Anderson Transitions Driven by Non-Hermitian Disorder, *Phys. Rev. Lett.* **126**, 090402 (2021).
- [30] X. Luo, T. Ohtsuki, and R. Shindou, Transfer matrix study of the Anderson transition in non-Hermitian systems, *Phys. Rev. B* **104**, 104203 (2021).
- [31] X. Luo, Z. Xiao, K. Kawabata, T. Ohtsuki, and R. Shindou, Unifying the Anderson transitions in Hermitian and non-Hermitian systems, *Phys. Rev. Research* **4**, L022035 (2022).
- [32] H. Liu, J.-S. You, S. Ryu, and I. C. Fulga, Supermetal-insulator transition in a non-Hermitian network model, *Phys. Rev. B* **104**, 155412 (2021).
- [33] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).
- [34] N. Okuma and M. Sato, Non-Hermitian Topological Phenomena: A Review, *Annu. Rev. Condens. Matter Phys.* **14**, 83 (2023).
- [35] A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, *Phys. Rev. B* **55**, 1142 (1997).
- [36] F. Evers and A. D. Mirlin, Anderson transitions, *Rev. Mod. Phys.* **80**, 1355 (2008).
- [37] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Classification of topological quantum matter with symmetries, *Rev. Mod. Phys.* **88**, 035005 (2016).
- [38] F. Haake, S. Gnutzmann, and M. Kuš, *Quantum Signatures of Chaos* (Springer, Cham, 2018).
- [39] D. Bernard and A. LeClair, A Classification of Non-Hermitian Random Matrices, in *Statistical Field Theories*, edited by A. Cappelli and G. Mussardo (Springer, Dordrecht, 2002) pp. 207–214.
- [40] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and Topology in Non-Hermitian Physics, *Phys. Rev. X* **9**, 041015 (2019).
- [41] R. Grobe, F. Haake, and H.-J. Sommers, Quantum Distinction of Regular and Chaotic Dissipative Motion, *Phys. Rev. Lett.* **61**, 1899 (1988).
- [42] R. Grobe and F. Haake, Universality of cubic-level repulsion for dissipative quantum chaos, *Phys. Rev. Lett.* **62**, 2893 (1989).

- [43] Z. Xu, L. P. García-Pintos, A. Chenu, and A. del Campo, Extreme Decoherence and Quantum Chaos, *Phys. Rev. Lett.* **122**, 014103 (2019).
- [44] R. Hamazaki, K. Kawabata, and M. Ueda, Non-Hermitian Many-Body Localization, *Phys. Rev. Lett.* **123**, 090603 (2019).
- [45] S. Denisov, T. Laptyeva, W. Tarnowski, D. Chruściński, and K. Życzkowski, Universal Spectra of Random Lindblad Operators, *Phys. Rev. Lett.* **123**, 140403 (2019).
- [46] T. Can, V. Oganesyan, D. Orgad, and S. Gopalakrishnan, Spectral Gaps and Midgap States in Random Quantum Master Equations, *Phys. Rev. Lett.* **123**, 234103 (2019).
- [47] T. Can, Random Lindblad dynamics, *J. Phys. A* **52**, 485302 (2019).
- [48] R. Hamazaki, K. Kawabata, N. Kura, and M. Ueda, Universality classes of non-Hermitian random matrices, *Phys. Rev. Research* **2**, 023286 (2020).
- [49] G. Akemann, M. Kieburg, A. Mielke, and T. Prosen, Universal Signature from Integrability to Chaos in Dissipative Open Quantum Systems, *Phys. Rev. Lett.* **123**, 254101 (2019).
- [50] L. Sá, P. Ribeiro, and T. Prosen, Complex Spacing Ratios: A Signature of Dissipative Quantum Chaos, *Phys. Rev. X* **10**, 021019 (2020).
- [51] K. Wang, F. Piazza, and D. J. Luitz, Hierarchy of Relaxation Timescales in Local Random Liouvillians, *Phys. Rev. Lett.* **124**, 100604 (2020).
- [52] Z. Xu, A. Chenu, T. Prosen, and A. del Campo, Thermofield dynamics: Quantum chaos versus decoherence, *Phys. Rev. B* **103**, 064309 (2021).
- [53] A. M. García-García, Y. Jia, D. Rosa, and J. J. M. Verbaarschot, Dominance of Replica Off-Diagonal Configurations and Phase Transitions in a *PT* Symmetric Sachdev-Ye-Kitaev Model, *Phys. Rev. Lett.* **128**, 081601 (2022).
- [54] J. Li, T. Prosen, and A. Chan, Spectral Statistics of Non-Hermitian Matrices and Dissipative Quantum Chaos, *Phys. Rev. Lett.* **127**, 170602 (2021).
- [55] J. Cornelius, Z. Xu, A. Saxena, A. Chenu, and A. del Campo, Spectral Filtering Induced by Non-Hermitian Evolution with Balanced Gain and Loss: Enhancing Quantum Chaos, *Phys. Rev. Lett.* **128**, 190402 (2022).
