CMPT 125: Introduction to Computing Science and Programming II

Spring 2023

Week 5: Big-O notation, searching algorithms

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Fact of the day

The first hard drive made was in 1979 and could only hold 5MB of data

Recap from Last Lecture

- Measuring performance of algorithms
  - #1 Criteria: It has to be correct
  - #2 Criteria: easy to code & debug, needs less memory, takes less time, …etc.
- Big-O notation
  - A way to compare estimate of critical operations of algorithm to reference functions
  - Typically in terms of the input size

Review from Last Lecture (I)

- What happens to the time complexity if the for loop in Line 5 (and thus Line 6) in Example 3 of the Calculating Time Complexity slides is indented to be part of the for-loop in Line 3?

```
1       count = 0
2       for i in 1…n:
3         for j in 1…n:
4           count = count + 10
5           for k in 1…n:
6             count += 2
```

Recall:
- Count the number of times a critical operation is executed
- Disregard “constants”
- Disregard “lower exponent terms”

- Critical operations?
  - addition/assignments (lines 1,4&6)
- What are the constants?
  - Line 1
- How many addition/assignments that matters are executed?
  - $((n+1)* n) * n = n^3 + n^2$ (line 6 repeated n times in k loop, together with line 4 they repeat n times in j loop, all repeated n times in i loop)
- What is the time complexity?
  - $O(n^3)$

Review from Last Lecture (2)

- This is the pseudo-code for the **Selection Sort**. What is its time complexity?

```
1    for i = 0 to N-1
2      minIndex ← i
3      for j = i+1 to N-1
4        if A[j] < A[minIndex]
5          minIndex ← j
6        end if
7      end for
8      Swap A[i], A[minIndex]
9    end for
```

- Critical operations?
  - assignment, comparison, swap (lines 2, 4, 5, 8)
- What are the constants?
  - No constants
- How many assignments/comparisons/swap that matters are executed?
  - $[1 + (N-1)*2 + C] + [1 + (N-2)*2 + C] + [1 + (N-3)*2 + C] + \ldots + [1 + 0*2 + C]$
    $= 1*N + [(N-1)+(N-2)+(N-3)+\ldots+0] + C*N$
    $= (N+1)*N/2 + CN$
- What is the time complexity?
  - $O(N^2)$

A More Sophisticated Example – Checking Duplicates

outside loop: 3 ops

outer loop: 4*n ops (condition, assignment, increment)

```
//main idea: each round expand the search space by 1, check left
bool check_duplicates(const int* arr, int n) {
  int i=0, j;
  bool found = false;
  while (i<n && !found) {
    j = 0;
    while (j<i && !found) {
      if (arr[i] == arr[j]) {
        found = true;
      }
      j++;
    }
    i++;
  }
  return found;
}
```

inner loop: 4*i ops (condition, increment)

Outside loop: 3
Outer loop: 4n
Inner loop = 4*1 + 4*2 + 4*3+ … + 4*(n-1) = $2n^2$ -2n

Total: ≅ $2n^2 + 2n + 3 = O(n^2)$

Today

- Big-O notation (cont’d)
  - other mathematical variations
  - ways to derive time complexity on recursive algorithms
- Complexity analysis of sorting algorithms
  - Selection Sort
  - Mergesort
- Complexity analysis of searching algorithms
  - Linear Search
  - Binary Search, under one condition

Other Variations of the Big-O Mathematical Definition

- Let f(N) and g(N) are two functions (mathematical functions, not programming functions) with N being integers

The original

f = O(g) if there exists a large enough constant C such that f(N) <= C*g(N) for all sufficiently large N

- This is equivalent to say (this allows us to find C easier in some cases):

f = O(g) if there exists a large enough constant C such that f(N)/g(N) <= C for all sufficiently large N

- Or:

f = O(g) if $\lim_{N \to \infty} \left(\frac{f(N)}{g(N)} \right) < \boldsymbol{infinity}$

