From Implied Volatility Surface To Quantitative Options Relative Value Trading *

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Abstract

The only thing one can say about financial markets is that parsimonious information on option prices is available in time and space, and that we can only use the No-Dominance law (or stronger version of No-Arbitrage) to account for it. Thus, one requires a consistent model to assess relative value between them. We describe a single parametric model for the entire volatility surface with interpolation and extrapolation technique generating a smooth and robust implied volatility surface without arbitrage in space and time. Prices can now be generated such that the No-Dominance principle is preserved, and one can safely assess relative value between them. In order to perform statistical analysis of the relationships between points on the implied volatility surface (IVS), we are left with finding a way of modeling dynamically the agents rational anticipations. We assume that the volatility surface is dynamically modified according to the stock price realisations. Having related the stock price level to the implied volatility surface, we use their respective historic evolution to characterise the transition probabilities, that is, the conditional densities. A statistical technique is used to regress the observed implied smile against the realised stock level. Therefore, the current stock evolution directly influences its future increment which means that, given the stock price at a future time, the conditional density is known.

Implied Volatility Surface, Calibration, Options Relative Value, Quantitative Strategies, Statistical Dynamics Of The Smile

1 Introduction

The Black-Scholes model [1973] for pricing European options assumes a continuous-time economy where trading can take place continuously with no differences between lending and borrowing rates, no taxes and short-sale constraints. Investors require no compensation for taking risk, and can construct a self-financing riskless hedge which must be continuously adjusted as the asset price changes over time. In that model, the volatility is a parameter quantifying the risk associated to the returns of the underlying asset, and it is the only unknown variable. However, since the market crash of October 1987, options with different strikes and expirations exhibit different Black-Scholes implied volatilities (IV). Hence, the Black-Scholes formula can be used as a mapping device from the space of option prices to a single real number called the implied volatilities. The out-of-the-money (OTM) put prices have been viewed as an insurance product against substantial downward movements of the stock price and have been overpriced relative to OTM calls that will

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pay off only if the market rises substantially. As a result, the implicit distribution inferred from option prices is substantially negatively skewed compared to the lognormal distribution inferred from the Black-Scholes model. That is, given the Black-Scholes assumptions of lognormally distributed returns, the market assumes a higher return than the risk-free rate in the tails of the distributions.

In principle, one should use the future volatility in the Black-Scholes formula, but its value is not known and needs to be estimated. As a result, practitioners use the implied volatility when managing their books, assuming that the IV bears valuable information on the asset price process and its dynamics. Hence, acknowledging the limitations of the Black-Scholes model, traders keep having to change the volatility assumption in order to match market prices. Therefore, as explained by Fengler [2005], the information content of the IV and its capability of being a predictor for future asset price volatility is of particular importance on trading markets. It is assumed that the IV bears valuable information on the asset price process and its dynamics which can be exploited in models for the pricing and hedging of other complex derivatives. One of the reason being the existence of highly liquid option and futures markets dated back from the early nineteen-nineties. In an efficient market, options instantaneously adjust to new information such that the IV predictions do not depend on the historical price or volatility series. The overall consensus of the literature is that IV based predictors do contain a substantial amount of information on future volatility and are better than only time series based methods. Nonetheless most authors conclude that IV is a biased predictor.

As opposed to traditional fields of Applied Mathematics where fundamental laws exist, financial markets only obey the No-Dominance principle (monotonicity rule), so that modeling is based upon comparison between assets (information driven). For instance the Black-Scholes model derives the price of an option by comparison to the underlying asset price. Hence, we need to incorporate the relevant information in the model (objective driven). Information being different from one market to another, the relevant criterion depends on the objective for which the model is built, such as prediction, hedging, risk management etc. Therefore, the nature of the model depends on the corresponding market and its final objective. In the same spirit as Blyth [2004], we want to establish a consistent framework for relative-value volatility trading that enables identification of value within the universe of equity, FX, or commodity liquid option products. The logicist approach is concerned with reasoning and more specifically with what we can say about the derivatives markets (see Dempster [1998]). It requires choices, necessarily subjective, driven by the judgement of the investigator, about which objective features to include or omit in the modeling formulation. Careful assessment is required. This is to oppose to the pragmatic approach which is usually adopted whereby the simplest model that gives a price for the relevant instrument is used for pricing and risk management. Following the logicist approach, the only thing we can say about financial markets is that parsimonious information on option prices is available in time and space, and that we can only use the No-Dominance law (or stronger version of No-Arbitrage) to account for it. Further, options on a single stock with different strikes and maturities are functions of the same IV surface and depend in non-trivial ways on the same underlying dynamics. As a result, when analysing volatility surfaces, it makes little sense to consider an option with maturity T and strike K as an independent instrument with its own volatility. Thus, one requires a consistent model to assess relative value between them.

To perfectly know the future expected dynamics of the underlying stock price requires the knowledge of call prices for all strikes and maturities, but since in practise we can only observe a few market prices from standard strikes and maturities, the market is therefore incomplete and there are more than one acceptable price (volatility) surfaces satisfying the no-arbitrage conditions. As a result, proper option portfolio management requires suitably chosen interpolation and extrapolation techniques of the IV surface, forcing practitioners to use a fully specified model. The standard approach is to interpolate and extrapolate market prices or volatilities to complete the market. Practitioners use the Black-Scholes implied volatility and smooth the prices in a parabolic way to generate the missing prices. Due to the difficulty of defining a single parametric function for the entire volatility surface, a typical solution is to estimate each smile independently with some nonlinear function. Then, the IV surface is reconstructed by interpolating total variances along the forward moneyness. One way forward is to interpolate the smile within the region of observed prices with a polynomial as was done by Malz [1997] and to cut the volatility outside that region. Similarly, Daglish, Hull and Suo [2006] performed a Taylor expansion up to the second order of the implied volatility surface around the money forward level. Alternatively, one can fit with little control a parametric form for the implied volatility derived from a model which is usually the result of an asymptotic expansion of a stochastic volatility model, see for examples Hagan et al. [2002] or Gatheral [2006]. However, direct interpolation and extrapolation of implied volatility surfaces does not guarantee a resulting smooth risk-neutral density, hence a proper local volatility surface.

In a market with a limited number of prices, a model of interpolation and extrapolation of the volatility surface should have few parameters with the ability of mapping a large family of surfaces. Among the different techniques proposed for obtaining a smooth volatility surface from market prices, Rebonato et al. [2004] argued that modeling directly the density was the most desirable approach. They extended the mixture of normals approach proposed by Alexander [2001], obtaining a density with non-zero skew and satisfying the risk-neutral forward condition while retaining an unconstrained numerical search. However, in markets with long maturity products and discrete dividends such as the Japanese market, it is important for model pricing to obtain a reliable volatility surface satisfying the no-arbitrage constraint not only in space but also in time. So, we intend to generate a surface without arbitrage in time and in space as closely as possible from the market data.

Interpolation techniques to recover a globally arbitrage-free call price function have been proposed by Kahalé [2004] where he considered a piecewise convex polynomials, and by Wang et al. [2004] who suggested the use of a cubic B-spline interpolation. Later, Fengler [2009] considered smoothing call prices with a natural cubic splines by choosing to minimise a penalised sum of squares resulting in an iterative quadratic minimisation problem under constraints. Similarly, we propose in Section (5.2) to impose smoothness and value constraints directly on the market prices and their resulting implied volatility surface. We impose the market prices to satisfy the no-arbitrage conditions and we smooth the implied volatility surface by fitting a special functional form to the observed market prices. We consider in Section (5) a parametric representation of the market call prices under constraints in order to smooth the data and get nice probability distribution function (pdf). We use the Differential Evolution algorithm described by Bloch et al. [2011] to calibrate the model's parameters to a finite set of option prices. Using the properties of our parametric model, in Section (5.3) we compute analytically the Greeks of the model. Similarly to Ingersoll [1998], we let the Digital Bond be the value at time t of receiving one dollar at the maturity T if and only if a probabilist event occurs, while the Digital Share is the value at time t of receiving one share of the stock at the maturity T if and only if a probabilist event occurs. In Section (5.4), we obtain analytical solution to the Digital Bond and the Digital Share in the special case where the instantaneous volatility of the stock price is a deterministic function of time and the stock price. In Section (5.5), using closed-form solutions for European options and Digital options, we infer analytically the skew and curvature of the parametric model for all strikes and maturities.

Defining in Section (2) option relative value, its principle which consists in taking advantage of price differences between related financial instruments by simultaneously buying and selling the different instruments, can be seen as a directional bet on the expected future dynamics of the underlying stock price. Hence, one need to provide traders with tools capable of properly quantifying market arbitrages to option prices, together with tools modeling dynamically the agents rational anticipations. Since parsimonious information is available in time and space, and since we can only use the No-Dominance law to account for it, our parametric model reaches its objectives by fitting globally the market prices such that the no-arbitrage in time and space is satisfied. We are left with finding a way of modeling dynamically the agents belief of future evolution of the stock price returns. Bloch et al. [2002] showed that there exists a link between the implied volatility surface and the spot level over time. In Section (6) we take the spot level as driving factor, and assume that the volatility surface is dynamically modified according to the realisation of stock prices. Having related the stock price level to the volatility surface, we use the historic evolution of the implied volatility surface to characterise the transition probabilities, that is, the conditional densities. Then from the implied marginal distributions observed on the market, we infer the joint distributions which are consistent with the relationship between implied volatility and stock price level over time and satisfy the absence of arbitrage (AOA) constraint. This hypothesis implies that European call prices are not time-dependent, that is, do not depend on current time but are stock dependent. Therefore, the current stock evolution directly influences its future increment which means that, given the stock price at a future time, the conditional density is known.

2 Defining option relative value

Relative value is the attractiveness of one instrument relative to another measured in terms of risk, liquidity, and return. Given that options are a derivative instrument, meaning they derive their value from an underlying security, options themselves have value relative to other options. Hence, when comparing two options, one option's value can be deduced from or defined relative to another option's value. In practise, even though options are quoted on the basis of price, option traders assess relative value on the basis of volatility. That is, knowing the price of the option, one can solve for the IV of the underlying stock by inverting the Black-Scholes formula.

2.1 From pair trading to skew trading

2.1.1 Pair trading

One of the simplest relative-value arbitrage called pair trading is an investment strategy that seeks to take advantage of price differences between related financial instruments by simultaneously buying and selling the different instruments, thereby allowing investors to potentially profit from the relative value of the two products. The simultaneous purchase and sale of two similar products whose prices are not in synchrony with what the trader believes to be "true value" is called an arbitrage in the hedge funds world. For example, acting on the assumption that option prices will revert to their true value over time, traders will sell short the overpriced security and buy the underpriced one. Once prices revert to their true value, the trade can be liquidated at a profit. It is clear from this example that what hedge funds call an arbitrage is simply the fact that their views about the future performance of the underlying asset differ from the market's view. It is a purely directional bet on the expected future dynamics of the underlying stock price. However, in the classical option pricing theory, arbitrages are quite different (see Shreve [2004]). One can reconcile the two approaches by providing the traders with tools capable of properly quantifying market arbitrages to option prices, together with tools modeling dynamically the agents rational anticipations.