- [56] A. M. García-García, L. Sá, and J. J. M. Verbaarschot, Symmetry Classification and Universality in Non-Hermitian Many-Body Quantum Chaos by the Sachdev-Ye-Kitaev Model, *Phys. Rev. X* **12**, 021040 (2022).
- [57] M. Prasad, H. K. Yadalam, C. Aron, and M. Kulkarni, Dissipative quantum dynamics, phase transitions, and non-Hermitian random matrices, *Phys. Rev. A* **105**, L050201 (2022).
- [58] L. Sá, P. Ribeiro, and T. Prosen, Lindbladian dissipation of strongly-correlated quantum matter, *Phys. Rev. Research* **4**, L022068 (2022).
- [59] A. Kulkarni, T. Numasawa, and S. Ryu, Lindbladian dynamics of the Sachdev-Ye-Kitaev model, *Phys. Rev. B* **106**, 075138 (2022).
- [60] A. M. García-García, Y. Jia, D. Rosa, and J. J. M. Verbaarschot, Replica symmetry breaking in random non-Hermitian systems, *Phys. Rev. D* **105**, 126027 (2022).
- [61] G. Cipolloni and J. Kudler-Flam, Entanglement Entropy of Non-Hermitian Eigenstates and the Ginibre Ensemble, *Phys. Rev. Lett.* **130**, 010401 (2023).
- [62] Z. Xiao, K. Kawabata, X. Luo, T. Ohtsuki, and R. Shindou, Level statistics of real eigenvalues in non-Hermitian systems, *Phys. Rev. Research* **4**, 043196 (2022).
- [63] S. Shivam, A. De Luca, D. A. Huse, and A. Chan, Many-Body Quantum Chaos and Emergence of Ginibre Ensemble, *Phys. Rev. Lett.* **130**, 140403 (2023).
- [64] S. Ghosh, S. Gupta, and M. Kulkarni, Spectral properties of disordered interacting non-Hermitian systems, *Phys. Rev. B* **106**, 134202 (2022).
- [65] L. Sá, P. Ribeiro, and T. Prosen, Symmetry Classification of Many-Body Lindbladians: Tenfold Way and Beyond, *Phys. Rev. X* **13**, 031019 (2023).
- [66] K. Kawabata, A. Kulkarni, J. Li, T. Numasawa, and S. Ryu, Symmetry of Open Quantum Systems: Classification of Dissipative Quantum Chaos, *PRX Quantum* **4**, 030328 (2023).
- [67] G. Cipolloni and J. Kudler-Flam, Non-Hermitian Hamiltonians violate the eigenstate thermalization hypothesis, *Phys. Rev. B* **109**, L020201 (2024).
- [68] K. Kawabata, Z. Xiao, T. Ohtsuki, and R. Shindou, Singular-Value Statistics of Non-Hermitian Random Matrices and Open Quantum Systems, *PRX Quantum* **4**, 040312 (2023).
- [69] Z. Xiao, R. Shindou, and K. Kawabata, Universal hard-edge statistics of non-Hermitian random matrices, *Phys. Rev. Research* **6**, 023303 (2024).
- [70] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1996).
- [71] S. M. Nishigaki and A. Kamenev, Replica treatment of non-Hermitian disordered Hamiltonians, *J. Phys. A* **35**, 4571 (2002).
- [72] Z. Chen, K. Kawabata, A. Kulkarni, and S. Ryu (in preparation).
- [73] A. Edelman and S. Jeong, Fifty Three Matrix Factorizations: A Systematic Approach, *SIAM J. Matrix Anal. Appl.* **44**, 415 (2023).
- [74] Y. Liu, J. Kudler-Flam, and K. Kawabata, Symmetry classification of typical quantum entanglement, *Phys. Rev. B* **108**, 085109 (2023).
- [75] I. V. Yurkevich and I. V. Lerner, Nonperturbative results for level correlations from the replica nonlinear  $\sigma$  model, *Phys. Rev. B* **60**, 3955 (1999).
- [76] J. J. M. Verbaarschot and M. R. Zirnbauer, Critique of the replica trick, *J. Phys. A* **18**, 1093 (1985).
- [77] A. Kamenev and M. Mézard, Wigner-Dyson statistics from the replica method, *J. Phys. A* **32**, 4373 (1999).

- [78] E. Kan zieper, Replica Field Theories, Painlevé Transcendents, and Exact Correlation Functions, *Phys. Rev. Lett.* **89**, 250201 (2002).
- [79] K. Splittorff and J. J. M. Verbaarschot, Replica Limit of the Toda Lattice Equation, *Phys. Rev. Lett.* **90**, 041601 (2003).
- [80] A. K. Kulkarni, *Internal symmetry and dynamical phenomena in open quantum systems* (Non-Hermitian topology, geometry and symmetry across physical platforms, October 7th, 2024).
- [81] G. Akemann, N. Aygün, M. Kieburg, and P. Päßler, Complex symmetric, self-dual, and Ginibre random matrices: Analytical results for three classes of bulk and edge statistics, [arXiv:2410.21032](https://arxiv.org/abs/2410.21032).