Some Simple Rules for Deriving Big-O Mathematically

- For fixed values a & b where $0 < a < b$ (e.g., $a=2$, $b=4$)
  - $N^a = O(N^b)$ (but the reverse is NOT true!)
- $\log(N)$ is smaller than any positive power of N, i.e., $\log(N) = O(N^a)$, even when a is < 1
- If you see $\log_2(N)$, or with any base, you can just write $\log(N)$, because $\log_2(N) = \log_m(N) * \log_2(m)$
- If $f = O(g)$ and $g = O(h)$, then $f = O(h)$
  - Further more, $f + g = O(h)$, because $f + g \le 2\max(f, g) = O(\max(f, g))$
- If $f_1 = O(g)$ and $f_2 = O(g)$, then $f_1 + f_2 = O(g)$

One More Example

- Suppose $f(N) = 10*2^N + 4N^4 + 3$, prove that $f = O(2^N)$
  Use the fact that $N^4 < 2^N$ for all $N>20$

- Prove:
  $f(N) = 10*2^N + 4N^4 + 3 < 10*2^N + 4*2^N + 3*2^N$ (for $N>20$) $= 17*2^N$
  That is, there exists a large enough constant, $C=17$, that results in $f(N) <= C* 2^N$ for all sufficiently large $N>20$.
  Therefore $f = O(2^N)$

Why So Much Math??

- Computer Science and Mathematics have a lot of overlaps
  - You use math to calculate complexity of algorithms so you can tell if which one is good at what situation
  - When you have an algorithm, you can often use math/logic to prove its correctness (e.g., induction)
  - In many cases what your computer is doing is just calculations, for example:
    - Rotating an image (transforming pixel locations)
    - Encrypting/Decrypting data (a series of math operations)
    - Machine learning (extract patterns and calculate the most likely match)
- If you understand math, you can understand CS better

Big-O Notations for Recursive Algorithms

- We can use **algebraic mechanisms** to find the time complexities of recursive algorithms
- For example, the time complexity of our recursive Selection Sort can be expressed as:

$$T(N) = N + C + T(N-1)$$

- where N is the search space (number of elements) and C is a constant including all other operations
- The recursive call works on 1 less element, hence N-1

```c
void selectionSortRecursive(int array[], unsigned size, unsigned int start) {
    //only need to sort when there is more than 1 element (start is at most size-1)
    if (size-start > 1) {
        unsigned int minIndex = start;
        for (int i=start+1; i<size; i++) {
            if (array[i] < array[minIndex]) {
                minIndex = i;
            }
        }

        //swap the smallest value to the front of the search space
        int temp = array[minIndex];
        array[minIndex] = array[start];
        array[start] = temp;
        printf("swapping %d with %d\n", array[minIndex], array[start]);

        //repeat starting from the next element
        selectionSortRecursive(array, size, start+1);
    }
}
```

Big-O Notations for Recursive Algorithms (Cont'd)

$$
\begin{aligned}
T(N) &= N + C + T(N\text{-}1) \\
&= N + C + [(N\text{-}1) + C + T(N\text{-}2)] \\
&= N + C + (N\text{-}1) + C + [(N\text{-}2) + C + T(N\text{-}3)] \\
&= \ldots \\
&= N + C + (N\text{-}1) + C + (N\text{-}2) + C + \ldots + [(N\text{-}(N\text{-}2)) + C + T(N\text{-}(N\text{-}1))] \\
&= N + (N\text{-}2) + (N\text{-}3) + \ldots + 2 + (N\text{-}1)*C + T(1) \\
&= N(N+1)/2 - 1 + CN - C + T(1) \\
&= O(N^2)
\end{aligned}
$$

Drop the multiplicative constant

Drop the lower order terms

Mergesort Time Complexity

- Recall Mergesort is a recursive algorithm

```
mergeSort(array)
    if (array has 2 or more elements):
        sortedLeft = mergeSort(left half of array)
        sortedRight = mergeSort(right half of array)
        result = merge(sortedLeft , sortedRight)
    else:
        result = array  # the array is already sorted, 1 element only
    return result
```

- The merge part takes O(N) of time because it simply scans the 2 smaller sorted arrays and fill up the larger array; and when N is 1, it is the base case where nothing needs to be sorted, thus takes O(1) of time

Mergesort Time Complexity (Cont'd)

- The time complexity of Mergesort can be expressed as

$T(N) = 2T(N/2) + N + C$ (where N is the number of elements in the current call of the function, C is a constant)