2.1.2 Skew trading

The IV being a biased predictor of the future volatility, it bears valuable information on the asset price process and its dynamics. That is, the options market provides a remarkable outlook on future expectations of the value or performance of an underlying asset, allowing traders to compare various strike prices over different maturities. Since supply and demand ultimately drive prices, traders can learn which options are cheap or expensive relative to others, as measured by the implied volatility of each option. This relative value is defined as options skewness, or skew, and can be used to identify trading opportunities. Trading "with" the skew is defined as buying higher valuation options and selling lower valuation options, while trading "against" the skew is defined as buying lower valuation options and selling higher valuation options. Traders can then use quantitative tools and decide to either trade "with" or "against" the skew or smile. Even though there are different explanations for the skew, one of the most straightforward is leverage. Skew is priced to reflect the market's assessment of future risk, which takes into account an asset's current price, pricing trends, and the potential for a sudden price jump in either direction. The basis of skew is that even though options are founded on a risk-neutral concept, market participants have risk profiles that affect the supply-demand relationship of the options market. For example, many equity option traders tend to sell upside calls and purchase downside puts (termed options collar) to reduce their overall risk exposure. Given that many participants have predetermined preferences, options with lower strikes tend to have higher implied volatilities relative to options with higher strikes. Consequently, if you believe that the skew is wrongly priced, then the market is either underestimating or overestimating the probability of a large upside or downside move in the underlying instrument. Hence, one can look at the volatility skews of an index or a single stock over different maturities and compare their relative predicting power for the movement of the underlying index or stock. For example, in the case where the short term months have very low implied volatilities relative to intermediate term options, it is usually the case that the options market is expecting news likely to move the index or stock in a dramatic way in a few months' time, but not in the shorter term. The trading strategies should be based on the trader's prediction for market movements, or lack thereof, relative to market expectations (in the time period selected). However, one must recognise the conditions where a strategy might fail.

2.1.3 Taking a view on the Skew

When trading "with" the skew, the market is willing to overpay for a certain strike price or time frame. A trader can purchase the higher implied volatility and sell a different strike price or month at a lower implied volatility. He chooses to purchase higher-valued options and sell lower-valued options because his market forecast could mirror the options market (more demand relative to supply creates the skew) where he gives away theoretical edge. Alternatively, he can trade "against" the skew, assuming he forecasted a symmetric event and wanted to gain theoretical edge by selling higher-valued options and buying lower-valued options. Trading against the skew is a "reversion to the mean" strategy, meaning that we are implementing an options strategy that benefits from a more normalised trading scenario (think of symmetrical rather than asymmetric underlying moves). Recall, "cheap" or "expensive" attributions are based on a symmetric mathematical options model, such as the Black-Scholes formula. For example, if the trader believes the market will remain calm in the near-term but that, in the coming months, it's going to become more volatile, he can implement a long calendar position. This is considered trading "with" the skew if the shorter-dated options are priced much cheaper than the longer-dated options. There are many variations of trading "with" the skew. The only requirement is that the option you sold has a lower implied volatility than the option you purchased. To conclude, since the IV is an indicator of potential outcomes for an underlying asset, with insight into market expectations, traders can choose to trade "with" the skew (smile) or "against" it.

2.2 Accounting for the dynamics of the IV surface

The main idea behind relative value is that the goodness of the model is not so important as long as it is applied consistently across all option prices. That is, one focus on the price of options relative to each other. In general, many traders believe that to take advantage of skew opportunities they should plan to buy undervalued options and sell overvalued options. That is, whether the individual options in a specific market are out of line relative to each other, and not whether the market is out of line with the model. But, what does it mean to say that an option is undervalued or overvalued? The valuation of most assets is normally approached on a relative basis. In practise, most hedge funds assume stock prices are lognormally distributed, but the actual price distributions in all markets tend to have fatter tails than suggested by the lognormal distribution. As a result, in the real world extreme price movements occur far more frequently than implied by the standard Black-Scholes assumptions. Hence, in reality it makes more sense to be a buyer of deep OTM options than might be suggested by the model. Consequently, if you blindly believe the lognormal assumption, you would be selling more OTM options than you should, leading to potential big losses. Hence, traders should be wary of making blind assumptions about options being undervalued or overvalued.

2.3 Relating long-dated options to short-dated options

2.3.1 A mean-reverting volatility process

Central to trading options is some understanding of the fundamental forces that affect market volatility. Volatility in financial markets tends to be a mean reverting process. One of the reason for volatility to be mean-reverting is that it can not be explosive, and implied volatilities can not deviate too far from their central tendency. As a result, the further away volatility gets from its long term average or normal level, the greater the likelihood that it will move back towards its mean. This property violates the constant volatility assumption of some major options valuation models, such as the BS model. Thus, an understanding of the mean reverting nature of the volatility process enables someone taking an option's position to recover the market's implicit beliefs about the dynamics of volatility as the option moves through time. If volatility is a mean-reverting process, then clearly buying or selling it at extreme levels should be a profitable trading strategy. However, an options position has both a spot directional and a volatility component, making the strategy less trivial than expected. While volatility drops, as it should in a mean reverting process, the move in the spot price may rendered the position unprofitable. In addition, implied volatilities have to drop faster than the forward volatility curve anticipates for the position to make money.

2.3.2 Modeling the term-structure of volatility

Options at different dates or tenors T_i for $i = 1, .., n$ reveal a term structure of implied volatilities that is a useful tool in relative value analysis. Long dated options are priced relative to the value of short dated options. The effect is similar to that observed in fixed income yield curves. In general, long dated implied volatilities do not fully respond to transitory movements in short-dated implied volatility because of the mean reverting nature of the series. That is, a shock to short-dated volatility is likely to dissipate over a longer time interval as the market reverts to its normal or average level. Finally, greater volatility of short-dated volatility reduces the value of short dated volatility as a benchmark for valuing long dated volatility. As a result, it tends to lower long-dated implied volatilities. Using this intuition, practitioners (see Simpson [2003]) have devised statistical models where the term-structure of the IV is a function of two factors, the short-dated volatility and the volatility of short-dated volatility

$$\sigma_l(t) = \alpha + \beta_1 \sigma_s(t) + \beta_2 \sigma_s^2(t) + \epsilon$$

where $\sigma_l(t)$ is the volatility of the long-dated options and $\sigma_s(t)$ is that of the short-dated option. The parameters of the model are α and β_i for $i = 1, 2$ and ϵ is a white noise. Deviations from the expected values from this econometric model suggest a useful approach to determining whether implied volatilities in a specific portion of the yield curve are expensive or cheap.

2.4 Some trading strategies

We saw in the Section (2.1) that having insight into market expectations of the future dynamics of the underlying price returns enables traders to define trading strategies. It requires skills originating from risk management and leveraged to systematic speculation, such as single surface risk management as well as delta and gamma trading. Other examples include vega trading, dispersion and correlation trading. As explained by Hutchinson [2011], there exists a large number of option value strategies among which the most famous one are

1. Volatility Surface Relative Value. It optimises a portfolio of long and short option positions, on the same underlying security, to harvest gains from changes in the shape of the volatility surface.

2. Implied Correlation / Dispersion. It trades the relative value relationship between the implied volatility of an Index option and the implied volatilities of the Component Stocks that comprise the Index.
3. Capital Structure Arbitrage. It trades the relative value relationship between options on various segments of the capital structure of an individual company, or the options embedded in the same.
4. Generic Volatility Long/Short. It trades the relative value relationship between options on a full array of securities, and is not limited to one underlying security or issuer of securities.
5. Directional Volatility. It creates an option portfolio that exhibits a continuous short or long exposure to changes in implied volatility.

Options on single stocks or indices with different strikes and maturities being functions of the same IV surface and depending in non-trivial ways on the same underlying dynamics, one requires a consistent model to assess relative value between them. Since parsimonious information is available in time and space, and since we can only use the No-Dominance law to account for it, our model must fit globally the market prices such that the no-arbitrage in time and space is satisfied. We are left with finding a way of modeling dynamically the agents rational anticipations in order to infer relative value between option prices and take advantage of their differences over time.

2.5 Some fundamental analysis

2.5.1 Volatility surface relative value

As discussed previously in Section (2.4) many relative value strategies exist, and we are going to concentrate on volatility surface relative value. Note, as the IV surface is a three-dimensional array obtained by plotting the inverted option prices along two axes, the time to maturity and the strike prices, one must rely on tools quantifying the relative value of option prices not only on space but also on time. Since the IV surface changes shape as investors change their expectation of risk to come, and thus the price they are willing to pay for options on the surface, one can evaluates the IV surface and optimises a portfolio of long and short option positions. For example, point pairs on the surface may be initiated when the pair relationship is at statistical extreme, or it reflects misplaced expectations. In that setting the investment strategy relies on some fundamental analysis consisting of

- examination of the IVS
- analysis of option market values relative to model values
- statistical analysis of relationships between points on the IVS
- objective consideration of factors affecting expectations of risk (IVS dynamics)

Then the trades are implemented and executed based on some targeted opportunities. Notes, trades can also be identified for risk management purposes by offsetting aggregated portfolio sensitivities (Greeks). In general the market volatility surface is analysed relative to a model volatility surface and an electronic eye is used to identify and display trade opportunities. Using quantitative tools, traders should be able to assess opportunities, identify hedges and execute orders. Therefore, one can identify potential opportunities by profiling a smile of IV generated by proprietary modeling against the IV of options actually observed in the market place. For instance, one can consider cheap IV versus model as well as the expensive one against model. For that relation to exist the model must be arbitrage-free, not just in space but also in time. In addition, the market surface can also be described statistically by looking at the relationship between each point. For example, one can look at the ratios and spreads of 10% OTM puts for various maturities, placed in one-year historical context via a percentile calculation. A pair with a Vol Ratio Percentile of less than 15% would be coloured blue or green, while high ratios would be coloured red.

2.5.2 Dispersion relative value

Assuming that we know the IV of a portfolio (or index) together with the weights and IV of the individual stocks in the portfolio, we can solve for the expected correlation also called the implied correlation (IC). The IC is traded via positions in single stock options offset by index option positions. As the relationship between index volatilities and their component stock volatilities change quite a lot over time, so does the implied correlation. The IC is usually viewed by practitioners as one of the option market's expectation of systematic risk. In principle it is increasing when investors sell stocks, and decrease as more benign expectations filter in.

The growth of the variance swap (VS) market and the success of the VIX raised the profile of volatility trading as an asset being negatively correlated to the underlying equity market. Rather than solving the difficult questions of knowing which volatility to own, and how to mitigate the expense of decay, the VS is a liquid, non-strike dependent hedge. One strategy consists in owning a basket of single stock volatility, assuming that volatilities would increase when the underlying shares sell off, and then take advantage of the elevated implied correlation. Rather than selling the basket of single volatility, the relatively more elevated index volatility is sold against the basket of single stock volatilities. Based on empirical studies, it is assumed that a basket of higher beta single stock volatilities may have a larger absolute move up than index volatility during a sell-off. However, single stock volatility having wider bid/offer spreads, exhibit a greater expected cost over the course of a year, and potentially unmanageable bid/offer spreads. In addition, the relationship between single stock volatilities and index volatility may not perform as historically expected, leading to basis risk. Nonetheless, some opportunities each year to sell elevated index volatility present the possibility of significantly mitigating the expense.

3 Understanding the mechanisms behind option relative value

Given the definition of option relative value in Section (2), one must be able to evaluate risk, liquidity, and returns from the financial instruments to infer their attractiveness.

3.1 The inconsistencies of the BS model

The Black-Scholes model [1973] for pricing European options assumes a continuous-time economy where trading can take place continuously with no differences between lending and borrowing rates, no taxes and short-sale constraints. Investors require no compensation for taking risk, and can construct a self-financing riskless hedge which must be continuously adjusted as the asset price changes over time. To the extent that the world deviates from the BS assumptions of constant volatility and a lognormal distribution to price changes, the BS model will be biased in certain, often predictable ways (see Hull and White [1987]). In reality, since the volatility is not constant it has a major impact on the values of certain options, especially those options that are away from the money, because the dynamics of the volatility process rapidly change the probability that a given out-of-the-money (OTM) option can reach the exercise price. Hence, the BS model consistently underestimates the value of an option to the extent that volatility is stochastic rather than constant as assumed. It is well known in the literature that both the crash fears and the volatility evolution are explanatory factors for the negatively skewed implicit distribution and that each of them implies a different relationship between the option maturity and the implicit skewness. Stochastic diffusion models imply a direct relationship between the option maturity and the magnitude of the implicit skew while that relationship is inversed in a finite variation jump model. This is because jump components address moneyness biases while having stochastic latent variables allows distributions to evolve stochastically over time. A second major assumption of the BS model is that the underlying returns are normally distributed with a variance proportional to the length of time over which the asset trades. However, a number of academic studies show that the underlying price movements are neither normally nor lognormally distributed. Many financial assets exhibit more skewness and kurtosis than it is consistent with the Geometric Brownian Motion model of Black-Scholes. According to Bakshi et al. [1997] only a combination of jump and stochastic volatility models is capable of capturing the IVS. Similarly, Bates [2000] studied empirically the impact of each explanatory factors on the shifted distribution and concluded that one need a combination of both to recover a good fit to the market distribution. Again, the problem with the distributional assumption of the BS model means that it generally underestimates equity or FX option values because the likelihood of having an extreme price movement is greater than the model expects.

3.2 Accounting for the market price of risk

When in incomplete market, we can not perfectly hedge an equity derivative, but we can divide the asset into two pieces, one which is hedgeable and the other being totally unhedgeable where the hedgeable portion can be priced using arbitrage-free arguments. However, market practise consists in using the Black-Scholes formula as a tool for representing market risk via the implied volatility surface and then devise hedging strategies based on the first component of the option value. Even though most if not all the assumptions in the Black-Scholes model do not correspond to market realities, it gives a robust representation of market's behaviour and should be used to guide us toward the correct market value (see El Karoui et al. [1998]). In that model, volatility is the only unknown variable when pricing options and as such it is the only parameter quantifying the risk associated to the returns of the underlying asset. In the Black-Scholes formula, the option price is equivalent to the cost of continuously hedging the option, but in practise dynamic hedging is not a risk-free proposition. Moreover, additional risks must be taken into consideration such as changes in volatility, changes in interest rates, changes in dividends, trading costs and liquidity. As a results, all these risks are incorporated into the implied volatility in such a way that the skew can be seen as the view from the market that options with different strikes and different expiries have different risks and should be valued accordingly. That is, one expect a normal behaviour of the stock prices near ATM options which can be reasonably hedged, but when the stock prices exhibit large downward movements, the fear of non-headgeable jumps dominate. For example, in the case of short maturity options where OTM put prices should have a zero market value, they actually exhibit positive values representing exclusively a market risk premium. As discussed by Figlewski [1989] the IV is a free parameter containing expected volatility and everything else that affects option demand and supply but it is not the model, making it very difficult to disentangle the different factors. Once the market price of risk has been implicitly entered into the pricing equation, the traders are left to define their hedging strategies. For example, when delta hedging, the traders make the derivative of the portfolio with respect to the stock price zero at a point, but the portfolio still change in value if the asset moves a short distance from that point. However, if the distance is small, the change in value will be proportional to its square, whereas for a non delta hedged portfolio the change will be linear in the distance (see Fengler [2005]). Fortunately, one simplification to the trader's position arises from the fact that he will have bought and sold many different options on the same underlying. Each of these options has a delta and since in a complete market the model is linear, the traders can hedge them all by simply adding their deltas together. As the deltas of long and short positions have opposite signs, and if the portfolio is a mixture of such positions, the deltas will at least partially cancel each other.

3.3 Modeling the asset returns

Alternative explanations, from jump-diffusion process for the dynamics of the underlying stock, for the divergence between the risk-neutral distributions and observed returns include peso problems, risk premia (see Lettau et al. [2003] and Novales et al. [2003]) and option mispricing but no consensus has yet been reached. The notion that equity returns exhibit stochastic volatility is well documented in the literature, and evidence indicates the existence of a negative volatility risk premium in the options market (see Bakshi et al. [2003]). CAPM suggests that the only common risk factor relevant to the pricing of any asset is its covariance with the market portfolio, making beta the right measure of risk. However, excess returns on the traded index options and on the market portfolio explain this variation, implying that options are non-redundant securities. As a result, Detemple et al. [1991] argued that there is a general interaction between the returns of risky assets and the returns of options, implying that option returns should help explain stock returns. That is, option returns should appear as factors in explaining the cross section of asset returns. For example, Bekaert et al. [2000] investigated the leverage effect and the time-varying risk premium explanations of the asymmetric volatility phenomenon at both the market and firm level. They found covariance asymmetry to be the main mechanism behind the asymmetry for the high and medium leverage portfolios. Negative shocks increase conditional covariances substantially, whereas positive shocks have a mixed impact on conditional covariances. While the above evidence indicates that volatility risk is priced in options market, Arisoy et al. [2006] used straddle returns (volatility trade) on the $S\&P$ 500 index and showed that it is also priced in securities markets. As a result, one can assume the existence of arbitrage over short maturity option prices by assuming a non-null market risk premium. Similarly to the commodity markets where the holder of the spot is compensated for holding one unit of inventory in case of shortage, we can compensate the holder of the spot in the equity market for the risk of a large downward jumps.

4 The equity setup

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}_t$ is a right continuous filtration including all $\mathbb{P}$ negligible sets in $\mathcal{F}$. For simplicity, we let the market be complete and assume that there exists an equivalent martingale measure $\mathbb{Q}$ as defined in a mixed diffusion model by Bellamy and Jeanblanc [2000]. We let the underlying process $(S_t)_{t\geq 0}$ be a one-dimensional Ito process valued in the open subset D with dynamics under the risk-neutral measure $\mathbb{Q}$ being

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_S(t) \tag{4.1}$$

where the drift $\mu : D \to \mathbb{R}$ as well as the diffusion $\sigma : D \to \mathbb{R}$ are regular enough to have a unique strong solution valued in D. There are various ways of modeling discrete dividends, all resulting in different prices for path dependent options (see Bos et al. [2003]). We are going to introduce some notations on discrete dividends, and introduce the pure stock process with its corresponding pure implied volatility.

4.1 Discrete dividends

We are now introducing discrete dividends to the stock price and describe our framework. Given $t_0 = 0$ and $\mu_t = (r(t) - q(t))$ where $r(t)$ and $q(t)$ are deterministic functions of time, we let $C(t_0, t) = \frac{Re(t_0,t)}{P(t_0,t)} = e^{\int_{t_0}^t \mu_s ds}$ be the capitalisation factor from time zero until time t and D_t be the dividends paid till time t given by

$$D_t = \sum_{i=0}^{\infty} \mathcal{H}(t - t_{d_i}) d_i C(t_{d_i}, t)$$

with dynamics given by

$$dD_t = \mu_t D_t dt + D_t^{'} dt$$

where $D_t^{'} = \sum_{i=0}^{\infty} \delta(t - t_{d_i}) d_i C(t_{d_i}, t)$. Similarly, the dividends paid from time t till time T and capitalised at maturity T are given by

$$D(t, T) = D_T - C(t, T) D_t$$

Discounting these dividends to time t, we get the present value $D_{PV}(t; t, T)$ of the dividends paid between t and T

$$D_{PV}(t;t,T) = \frac{D(t,T)}{C(t,T)} = \frac{D_T}{C(t,T)} - D_t$$

Having made an assumtion on discrete dividends, for simplicity of exposition, we are going to consider the dynamics of the stock price in the Spot model.

4.2 The Spot model

We can always rewrite the spot price with discrete dividends in terms of a pure process Z with no discrete dividends and no rates with the transformation

$$S_t = a(t)Z_t + b(t) \tag{4.2}$$

where $a(t)$ and $b(t)$ are deterministic function of time (see Overhaus et al. [2002]). We assume that the dynamics of the stock price $(S_t)_{0 \leq t \leq T}$ in the Spot model and under the risk-neutral measure are given by

$$\begin{aligned} S_t &= Z_t - D_t \\ \frac{dZ_t}{Z_t} &= \mu(t)dt + \sigma_Z(t,Z)dW_Z(t) \\ Z_{t_0} &= S_{t_0} \end{aligned}$$

In that setting the price of a European call option with strike K and maturity T is

$$C_S(t_0;K,T) = P(t_0,T)E_{t_0}[(Z_T - K^{'})^+]$$

where $K^{'} = K + D_T$. This model is popular mainly due to the fact that when adding the already paid dividends to the strike and considering the special case where the volatility σ_Z is a consant, one can use the Black-Scoles formula to compute the call option price.

4.3 Pricing simple products : Digital contracts

Similarly to Ingersoll [1998], we let $D(S,t,T;\xi)$ be the value at time t of receiving \$1 at the maturity T if and only if the event ξ occurs. Such a contract is called a binary option or a cash-or-nothing option and we will call it a Digital Bond. We also let $S(S,t,T;\xi)$ be the value at time t of receiving one share of the stock at the maturity T (excluding any intervening dividends) if and only if the event ξ occurs. Such a contract is called a all-or-nothing share and we will call it a Digital Stock. Given the process $(S_t)_{t \geq 0}$ and the appropriate measure, the price of a call option can be expressed in terms of those quantities as

$$\begin{aligned} C(t,S_t,T,K) &= P(t,T)E_t[(S_T - K)^+] = P(t,T)E_t[(S_T - K)I_{S_T > K}] \\ &= P(t,T)\big(E_t[S_T I_{S_T > K}] - KE_t[I_{S_T > K}]\big) \end{aligned}$$

and

$$P(t,T)E_t[S_T I_{S_T > K}] = C(t,S_t,T,K) + KP(t,T)E_t[I_{S_T > K}]$$

Now from the definition of a digital option we have $E_t[I_{S_T > K}] = -\frac{1}{P(t,T)}\partial_K C(t,S_t,T,K)$ which is approximated with call-spread option, and the Digital Stock becomes

$$S(S,t,T;S_T > K) = P(t,T)E_t[S_T I_{S_T > K}] = C(t,S_t,T,K) - K\partial_K C(t,S_t,T,K)$$

and can be statically replicated with a call option and a call-spread. Therefore, the pricing of other European derivatives with piecewise linear and path-independent payoffs only requires valuing Digital Bond and Share with event $\xi = \{L < S_T < H\}$ for some constants L and H. For example, the call option price is

$$C(t,T,K) = S(S,t,T;S_T > K) - KD(S,t,T;S_T > K) \quad (4.3)$$

while the put option price is

$$P(t,T,K) = KD(S,t,T;S_T < K) - S(S,t,T;S_T < K)$$

In the special case where the rate, repo and volatility are constants, Cox and Ross [1976] showed that the Digital Bond and Digital Share with event $\xi = \{S_T > K\}$ where K is the strike price could be valued under the risk-neutral measure with the Black-Scholes formula, that is

$$\begin{aligned} D(S,t,T;\xi) &= E^Q[e^{-\int_t^T r_s ds} I_{S_T > K} | \mathcal{F}_t] = e^{-r(T-t)} N(d_2(T-t, S_t e^{(r-q)(T-t)}, K)) \\ S(S,t,T;\xi) &= E^Q[e^{-\int_t^T r_s ds} S_T I_{S_T > K} | \mathcal{F}_t] = S_t e^{-q(T-t)} N(d_1(T-t, S_t e^{(r-q)(T-t)}, K)) \end{aligned}$$

where

$$d_2(t,x,y) = \frac{1}{\sigma\sqrt{t}} \log \frac{x}{y} - \frac{1}{2}\sigma\sqrt{t} \text{ and } d_1(t,x,y) = d_2(t,x,y) + \sigma\sqrt{t}$$

However, when the volatility of the stock price is stochastic, and more generally when the instantaneous volatility of the stock price, the spot rate and repo rate are stochastic, one can no-longer use the Black-Sholes formula. Under general Markov processes for the model parameters, the conditional probabilities of the Digital Bond and Digital Share are difficult to solve analytically under any probability measure, and numerical tools must be used. We are going to show that in the special case where the instantaneous volatility of the stock price is a deterministic function of time and the stock price, one can obtain analytical solution to the Digital Bond and the Digital Share.

5 The choice of a volatility model

Since in practise we can only observe a few market prices from standard strikes and maturities with wide or narrow spreads depending on the liquidity on the market and the volume traded, the market is incomplete and there are more than one acceptable price (volatility) surface satisfying the no-arbitrage conditions. Hence, multiple risk-neutral distributions can fit the option prices so that one needs some additional criteria to generate a unique probability distribution function (pdf). To do so, one can either impose a functional form to the probability distribution and estimate its parameters using option data, or one can choose non-parametric methods obtaining perfect fit to market data. However, non-parametric methods are less adapted to the extrapolation problem than the parametric ones, and they tolerate less control over the generated volatility surface. We are therefore going to concentrate on parametric methods.

5.1 The standard approach

The standard approach is to interpolate and extrapolate market prices or volatilities to complete the market. Practitioners use the Black-Scholes implied volatility and smooth the prices in a parabolic way to generate the missing prices. However, direct interpolation and extrapolation of implied volatility surfaces does not guarantee a resulting smooth risk-neutral density, hence a proper local volatility surface. As described by Daglish et al. [2006], a natural choice would be to perform a Taylor expansion up to the second order of the implied volatility surface around the money forward level. In general, the implied volatility is modeled with a functional form of the smile around the money forward as

$$\Sigma(t,S_t;K,T-t) = f\left(t,T-t,\, X_t^{T-t}, Y_t^{T-t}, Z_t^{T-t}; K\right)$$

where the processes X_t^{T-t}, Y_t^{T-t} and Z_t^{T-t} represent respectively the at-the-money volatility, the skew and the curvature of the smile at time t for call options with maturity T *i.e.* time to maturity $T-t$. Using historical data, some authors studied the dynamics of these process. For instance, assuming an Ornstein-Uhlenbeck dynamic for the processes Cont et al. [2002] explained the deformation of the volatility surface. Alternatively, Bloch et al. [2002] assumed the parameters to be led by the spot process holding the whole market risk. When it comes to generating a volatility surface, the evaluation time t is fixed and practitioners estimate one set of parameters X, Y and Z per trading maturities. Then they rely on some interpolation and extrapolation of parameters in time with no guarantee of satisfying the calendar spread. In the presence of discrete dividends, it is not an easy task to satisfy the time constraint given by the calendar spread. For example, we consider the functional form

$$f(t,T-t,X,Y,Z;K) = X - Y\ln\left(\frac{KP(t,T)}{S_t}\right) + Z\left(\ln\left(\frac{KP(t,T)}{S_t}\right)\right)^2 \tag{5.4}$$

with $P(t,T) = e^{-r(T-t)}$ and $X = \Sigma(t, KP(t,T); KP(t,T), T-t)$, $Y = \frac{\partial\Sigma(t,S_t;K,T-t)}{\partial K}$ and $Z = \frac{\partial^2\Sigma(t,S_t;K,T-t)}{\partial K^2}$. Note, whatever the shape we take, the smile needs to be caped and floored in the lowest and highest strikes in order to avoid any arbitrage opportunity. If the slope of the smile is to high the implicit *pdf* can be negative. Quantitatively speaking, the implied risk-neutral probability density is given by

$$\phi(S_T,T,S_t,t) = \frac{1}{S_T\Sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}d_1^2}\left(\left(1+S_Td_1\sqrt{T-t}\frac{\partial\Sigma}{\partial K}\right)^2 + S_T^2(T-t)\Sigma\left(\frac{\partial^2\Sigma}{\partial K^2} - d_1\left(\frac{\partial\Sigma}{\partial K}\right)^2\sqrt{T-t}\right)\right)$$

with $d_1 = \frac{\ln(S_t/S_T)+(r+\frac{1}{2}\Sigma^2)(T-t)}{\Sigma\sqrt{T-t}}$. Therefore, if the quantity

$$\frac{\partial^2\Sigma}{\partial K^2} - d_1\left(\frac{\partial\Sigma}{\partial K}\right)^2\sqrt{T-t}$$

is too negative, which happens when the slope of the smile $\frac{\partial\Sigma}{\partial K}$ is to high, the pdf can be negative. This means that the European call prices are not the result of the expectancy under the risk-neutral measure of future pay-off and therefore allow for arbitrage opportunities [1].

5.2 The parametric model

We are going to use a parametric representation of the market call prices in order to smooth the data and get nice probability distribution function (pdf). In order to get a model with a varying number of parameters to control the level of accuracy, we consider a weighted sum of interpolation functions taken in a parametric family. We want each function to satisfy the no-free lunch constraints in such way that they are preserved in the weighted sum. Several families can satisfy the No-Free-Lunch constraints, for instance a sum of lognormal distribution, but in order to match a wide variety of volatility surfaces the model has to produce prices that lead to risk-neutral pdf of the asset prices with a pronounced skew. Since we can always convert a density into call prices, we can then convert a mixture of normal densities into a linear combination of Black-Scholes formula, see Brigo et al. [2000]. Therefore, to obtain a pronounced skew we consider a sum of shifted log-normal distributions, that is, using the Black-Scholes formula with shifted strike (modified by the parameters $\mu_i(t)$) as an interpolation function. In our parametric model, the market option price $C_M(K,t)$ of strike K and maturity t is estimated at time $t_0 = 0$ by the weighted sum

[1] In reality the slope of the smile can be higher than the one calculated using the risk-neutral measure as the market is incomplete. However we restrict ourselves to the complete market assumption.

$$C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\sum_{i=1}^{n}a_i(t)Call_{BS}(t_0,S_0,R_t,P_t,\overline{K}(K,t),t,\Sigma_i(t)) \tag{5.5}$$

where $a_i(t)$ for $i=1,..,n$ are the weights, and $\overline{K}(K,t)=K^{'}(K,t)(1+\mu_i(t))$ with $K^{'}(K,t)=K+D_t$. In that setting $R_t=Re(0,t)$ is the repo factor in the range $[0,t]$, $P_t=P(0,t)=e^{-rt}$ is the zero-coupon bond price, $C_t=C(0,t)=\frac{R_t}{P_t}$ is the cost of carry and $D_t=D(0,t)$ is the compounded sum of discrete dividends between $[0,t]$ defined in Section (4.1). The no-arbitrage theory imposes time and space constraints on market prices. Introducing the time dependent parameters $a_i(t)$ and $\mu_i(t)$, the simplest way of ensuring these constraints is to take the same time dependency for each μ, that is, $\mu_i(t)=\mu_i f(t,\beta_i)$. As the pdf of the equity price should tend toward a single Dirac when $t\to 0$, to get control on $\mu_i(t)$ we choose to let the function $f(t,x)$ tend to 1 when $t\to+\infty$, getting

$$f(t,x)=1-\frac{2}{1+(1+\frac{t}{x})^2}$$

with $f^{'}(t,x)=[1-f(t,x)]^2\frac{1}{x}(1+\frac{t}{x})$. Moreover, to keep manageable the no-free lunch constraints, we make the weight $a_i(t)$ proportional to $\frac{a_i^0}{f(t,\beta_i)}$ for some constant $a_i^0>0$ getting the representation

$$\mu_i(t)=\mu_i^0 f(t,\beta_i) \text{ and } a_i(t)=\frac{a_i^0}{f(t,\beta_i)\times norm}$$

where $norm=\sum_{i=1}^{n}\frac{a_i^0}{f(t,\beta_i)}$. As a result, with seperable functions of time, the no-free lunch constraints simplify to

$$\begin{aligned} a_i^0 &\geq 0 \\ \sum_{i=1}^{n}a_i^0\mu_i^0 &= 0 \\ \mu_i^0 &\geq -1 \end{aligned} \tag{5.6}$$

At last, the no-arbitrage condition with respect to time holds if and only if the total variance $\nu(K,t)=\Sigma_{imp}^2(K,t)t$ is an increasing function of time t. Since we chose to model directly the square-root of the average variance, to guarantee the positivity of the local volatility we must verify

$$\Sigma_i^2(t)+2\Sigma_i(t)t\partial_t\Sigma_i(t)\geq 0$$

In the special case where the user has no information on the term-structure of the implied volatility surface, we set $\Sigma_i(t)=d_i$ where $d_i>0$. On all the other cases, the user can choose among different term-structures based on his information of the implied volatility surface. To get a general volatility function capable of generating both an upward hump or a downward one, we consider the function

$$\Sigma_i(t)=\big(a_i+b_i\ln(1+e_it)\big)e^{-c_it}+d_i$$

where $c_i>0$, $d_i>0$ and $a_i\in\mathbb{R}$, $b_i\in\mathbb{R}$ and $e_i\in]-\frac{1}{t},\infty[$. The derivative of the function with respect to time t is

$$\Sigma_i^{'}(t)=\big(-a_ic_i+b_i\frac{e_i}{1+e_it}-b_ic_i\ln(1+e_it)\big)e^{-c_it}$$

and the no-arbitrage constraint must satisfy

$$e^{-c_i t}\big[(1-2tc_i)\big(a_i+b_i\ln(1+e_i t)\big)+2tb_i\frac{e_i}{1+e_i t}\big]+d_i\geq 0$$

Since $a_i \in \mathbb{R}$, at time $t=0$ the left hand side of the inequality can becomes negative. Consequently, we must impose the constraint

$$a_i+d_i>0$$

to get the constraint satisfied at $t=0$. Further, when $t>\frac{1}{c_i}$ then for c_i sufficiently large the constant d_i will dominates $\big(a_i+b_i\ln(1+e_i t)\big)$ ensuring positivity of the left hand side. Hence, at time $t=\frac{1}{c_i}$ we must impose

$$d_i\geq\big(a_i-2\frac{b_i}{c_i}\frac{e_i}{1+\frac{e_i}{c_i}}+b_i\ln(1+\frac{e_i}{c_i})\big)e^{-1}$$

5.3 Computing the Greeks analytically for European options

Market risk management of an option portfolio requires simultaneous real-time transparency of the several different elements that can influence an option price. Option prices have sensitivities or Greeks to movements in market data such as the underlying security, the expected volatility of the underlying security, dividends, the passage of time, and interest rates. Assuming that the dynamics of the spot price with discrete dividends follow the Spot model described in Section (4.2) with $a(t,T)=1$ and $b(t,T)=-D(t,T)$, then our parametric model in Equation (5.9) corresponds to a weighted sum of Black-Scholes formulas in the Z-space. In that setting, we can compute analytically the Greeks of the parametric model. The derivative of a call option price with respect to the spot is

$$\frac{\partial}{\partial S_0}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)\frac{\partial}{\partial S_0}Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t))$$

where $\tilde{K}(K,t)=P(t_0,t)\overline{K}(K,t)$ and $\overline{K}(K,t)=K^{'}(K,t)(1+\mu_i(t))$ and $\overline{a}_i(t)=\frac{a_i^0}{f(t,\beta_i)}$. Differentiating one more time the call price with respect to the spot, we get

$$\frac{\partial^2}{\partial S_0^2}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)\frac{\partial^2}{\partial S_0^2}Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t))$$

Differentiating the call price with respect to the strike, we get

$$\frac{\partial}{\partial K}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)\frac{\partial}{\partial\tilde{K}}Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t))\frac{\partial\tilde{K}}{\partial K}\tag{5.7}$$

where $\frac{\partial\tilde{K}}{\partial K}=P(t_0,t)(1+\mu_i(t))$. Given the derivative of the Black-Scholes formula with respect to the strike K, that derivative simplifies to

$$\frac{\partial}{\partial K}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=-\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)P(t_0,t)(1+\mu_i(t))N(d_2^i(t-t_0,x,\tilde{K}(K,t)))$$

Differentiating one more time the call price with respect to the strike, we get

$$\frac{\partial^2}{\partial K^2}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)\frac{\partial^2}{\partial(\tilde{K})^2}Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t))(\frac{\partial\tilde{K}}{\partial K})^2 \quad (5.8)$$

which we can write in terms of the vega Black-Scholes as

$$\frac{\partial^2}{\partial K^2}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)\big[P(t_0,t)(1+\mu_i(t))\big]^2\frac{1}{\tilde{K}^2(K,t)\Sigma_i(t)(t-t_0)}Vega(\tilde{K}(K,t),t;\Sigma_i(t))$$

Differentiating the call price with respect to time t_0, we get

$$\begin{aligned}
& \frac{\partial}{\partial t_0}C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{d}{dt_0}(\frac{1}{norm})C_M(t_0,S_0,P_t,R_t,D_t;K,t)norm \\
+ \quad & \frac{1}{norm}\Big(\sum_{i=1}^n\frac{d}{dt_0}\overline{a}_i(t)Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t)) \\
+ \quad & \sum_{i=1}^n\overline{a}_i(t)\frac{d}{dt_0}\tilde{K}(K,t)\frac{\partial Call_{BS}}{\partial\tilde{K}}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t)) \\
+ \quad & \sum_{i=1}^n\overline{a}_i(t)\frac{d}{dt_0}\Sigma_i(t)\frac{\partial Call_{BS}}{\partial\Sigma_i}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t)) \\
+ \quad & \sum_{i=1}^n\overline{a}_i(t)\frac{\partial Call_{BS}}{\partial t_0}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t))\Big)
\end{aligned}$$

where

$$\frac{d}{dt_0}\tilde{K}(K,t)=\frac{d}{dt_0}D(t_0,t)P(t_0,t)(1+\mu_i(t))+r_{t_0}\tilde{K}(K,t)+P(t_0,t)(K+D(t_0,t))\frac{d}{dt_0}(1+\mu_i(t))$$

with $\frac{d}{dt_0}\mu_i(t)=\mu_i^0\frac{d}{dt_0}f(t,\beta_i)$. Note, when there is no dividends between t_0 and $t_0+\epsilon$ then the cumulative dividends term $D(t_0,t)=D_t$ does not depend on t_0 and the derivative becomes $\frac{d}{dt_0}\overline{K}(K,t)=r_{t_0}\tilde{K}(K,t)+P(t_0,t)(K+D(t_0,t))\frac{d}{dt_0}(1+\mu_i(t))$. Note, replacing the Greeks of our model in Dupire's forward Equation (see Dupire [1994]), we obtain a smooth and robust deterministic local volatility.

5.4 Digital contracts in the parametric model

Given our discrete dividends assumption on the dynamics of the underlying stock in Section (4.1), we should always consider the modified strike $K'(K,t)=K+D_t$ with event $\xi=\{Z_t>K'\}$ when pricing a Digital contracts. We are now going to obtain analytical solution to the Digital Bond and the Digital Share in the special case where the instantaneous volatility of the stock price is a deterministic function of time and the stock price. Setting the interest rate to zero and multiplying the strike with the discount factor, the parametric model for a call option price of maturity t becomes

$$C_M(t_0,S_0,P_t,R_t,D_t;K,t)=\frac{1}{norm}\sum_{i=1}^n\overline{a}_i(t)Call_{BS}(t_0,S_0,R_t,1,\tilde{K}(K,t),t,\Sigma_i(t)) \quad (5.9)$$

where $\tilde{K}(K,t) = P(t_0,t)\overline{K}(K,t)$ and $\overline{K}(K,t) = K^{'}(K,t)(1+\mu_i(t))$ and $\overline{a}_i(t) = \frac{a_i^0}{f(t,\beta_i)}$ for $i = 1,..,n$. As a result, to each modified Black-Scholes formula (see Annexe (A.2)) corresponds the event $\xi_i = \{Z_t > \overline{K}\}$. Given the definition of the Black-Scholes formula, we can always rewrite the model call price as

$$
\begin{aligned}
C_M(t_0,S_0,P_t,R_t,D_t;K,t) &= xRe(t_0,t)\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)N(d_1^i(t-t_0,x,\tilde{K}(K,t)) \\
&- K^{'}(K,t)\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_iP(t_0,t)(1+\mu_i(t))N(d_2^i(t-t_0,x,\tilde{K}(K,t)))
\end{aligned}
$$

Given the derivative of the parametric model with respect to the strike estimated in Equation (5.7), and combining terms together, the parametric model for a call option becomes

$$
C_M(t_0,S_0,P_t,R_t,D_t;K,t) = xRe(t_0,t)\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)N(d_1^i(t-t_0,x,\tilde{K}(K,t))+K^{'}(K,t)\frac{\partial}{\partial K}C_M(t_0,S_0,P_t,R_t,D_t;K,t)
$$

Since the Digital Bond can be expressed in terms of a digital option as $D(S,t_0,t;\xi) = -\partial_K C(t_0,S_{t_0},t,K)$, its value in the parametric model is

$$
D_M(S,t_0,t;\xi) = \frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)P(t_0,t)(1+\mu_i(t))N(d_2^i(t-t_0,x,\tilde{K}(K,t))) \tag{5.10}
$$

which we can write as a shifted weighted sum of digital options on a shifted strike

$$
D_M(S,t_0,t;\xi) = \frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)(1+\mu_i(t))D(S,t_0,t;\xi_i)
$$

Similarly, by analogy to the call price in Equation (4.3), the Digital Share is

$$
S_M(S,t_0,t;\xi) = xRe(t_0,t)\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)N(d_1^i(t-t_0,x,\tilde{K}(K,t))
$$

so that the call option becomes

$$
C_M(t_0,S_0,P_t,R_t,D_t;K,t) = S_M(S,t_0,t;Z_t > K^{'}) - K^{'}(K,t)D_M(S,t_0,t;Z_t > K^{'})
$$

Given $\frac{\partial}{\partial d_2}N(d_2) = \frac{xRe(t_0,t)}{KP(t_0,t)}\frac{\partial}{\partial d_1}N(d_1)$ and $\frac{dd_2}{dx} = \frac{1}{x\sigma\sqrt{t-t_0}}$ the delta of a Digital option is

$$
\frac{\partial}{\partial KS}C_M(t_0,S_0,P_t,R_t,D_t;K,t) = -\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)P(t_0,t)(1+\mu_i(t))n(d_2^i(t-t_0,x,\tilde{K}(K,t)))\frac{1}{x\Sigma_i(t)\sqrt{t-t_0}}
$$

5.5 Computing the Skew and Curvature analytically

For every model that one can define, we need to estimate the vector Ψ of model parameters from either the market prices or their implied volatility quotes. When the parametric model is calibrated to the market quotes and the optimum vector Ψ^* is obtained, the model call and put prices must equate the market ones. Since our model can retrieve analytically European prices as well as Digital price we show how to infer analytically the skew and curvature of the IV surface for all strikes and maturities.

5.5.1 Computing the Skew

When the parametric model is calibrated to the market quotes and the optimum vector Ψ^* is obtained, the model Digital Bond in Equation (5.10) must equate the market digital price in Equation (C.16). As a result, we can infer analytically the skew of the parametric model for the strike K and the maturity t

$$Skew(K,t) = \frac{1}{Vega(K,t;\Sigma_{BS}(K,t))}\left[-\frac{\partial}{\partial K}C_{BS}(K,t;\Sigma_{BS}) - D_M(S,t_0,t;\xi)\right]$$

Given the definition of the Black-Scholes digital option, we get

$$Skew(K,t) = \frac{1}{Vega(K,t;\Sigma_{BS}(K,t))}\left[D_{BS}(S,t_0,t;\xi) - D_M(S,t_0,t;\xi)\right]$$

which gives

$$Skew(K,t) = \frac{1}{Vega(K,t;\Sigma_{BS}(K,t))}\left[D_{BS}(S,t_0,t;\xi) - \frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)(1+\mu_i(t))D(S,t_0,t;\xi_i;\Sigma_i))\right]$$

When the shift terms are set to zero, $\mu_i(t)=0$ for $i=1,..,n$, the Digital Bond simplifies to $\frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)D(S,t_0,t;\xi;\Sigma_i)$ which is a weighted sum of digital option on a GBM. Note, it still has a skew but much less prononced.

5.5.2 Computing the Curvature

When the parametric model is calibrated to the market quotes, the parametric density in Equation (5.8) must equate the market density in Equation (C.17). From the formula of the convexity of the smile expressed in prices in Equation (B.15), we can infer analytically the curvature of the parametric model for the strike K and the maturity t

$$\partial_{KK}\Sigma(K,t) = \\ \frac{\partial_{KK}C_M(t_0,S_0;K,t)}{Vega(K,t;\Sigma_{BS}(K,t))} - \frac{1}{K^2\Sigma(K,t)(t-t_0)}\left[1+2Kd_1\sqrt{t-t_0}Skew(K,t) + K^2d_1d_2(t-t_0)(Skew(K,t))^2\right]$$

Given the optimum vector Ψ^* of model parameters, we can compute analytically the European call and put prices for all maturity t and strike K. Inverting the Black-Scholes formula, we recover the implied volatility surface $\Sigma(K,t)$. We can then use that surface to compute exactly the skew and curvature of the parametric IV surface. Alternatively, one can approximate the skew and curvature of the IV surface around the money by considering the ATM volatility $\hat{\sigma} = \hat{\Sigma}(t) = \frac{1}{norm}\sum_{i=1}^{n}\overline{a}_i(t)\Sigma_i(t)$. As a result, given a volatility function $\Sigma(K,t)$ obtained with a parametric function such as a SVI or SABR, one can directly fit our parametric model to volatilities without inverting the Black-Scoles formula.

6 Modeling statistically the IV surface

Even though the implied volatility surface (IVS) characterises agents belief of future evolution of the stock price returns, it is evident that today's market prices do not provide us with the right future anticipations of the stock price process. This is because the implied volatility surface is neither stationary nor Markovian but stochastic. We saw earlier that changes of the implied volatility surface have a significant influence on the value of the option position. For instance, an incorrect estimate of the IV surface and its expected shifts could lead to a significant miss pricing of the options. Further, model based quantitative forecasts can provide financial institutions with a valuable estimate of a future market trend. Hence, in order to perform statistical analysis of the relationships between points on the IVS, we are left with finding a way of modeling dynamically the agents rational anticipations.

6.1 Existing models of the IVS and their dynamics

We briefly present the work done by Ladokhin [2009] who described a number of representations of the IV surface and performed a comparison analysis of their dynamics. He discussed and compared several models of the approximation of the surfaces, as well as several approaches to the dynamics of these surfaces including different types of polynomial fitting as well as a stochastic volatility model. Among the polynomial models describing the whole volatility surface with one equation are the Cubic model and the Spline model. In the former the IV is a cubic function of moneyness $X = \ln \frac{K}{S}$ (or the time adjusted moneyness $X = \frac{1}{\sqrt{\tau}} \ln \frac{K}{S}$) and a quadratic function of time to expiry τ

$$\Sigma(K,T) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 \tau + a_5 \tau^2$$

where a_i for $i = 0,..,5$ are parameters to be estimated. Similarly, one can build a Spline model of the volatility surface as

$$\Sigma(K,T) = a_0 + a_1 X + a_2 X^2 + a_3 \tau + a_4 \tau^2 + D\big(a_5 + a_6 X + a_7 X^2 + a_8 \tau + a_9 \tau^2\big)$$

with

$$D = \begin{cases} 0 \text{ if } X < 0 \\ 1 \text{ if } X \geq 0 \end{cases}$$

together with the constraints

$$a_5 + a_6.0 + a_7.0^2 + a_8 \tau + a_9 \tau^2 = 0 \ , \ D a_6 = 0$$

for the volatility function to be continuous and differentiable. Contrary to the polynomial models, stochastic volatility models such as the SABR model (see Hagan et al. [2002]) or the SVI model (see Gatheral [2006]), assume some behaviour of the underlying asset and connections with the values of the implied volatility. In those models the shape of the volatility skew is derived analytically from these assumptions. In the case of the SABR model, it can either be used as a model for a whole volatility surface or for the skew. Under the first approach, the parameters α, β, ρ and ν are calibrated for all given times to expiration τ_i for $i = 1, 2, ...$ In the second approach, one fit a SABR skew for each observed time to expiration, and then interpolate the values of implied volatility for any arbitrary τ. The piecewise SABR (PSABR) parameters α_i, β_i, ρ_i and ν_i are calculated separately for each time to expiration τ_i, and the IV surface is built as a linear approximation of separate skews.

While a model with a large number of parameters may calibrate well the volatility surface on a given day, the same model parameters may give poor results on the next day. On the other hand, any risk management system tries to estimate the future (short term forecast) behaviour of the volatility surface. As a result, one need a model that accurately fit market prices with stable and robust parameters. Ladokhin [2009] focused on two different approaches to model the dynamics of the IV surfaces, one applicable to the Cubic and the Spline model and the other used for SABR models. In the first approach, the dynamics of the surface is treated as dynamics of these parameters. To reduce the dimensionality of the problem, he applied Principal Component Analysis (PCA) to the values of the parameters a_i for $i = 0,..,n$. This is to switch to another space of the no-correlated factors, that fully describe the dynamic of the implied volatility surface. Since the first few principal components explain most of the variance of the calibrated parameters, the dynamics of the volatility surface over time is explained by the dynamics of the first two principal components modeled with an Autoregressive moving average model (ARMA). In the second approach, the SABR model already assumes certain dynamics of the volatility and the underlying asset expressed by a system of stochastic differential equations. Consequently, given the calibrated parameters, the spot price can be simulated with the Monte Carlo method, and for each path, the IV surface generated. The forecast of the implied volatility surface is an average surface over the simulated paths.

Strong of these dynamics, Ladokhin [2009] used the rolling horizon technique to build a 1 and 5 day forecasts of the volatility skews. Setting $N = 100$, he used observations from days $t - N$ till $t - 1$ to calibrate the dynamic models in order to build a forecast of the skew for day t. Then, at time $t+1$ the horizon is rolled so that days $t - N + 1$ till t are used for calibration. An equivalent technique is used for the five days ahead forecast. He then tested how the models can hold the volatility skew pattern by performing a static test. Calibrating the models to observed IV surface at date $t - 1$ or $t - 5$ he calculated the weighted mean square error (WMSE) between the model results and the IV surface on day t. Cubic and Spline models approximate the implied volatility surface with the function of a certain form. The Spline model is perhaps, the most effective to minimise the fitting error. Good performance of the dynamic version of the Spline model is an empirical evidence of the dependence of the dynamics of the surface of two principal components. Both of these models use much less parameters, than the PSABR model. Even though the SABR and the PSABR assume a certain model for the joint dynamics of the volatility and the underlying asset, their fitting results have a higher error than the polynomial models. Nonetheless, the PSABR tends to model the skew rather effectively in case of insufficient or bad data. Because these models assume some shape of the IV surface, they are predisposed to give a more theoretical shape of the skew resulting in lower relative forecasting error. As a result, models relating prices in time and space perform better on an incomplete market or with missing data. To conclude, no single method exhibits superior accuracy in the analysis of every data set. Some methods perform better for certain underlying assets, while other methods are more suitable for the other.

6.2 Evolving the IV surface

Having defined in Section (5.2) a consistent model with a global fit to prices under constraints, prices can now be generated such that the No-Dominance principle is preserved, and one can safely assess relative value between them. Bloch [2010] showed that the dynamics implied by the single parametric model for the entire volatility surface in Equation (5.9) were those of a mixture diffusion process associated to an uncertain-volatility model. That is, similarly to the SABR models, our model already assumes some dynamics of the volatility and the underlying asset expressed by a system of stochastic differential equations. In that model, the instantaneous volatility being a deterministic function of the spot price and time, we can simulate the underlying process with a local volatility model. However, the implied volatility surface being neither stationary nor Markovian but stochastic, we are going to combine the two different approaches to model the dynamics of the IV surfaces described in Section (6.1) by providing memory to the parameters of our model. Hence, we now consider the implied volatility surface to be a stochastic process driving the option prices and choose to model its dynamics in its full term and strike structure. Option prices deriving their values from an underlying security, it is natural to use mathematical tools to infer their dynamics from that of the underlying stock process. One approach is to model the implied volatility with dynamics based on a statistical analysis of its behaviour through time. It naturally leads us to model the stock price process discreetly with Markov chains. We impose that future smile surfaces should be compatible with today's prices of calls and puts. Formalising the Kolmogorov-Compatibility condition, we impose that the future density is actually a conditional density. Knowing that when the number of fixing dates in a model is finite there is an infinity of conditional densities, we choose to satisfy this infinity of solution by giving the forward smile a shape consistent with its historical evolution. Following Bloch et al. [2003], we assume that the volatility surface is dynamically modified according to the stock price realisations. Having related the stock price level to the implied volatility surface, we use their respective historic evolution to characterise the transition probabilities, that is, the conditional densities. A statistical technique is used to regress the observed implied smile against the realised stock level. Then from the implied marginal distributions observed on the market, we infer the joint distributions which are consistent with the relationship between implied volatility and stock price level over time and satisfy the AOA constraint. It implies that European call prices are not time-dependent, that is, do not depend on current time but are stock dependent. Therefore, the current stock evolution directly influences its future increment which means that, given the stock price S_{t_1} at time

t_1, the conditional density $\phi(., T, S_{t_1}, t_1)$ is *known*.

6.3 The forward IV surface

In order to satisfy the Kolmogorov probability, that is, assure that the forward smile implies a compatible conditional *pdf*, the model must fit European call prices observed at evaluation time t_0. In continuous time models with volatility a deterministic function of time and stock price such as the Dupire model, the volatility surface is assumed Markovian and stationary, giving a unique solution to the forward volatility. However the prices of forward start call options given by Dupire's model are lower than the ones observed in the market. In view of getting higher forward smile, practitioners either add a stochastic process into the local volatility or combine the local volatility with a jump process. It means that there is a risk attached to such products which is not taken into account in today's information. Bloch et al. [2003] showed that this risk translated into a forward volatility being higher than the spot volatility. Consequently, in order to price correctly a forward start option, a model must be calibrated to some kind of evolution of the implied volatility surface in such a way that the forward volatility that we get is higher than today's volatility. Knowing that when the number of fixing dates in a model is finite there is an infinity of conditional densities, we choose to satisfy this infinity of solution by giving the forward smile a shape consistent with its historical evolution. That is, the past evolution of the implied smile regressed against its stock price level would help explaining its future dynamics. Therefore, we are going to define a shape for the volatility surface together with its dynamics by using a statistical method, such that we can price vanilla options and then deduce a proper forward smile for the pricing of forward start options.

6.4 Statistical Dynamics

6.4.1 Computing the smile parameters

In this section, we are going to model the implied volatility Σ with dynamics based on a statistical analysis of its behaviour through time. We first need to infer a shape of the volatility surface in accordance with market observable data. For the sake of clarity we will deal with the polynomial defined in Equation (5.4) which is a special case of the Spline model presented in Section (6.1). Obviously the factor X_t^{T-t} is set equal to the observed at-the-money volatility *i.e.*

$$X_t^{T-t} = \Sigma(t, S_t; S_t, T-t)$$

Then, there exists several methods to fit the skew factor Y_t^{T-t} and the curvature factor Z_t^{T-t}. The simplest method, based on finite differences, leads to computing Y_t^{T-t} and Z_t^{T-t} such that

$$\begin{aligned}
Y_t^{T-t} &= -\frac{\Sigma(t, S_t; K^+, T-t) - \Sigma(t, S_t; K^-, T-t)}{K^+ - K^-} \\
Z_t^{T-t} &= \frac{\Sigma(t, S_t; K^+, T-t) + \Sigma(t, S_t; K^-, T-t) - 2\Sigma(t, S_t; S_t, T-t)}{\left(\frac{K^+ - K^-}{2}\right)^2}
\end{aligned}$$

for some strikes K^- and K^+ close enough to the money and such that $K^- < S_t < K^+$. One possible setting would be to take $K^- = 0.8S_t$ and $K^+ = 1.2S_t$. A much better approach, satisfying the no-arbitrage principle, consists in considering the properties of our parametric model, and directly use the analytical skew and curvature defined in Section (5.5). That is, we project the complexity of the single parametric model in Equation (5.9) into a simplified version of the Spline model.

6.4.2 Providing some dynamics

We are now going to provide these parameters with some dynamics. The fundamental idea (*c.f.* Bloch et al. [2002]) is to assume that these three parameters are driven by the spot process S_t, for the spot process holds the whole market risk. That is, the parameters of our model have memory given by the realised trajectory of the spot price. As such, they are now factors expressed as

$$\begin{aligned} X_t^{T-t} &\equiv X_t^{T-t}(S_t) \\ Y_t^{T-t} &\equiv Y_t^{T-t}(S_t) \\ Z_t^{T-t} &\equiv Z_t^{T-t}(S_t) \end{aligned}$$

and we need to infer a shape for the spot functions $X_t^{T-t}(S_t)$, $Y_t^{T-t}(S_t)$ and $Z_t^{T-t}(S_t)$. Assuming a continuum of maturities, we consider that the maturity effect, as traders are used to specify it, is proportional to $\frac{1}{(T-t)^\theta}$ for some parameter θ. Plotting X_t^{T-t}, Y_t^{T-t} and Z_t^{T-t} against S_t on a window of historical data, we consider the following shape

$$\begin{aligned} X_t^{T-t} &= \tfrac{\exp(a_x+b_xS_t+\varepsilon_t^x)}{(T-t)^{\theta_x}}, \\ Y_t^{T-t} &= \tfrac{\exp(a_y+b_yS_t+\varepsilon_t^y)}{(T-t)^{\theta_y}}, \\ Z_t^{T-t} &= \tfrac{\exp(a_z+b_zS_t+\varepsilon_t^z)}{(T-t)^{\theta_z}}. \end{aligned}$$

which we rewrite as

$$\begin{aligned} \log(X_t^{T-t}) &= a_x + b_xS_t - \theta_x \log(T-t) + \varepsilon_t^x, \\ \log(Y_t^{T-t}) &= a_y + b_yS_t - \theta_y \log(T-t) + \varepsilon_t^y, \\ \log(Z_t^{T-t}) &= a_z + b_zS_t - \theta_z \log(T-t) + \varepsilon_t^z. \end{aligned}$$

At this point we need to estimate the model parameters. A first and common method would consist in computing the ordinary least squares (OLS) *i.e.* minimising the $\mathcal{L}^2(\mathbb{R})$ norm of residuals. Note, when using the OLS method it implicitly means that we suppose that the processes ε_t^x, ε_t^y and ε_t^z are white noises and then that an hexogen perturbation of the stock has no consequence on the future option values. One need to perform the *generalized Durbin-Watson tests* on the data to see if the hypothesis for white noise is accepted or reject. In the case where it is rejected, it is not desirable to use ordinary regression analysis for the data we are dealing with since the assumptions on which the classical linear regression model is based will be obviously violated.

Violation of the independent errors assumption has three important consequences for ordinary regression. First, statistical tests of the significance of the parameters and the confidence limits for the predicted values are not correct. Second, the estimates of the regression coefficients are not as efficient as they would be if the autocorrelation were taken into account. Third, since the ordinary regression residuals are not independent, they contain information that can be used to improve the prediction of future values. One way forward is to introduce some dynamics on the errors in order to capture this effect.

We now have a consistent and relevant model to explain the dynamics of the volatility surface over time. Tests need to be performed to measure the capability of the model to forecast the volatility ATM, the skew and the curve of the volatility smile. To do so, we can reproduce the tests presented in Section (6.1). For instance, one can perform a daily forecast moving test also called rolling horizon technique. Starting from an initial historical table of 300 observations (out of 368), we use the model calibrated to these 300 observations to forecast the volatility parameters (ATM, skew, curve) over the next five days. We then compare these forecasted values to the market data by computing the mean over the five days of the relative errors [2] expressed in percentage. Then we add one day to the historical data and start again. We repeat the operation ten times (till we have an historical basis with 309 observations) and give the results for the ten iterations. We can also show how the model forecast the volatility ATM on a window of time given that it was calibrated to market data on a previous window of time. If the model is stable over time it will not need to be recalibrated too often.

One application of our statistical deterministic implied volatility model is to run a computer program generating theoretical value to option prices within a range of stock prices. Any time an option pair is out of line with the theoretical model, one can buy or sell it to realise a profit.

7 Conclusion

Our goal was to devise a consistent volatility model to assess relative value between option prices, and to provide the implied volatility surface with general dynamics. We described a single parametric model for the entire volatility surface with interpolation and extrapolation technique generating a smooth and robust implied volatility surface without arbitrage in space and time. Marking option prices on indices and single stocks, one can safely use the model to devise relative value trading strategies. Greeks and stress scenarios are calculated analytically in the parametric model without recalibration of the model parameters. Further, getting analytical solutions to Digital options we can use them to recover analytically the Skew and Curvature of the IV surface for all strikes and maturities. To perform statistical analysis of the relationships between points on the IV surface we modeled dynamically the agents belief of future evolution of the stock price returns. Using the Skew and Curvature inferred from our model, we introduced a possible shape for the volatility surface by linking its future evolution to an observable stochastic process and by adding noises. Based on empirical results showing the existence of a strong link between the implied volatility surface and the spot level over time, we took the spot level as driving factor. It implied that the volatility surface was dynamically modified according to the realisation of stock prices which is a way of modeling dynamically the agents rational anticipations. Having related the stock price level to the volatility surface, we used the historic evolution of the implied volatility surface to characterise the transition probabilities, that is, the conditional densities. This hypothesis implies that European call prices are not time-dependent, that is, do not depend on current time but are stock dependent. Therefore, the current stock evolution directly influences its future increment which means that, given the stock price at a future time, the conditional density is known.

[2] $100 * \frac{\text{forecasted value-realised value}}{\text{realised value}}$

Annexes

A The Black-Scholes formula

A.1 Some Greeks

We describe a few Greeks in the Black-Scholes formula that will be used later on to devise our parametric model. The option in the Black-Scholes model is hedged with a portfolio containing

$$\Delta(t,S_t) = \partial_x C_{BS}(t,S_t,K,T) = e^{-q(T-t)} N(d_1(T-t, S_t e^{(r-q)(T-t)}, K)) \tag{A.11}$$

stocks. Similarly, the price of a put option is given by

$$\begin{aligned} P_{BS}(t,x,K,T) &= -xe^{-q(T-t)} N(-d_1(T-t, xe^{(r-q)(T-t)}, K)) + Ke^{-r(T-t)} N(-d_2(T-t, xe^{(r-q)(T-t)}, K)) \\ &= Ke^{-r(T-t)} N(d_1(T-t, K, xe^{(r-q)(T-t)})) - xe^{-q(T-t)} N(d_2(T-t, K, xe^{(r-q)(T-t)})) \end{aligned}$$

with delta

$$\begin{aligned} \Delta(t,S_t) &= \partial_x P_{BS}(t,S_t,K,T) = -e^{-q(T-t)} N(-d_1(T-t, S_t e^{(r-q)(T-t)}, K)) \\ &= -e^{-q(T-t)} N(d_2(T-t, K, S_t e^{(r-q)(T-t)})) \end{aligned}$$

since

$$d_2(t,x,y,\sigma^2) = -d_2(t,y,x,\sigma^2)$$

The vega in the Black-Scholes model is

$$Vega = xRe(t,T)\sqrt{T-t}n(d_1)$$

where $n(x) = \frac{\partial}{\partial x} N(x)$ and $N^{''}(x) = -xn(x)$. Also, we have the limit cases $\lim_{(T-t)\to 0} Vega = 0$ and $\lim_{(T-t)\to\infty} Vega = 0$. Differentiating one more time the Vega with respect to volatility which is called the Volga, and using the relations $\frac{dd_1}{d\sigma} = -\frac{1}{\sigma} d_2$ or $\frac{dd_2}{d\sigma} = -\frac{1}{\sigma} d_1$, we get

$$\frac{\partial}{\partial\sigma} Vega = Volga = xRe(t,T)\sqrt{T-t}\frac{1}{\sigma} d_1 d_2 n(d_1) = \frac{1}{\sigma} d_1 d_2 Vega \tag{A.12}$$

Again, differentiating the Volga with respect to volatility, we get

$$\frac{\partial}{\partial\sigma} Volga = \left(-d_1 d_2 - d_2^2 - d_1^2 + d_1^2 d_2^2\right) \frac{1}{\sigma^2} Vega$$

Now we differentiate the Vega with respect to the strike K, getting

$$\frac{\partial}{\partial K} Vega = \frac{d_1}{K\sigma\sqrt{T-t}} Vega$$

which we differentiate one more time with respect to the strike, getting

$$\frac{\partial^2}{\partial K^2} Vega = \frac{d_1 d_2}{K^2\sigma^2(T-t)} Vega - \frac{1}{K^2\sigma^2(T-t)} Vega = \frac{(d_1 d_2 - 1)}{K^2\sigma^2(T-t)} Vega$$

To conclude, we need to differentiate the Vega with respect to maturity T, getting

$$\frac{\partial}{\partial T}Vega = -q(T)Vega + \frac{1}{2(T-t)}Vega + d_1\big(\frac{1}{2(T-t)}d_2 - \frac{(r(T)-q(T))}{\sigma\sqrt{T-t}}\big)Vega$$

We now differentiate the call price with respect to the stike

$$\frac{\partial}{\partial K}C_{BS}(t,x,K,T) = -P(t,T)N(d_2(T-t,x,K))$$

Differentiating the call price twice with respect to the strike we get

$$\frac{\partial^2}{\partial K^2}C_{BS}(t,x,K,T) = \frac{P(t,T)}{K\sigma\sqrt{T-t}}n(d_2(T-t,x,k))$$

which we rewrite as

$$\frac{\partial^2}{\partial K^2}C_{BS}(t,x,K,T) = \frac{xRe(t,T)}{K^2\sigma\sqrt{T-t}}n(d_1(T-t,x,k)) = \frac{1}{K^2\sigma(T-t)}Vega(K,T;\sigma)$$

Setting $x = S_t$, we differentiate the call price with respect to maturity T getting

$$\frac{\partial}{\partial T}C_{BS}(t,S_t,K,T) = \frac{\sigma}{2(T-t)}Vega + r_T KP(t,T)N(d_2) - q_T xRe(t,T)N(d_1)$$

We define $X_{BS}(t,S_t,K,T)$ as

$$X_{BS}(t,S_t,K,T) = \frac{\sigma}{2(T-t)}Vega + r_T KP(t,T)N(d_2)$$

which is always positive, and rewrite the above derivative as

$$\frac{\partial}{\partial T}C_{BS}(t,S_t,K,T) = X_{BS}(t,S_t,K,T) - q_T xRe(t,T)N(d_1)$$

Similarly, the derivative of the put price with respect to maturity T is

$$\frac{\partial}{\partial T}P_{BS}(t,S_t,K,T) = -\frac{\sigma}{2(T-t)}Vega + r_T KP(t,T)N(d_1) - q_T xRe(t,T)N(d_2)$$

Since $\frac{d\tau}{dt} = -1$ where $\tau = T-t$, the derivative of the price with respect to time t is given by $\frac{\partial C_{BS}(t,S_t,K,T)}{\partial \tau}\frac{d\tau}{dt}$. From the relation

$$r_t S_t \Delta(t,S_t) - r_t C_{BS}(t,S_t,K,T) = r_t KP(t,T)N(d_2)$$

the Theta becomes

$$\frac{\partial}{\partial t}C_{BS}(t,S_t,K,T) = -\frac{\sigma}{2(T-t)}Vega - r_t x\Delta(t,S_t) + r_t C_{BS}(t,S_t,K,T) + q_t xRe(t,T)N(d_1)$$

A.2 Pricing the modified Black-Scholes call option

Given the stock price $(S_t)_{t\in[0,T]}$, the price of the modified Black-Scholes call option with strike K, maturity T and shift μ_T under the risk-neutral probability measure $\mathbb{Q}$ is

$$\begin{aligned} C_{t_0} &= E^Q[e^{-\int_{t_0}^T r_s ds}(S_T - K(1+\mu_T))^+|\mathcal{F}_{t_0}] \\ &= E^Q[e^{-\int_{t_0}^T r_s ds}S_T I_{\{S_T>\overline{K}\}}|\mathcal{F}_{t_0}] - \overline{K}E^Q[e^{-\int_{t_0}^T r_s ds}I_{\{S_T>\overline{K}\}}|\mathcal{F}_{t_0}] \end{aligned} \tag{A.13}$$

where $\overline{K} = K(1+\mu_T)$. Assuming a deterministic convenience yield, we let $X_t = S_t D(t,T)$ be the present value of the stock price S_T seen at time t, where $D(t,T) = \frac{\gamma(t)}{\gamma(T)}$ and such that when $t = T$ we get $X_T = S_T$. Hence, in the first expectation of the call price, we can take the process X_t as Numeraire with the density $Z_X(t,T) = \frac{X_T \beta(t)}{\beta(T) X_t}$, and since $X_T = S_T$ we can re-express the first expectation term as

$$E^Q[\frac{\beta(t_0)}{\beta(T)} X_T I_{\{X_T > \overline{K}\}} | \mathcal{F}_{t_0}] = X_{t_0} E^Q[Z_X(T) I_{\{X_T > \overline{K}\}} | \mathcal{F}_{t_0}]$$

while in the second expectation we take the bond as Numeraire with density $Z_P(t) = \frac{P(t,T)}{\beta(t) P(t_0,T)}$. Hence, we can re-write the call price as

$$\begin{aligned} C_{t_0}(K,T) &= X_0 E^Q[Z_X(T) I_\xi | \mathcal{F}_{t_0}] - \overline{K} P(t_0,T) E^Q[Z_P(T) I_\xi | \mathcal{F}_{t_0}] \\ &= X_0 P^X(X_T > \overline{K}) - \overline{K} P(t_0,T) P^T(X_T > \overline{K}) \end{aligned}$$

where

$$\xi = \{X_T > \overline{K}\}$$

Since $X_T = F(T,T)$ where $F(t,T) = \frac{X_t}{P(t,T)}$ and since $F(t,T)$ is a martingale under $\mathbb{P}^T$ and $\frac{1}{F(t,T)}$ is a martingale under $\mathbb{P}^X$, the call price becomes

$$\begin{aligned} C(t_0;K,T) &= X_{t_0} P^X(F(T,T) > \overline{K}) - \overline{K} P(t_0,T) P^T(F(T,T) > \overline{K}) \\ &= X_{t_0} P^X(\frac{1}{F(T,T)} < \frac{1}{\overline{K}}) - \overline{K} P(t_0,T) P^T(F(T,T) > \overline{K}) \end{aligned}$$

Expanding the solution of the forward price, we get

$$\begin{aligned} C(t_0;K,T) &= X_{t_0} P^X(-\frac{1}{2}\sigma_F^2(T-t_0) + \sigma_F\sqrt{(T-t_0)}y < \log\frac{F(t_0,T)}{\overline{K}}) \\ &- \overline{K} P(t_0,T) P^T(-\frac{1}{2}\sigma_F^2(T-t_0) + \sigma_F\sqrt{(T-t_0)}y > \log\frac{\overline{K}}{F(t_0,T)}) \\ &= S_{t_0} D(t_0,T) N(\overline{d}_1) - \overline{K} P(t_0,T) N(\overline{d}_2) \end{aligned}$$

where

$$\overline{d}_1 = \frac{1}{\sigma_F\sqrt{(T-t_0)}} \log\frac{F(t_0,T)}{\overline{K}} + \frac{1}{2}\sigma_F\sqrt{(T-t_0)} \text{ and } \overline{d}_2 = \overline{d}_1 - \sigma_F\sqrt{(T-t_0)}$$

from symmetry of the Brownian motion. Similarly to the displaced diffusion model, the present model is capable of generating asymmetric distribution to recover the market implied skew by using the standard Black-Scholes formula with appropriate input parameters.

Assumption A.1 *We assume that we can approximate the modified Black-Scholes price by*

$$Y = Y(0) + Z + H \tag{A.14}$$

where $Y(0) = C_{BS}(t,x,K,T;\sigma)$ with $\sigma = I_0(t,T)$, and such that Z and H account respectively for the skew and the curvature.

To do, so we must express the modified Black-Scholes price in term of the Black-Scholes price plus some extra terms

$$C(t_0; K, T) = S_{t_0} D(0,T) N(\overline{d}_1) - K P(t_0,T) N(\overline{d}_2) - K \mu_T P(t_0,T) N(\overline{d}_2)$$

Expanding the $\overline{d}_1$ term, we get

$$\overline{d}_1 = d_1 - \frac{1}{\sigma_F \sqrt{(T-t_0)}} \log\left(1+\mu_T\right)$$

so that the modified Black-Scholes price becomes

$$\begin{aligned} C(t_0; K, T) &= S_{t_0} D(0,T) N\big(d_1 - \frac{1}{\sigma_F \sqrt{(T-t_0)}} \log\left(1+\mu_T\right)\big) - K P(t_0,T) N\big(d_2 - \frac{1}{\sigma_F \sqrt{(T-t_0)}} \log\left(1+\mu_T\right)\big) \\ &- K \mu_T P(t_0,T) N(\overline{d}_2) \end{aligned}$$

B Expressing the convexity of the smile in prices

The implied volatility being a function of the strike K and the maturity T, we use the chain rule to express the derivatives of the market prices in terms of the Black-Scholes Greeks

$$\begin{aligned} &\partial_T C(t,S_t,K,T) = \partial_T C_{BS}(t,S_t,K,T) + \partial_\Sigma C_{BS}(t,S_t,K,T) \partial_T \Sigma(T,K) \\ &\partial_K C(t,S_t,K,T) = \partial_K C_{BS}(t,S_t,K,T) + \partial_\Sigma C_{BS}(t,S_t,K,T) \partial_K \Sigma(T,K) \\ &\partial_{KK} C(t,S_t,K,T) = \partial_{KK} C_{BS}(t,S_t,K,T) + 2\partial_{\Sigma K} C_{BS}(t,S_t,K,T) \partial_K \Sigma(T,K) + \partial_\Sigma C_{BS}(t,S_t,K,T) \partial_{KK} \Sigma(T,K) \\ + \; &\partial_{\Sigma\Sigma} C_{BS}(t,S_t,K,T) (\partial_K \Sigma(T,K))^2 \end{aligned}$$

Concentrating on the density, we get

$$\begin{aligned} \partial_{KK} C(t,S_t,T,K) &= \frac{1}{K^2 \Sigma(T,K)(T-t)} Vega(K,T) + 2\partial_{\Sigma K} C_{BS}(t,S_t,K,T) \partial_K \Sigma(T,K) \\ + \; &\partial_\Sigma C_{BS}(t,S_t,K,T) \partial_{KK} \Sigma(T,K) + \partial_{\Sigma\Sigma} C_{BS}(t,S_t,K,T) (\partial_K \Sigma(T,K))^2 \end{aligned}$$

where $Vega(K,T)$ is the BS vega with volatility $\Sigma(K,T)$. Using the Black-Scholes Greeks, the density becomes

$$\begin{aligned} \partial_{KK} C(t,S_t,T,K) &= \frac{1}{K^2 \Sigma(T,K)(T-t)} Vega(K,T) + \frac{2d_1}{K\Sigma(T,K)\sqrt{T-t}} Vega(K,T) \partial_K \Sigma(T,K) \\ + \; &\partial_\Sigma C_{BS}(t,S_t,K,T) \partial_{KK} \Sigma(T,K) + \frac{1}{\Sigma(T,K)} d_1 d_2 Vega(K,T) (\partial_K \Sigma(T,K))^2 \end{aligned}$$

We multiply by $\frac{K^2}{2}$ and factorise, getting

$$\begin{aligned} \frac{K^2}{2} \partial_{KK} C(t,S_t,T,K) &= \frac{Vega(K,T)}{2} \Big(\frac{1}{\Sigma(T,K)(T-t)} + K \frac{2d_1}{\Sigma(T,K)\sqrt{T-t}} \partial_K \Sigma(T,K) \\ + \; &K^2 \partial_{KK} \Sigma(T,K) + K^2 \frac{d_1 d_2}{\Sigma(T,K)} (\partial_K \Sigma(T,K))^2 \Big) \end{aligned}$$

which we can write as

$$\frac{Vega(K,T)}{2\Sigma(T,K)(T-t)}\Big[1+\Sigma(T,K)(T-t)K^2\Big(\frac{2d_1}{K\Sigma(T,K)\sqrt{T-t}}\partial_K\Sigma(T,K)+\partial_{KK}\Sigma(T,K)+\frac{d_1 d_2}{\Sigma(T,K)}(\partial_K\Sigma(T,K))^2\Big)\Big]$$

or

$$\begin{aligned}&\frac{K^2}{2}\partial_{KK}C(t,S_t,T,K)=\\&\frac{Vega(K,T)}{2\Sigma(T,K)(T-t)}\Big[1+2Kd_1\sqrt{T-t}\partial_K\Sigma(K,T)+K^2 d_1 d_2(T-t)(\partial_K\Sigma(K,T))^2+K^2\Sigma(K,T)(T-t)\partial_{KK}\Sigma(K,T)\Big]\end{aligned}$$

Rearranging, we get the convexity of the smile expressed in prices as

$$\begin{aligned}&\partial_{KK}\Sigma(K,T)= \\&\frac{\partial_{KK}C(t,S_t,T,K)}{Vega(K,T)}-\frac{1}{K^2\Sigma(T,K)(T-t)}\Big[1+2Kd_1\sqrt{T-t}\partial_K\Sigma(K,T)+K^2 d_1 d_2(T-t)(\partial_K\Sigma(K,T))^2\Big]\end{aligned} \tag{B.15}$$

C Pricing exotic options

C.1 The density

For all positive or bounded function h we have

$$E[h(S_t^x)]=\int h(y)\phi(t,x,y)dy$$

where $\phi(t,x,y)$ is the density function of S_t. In the special case where S_t follow a GBM we get

$$\begin{aligned}\phi_{\mu,\sigma^2}(t,x,y)&=\frac{1}{\sigma y\sqrt{2\pi t}}e^{-\frac{1}{2}d_2(t,xe^{\mu t},y)^2}\\d_2(t,x,y)&=\frac{1}{\sigma\sqrt{t}}\log\frac{x}{y}-\frac{1}{2}\sigma\sqrt{t}\end{aligned}$$

where $\nu^2 t=\sigma^2 t$ is the total variance.

C.2 The digital option

Given the stock price $(S_t)_{t\geq 0}$, a Digital option $D(K,T)$ for strike K and maturity T pays \$1 when the stock price S_T is greater than the strike K, and zero otherwise. The price of the Digital option is

$$D(K,T)=\lim_{\Delta K\to 0}\frac{C(K,T)-C(K+\Delta K,T)}{\Delta K}=-\frac{\partial}{\partial K}C(K,T)$$

Given $C(K,T)=C_{BS}(K,T;\Sigma_{BS}(K,T))$ where $\Sigma_{BS}(K,T)$ is the BS implied volatility for strike K and maturity T, and using the chain rule, the Digital option becomes

$$D(K,T)=-\frac{\partial}{\partial K}C_{BS}(K,T;\Sigma_{BS}(K,T))=-\frac{\partial}{\partial K}C_{BS}(K,T;\Sigma_{BS})-\frac{\partial}{\partial\Sigma}C_{BS}(K,T;\Sigma(K,T))\frac{\partial}{\partial K}\Sigma(K,T)$$

We can express the Digital option in terms of the Vega and the Skew as

$$D(K,T) = -\frac{\partial}{\partial K}C_{BS}(K,T;\Sigma_{BS}) - Vega(K,T)Skew(K,T) \tag{C.16}$$

where $Vega(K,T;\Sigma_{BS}(K,T))$ is the Black-Scholes vega for the strike K and maturity T, and $\partial_K C_{BS}(K,T;\Sigma_{BS})$ is the BS digital price for the volatility $\Sigma_{BS}(K,T)$. In the special case where $r = q = 0$, $T = 1$ and for $S_0 = 100$ and $K = 100$, given a skew of 2.5% per 10% change in the strike and an ATM volatility $\Sigma_{ATM} = 25\%$ we get

$$\begin{aligned} D(100,1) &= N[-\frac{\Sigma_{ATM}}{2}] - S_0 n[\frac{\Sigma_{ATM}}{2}]\frac{-0.025}{0.1S_0} \\ &\approx 0.45 + 0.25 \times 4 = 0.55 \end{aligned}$$

Ignoring the skew, the price is 45% of notional which is significantly lower than 55% of notional when the skew is included.

C.3 The Butterfly option

Assuming that the volatility surface has been constructed from European option prices, we consider a butterfly strategy centered at K where we are long a call option with strike $K - \Delta K$, long a call option with strike $K + \Delta K$, and short two call options with strike K. The value of the butterfly for strike K and maturity T is

$$B(t_0,K,T) = C(K-\Delta K,T) - 2C(K,T) + C(K+\Delta K,T) \approx P(t_0,T)\phi(t_0;K,T)(\Delta K)^2$$

where $\phi(t_0;K,T)$ is the probability density function (PDF) of S_T evaluated at strike K. As a result, we have

$$\phi(t_0;K,T) \approx \frac{1}{P(t_0,T)}\frac{C(K-\Delta K,T) - 2C(K,T) + C(K+\Delta K,T)}{(\Delta K)^2}$$

and letting $\Delta K \to 0$, the density becomes

$$\phi(t_0;T,K) = \frac{1}{P(t_0,T)}\frac{\partial^2}{\partial K^2}C(K,T) \tag{C.17}$$

Hence, for any time T one can recover the marginal risk-neutral distribution of the stock price from the volatility surface. However, it tells us nothing about the joint distribution of the stock price at multiple times $T_1, .., T_n$. This is because the volatility surface is constructed from European options prices which only depend on the marginal distribution of S_T.

References

[2001] Alexander C., Market models - a guide to financial data analysis, John Wiley and Sons, New York.

[2006] Arisoy Y.E., Altay-Salih A. and Akdeniz L., Is volatility risk priced in the securities market? Evidence from S&P 500 index options. Working Paper, Bilkent University, Faculty of Business Administration.

[1997] Bakshi G., Cao C. and Chen Z., Empirical performance of alternative option pricing models, *Journal of Finance*, **52** (5), pp 2003–2049.

[2003] Bakshi G., and Kapadia N., Delta-hedged gains and the negative market volatility risk premium. *Review of Financial Studies*, **16**, pp 527–566.

[2000] Bates D., Post-87 crash fears in the S & P 500 futures option market. *Journal of Econometrics*, **94**, 181-238.

[2000] Bekaert G., Wu G., Asymmetric volatility and risk in equity markets. *The Review of Financial Studies*, **13**, No. 1: 1–42.

[2000] Bellamy N., Jeanblanc M., Incompleteness of markets driven by a mixed diffusion. *Finance and Stochastics*, **4**, Number 2.

[2001] Beneder R., Vorst T., Options on dividends paying stocks. International Conference on Mathematical Finance, Shanghai.

[2004] Bergenthum J. and Ruschendorf L., Comparison of option prices in semimartingale models, University of Freiburg, Eckerstr.

[1973] Black F., and Scholes M., The pricing of options and corporate liabilities. *Journal of Political Economics*, **81**, 637–659.

[2002] Bloch D., Miralles P., Statistical dynamics of the smile. Technical Report, DrKW.

[2003] Bloch D., Aube JD., A statistical deterministic implied volatility model. Technical Report, Credit Lyonnais.

[2010] Bloch D., A practical guide to implied and local volatility. Working Paper, Mizuho Securities.

[2011] Bloch D., and Coello Coello C.A., Smiling at evolution. *Applied Soft Computing*, **11** (8), pp. 5724–5734.

[2004] Blyth S., Practical relative-value volatility trading. *Risk*, **5**, pp. 91–96.

[2003] Bos R., Gairat A. and Shepeleva A., Dealing with discrete dividends. *Risk*, January, pp 109–112.

[1988] Brenner M., Subrahmanyan M.G., A simple formula to compute the implied standard deviation. *Financial Analysts Journal*, **44**, No. 5, 80–83

[2000] Brigo V, Mercurio F., A mixed-up smile, *Risk*, September, pp 123–126.

[2008] