CHAPTER 11

Portfolio Constraints and Information Ratio

BESIDES THE PORTFOLIO TURNOVER CONSTRAINTS discussed in Chapter 8, there are other forms of portfolio constraints that portfolio managers in practice have to abide by. One such form of constraint is risk exposure constraint. We have discussed this when we developed the risk-adjusted information coefficients, which analyzed factors with their exposure to risk factors being neutralized. The reason for neutralizing or limiting exposure to these factors, such as market, size, growth, etc. (see Chapter 3 for more), is to control systematic risk of active portfolios and to generate excess returns that are stock specific and have low correlation with market returns.

Another form of constraint is the holding constraint for stocks, which has several variations. For example, one can require that any individual stock holding in a portfolio be no more than a certain percentage of the portfolio. In terms of active weights, one can require that any individual active weight be less than a certain percentage. These constraints are aimed at controlling the specific risk of individual holdings and limiting the damage that the poor performance of any single stock to the total portfolio. Holding constraints can also be placed on an aggregated basis such as sector bounds for an active portfolio. A typical sector constraint can be ±2% for sector bets, and for a global equity portfolio, it can be ±2% for country bets.

However, by far the most prevalent form of holding constraint is the long-only constraint, which requires portfolios to be long in all stocks, i.e., the weights have to be nonnegative. In other words, it prohibits one from shorting stocks. Thus, the constraint is often referred to as a *no-short rule*. In the U.S. and around the world, the overwhelming majority of equity portfolios were managed as long-only products before equity long-short hedge funds became more acceptable in the late 1990s and early 2000s, even though they had existed since the 1960s. However, these hedge funds are generally only available to institutional investors and high-net-worth individuals. Mutual funds, which are a typical choice for most retail investors, are still almost exclusively long-only funds. Given the influence of the long-only constraints in the investment industry, one can ask: "Is the no-short rule a good rule?"

Generally, the answer is no, because it hinders managers' ability to generate excess returns. However, to some, shorting is associated with leverage and even appears unpatriotic. From a risk perspective, shorting stock outright can be a risky proposition. In contrast to buying a stock, where one can only lose 100% of the investment, shorting stock can lead to losses well above the initial investment[1]. However, these risks are well controlled in a risk-managed portfolio.

The no-short rule limits investment opportunities to generate returns. Consider the goal of active investment: beating the market-cap weighted benchmark subject to typical tracking error constraints. The cap-weighted index Goliaths are heavily weighted toward a set of large cap stocks. For example, the largest 4% of the Standard & Poor's (S&P) 500 names comprise about 70% of the index weight. In contrast, the smallest 25% comprise only 4% of the index. If the active manager's skill ability is equal across all cap ranges, how can he win? He cannot efficiently express his beliefs in specific stocks. With notional limits (no negative weights) on many of the "bad" ones, there is insufficient funding for the "good" ones. For example, managers can only underweight the small stocks by a few basis points (their weight in the index) when they have a negative forecast. This implies long-only managers can only add real value from their views on small stocks half of the time: when the forecast is positive! Given the fact that most capitalization-weighted benchmarks have a large portion of stocks with small benchmark weights, the impact of the long-only constraint on the portfolio return could potentially be significant. Thus, it is important for both portfolio managers and investors to analyze and estimate the magnitude of the likely impact.

A more recent solution is to make partial relaxation of the long-only constraint that resides in the traditional investment guideline. In this way, the resulting portfolios can invest in both long and short, and continue to manage against their respective benchmarks. We refer hereafter to these as *constrained long-short portfolios.* For example, the manager might buy a 125% exposure in long-equity positions and sell a 25% exposure in short-equity positions with the net result being 100% long systematic risk. However, the total leverage to the alpha source is 150% (125% long and 25% short). Although the constrained long-short portfolios might be suboptimal compared to the market neutral portfolio (with derivatives), it offers considerable benefit over “handcuffed” long-only portfolios.

We shall provide results on long-only and constrained long-short portfolios in this chapter. This analysis presents an analytical challenge because the long-only constraint, or range constraint on portfolio weights, is an inequality constraint. With equality constraints such as risk neutral or sector neutral, we can find exact solutions to the optimal long-short portfolio weights. Our analysis so far has been based on the long-short portfolio setting, and we can establish an analytical relationship between the risk-adjusted information coefficient (IC) and the portfolio excess return. In contrast, with an inequality constraint, an analytical solution for the optimal weights does not exist, and a solution can only be found through numerical means.

We present an efficient numerical method for solving the mean–variance optimization problems with range constraints, making it possible to analyze the impact of the long-only constraint, or any other form of range constraints, very efficiently. It can be seen that the impact varies with different factors, even though it is generally negative in the form of a lower information ratio (IR). A closely related question is, how IR improves as we loosen the long-only constraint to allow short positions.

11.1 SECTOR NEUTRAL CONSTRAINT

We first analyze the impact of the sector neutral constraint on alpha factors. As we stated earlier in Chapter 5, for value factors such as earnings yield or book-to-price, one typically needs to employ them on a sector-relative basis. There are at least two reasons for this. One is that some sectors, such as technology, always look more expensive than other sectors, such as utilities, due to their higher growth prospects. Therefore, using value factors without any adjustment would cause a permanent underweight in the technology sector and a permanent overweight in the utility sector. The second, but related, reason is that these factors appear to be much less effective in predicting sector returns than relative stock returns within sectors.

11.1.1 Return Decomposition

We can analyze a factor's sector selection and stock selection ability by decomposing its excess returns. From Chapter 4, Equation 4.19, we have

$$\alpha_t = \sum_{i=1}^{N} w_i r_i = \lambda^{-1} \sum_{i=1}^{N} F_i R_i \ , \tag{11.1}$$

where F is the risk-adjusted forecast, R is the risk-adjusted return, λ is the risk-aversion parameter that calibrates the targeted tracking error, and N is the number of stocks. Suppose the stock universe consists of S sectors, $s = 1,2,\cdots,S$, and in sector s there are N_s stocks, such that

$$\sum_{s=1}^{S} N_s = N_1 + N_2 + \cdots N_S = N \ . \tag{11.2}$$

We can then rewrite (11.1) into a summation over sectors, i.e.,

$$\alpha_t = \lambda^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_s} F_{si} R_{si} \ , \tag{11.3}$$

where F_{si} and R_{si} are the risk-adjusted forecast and return of the i-th stock in s-th sector. We define the sector mean of forecasts and returns as

$$\bar{F}_s = \frac{1}{N_s} \sum_{i=1}^{N_s} F_{si}, \text{ and } \bar{R}_s = \frac{1}{N_s} \sum_{i=1}^{N_s} R_{si} \ . \tag{11.4}$$

The overall averages are given by

$$\bar{F} = \sum_{s=1}^{S} \frac{N_s}{N} \bar{F}_s, \text{ and } \bar{R} = \sum_{s=1}^{S} \frac{N_s}{N} \bar{R}_s \ , \tag{11.5}$$

and they are often close to zero in practice. Equation 11.3 can be written as

$$\begin{aligned} \alpha_t &= \lambda^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_s} \left(F_{si} - \bar{F}_s + \bar{F}_s\right)\left(R_{si} - \bar{R}_s + \bar{R}_s\right) \\ &= \lambda^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_s} \left[\left(F_{si} - \bar{F}_s\right)\left(R_{si} - \bar{R}_s\right) + \bar{R}_s\left(F_{si} - \bar{F}_s\right) + \bar{F}_s\left(R_{si} - \bar{R}_s\right) + \bar{F}_s \bar{R}_s\right] \end{aligned} \tag{11.6}$$

The second and third terms vanish by the definition of the averages. Therefore, we have

$$\alpha_t = \lambda^{-1} \sum_{s=1}^{S} \sum_{i=1}^{N_s} \left[\left(F_{si} - \bar{F}_s \right) \left(R_{si} - \bar{R}_s \right) \right] + \lambda^{-1} \sum_{s=1}^{S} N_s \bar{F}_s \bar{R}_s \,. \tag{11.7}$$

The interpretation of the first term is straightforward: it is the excess return generated by the sector-relative risk-adjusted forecast. The second term is related to the sector excess return, which can be rewritten as

$$\lambda^{-1} \sum_{s=1}^{S} N_s \bar{F}_s \bar{R}_s = \lambda^{-1} N \sum_{s=1}^{S} \frac{N_s}{N} \bar{F}_s \bar{R}_s \approx \lambda^{-1} N \sum_{s=1}^{S} \frac{N_s}{N} \left(\bar{F}_s - \bar{F} \right) \left(\bar{R}_s - \bar{R} \right). \tag{11.8}$$

Thus, it is proportional to a weighted covariance between the aggregated sector forecast and the aggregated sector return, or excess return generated by the forecast on a sector level. Hence, we can write the excess return as the sum of the sector-relative excess return and the sector excess return and use this framework to analyze individual alpha factors.

Example 11.1

Table 11.1 provides a simple illustration with two sectors and three stocks in each sector. In sector 1, stock 1 has the lowest forecast while stock 3 has the highest forecast. This is also true in sector 2. We observe that the actual returns in both sectors have the same ranking. Hence, we conclude that within each sector the forecasts must have positive excess returns. The average forecast is –1 for sector 1 and 1 for sector 2, respectively, predicting a higher return for sector 2; instead, the average return is 5% for sector 1 and –5% for sector 2. In this case, the prediction for sector returns is wrong. Note the following remark:

- The decomposition of excess return essentially involves the decomposition of the covariance between the forecasts and the actual returns. Similarly, the variance of active returns can be decomposed into (a) stock return variance within sectors and (b) sector return variance (see Problem 11.2). This decomposition can shed light on the relative investment opportunities in "pure" stock selection and in sector allocation. For global equity portfolios that are managed with country allocation and stock selection, a similar analysis applies.

TABLE 11.1 An Example of Two Sectors and Three Stocks in Each Sector

Sector	Stock	F	R (%)	$F - F_s$
1	1	−1.50	0.0	−0.50
1	2	−1.00	5.0	0.00
1	3	−0.50	10.0	0.50
2	1	0.50	−10.0	−0.50
2	2	1.00	−5.0	0.00
2	3	1.50	0.0	0.50

11.1.2 Sector Constraint on Individual Factors

Table 11.2 shows the empirical results for the set of quantitative factors outlined in Chapter 5. Portfolio alpha (overall) is decomposed into stock selection alpha and sector timing alpha according to Equation 11.7. IR is the ratio of average return divided by the standard deviation of returns for each of the three alpha streams through time.

In general, sector timing alpha is of the same sign as the stock selection alpha, meaning that taking sector bets does increase alpha. However, the levels of the two sets of IR are quite different, with the stock selection IR consistently higher than the sector timing IR. This indicates that quantitative factors are better at selecting stocks bottom-up than making top-down sector calls.

One factor warrants closer examination: the short-term price momentum reversal factor (ret1). The stock selection and sector timing alphas have different signs, and the short-term momentum reversal phenomenon is much more pronounced within each sector rather then within the whole market. The IR of ret1 without sector neutralization is 0.44 (using positive number for IR), whereas it is 0.76 with sector neutralization. More interestingly, short-term sector momentum actually exhibits continuation rather than reversal; that is, sectors that outperformed in the last month tend to be winners again in the next 3 months, whereas stocks that outperformed in the last month tend to be losers in the next 3 months. Note the following remark:

- In general, factors that forecast stock returns are not strong in determining sector returns. Hence, in order to build effective sector forecasting models and implement sector rotation strategies, one needs to search for additional factors and possibly alternative modeling processes.

TABLE 11.2 Empirical Result in the U.S. Market Using R3000 as the Universe

		Overall		Stock Selection		Sector Timing	
		Alpha	**IR**	**Alpha**	**IR**	**Alpha**	**IR**
Value	CFO2EV	6.67%	1.11	6.39%	0.94	0.27%	0.20
	EBITDA2EV	5.26%	0.73	4.73%	0.62	0.54%	0.41
	E2PFY0	3.90%	0.58	3.35%	0.47	0.56%	0.38
	E2PFY1	3.31%	0.37	2.84%	0.31	0.48%	0.36
	BB2P	2.65%	0.30	1.96%	0.25	0.69%	0.28
	BB2EV	4.24%	0.65	3.79%	0.64	0.45%	0.28
	B2P	1.43%	0.15	1.05%	0.11	0.38%	0.31
	S2EV	3.67%	0.40	3.44%	0.35	0.23%	0.19
Fundamental	RNOA	3.05%	0.42	2.83%	0.39	0.21%	0.18
	CFROI	5.43%	0.91	5.35%	0.97	0.08%	0.08
	OL	3.66%	0.91	3.62%	0.95	0.04%	0.04
	OLinc	3.60%	1.07	3.59%	1.04	0.02%	0.05
	Wcinc	–3.97%	–0.90	–3.92%	–0.89	–0.05%	–0.08
	NCOinc	–3.15%	–0.68	–3.04%	–0.66	–0.10%	–0.10
	icapx	–3.00%	–0.70	–2.95%	–0.70	–0.05%	–0.10
	capxG	–1.99%	–0.50	–2.00%	–0.50	0.01%	0.01
	XF	–4.50%	–0.95	–4.25%	–1.00	–0.25%	–0.18
	shareInc	–2.28%	–0.52	–2.07%	–0.52	–0.21%	–0.12
Momentum	ret1	–4.36%	–0.44	–6.60%	–0.76	2.24%	0.72
	ret9	2.95%	0.22	3.19%	0.25	–0.24%	–0.06
	adjRet9	6.29%	0.49	5.22%	0.51	1.08%	0.24
	earnRev9	3.90%	0.38	4.25%	0.56	–0.35%	–0.10
	earnDiff9	5.10%	0.46	5.52%	0.67	–0.42%	–0.11

11.2 LONG/SHORT RATIO OF AN UNCONSTRAINED PORTFOLIO

Before analyzing the impact of long-only and other types of range constraints, we will first study the long/short ratio of an unconstrained active portfolio vs. a benchmark, because it represents the optimal setting of generating excess returns. In this case, as the portfolio is unconstrained, the active portfolio should be just the long-short portfolio. The benchmark has no effect on the active portfolio, but it becomes relevant when we aggregate the active weight with the benchmark weights to obtain the total portfolio weights. The distribution of the benchmark weights plays a role in determining the long/short ratio of portfolios that are managed against that benchmark. Therefore, we will first examine that distribution empirically and present a statistical model for it.

11.2.1 Distribution of Benchmark Weights

Almost all capitalization-based benchmarks, to varying degrees, have more stocks with small weights than large weights. Over time, the distribution might change, for example, due to stocks' relative performance. However, the overall shape remains intact. Consider the S&P 500 index at February 2006. The stock with the largest weight was Exxon Mobil at 3.347%, and the stock with the smallest weight was Dana Corp (now bankrupt) at 0.006%, or 0.6 bps (basis points). The mean weight is 0.200%, whereas the median is 0.100%, demonstrating the skewness of the distribution. The top 10 names accounts for roughly 20% of the index weight, whereas the bottom half of the stocks accounts for only 13.5%. Figure 11.1 shows the histogram of the benchmark weights. It can be seen that there are only a handful of stocks with weights above 1%.

Another way of analyzing the distribution of benchmark weights is the cumulative sum of ranked stock weights. Figure 11.2 displays the sum as a function of the number of stocks included; the thick line is for the S&P 500 index, whereas the thin, dashed line is based on a fitted model with lognormal distributions that is described below. The function rises very rapidly at first and approaches 1 at a very slow rate in the end.

The model of the benchmark weights shown in Figure 11.2 is based on a lognormal distribution. For a random variable $x > 0$, it follows a lognormal distribution if ln(x) is normally distributed. The probability density is given by:

$$p\left(x\,|\,\mu,\sigma\right)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\left[-\frac{\left(\ln x-\mu\right)^2}{2\sigma^2}\right] \tag{11.9}$$

Figure 11.3a shows the probability density with $\mu=0$, and $\sigma=1.195$. The shape of the distribution resembles that of Figure 11.1, but the range is much too wide. The lognormal distribution, often used to model percentage changes in stock price, ranges from zero to infinity. As the benchmark weights are restricted to (0,1), we need to rescale the lognormal distribution to suit our purpose. If we rescale x by a factor of k, then the new density function should be

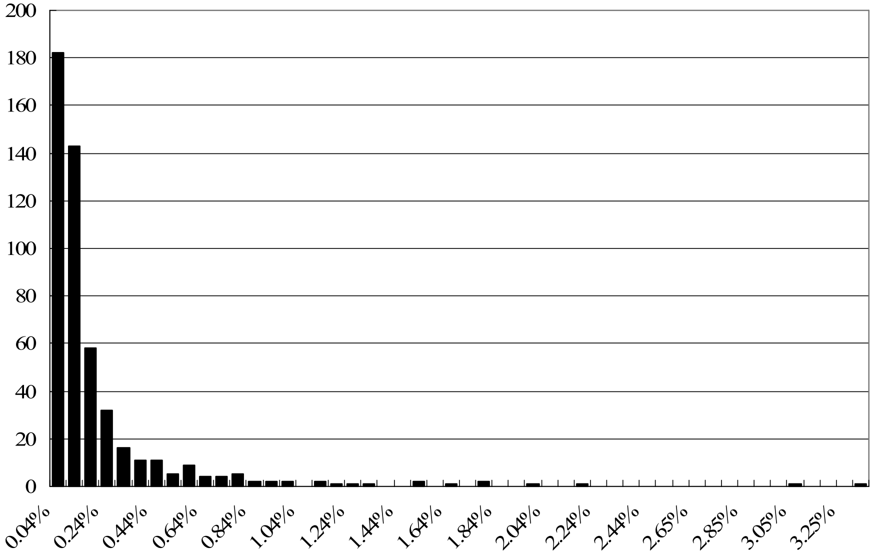


FIGURE 11.1. Histogram of benchmark weights in S&P 500 index as of February 2006.

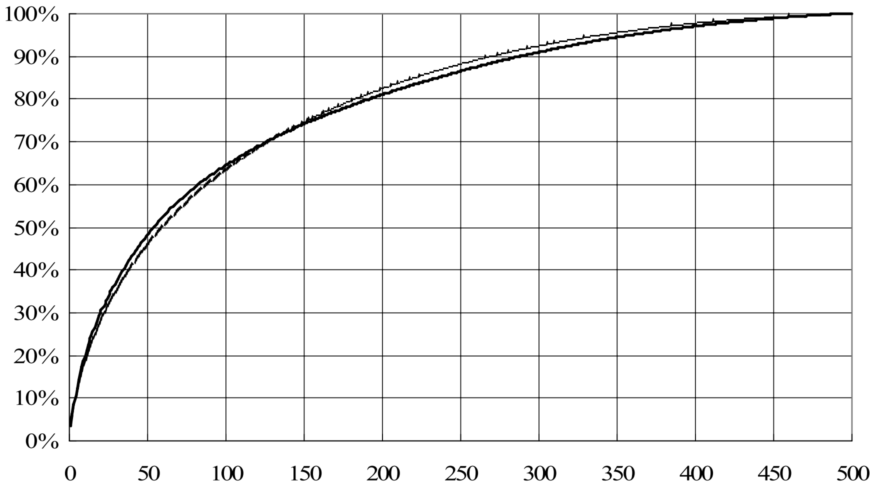
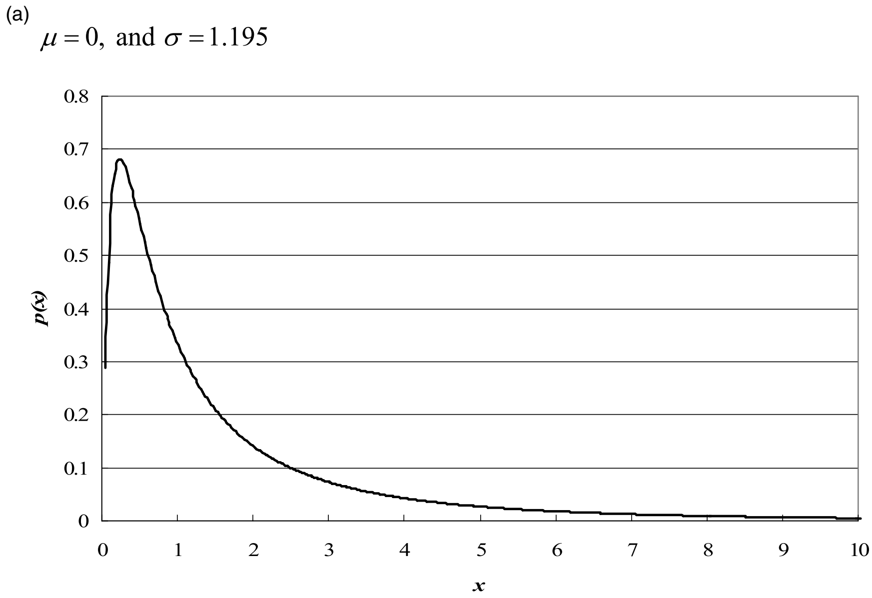


FIGURE 11.2. Cumulative weights of ranked benchmark weight: the solid line is for the S&P 500 index and the thin, dashed line is for the model.

$$\tilde{p}\left(x\,|\,\mu,\sigma,k\right)=k\cdot p\left(kx\,|\,\mu,\sigma\right). \tag{11.10}$$



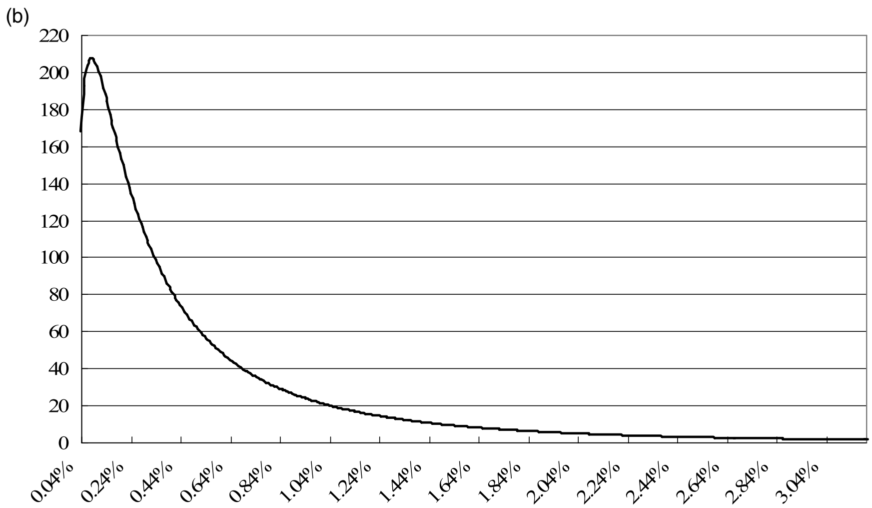


FIGURE 11.3. (a) Probability density function of the lognormal distribution function with $\mu = 0$, and $\sigma = 1.195$. (b) Scaled lognormal distribution of (a) with $k = 305$.

Figure 11.3b shows the scaled density function with factor k = 305. The graph now resembles the histogram of S&P 500 index weights in Figure 11.1.

11.2.2 Simulation of Benchmark Weights

Grinold and Kahn (2000) provided an algorithm to simulate benchmark weights based on a scaled lognormal distribution. For a given number of stocks N in the benchmark, a parameter c is used to characterize the concentration of the index. If $c = 0$, the index is equally weighted. As c increases, the index becomes more concentrated. The algorithm has four steps:

1. Discretize the probability interval (0,1) with $p_i = 1 - \frac{i - 0.5}{N}$, $i = 1, \cdots, N$.
2. Find the value of the standard normal variable that has the cumulative probability p_i, i.e., $y_i = \Phi^{-1}(p_i)$, where Φ^{-1} is the inverse of the cumulative density function.
3. Transform y_i to a lognormal variable using $s_i = \exp(cy_i)$, c being the concentration parameter.
4. Scale s_i to obtain benchmark weight $b_i = s_i \Big/ \sum_{i=1}^{N} s_i$.

Figure 11.4 shows the simulated benchmark weights for several values of c. The curves are the cumulative total of weights ranked in descending order. The curve for c = 0, i.e., an equally weight benchmark, is a straight line. As c increases, the benchmark becomes top heavy with a few stocks occupying more weight within the benchmark.

11.2.3 Long/Short Ratio of a Single Stock

Our approach to obtaining the long/short ratio of a portfolio is to calculate the long/short ratio of a single stock and then sum up across the benchmark. From Chapter 4, we know that the long-short portfolio weights are $w_i = \lambda^{-1} F_i / \sigma_i$, where F_i is the risk-adjusted forecast, σ_i is the stock-specific risk, and λ is the risk-aversion parameter. The risk-aversion parameter is related to the target tracking error by

$$\frac{1}{\lambda} = \frac{\sigma_{\text{target}}}{\sqrt{N}}. \tag{11.11}$$

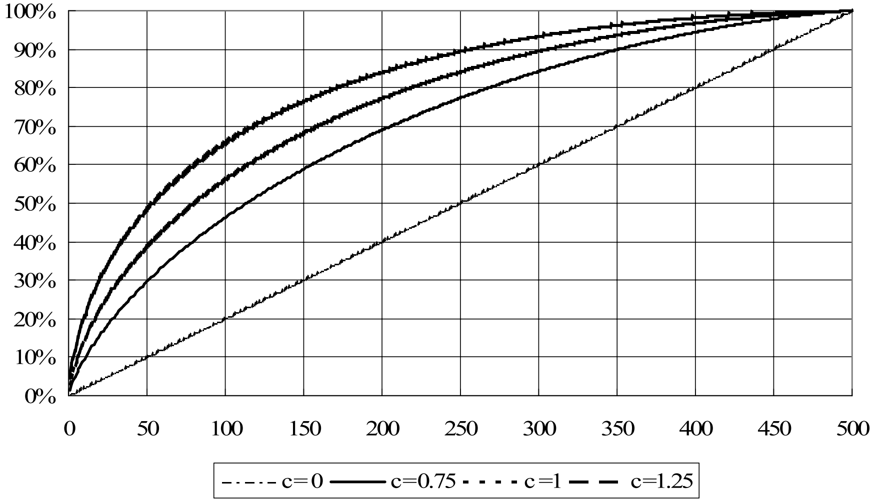


FIGURE 11.4. Cumulative weights of ranked benchmark stocks for different values of *c*.

We have assumed that the risk-adjusted forecast is standardized, i.e., dis(F) = 1 and is of zero mean, and N is the number of stocks. Hence the active weight is given by

$$w_i = \frac{\sigma_{\text{target}} F_i}{\sqrt{N}\sigma_i}. \tag{11.12}$$

The benchmark weights are b_i with $\sum_{i=1}^{N} b_i = 1$, and $b_i \geq 0$.

- Normally, benchmark weights are all positive. We will allow b_i to be zero if the stock is an out-of-benchmark bet. Hence, in our notation, the stock universe includes stocks both in and out of the benchmark.

Given the active weight (11.12) and the benchmark weight b_i, the total portfolio weight in a stock is $W_i = w_i + b_i$. If $W_i > 0$, it is a long position and if $W_i < 0$, it is a short position.

If we assume that the risk-adjusted forecast is normally distributed for stock i, according to (11.12), the active weight follows a normal distribution with zero mean and standard deviation

$$s_i = \frac{\sigma_{\text{target}}}{\sqrt{N}\sigma_i}. \tag{11.13}$$

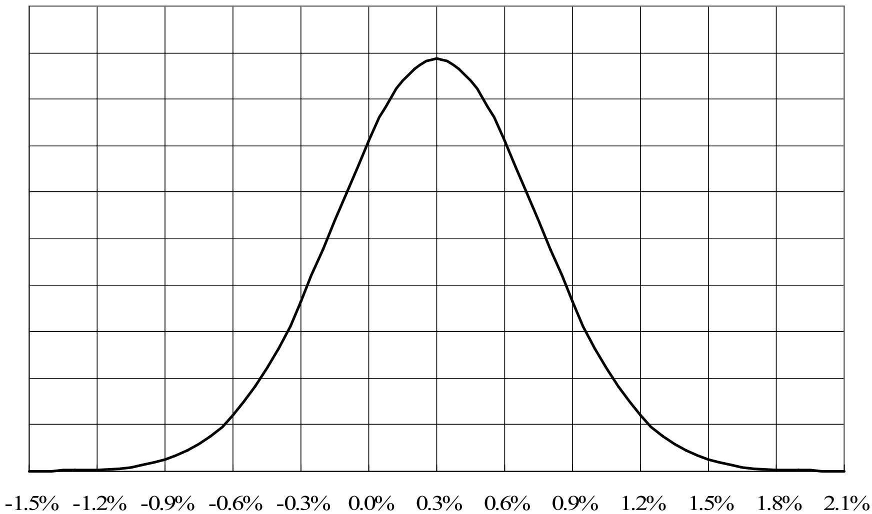


FIGURE 11.5. The probability density function of the total weight of a stock with 0.3% benchmark weight and 30% specific risk.

Hence, the total weight $W_i = w_i + b_i$ follows a normal distribution with mean b_i and standard deviation s_i.

Example 11.2

Consider an active portfolio with 3% targeted tracking error with 500 stocks. If the stock-specific risk is 30%, then

$$s_i = \frac{3\%}{\sqrt{500} \cdot 30\%} = 0.45\% .$$

The active position has a standard deviation of 45 bps. If the benchmark weight of the stock is 0.3%, or 30 bps, the density distribution of the total weight looks as in Figure 11.5.

The probability of W_i being a short position is given by

$$P\left(W_i < 0\right) = \frac{1}{\sqrt{2\pi} s_i} \int_{-\infty}^{0} \exp\left[-\frac{\left(x - b_i\right)^2}{2s_i^2}\right] dx . \tag{11.14}$$

It is simply the cumulative distribution function of W_i evaluated at 0. Because $b_i \geq 0$, (11.14) is always less than one half. If $b_i = 0$, the probability is exactly one half. This is relevant for stocks out of the benchmark, and it is also true for long-short portfolios without a benchmark. For the stock considered in Example 11.2, the probability of it being in a short position is about 25%. We are likely to prefer a short position for a given stock if the following conditions are met: (1) the lower the forecast, (2) the smaller the benchmark weight, (3) the smaller the specific risk, (4) the lower the risk-aversion parameter, and (5) the higher the target tracking error, *ceteris paribus*.

- We note that the probability is for multiple periods. At any given period, depending on the forecast for the stock, the position could be either positive (long) or negative (short). This is true for all stocks.

11.2.4 Portfolio Average Long/Short Ratio

The total short position of the whole portfolio is simply the sum of short positions, i.e.,

$$S = \sum_{W_i<0} W_i = \sum_{w_i+b_i<0} \left(w_i + b_i\right). \tag{11.15}$$

Similarly, the total long is

$$L = \sum_{W_i>0} W_i = \sum_{w_i+b_i>0} \left(w_i + b_i\right). \tag{11.16}$$

In our notation, short positions are weights that are negative. Because the active weights are dollar neutral, the sum of total long and total short should be just the total benchmark weights, i.e., $L + S = 1$. However, in any given period, the total long and short are not fixed. For instance, if the forecasts happen to be high for small stocks and low for large stocks in that period, then the total short would be lower, as we are more likely to overweight small stocks and underweight large ones, reducing the chance of negative positions. The situation would be reversed if the forecasts happen to be high for large stocks but low for small stocks. Then, we are likely to underweight small stocks, often leading to short positions.

We are interested in the averages of the total long and short positions. For the shorts, we have

$$\bar{S} = \sum_{i=1}^{N} \mathrm{E}\left(w_i + b_i \mid w_i + b_i < 0\right). \tag{11.17}$$

We simply calculate the average short position for each stock and sum them up. As the weight of stock i follows a normal distribution, we have

$$\begin{aligned}E\left(w_i+b_i \mid w_i+b_i<0\right) &= \frac{1}{\sqrt{2\pi}s_i}\int_{-\infty}^{-b_i}\left(x+b_i\right)\exp\left(-\frac{x^2}{2s_i^2}\right)dx \\ &= -\frac{s_i}{\sqrt{2\pi}}\exp\left(-\frac{b_i^2}{2s_i^2}\right)+b_i\cdot \mathrm{cdf}\left(-b_i,0,s_i\right).\end{aligned} \tag{11.18}$$

The function cdf is the cumulative density function evaluated at $-b_i$ for the normal distribution with zero mean and standard deviation s_i.

Example 11.3

Consider the case of the stock in Example 11.2. The benchmark weight is 0.3%, or 30 bps. The standard deviation of the active position is 0.45%, or 45 bps. Substituting them into (11.18), we obtain the average short position of –0.07%, or –7 bps.

Example 11.4

For out-of-benchmark stocks or long-short portfolio, we have $b_i = 0$. Then

$$E\left(w_i \mid w_i<0\right) = -\frac{s_i}{\sqrt{2\pi}} = -\frac{\sigma_{\text{target}}}{\sqrt{2\pi}\sqrt{N}\sigma_i}.$$

Assuming constant specific risk $\sigma_i = \sigma_0$, then $\bar{S} = -\frac{\sqrt{N}\sigma_{\text{target}}}{\sqrt{2\pi}\sigma_0}$.

With simulated benchmark weights b_i, Equation 11.17 and Equation 11.18 give rise to the average long/short ratio for the total portfolio, which is a function of two parameters: the concentration parameter c, and the targeted tracking error σ_{target}. Similar results have been obtained by Clarke et al. (2004). Figure 11.6 show the results for a fixed value of c and varying targeted tracking error. It plots four curves. First, the curve for long plus short (L+S) is always at 100%. The next two curves are for both long and short. As the tracking error increases, the long and short both increase in magnitude, with long exceeding 100% and short becoming more negative. The rate of increase for both sides is roughly linear. The fourth curve is for the total leverage (L-S), and it sits on the top. When the tracking error is small, at 0.5%, the total leverage is only 104%. When the

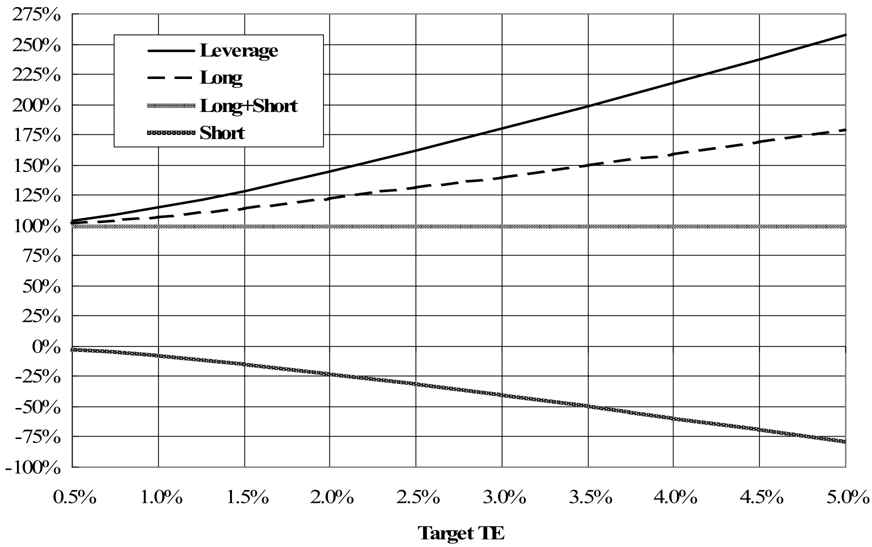


FIGURE 11.6. The long/short ratio of active portfolios with 500 stocks with $c = 1.2$ and specific risk at 40% for all stocks.

tracking error is at 2.5%, the long is 131%, the short is –31%, and the total leverage is 162%. When the tracking error reaches 5%, the long/short ratio is 179%/–79%, and the total leverage is 258%. In this case, if an investor has $100 in capital, he would buy $179 worth of stocks (long) and borrow and sell $79 worth of other stocks. There should be no overlapping between the longs and the shorts.

Figure 11.7 shows the change in the long/short ratio as the benchmark index c changes. The tracking error is fixed at 2.5%, and again our benchmark has 500 stocks, and the specific risk is set at 40% for all stocks. As we can see from the graph, the long, the short, and the total leverage increase slowly as c increases. When c is zero for an equally weighted benchmark, the long/short ratio is 119%/–19% and the total leverage is 138%. When c increases and the benchmark becomes increasingly concentrated, the long/short ratio increases. At $c = 1.2$, the long/short ratio is 131%/–31% and the total leverage is 162%. As c reaches 1.5, the long/short ratio is 135%/–35% with a total leverage of 170%. So there is an increase of 8% in total leverage as c goes from 1.2 to 1.5.

Finally, Figure 11.8 shows a three-dimensional view of the total leverage as a function of both c and tracking error. The graph again shows that the total leverage increases rapidly with an increase in tracking error and the pace is much more gradual with an increase in benchmark index c.

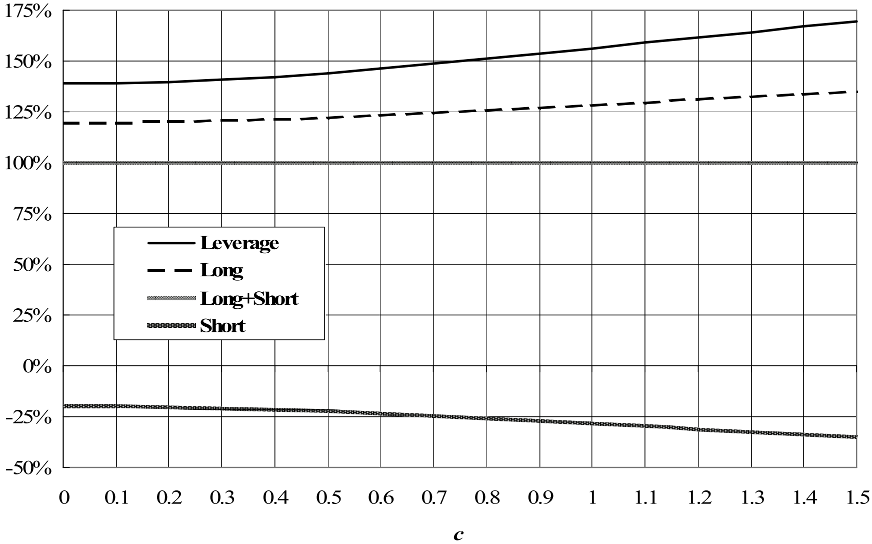


FIGURE 11.7. The long/short ratio of active portfolios with 500 stocks and specific risk at 40% for all stocks. The tracking error is 2.5%. The benchmark index *c* changes from 0 (equally weighted benchmark) to 1.5.

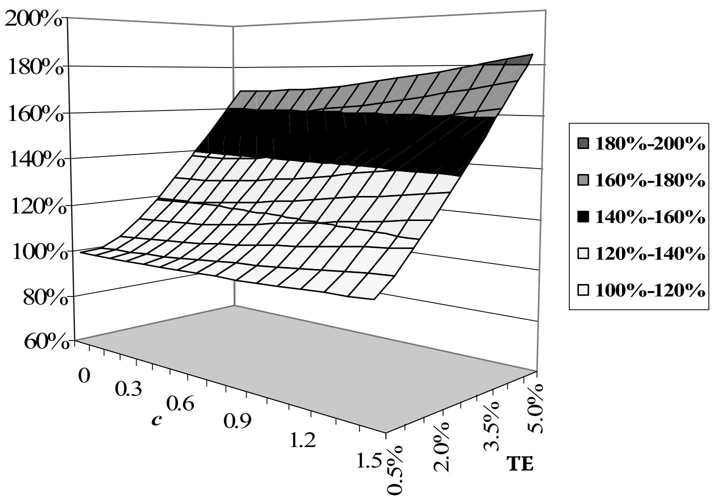


FIGURE 11.8. The total leverage of optimal portfolios as a function of both benchmark index *c* and tracking error. The benchmark has 500 stocks and specific risk is 40% for all stocks.

11.3 LONG-ONLY PORTFOLIOS

When the long-only constraint is placed on a portfolio, it is equivalent to a range constraint on active positions of all stocks: they must always be greater than the negative of their benchmark weights, i.e., $w_i \geq -b_i$. With the no-short rule, the portfolio's long/short ratio would be 100%/0%, which is obviously different from the long/short ratio of unconstrained portfolios. On the stock level, it is expected that the optimal weights of these two types of portfolios are different, resulting in different performance. For portfolios with low tracking errors, the difference in weights might not be so large. However, for portfolios with high tracking errors, the difference can be very significant. In this section, we shall analyze the impact of the long-only constraint on portfolio weights and performance of active strategies. In practice, most long-only portfolios are managed with maximum weight constraints in addition to the no-short constraint. The same is true for long-short portfolios, for which the range of stock weights is generally constrained. However, as there is no benchmark for long-short portfolios, the range is absolute, not relative to a benchmark.

The disadvantage of long-only portfolios managed against market-cap-weighted benchmarks has been stated previously at the stock level. The asymmetry also severely reduces the opportunity set for long-only managers who maintain minimal portfolio exposure to systematic size risk. With a size risk constraint, the active positions of a portfolio must be roughly balanced among stocks with similar market cap. Since this is not achievable among small stocks due to the long-only constraint, the portfolio is forced to take up more active positions and spend the majority of its active risk budgets among large stocks, where the market is probably more efficient and thus offers less alpha. We will demonstrate that an active portfolio with 3% targeted tracking error in the S&P 500 stock universe could have close to 50% of active risk in the S&P 100.

11.3.1 Constrained Long-Short Portfolios

Constrained long-short portfolios lie between long-only portfolios and unconstrained portfolios. Such portfolios, for example, might buy long 125% stocks and sell short 25% stocks, so the net result is still 100% with the total leverage ratio of 125% + 25% = 150%. Whereas the constrained long-short portfolios might still be suboptimal compared to unconstrained portfolios, they offer considerable benefit over long-only portfolios and have gained increasing acceptance with institutional investors.

With some ability to short, the constrained long-short portfolios alleviate some of the problems discussed previously. Therefore, in theory, one should expect them to deliver higher risk-adjusted returns than their long-only counterparts. However, there is an additional cost for the constrained long-short portfolios that is absent in the long-only portfolios that is due to the leverage. To see the leverage cost, it is important to understand the mechanism of long-short investing. Although standard financial theory often invokes the concept of a self-financing portfolio that implies costless leverage, in practice, leverage is not free. Suppose an investor has \$100. With long-only portfolios, the investor can buy \$100 worth of stocks and the leverage ratio is 1:1. As no borrowing is involved, there is no leverage cost. With a 125/25 portfolio, the investor buys \$100 worth of stocks with his own capital. He then borrows \$25 to buy an additional \$25 worth of stocks, and at the same time borrows \$25 worth of stocks to sell. From a pure theoretical standpoint, the short proceeds of \$25 would be used to buy the additional \$25 long with no additional cost. However, from a practical standpoint, used by prime brokers for pricing, the investor has bought \$25 worth of stocks on margin, whereas the short proceeds of \$25 is kept at the broker as collateral for the short positions. The short proceed earns an interest rebate from the brokers, but the rate is always lower than the financing cost on the long side. Therefore, the interest rate spread on the \$25 is a cost that the investor must bear.[2]

Example 11.5

Suppose the spread between the financing and the rebate is 1%, the additional cost for 125/25 portfolios would be 0.25% or 25 bps. Similarly, the additional cost for 150/50 portfolios would 0.5% or 50 bps.

11.3.2 Numerical Methods for MV Optimization with Range Constraints

An analytical solution does not exist for optimal weights of long-only portfolios, or range-constrained portfolios, in general. We shall carry out our analysis through numerical means. The problem falls in the general category of quadratic programming, in which we maximize a quadratic objective function subject to linear constraints, as well as range constraints. For large-scale problems with thousands of stocks, finding numerical solutions of general problems can be time consuming. However, there exists an efficient algorithm for the special case in which the covariance matrix is diagonal. This would be true if we neutralize all the systematic factor exposures and optimize with residual alphas and specific risks.

The algorithm is based on the Kuhn–Tucker condition for optimization with inequality constraints. The appendix provides a detailed description of the Kuhn–Tucker condition for the general optimization problem and its application to mean–variance optimization, which is to find the optimal active weights **w** in the following

$$\text{Maximize: } \mathbf{f}' \cdot \mathbf{w}$$

$$\text{Subject to:}$$

$$\mathbf{w}' \cdot \mathbf{\Sigma} \cdot \mathbf{w} = \sigma^2_{\text{target}} \tag{11.19}$$

$$\mathbf{w}' \cdot \mathbf{i} = 0, \text{ and } \mathbf{w}' \cdot \mathbf{B} = 0$$

$$w_i - U_i \le 0, \text{ and } L_i - w_i \le 0, \text{ for } i = 1, \cdots, N.$$

The vector **f** is the forecast vector, the covariance matrix $\mathbf{\Sigma} = \mathbf{B}\mathbf{\Sigma}_I\mathbf{B}' + \mathbf{S}$, and σ_{target} is the target tracking error. The equality constraints are dollar neutral and market neutral $\mathbf{w}' \cdot \mathbf{i} = 0$, and $\mathbf{w}' \cdot \mathbf{B} = 0$. The range constraints are

$$w_i - U_i \le 0, \text{ and } L_i - w_i \le 0, \text{ for } i = 1, \cdots, N.$$

The Kuhn–Tucker condition implies that the solution takes the following form:

$$\mathbf{w} = \frac{1}{2\lambda}\mathbf{S}^{-1}\mathbf{f}_{\text{adj}}, \text{ or}$$
$$w_i = \frac{f_i - l_0 - l_1 b_{1i} - \cdots l_K b_{Ki} - \tilde{l}_{1i} + \tilde{l}_{2i}}{2\lambda\sigma_i^2}. \tag{11.20}$$

In the solution, l_0 is the Lagrangian multiplier for the dollar neutral constraint; $l_1, \cdots, l_K$ are the Lagrangian multipliers for market neutral constraints; $\tilde{l}_{1i}$ and $\tilde{l}_{2i}$ are the Lagrangian multipliers for the upper and lower bounds, respectively; and λ is the Lagrangian multiplier for the tracking error constraint. As only one of $\tilde{l}_{1i}$ and $\tilde{l}_{2i}$ can be nonzero, we combine them into one: $\tilde{l}_i = \tilde{l}_{1i} - \tilde{l}_{2i}$.

Our numerical algorithm finds the optimal weights and the Lagrangian multipliers iteratively. At step n, we have the weight w_i^n and multipliers $l_0^n, l_1^n, \cdots, l_K^n$, $\tilde{l}_i^n$, λ^n. If the weights violate the range constraint, we proceed as follows:

- Apply range constraints to the weight $w_i^{new} = \max\left(\min\left(w_i^{new}, U_i\right), L_i\right)$.
- Update Lagrangian multipliers for range constraints with

$$\tilde{l}_i^{n+1} = f_i - l_0^n - l_1^n b_{1i} - \cdots l_K^n b_{Ki} - 2\lambda^n \sigma_i^2 w_i^{new}.$$

- Update Lagrangian multipliers for dollar neutral and beta-neutral constraints with the solution from the system of linear equations in which

$$\left\langle \mathbf{x}, \mathbf{y} \right\rangle = \sum_{i=1}^{N} x_i y_i / \sigma_i^2$$

(see Chapter 4) and $\tilde{\mathbf{l}}^{n+1}$ is the vector of newly updated Lagrangian multipliers from the previous step.

$$\begin{cases} l_0^{n+1}\left\langle \mathbf{i}, \mathbf{i} \right\rangle + l_1^{n+1}\left\langle \mathbf{i}, \mathbf{b}_1 \right\rangle + \cdots + l_K^{n+1}\left\langle \mathbf{i}, \mathbf{b}_K \right\rangle = \left\langle \mathbf{i}, \mathbf{f} - \tilde{\mathbf{l}}^{n+1} \right\rangle \\ l_0^{n+1}\left\langle \mathbf{b}_1, \mathbf{i} \right\rangle + l_1^{n+1}\left\langle \mathbf{b}_1, \mathbf{b}_1 \right\rangle + \cdots + l_K^{n+1}\left\langle \mathbf{b}_1, \mathbf{b}_K \right\rangle = \left\langle \mathbf{b}_1, \mathbf{f} - \tilde{\mathbf{l}}^{n+1} \right\rangle \\ \vdots \\ l_0^{n+1}\left\langle \mathbf{b}_K, \mathbf{i} \right\rangle + l_1^{n+1}\left\langle \mathbf{b}_K, \mathbf{b}_1 \right\rangle + \cdots + l_K^{n+1}\left\langle \mathbf{b}_K, \mathbf{b}_K \right\rangle = \left\langle \mathbf{b}_K, \mathbf{f} - \tilde{\mathbf{l}}^{n+1} \right\rangle \end{cases}$$

- Calculate the tracking error of $\mathbf{w}^{new}$ and update the Lagrangian multiplier for the tracking error

$$\sigma^{new} = \sqrt{\sum_{i=1}^{N} \left(w_i^{new}\right)^2 \sigma_i^2}, \qquad \lambda^{n+1} = \lambda^n \frac{\sigma^{new}}{\sigma_{\text{target}}}.$$

- Calculate the new weights w_i^{n+1} by

$$w_i^{n+1} = \frac{f_i - l_0^{n+1} - l_1^{n+1} b_{1i} - \cdots l_K^{n+1} b_{Ki} - \tilde{l}_i^{n+1}}{2\lambda^{n+1}\sigma_i^2}.$$

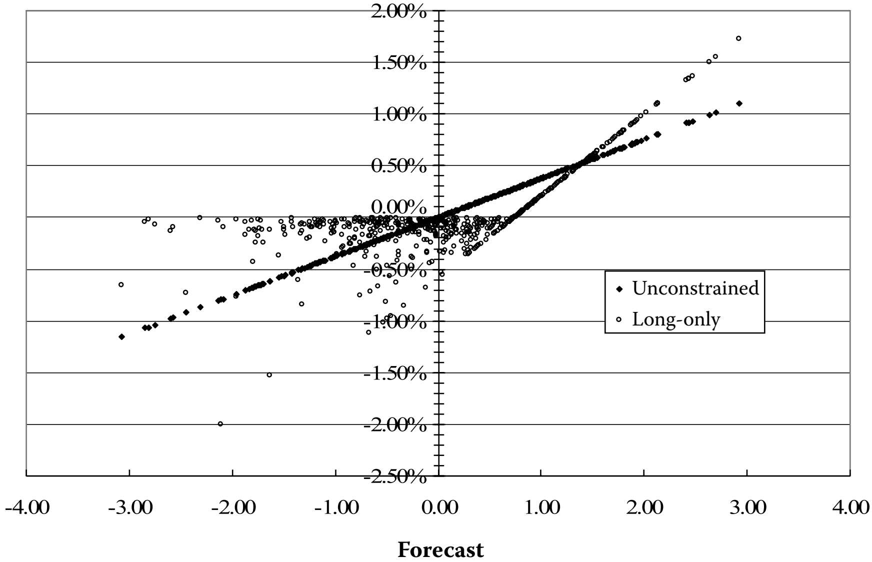


FIGURE 11.9. Optimal active weights of unconstrained and long-only portfolios.

After these steps, we have the weight w_i^{n+1} and multipliers $l_0^{n+1}, l_1^{n+1}, \cdots, l_K^{n+1}$, $\tilde{l}_i^{n+1}$, λ^{n+1}. The new weights are checked against the range constraints. If there is violation, the foregoing steps are repeated until there is no range violation.

Example 11.6

We use the preceding algorithm to find long-only optimal portfolio weights against a benchmark of 500 stocks that has a concentration index of c 1.2, and compare these weights to unconstrained optimal weights. Both portfolios have a targeted tracking error of 3%, and all stocks are assumed to have a specific risk of 35%. We also impose a maximum active weight of 2% for all stocks. The forecasts are simulated based on a standard normal distribution. Figure 11.9 plots the forecasts vs. both sets of optimal active weights. We first note that the unconstrained optimal weights form a straight line going through the origin. Indeed, they are proportional to the forecasts. The optimal weights of the long-only portfolio show several features: (1) There are many small negative weights. They belong to the active weights of stocks with tiny benchmark weights, due to the long-only constraint; (2) Positive active weights also seem to fall on a straight line, which has a steeper slope and a negative intercept on the y-axis. Some negative active weights also fall on this line. Mathematically, this is due to a smaller Lagrangian multiplier for the tracking error constraint in the long-only optimization than its counterpart in the unconstrained optimization (the slope is inversely proportional to λ in Equation 11.20). In addition, the Lagrangian multiplier for the dollar neutral constraint is positive. This implies that large positive, active weights are magnified whereas smaller positive ones are shrunk; and (3) Many stocks with positive forecasts will end up with negative active weights, as underweights in stocks with small benchmark weights are not sufficient to fund overweights. Note the following remark:

- In the unconstrained optimal portfolio, the active weights and the forecasts have perfect correlation. However, in the constrained portfolio, the correlation is less than perfect. This correlation can be used as a gauge of the stringency of the constraint. Alternatively, it measures the extent to which the forecasts are reflected in the portfolio. Clarke et al. (2002) coined the term *transfer coefficient* for a variation of this correlation. In our example, this correlation is about 0.7.

Figure 11.10 plots the active weights vs. the benchmark weights. In Figure 11.10a for an unconstrained portfolio, the active weights are independent of the benchmark. In Figure 11.10b, for the long-only portfolio, the active weights are bounded below by the benchmark, and there is a negative correlation between the two.

11.4 THE INFORMATION RATIO OF LONG-ONLY AND LONG-SHORT PORTFOLIOS

Unconstrained optimal portfolios have intrinsic long/short leverage ratios, depending on portfolio and benchmark characteristics such as target tracking error, benchmark concentration, stock-specific risks, and the number of stocks in the benchmark and portfolio. In theory, these long/short ratios are optimal for given portfolio mandates in terms of maximizing the IR. Range constraints such as long-only or limited shorting would reduce the theoretical IR.

With the numerical algorithm described earlier, we now analyze the information ratio of long-only, as well as constrained long-short portfolios. There are many practical reasons that might prevent portfolio managers from fully implementing the unconstrained optimal portfolios. Some constraints are institutional. For example, prime brokers might

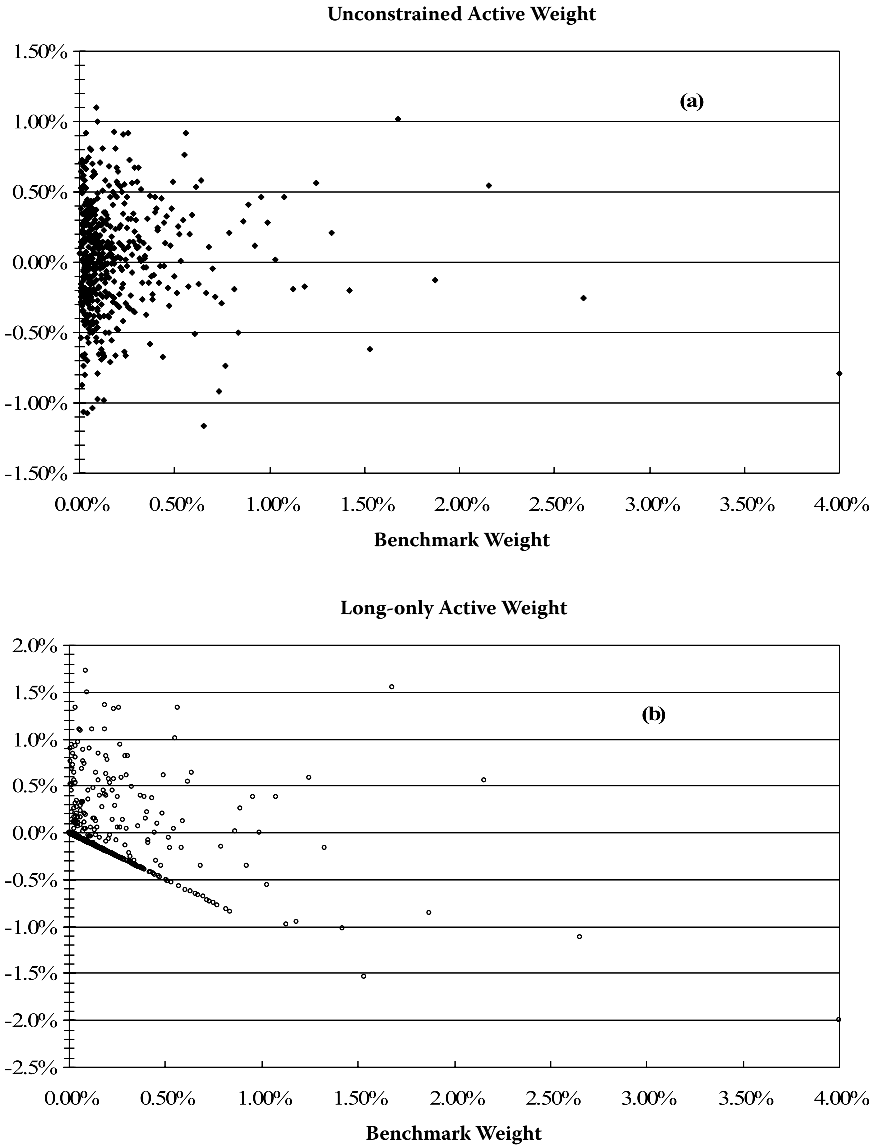


FIGURE 11.10. Optimal active weights vs. the benchmark weights: (a) for unconstrained portfolio and (b) for long-only portfolio.

place limits on the amount of leverage allowed in a portfolio; or it might be hard to borrow certain stocks, which reduces the amount of shorting. Some concerns are cost related. As mentioned earlier, the higher the leverage, the higher the financing cost. In addition, portfolios with higher leverage require higher turnover, resulting in higher transaction costs, a component often missed in some previous analysis of long-short portfolios (see Chapter 8). Therefore, there is a need to distinguish between theoretical IR and net IR that account for both leverage and transaction costs. However, note the following remark:

- Some other issues arise in long-short investing that must be considered. For example, the number of stocks in a long-short portfolio will be much higher than that in a long-only portfolio. This might not be a big issue for quantitative managers, but it could impose additional work on fundamental managers.

To better understand the benefit of constrained long-short portfolios compared to long-only portfolios, we carry out numerical simulations for long-only portfolios and long-short portfolios with varying amounts of short positions. In the simulation, we first calculate the "paper" or theoretical excess returns from portfolio weights and returns, and then deduct financing costs according to the portfolio's leverage and by transaction costs according to portfolio turnover.

11.4.1 Simulation Assumptions

Simulation results depend on a host of parameters, which are listed in detail as follows:

- *Investment universe and benchmark:* To be consistent with our discussion of unconstrained optimal portfolios, we choose a universe of 500 stocks and portfolios that are managed against a 500-stock index, with the index concentration being measured by the parameter c. Stock-specific risk is 35% for all stocks.
- *Tracking error target:* We choose a series of tracking error targets ranging from 1 to 5%.
- *Long/short ratio:* We impose the long/short ratio constraints through a range constraint on individual stocks. Starting from long-only portfolios, which have a constraint on the weights as $w_i \geq 0$, we gradually loosen the constraint to $w_i \geq -s$, where s is the short position allowed in individual stocks. For instance, if s = 0.1%, we can short each stock by a maximum of 10 bps. As s grows, the total short position grows and the portfolio would approach the unconstrained optimal portfolio. We also set the maximum active weight at ±3%.

- *Other portfolio constraints:* Besides targeted tracking error and range constraints on the individual stocks, the only other portfolio constraint is the dollar neutral constraint.

- *Forecasts:* We simulated forecast in the form of normally distributed *z*-scores. We also assume consecutive forecasts have autocorrelation ρ_f, which is one of the factors influencing portfolio turnover. The other factors are target tracking error and the leverage ratio (see Chapter 8).

- *Information coefficient and returns:* The risk-adjusted returns are simulated based on the IC — the cross-sectional correlation coefficient between the forecast and the returns. Two parameters characterize the random nature of IC: the average IC and the standard deviation of IC. The risk-adjusted return is also assumed to be normally distributed and its cross-sectional dispersion is unity (Qian and Hua 2004).

In each simulation, we first generate standardized forecasts and actual returns based on either a constant or stochastic IC. We then calculate excess returns of active portfolios that are managed against a benchmark with a specified concentration index and a series of targeted tracking errors that are optimized with different range constraints that lead to different long/short ratios. A theoretical IR can then be obtained from the time series of excess returns. In addition, we also obtain the average portfolio turnover and long/short ratio of these portfolios. We estimate transaction costs and leverage costs and subtract them from the theoretical excess return. Finally, "net" IR is calculated as the ratio of net excess return to the realized tracking error, not the target tracking error. We note that the realized tracking error is higher than the targeted tracking error when the IC has intertemporal variability (Qian and Hua 2004).

11.4.2 Simulation Results: Constant IC

Table 11.3 shows the results of one such simulation in which we assume that the IC is constant and the only source of time-series variation is sampling error. There are 11 portfolios across the table, ranging from the long-only portfolio (column 1) to the unconstrained portfolio (column 11). They all have the same target tracking error of 3%. We have assumed that the IC is constant at 0.1. As a result, the realized tracking error, or standard deviation of alpha, is also 3%. The theoretical IR of the unconstrained portfolio (column 11) is then the IC times the square root of *N*,

TABLE 11.3 Simulation Results for Long-Only Portfolios, Constrained Long-Short Portfolios, and Unconstrained Long-Short Portfolios

	Long-only										Unconstrained
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Avg Alpha	4.82%	5.30%	5.56%	5.81%	6.05%	6.24%	6.41%	6.54%	6.63%	6.69%	6.72%
Std Alpha	3.04%	3.03%	3.03%	3.02%	3.01%	3.00%	3.00%	2.99%	2.99%	3.00%	3.00%
Theoretical IR	1.59	1.75	1.84	1.92	2.01	2.08	2.14	2.19	2.22	2.23	2.24
Total Long	100%	109%	115%	121%	127%	133%	138%	143%	146%	147%	148%
Turnover	64%	71%	75%	79%	83%	86%	89%	91%	93%	94%	94%
Leverage Cost	0.00%	0.09%	0.15%	0.21%	0.27%	0.33%	0.38%	0.43%	0.46%	0.47%	0.48%
Transaction Cost	0.64%	0.71%	0.75%	0.79%	0.83%	0.86%	0.89%	0.91%	0.93%	0.94%	0.94%
Net Avg Alpha	4.18%	4.50%	4.66%	4.81%	4.94%	5.05%	5.13%	5.20%	5.25%	5.28%	5.30%
Net IR	1.38	1.48	1.54	1.59	1.64	1.68	1.71	1.74	1.75	1.76	1.77
Theoretical IR decay	0.71	0.78	0.82	0.86	0.90	0.93	0.95	0.98	0.99	1.00	1.00
Net IR Decay	0.78	0.84	0.87	0.90	0.93	0.95	0.97	0.98	0.99	1.00	1.00
Transfer Coefficient	0.70	0.78	0.82	0.85	0.89	0.92	0.95	0.97	0.98	0.99	1.00

Note: The number of stocks is 500; benchmark concentration index, 1.2; target tracking error, 3%; stock-specific risk, 35%; average IC, 0.1 with no intertemporal variation; forecast autocorrelation, 0.25; leverage cost, 1%; and the transaction cost, 1%.

Source: From Sorensen, E.H., Hua, R., and Qian, E., *Journal of Portfolio Management*, Vol. 33, No. 2, 1–9, Winter 2007. With permission.

equaling 2.24, whereas the theoretical IR of the long-only portfolio (column 1) is only 1.59.

The next two rows of Table 11.3 report the total long positions of the portfolios and their turnover. As we relax the short constraint, the total long and the total short both increase. Because the long minus short is always 100%, we omit the short from the table. For instance, the portfolio in column 5 is long 127% on average and its theoretical IR is 2.01. Table 11.3 shows that portfolio turnover increases with leverage. It averages 64% for the long-only portfolio and about 94% for the unconstrained portfolio. These numbers are based on our assumption of a forecast autocorrelation of 0.25. The turnover for the unconstrained portfolio is consistent with the results in Chapter 8. As we can see, the turnovers for the long-only portfolios are much lower. It is easy to understand that range constraints have a dampening effect on portfolio turnover, because they prohibit portfolios from adjusting fully to changes in forecasts, which is why they have a negative impact on investment performance (Qian et al. 2004). What is startling is that Table 11.3 shows that turnover is a linear function of leverage. The ratio of turnover to total long is about 0.64 for all portfolios.

To calculate the net average alpha, we assume that the spread between the long financing and the short rebate is 1%, and the transaction costs are 1% is for 100% turnover. These rates are reasonable and conservative estimates. In practice, the financing and rebate spread is subject to negotiation with prime brokers, and transaction costs depend on many factors such as commissions, bid/ask spreads, and market impact. Using the net average alpha, we then calculate the net IR. For the long-only portfolio, the IR drops from 1.59 to 1.38, a decrease of 0.21. For the unconstrained portfolio, the IR drops from 2.24 to 1.77, a much larger decrease of 0.47 due to the higher leverage cost and higher transaction costs.

Lastly, we will compute both theoretical and net *IR decay*, defined as the ratio of the IR of the constrained portfolios to that of the unconstrained portfolio. For instance, the long-only portfolio's theoretical IR is 71% of the unconstrained IR, but its net IR is 78% of the unconstrained net IR. Portfolio (column 6), with an average of 133% long, achieves about 95% of the unconstrained net IR. The last row of Table 11.3 shows the transfer coefficient (Clarke et al. 2002), defined as the correlation between the active weights in the constrained portfolios and the forecasts. In this case, the transfer coefficients are close to the theoretical IR decay but differ from the net IR decay.

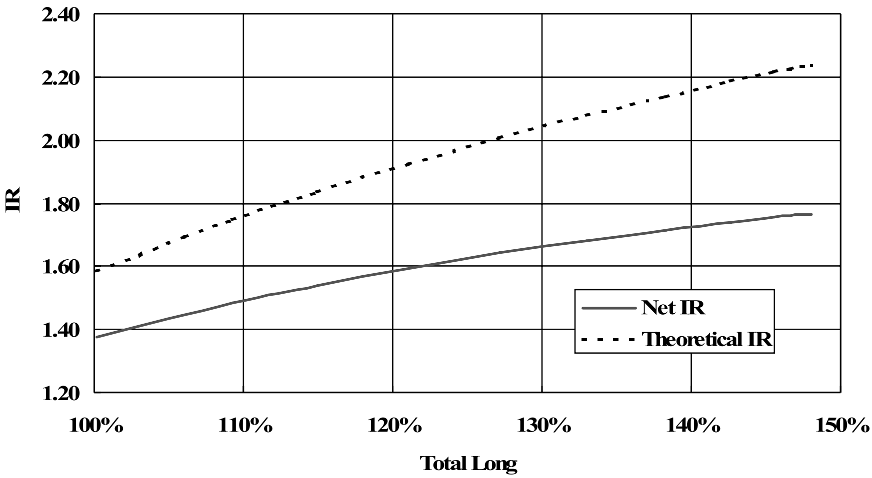


FIGURE 11.11. The theoretical and net IR as shown for Table11.3. (From Sorensen, E.H., Hua, R., and Qian, E., *Journal of Portfolio Management*, Vol. 33, No. 2, 1–9, Winter 2007. With permission.)

Figure 11.11 displays both the theoretical IR and net IR as a function of total long portfolio positions. We note two features of this graph. First, the rate of increase in IR with a loosening of short constraint is higher in terms of theoretical IR than in terms of net IR. This is due to the higher leverage and transaction costs associated with less constrained portfolios. Second, both curves are not straight lines. The marginal increase in IR seems to be the strongest for long-only portfolios, and it diminishes as the short constraints are relaxed further.

11.4.3 Risk Allocation of Long-Only and Long-Short Portfolios

One of the reasons for the low IR of the long-only portfolios is that they have inferior allocation of active risk. If a signal has uniform predictive power across stocks of all sizes, then the optimal allocation of active risk should be the same across the size spectrum. However, this is not the case for the long-only portfolios, because the constraint forces more active risk into stocks with large benchmark weights. Figure 11.12 shows the contribution to the active risk of 3% from 5 quintiles of 500 stocks in portfolios with different constraints. The long-only portfolio gets 45% of risk from the largest quintile, 17% in the second largest quintile, whereas the remaining 3 quintiles each contribute roughly 13%. As we loosen the short constraint,

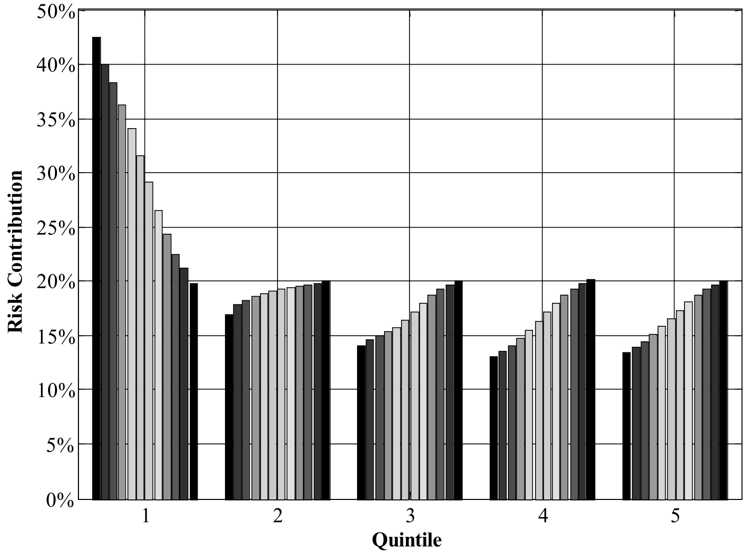


FIGURE 11.12. Risk contributions from quintiles of stocks. The active risk is 3%. There are 500 stocks and each quintile has 100 stocks: quintile 1 has the top 100 stocks of the largest weights, whereas quintile 5 has the bottom 100 stocks of the smallest weights. In each quintile, there are 11 portfolios (from left to right) ranging from the long-only portfolio to the unconstrained constrained. (From Sorensen, E.H., Hua, R., and Qian, E., *Journal of Portfolio Management*, Vol. 33, No. 2, 1–9, Winter 2007. With permission.)

the contribution from the 1st quintile decreases, whereas the rest contribute more, until we reach the unconstrained portfolio where all quintiles contribute the equal and optimal amount — 20% to the active risk.

11.4.4 Simulation Results: Stochastic IC

One of the underlying assumptions for the simulation in the previous section is the constancy of the IC. This assumption, however, is often violated in practice. As shown by Qian and Hua (2004), active investment strategies bring additional risk, which is not captured by generic risk models, and as a result the realized or *ex post* tracking error often exceeds the target or *ex ante* tracking error. This additional risk, referred to as *strategy risk*, can be represented by the intertemporal variation of IC, and the realized tracking error is then a function of the standard deviation of the IC that consists of both the intertemporal variation and the sampling error. The IR of an active investment strategy is then given by the ratio of average IC to the standard deviation of IC, i.e.,

$$IR = \frac{\overline{IC}}{\text{std}\left(IC\right)}.$$

For example, if the intertemporal variation of IC is 0.02, then the standard deviation of IC is

$$\text{std}\left(IC\right) = \sqrt{0.02^2 + \frac{1}{N}} = \sqrt{0.02^2 + \frac{1}{500}} = 0.049.$$

The IR of unconstrained portfolios with the additional strategy risk is then $IR = 0.1/0.049 = 2.04$, compared to the previous value of 2.24 when the IC was constant.

What is the information ratio of long-only and constrained long-short portfolios, if the IC is stochastic? Table 11.4 shows the simulation results that take into account the additional intertemporal variation of IC, in this case, at 0.02. First, notice the unconstrained portfolio (column 11) has a realized tracking error of 3.28%, even though the target is 3%, due to the additional strategy risk and the theoretical IR is 2.04, as indicated earlier. Second, we note that the realized tracking error for the long-only portfolio is 3.08%, not too different from the target. As a result, its IR is 1.52, only slightly lower than 1.59 in the previous case; and as we relax the no-short constraint, the realized tracking error increases. These results indicate that more stringent range constraints have the potential benefit of controlling *ex post* tracking error when there is additional strategy risk. In other words, relaxing long-only constraints could potentially lead to higher *ex post* tracking error, and portfolio managers must pay extra attention to risk management.

The other characteristics of the portfolios, such as total long and turnover, stay the same, so additional costs remain unchanged. However, the net IR is lower in Table 11.4 than in Table 11.3 due to the higher realized tracking error. Here, the net IR goes from 1.31 for the long-only portfolio to 1.61 for the long-short portfolio.

Table 11.4 also indicates that the transfer coefficient is no longer a reliable gauge of IR decay, even for the theoretical IR. For instance, the long-only portfolio has a transfer coefficient of 0.70, but the theoretical IR decay is slower at 0.74 and the net IR is 0.82. When strategy risk grows, we find that the difference between the transfer coefficient and IR decay grows as well.

TABLE 11.4 Simulation Results for Long-Only Portfolios, Constrained Long-Short Portfolios, and Unconstrained Long-Short Portfolios

	Long-Only									Unconstrained	
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Avg alpha	4.68%	5.18%	5.45%	5.71%	5.95%	6.17%	6.35%	6.49%	6.60%	6.66%	6.70%
Std alpha	3.08%	3.12%	3.15%	3.17%	3.19%	3.21%	3.22%	3.24%	3.26%	3.27%	3.28%
Theoretical IR	1.52	1.66	1.73	1.80	1.87	1.92	1.97	2.00	2.03	2.04	2.04
Total long	100%	109%	115%	121%	127%	133%	138%	143%	146%	147%	148%
Turnover	64%	71%	75%	79%	83%	87%	89%	91%	93%	94%	94%
Leverage cost	0.00%	0.09%	0.15%	0.21%	0.27%	0.33%	0.38%	0.43%	0.46%	0.47%	0.48%
Transaction cost	0.64%	0.71%	0.75%	0.79%	0.83%	0.87%	0.89%	0.91%	0.93%	0.94%	0.94%
Net avg alpha	4.04%	4.38%	4.54%	4.70%	4.85%	4.98%	5.08%	5.15%	5.21%	5.25%	5.28%
Net IR	1.31	1.40	1.44	1.48	1.52	1.55	1.58	1.59	1.60	1.61	1.61
Theoretical IR decay	0.74	0.81	0.84	0.88	0.91	0.94	0.96	0.98	0.99	0.99	1.00
Net IR decay	0.82	0.87	0.90	0.92	0.94	0.96	0.98	0.99	0.99	1.00	1.00
Transfer coefficient	0.70	0.78	0.82	0.85	0.89	0.92	0.95	0.97	0.98	0.99	1.00

Note: The intertemporal variation of IC is 0.02, and all other assumptions are the same as in Table 11.3.

PROBLEMS

11.1 Calculate the return decomposition for Example 11.1.

11.2 (Variance decomposition) Cross-sectional return variance is given by

$$\sigma_R^2 = \sum_{i=1}^{N} b_i \left(R_i - \bar{R}\right)^2 ,$$

where b_i could be the benchmark weight for cap-weighted variance or $b_i = 1/N$ for equally-weighted variance.

(a) Prove that the variance can be decomposed as

$$\sigma_R^2 = \sum_{s=1}^{S} \sum_{i=1}^{N_s} b_i \left(R_{si} - \bar{R}_s\right)^2 + \sum_{s=1}^{S} B_s \left(\bar{R}_s - \bar{R}\right)^2 , \qquad (11.21)$$

where $B_s = \sum b_i$ for stocks in the sector s, i.e., the sector weight.

(b) Interpret the decomposition as investment opportunities for stock selection and sector bets in terms of their relative magnitude.

11.3 Assume the benchmark weight of a stock is b_i, and its active weight of a stock is given by

$$w_i = \frac{\sigma_{\text{target}} F_i}{\sqrt{N} \sigma_i} .$$

Instead of the normal distribution, assume the factor F_i is uniformly distributed with zero mean and standard deviation one. This uniform distribution describes factors that are percentile ranking instead of normalized z-scores.

(a) Find the range of F_i and therefore the range of w_i.

(b) Find the probability that the total position $w_i + b_i$ is net short.

(c) Find the average long/short ratio for the stock.

11.4 Suppose the financing cost is the federal funds rate plus 50 bps and the short rebate is the federal funds rate minus 75 bps. What is the leverage cost for (a) a constrained 130/30 portfolio and (b) a market-neutral portfolio with 100 long and 100 short?

11.5 If the active weights are given by the Kuhn–Tucker condition, calculate the transfer coefficient.

11.6 A forecast model has an average IC of 0.1 for a universe of 500 stocks. Suppose the IC has no intertemporal variation so that the fundamental law of active management holds.

(a) What is the model's IR?

(b) Suppose the model is uniformly effective across all 500 stocks. What is the model's IR when applied to each quintile?

(c) What is the optimal allocation of active risk across the five quintiles if excess returns from five quintiles are uncorrelated?

APPENDIX
A11.1 MEAN–VARIANCE OPTIMIZATION WITH RANGE CONSTRAINTS

Given a forecast vector **f**, we maximize the following objective function to obtain portfolio weights **w**

$$\mathbf{f}'\cdot\mathbf{w}-\frac{1}{2}\lambda\cdot(\mathbf{w}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{w})\ . \tag{11.22}$$

In addition to the dollar neutral and market neutral constraints: $\mathbf{w}'\cdot\mathbf{i}=0$, and $\mathbf{w}'\cdot\mathbf{B}=0$, we also have range constraints on individual stocks: $\mathbf{l}\le\mathbf{w}\le\mathbf{u}$, where **l** and **u** are vectors of lower and upper bound for all stocks. As the range constraints are inequality constraints, there is no analytical solution for the optimization problem. However, a numerical solution can be found through Kuhn–Tucker conditions. For details, please refer to McCormick (1983).

A11.1.1 Kuhn–Tucker Conditions

Kuhn–Tucker conditions are for general optimization problems with inequality constraints. We first present the conditions for a general problem and then specify them for the mean–variance optimization with range constraints.

Suppose the problem is to maximize $p(\mathbf{w})$ subject to $g_j(\mathbf{w}) \le 0$ for $j = 1, \cdots, m$, then define the Lagrangian function L by

$$L(\mathbf{w}) = p(\mathbf{w}) - \sum_{j=1}^{m} l_j g_j(\mathbf{w}). \tag{11.23}$$

The Kuhn–Tucker conditions are

$$\frac{\partial L(\mathbf{w})}{\partial w_i} = \frac{\partial p(\mathbf{w})}{\partial w_i} - \sum_{j=1}^{m} l_j \frac{\partial g_j(\mathbf{w})}{\partial w_i} = 0, \text{ for } i = 1, \cdots, N, \tag{11.24}$$

and

$$g_j(\mathbf{w}) \le 0,\ l_j \ge 0,\ \text{ and } l_j g_j(\mathbf{w}) = 0,\ \text{for } j = 1, \cdots m. \tag{11.25}$$

We note that condition (11.24) is the same for equality constraints. However, condition (11.25) is different for inequality constraints, and states that (1) the inequality constraints must be satisfied, of course; (2) the Lagrangian multipliers must be nonnegative; and (3) either the Lagrangian multiplier is 0, or the constraints are binding.

A11.1.2 Kuhn–Tucker Conditions for Mean–Variance Optimization with Range Constraints

When the range of weight for a stock is constrained by $L_i \le w_i \le U_i$, we can represent the constraint with two inequality constraints: $w_i - U_i \le 0$, and $L_i - w_i \le 0$ in the form of $g(\mathbf{w}) \le 0$.

For a portfolio of N stocks, we could have a maximum of $2N$ inequality constraints:

$$w_i - U_i \le 0, \text{ and } L_i - w_i \le 0, \text{ for } i = 1, \cdots, N. \tag{11.26}$$

The objective function (11.22) also needs to be modified with the introduction of range constraints. Previously, the risk-aversion parameter was a free parameter used to achieve the targeted tracking error, because with dollar neutral and market neutral constraints the optimal weights are scalable. With inequality constraints, the optimal weights are no longer scalable. Hence, we need to set targeted tracking error as an additional constraint. The optimization problem becomes

Maximize: $\mathbf{f}'\cdot\mathbf{w}$

Subject to:

$$\mathbf{w}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{w}=\sigma_{\text{target}}^2 \tag{11.27}$$

$$\mathbf{w}'\cdot\mathbf{i}=0,\ \text{and}\ \mathbf{w}'\cdot\mathbf{B}=0$$

$$w_i-U_i\le 0,\ \text{and}\ L_i-w_i\le 0,\ \text{for}\ i=1,\cdots,N$$

The Lagrangian function for the problem is then

$$\begin{aligned}L(\mathbf{w})=\mathbf{f}'\cdot\mathbf{w}-\lambda\left(\mathbf{w}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{w}-\sigma_{\text{target}}^2\right)-l_0\left(\mathbf{w}'\cdot\mathbf{i}\right)-\sum_{i=1}^{K}l_i\left(\mathbf{w}'\cdot\mathbf{b}_i\right)\\-\sum_{j=1}^{N}\left[\tilde{l}_{j1}\left(w_j-U_j\right)+\tilde{l}_{j2}\left(L_j-w_j\right)\right]\end{aligned} \tag{11.28}$$

Now λ denotes the Lagrangian multiplier for the tracking error target constraint, l_0 is the Lagrangian multiplier for the dollar neutral constraint, l_i, $i=1,\cdots,K$ are the Lagrangian multipliers for the K risk factors, and $\tilde{l}_{j1}$ and $\tilde{l}_{j2}$, $j=1,\cdots,N$ are the Lagrangian multipliers for the range constraints on N stocks.

The Kuhn–Tucker condition for (11.28) is

$$\frac{\partial L(\mathbf{w})}{\partial\mathbf{w}}=\mathbf{f}-2\lambda\boldsymbol{\Sigma}\cdot\mathbf{w}-l_0\mathbf{i}-\sum_{i=1}^{K}l_i\mathbf{b}_i-\left(\tilde{\mathbf{l}}_1-\tilde{\mathbf{l}}_2\right)=0\,, \tag{11.29}$$

where $\tilde{\mathbf{l}}_1=\left(\tilde{l}_{11},\cdots,\tilde{l}_{1N}\right)'$ and $\tilde{\mathbf{l}}_2=\left(\tilde{l}_{21},\cdots,\tilde{l}_{2N}\right)'$ are vectors of Lagrangian multipliers. The equality constraints must be satisfied, i.e.,

$$\mathbf{w}'\cdot\boldsymbol{\Sigma}\cdot\mathbf{w}=\sigma_{\text{target}}^2,\ \mathbf{w}'\cdot\mathbf{i}=0,\ \text{and}\ \mathbf{w}'\cdot\mathbf{B}=0\,.$$

In addition, for the range constraints, we have

$$\begin{aligned} &\tilde{l}_{j1} \ge 0,\ w_j - U_j \le 0,\ \text{and}\ \tilde{l}_{j1}\left(w_j - U_j\right) = 0 \\ &\tilde{l}_{j2} \ge 0,\ L_j - w_j \le 0,\ \text{and}\ \tilde{l}_{j2}\left(L_j - w_j\right) = 0 \end{aligned} \tag{11.30}$$

Equation 11.29 can be solved as

$$\mathbf{w} = \frac{1}{2\lambda}\boldsymbol{\Sigma}^{-1}\left(\mathbf{f} - l_0\mathbf{i} - \sum_{i=1}^{K} l_i\mathbf{b}_i - \tilde{\mathbf{l}}_1 + \tilde{\mathbf{l}}_2\right) = \frac{1}{2\lambda}\boldsymbol{\Sigma}^{-1}\mathbf{f}_{\text{adj}}. \tag{11.31}$$

Hence, the optimal weights must be of the form of Equation 11.31, which resembles the optimal weights of unconstrained portfolios with forecasts adjusted for various constraints and then scaled by λ to give the targeted tracking error.

When the range constraint is nonbonding, i.e., $L_i < w_i < U_i$, we have $\tilde{l}_{j1} = 0$ and $\tilde{l}_{j2} = 0$ according to the condition (11.30). If $w_i = U_i$, i.e., the weight is at the upper bound, then $\tilde{l}_{j1} \ge 0$ and $\tilde{l}_{j2} = 0$. Similarly, if $w_i = L_i$, i.e., the weight is at the lower bound, then $\tilde{l}_{j1} = 0$ and $\tilde{l}_{j2} \ge 0$. Therefore, between $\tilde{l}_{j1}$ and $\tilde{l}_{j2}$ only one of them can be nonzero.

When the covariance matrix is that of a multifactor model, i.e., $\boldsymbol{\Sigma} = \mathbf{B}\boldsymbol{\Sigma}_I\mathbf{B}' + \mathbf{S}$, Equation 11.31 can be simplified to

$$\begin{aligned} &\mathbf{w} = \frac{1}{2\lambda}\mathbf{S}^{-1}\mathbf{f}_{\text{adj}},\ \text{or} \\ &w_i = \frac{f_i - l_0 - l_1 b_{1i} - \cdots l_K b_{Ki} - \tilde{l}_{1i} + \tilde{l}_{2i}}{2\lambda\sigma_i^2} \end{aligned} \tag{11.32}$$

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