$= 2[2T(N/2^2) + N/2 + C] + N + C$

$= 2^2T(N/2^2) + N + 2C + N + C$

$= 2^2[2T(N/2^3) + N/2^2 + C] + N + 2C + N + C$

$= 2^3T(N/2^3) + N + 2^2C + N + 2C + N + C$

$= \ldots$

$= 2^kT(N/2^k) + N + 2^{k-1}C + \ldots + N + 2^2C + N + 2C + N + C$

$= 2^kT(N/2^k) + kN + (2^{k-1} + \ldots + 1)*C$

Mergesort Time Complexity (Cont'd)

- The time complexity of Mergesort can be expressed as

$T(N) = 2T(N/2) + N + C$ (where N is the number of elements in the current call of the function, C is a constant)

$= 2[2T(N/2^2) + N/2 + C] + N + C$

$= \ldots$

$= 2^kT(N/2^k) + N + 2^{k-1}C + \ldots + N + 2^2C + N + 2C + N + C$

$= 2^kT(N/2^k) + kN + (2^{k-1} + \ldots + 1)*C$

Let $N = 2^k \rightarrow \log_2 N = k$, then

$T(N) = NT(N/N) + N\log_2 N + (2^k - 1)C$

$= NT(1) + N\log_2 N + (N - 1)C$

$= N * O(1) + N\log_2 N + (N - 1)C = O(N\log_2 N)$ (or just $O(N\log N)$)

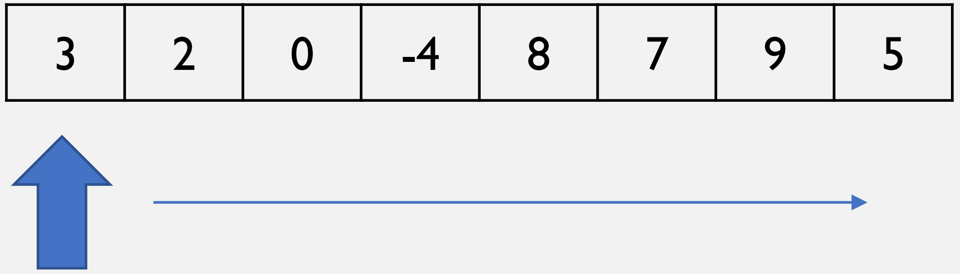
Searching Data in A Data Set

- It is a very common thing to do in computing science when handling a set of data
  - check if a record exists
  - look for duplicates
  - information lookup
  - …etc.
- Data are typically stored in a data structure, which may or may not have some special ordering
  - sorted from small to large or large to small based on a certain attribute
  - store based on time of insert
  - …etc.
- For now we assume data are stored in arrays, and we call the look up value "key"

Linear Search (Boolean)

Main idea: go through the array, upon finding a match with the key, return **true**. If reaches the end of the array, it means there is no match, return **false**.

```
for i = 0 to N-1
  if array[i] == key
    return true
  end if
end for
return false
```



Linear Search (Index)

Main idea: go through the array, upon finding a match with the key, return **index**. If reaches the end of the array, it means there is no match, return **-1**.

```
for i = 0 to N-1
  if array[i] == key
    return i
  end if
end for
return -1
```

Linear Search Time Complexity

- As mentioned in the previous lecture, some algorithms have different time complexities depending on the case
- Best case: every time the **first** array item matches with the key → O(1) (it is equivalent to "return array[0]")
- Worst case: everything the **last** array item matches with the key, or **no item** matches with the key → O(N)

```
for i = 0 to N-1
  if array[i] == key
    return i
  end if
end for
return -1
```

- We typically choose to report the worst case for a more conservative analysis (after all the best case doesn't happen that often, it's more likely that the match is randomly located in the array, so roughly N/2 comparisons)

Linear Search Is Slow

- The Linear Search algorithm is considered as a slow searching algorithm because it essentially is looking at each data once (it's considered the lower bound because one can't do worse than that)

- What if instead of a disorganized data set you have **an array where the items are sorted**?
  - Intuitively it'll be faster because you have a rough idea on which part of the array to look, e.g., if the key is small, the match can't be at the back in an ascendingly sorted array
    - The question is how much "back" you can skip looking for a match

Introducing Binary Search

- Suppose we have a pre-sorted (smallest to largest) array and we want to search with a key = 17

-2	-1	8	14	17	23	29	37	74	75	81	87	95

- Half of the array can be skipped by:
  - Check the middle item (if there are even number of items use the smaller one): 29
  - If the key is smaller than 29, then the item has to be in the left part (let's call this part the "active part")
    - otherwise the key is larger than 29, then the item has to be in the right part; or is a match, then we stop

Introducing Binary Search (Cont'd)

Main idea: Start with the whole array as the active part, look at the middle of the active part, if it is a match, done. Otherwise, decide which half is the new active part by comparing the middle item with the key and continue until there is no more items in the active part (not found).

-2	-1	8	14	17	23	29	37	74	75	81	87	95
-2	-1	8	14	17	23	29	37	74	75	81	87	95
-2	-1	8	14	17	23	29	37	74	75	81	87	95

Pseudo-code for Binary Search

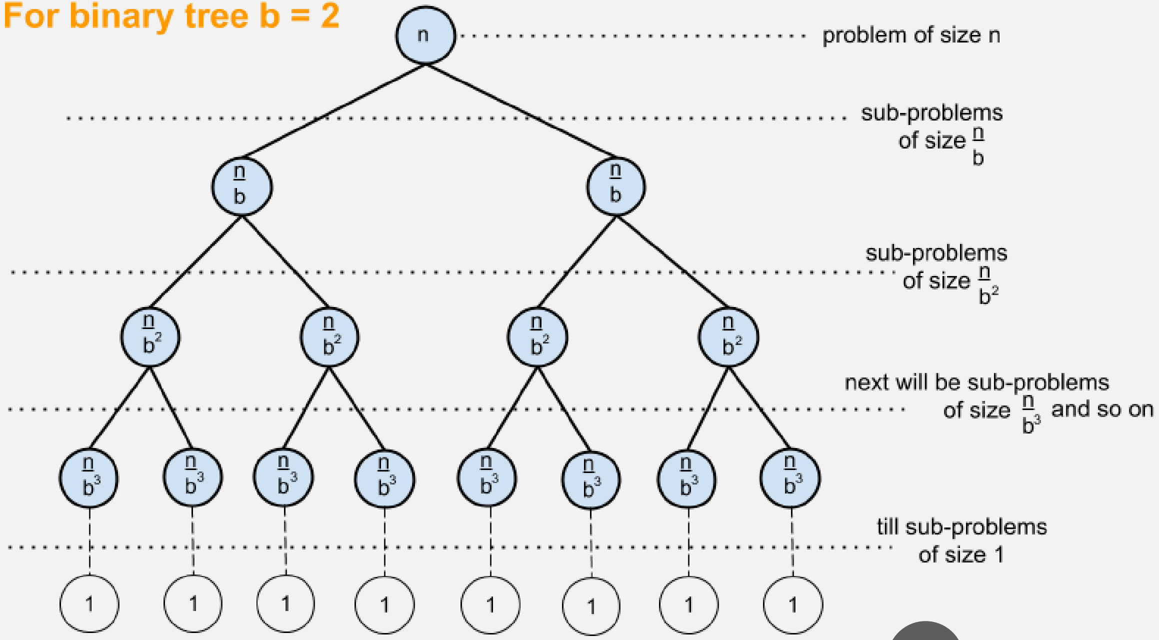
Assume array is ascendingly sorted. Let L and R the left/right boundary indexes of the active part (start as L = 0 and R = N-1)

```
while L <= R
  mid ← (L + R)/2
  if array[mid] == key
    return mid
  else if array[mid] < key
    L ← mid + 1
  else
    R ← mid - 1
  end if
end while
return -1
```

make the right half as the active part

make the left half as the active part

For binary tree b = 2



Binary Search Complexity

- In each iteration, the size of the active part is reduced by roughly half, and the loop ends when there is no more items to compare (unless the key has a match somewhere)
  - There is only a constant amount of critical operations in each iteration (say, C)
- Hence the total number of critical operations can be expressed as: C * number of iterations
  - This is equivalent to ask how many times can we divide the input size N by 2 until it becomes 1 (or 0)

$$\frac{N}{2^x} = 1$$

$$N = 2^x$$

$$x = \log_2 N$$

- Therefore, time complexity of Binary Search = O(logN)

Binary Search Only Works on Sorted Arrays

- Generally speaking, a comparison-based sorting algorithm can only be as fast as O(nlogn). So technically, to perform a binary search on an unsorted array of items the time complexity is O(nlogn) + O(logn) = O(nlogn)
- So why can't we just use the O(n) Linear Search Algorithm?

- Let's say you need to search 10000 times within 256 items
  - then you'll have 1 O(nlogn) sort + 10000 O(logn) searches ≈ 256*8 + 10000*8 ≈ 80k operations
  - instead of 10000 O(n) searches ≈ 10000*256 ≈ 2560k operations
- The efficiency comes from sorting a more stable set of data once and do an efficient search many times

Extra!

- Remember the fib(n) recursive function?
- Time complexity can be expressed by: $T(n) = T(n-1) + T(n-2) + C$

$T(n) = [T(n-2) + T(n-3) + C] + [T(n-3) + T(n-4) + C] + C$ ← this is very tedious ☹

- Let's consider T(n-1) and T(n-2), we can assume that T(n-1) has more operations to do than T(n-2), so

$T(n) > 2T(n-2) + C = 2[2T(n-4) + C] + C = 2^2[2T(n-6) + C] + 2C + C = \ldots = 2^kT(n-2k) + (2^k-1)C = O(2^{n/2})$

- On the other hand, we can also write this:

$T(n) < 2T(n-1) + C = 2[2T(n-1) + C] + C = 2^2[2T(n-2) + C] + 2C + C = \ldots = 2^kT(n-k+1) + (2^k-1)C = O(2^n)$

- We can conclude that $O(2^{n/2}) < T(n) < O(2^n)$, and thus T(n) is $O(2^n)$ (i.e., grows exponentially)

10Min Break And Practice

- Sort the functions in the increasing order:
  - $f_1(N) = N^2 + 100N$
  - $f_2(N) = 2^N + N^6 + 100N$
  - $f_3(N) = N^3 \log(N) + 400N^2$
  - $f_4(N) = 2N^3 + 100N + 10^8$
  - $f_5(N) = (\log(N))^{15}$
  - $f_6(N) = 99N + \log(N) + 4^N$
  - $f_7(N) = N \log(N) + 100N$
  - $f_8(N) = \log(N/2)$

Practice Answers

Sort the functions in the increasing order:

- $f_1(N) = N^2 + 100N$ - $O(N^2)$
- $f_2(N) = 2^N + N^6 + 100N$ - $O(2^N)$
- $f_3(N) = N^3 \log(N) + 400N^2$ - $O(N^3 \log(N))$
- $f_4(N) = 2N^3 + 100N + 10^8$ - $O(N^3)$
- $f_5(N) = (\log(N))^{15}$ - $(\log_{10}(N))^{15} << N \rightarrow O(N)$
- $f_6(N) = 99N + \log(N) + 4^N$ - $O(4^N)$
- $f_7(N) = N \log(N) + 100N$ - $O(N \log(N))$
- $f_8(N) = \log(N/2)$ - $O(\log(N))$

$f_8 < f_5 < f_7 < f_1 < f_4 < f_3 < f_2 < f_6$

Go to https://www.desmos.com/ and draw all these functions

Live Code Demo

- Function pointers (with arrays)
  - Implement the filter function
- File I/O
  - Using fscanf and fgets

Today's Review

- Big-O notation (cont'd)
  - other mathematical variations
  - using algebraic mechanics to derive time complexity of recursive algorithms
- Sorting algorithms
  - an $O(N^2)$ way (Selection Sort)
  - an $O(nlogn)$ way (Mergesort)
- Searching algorithms
  - an $O(n)$ way (Linear Search)
  - an $O(logn)$ way (Binary Search, under one condition)

Homework!

- Linear Search typically returns upon the first match. What if you want a Linear Search function that finds all the matches are? How do you write it in C?
- Suppose instead of ascendingly sorted we have an array of descendingly sorted items, how would you change the Binary Search algorithm to keep it working?
- Binary Search can also be written as a recursive function, try to write the code for it!

- Watch a video for a more detailed proof for time complexity of the Mergesort
  https://stream.sfu.ca/Media/Play/70de71db57474331a42982630605837d1d
- Watch a visualization of the Binary Search algorithm
  https://www.cs.usfca.edu/~galles/visualization/Search.html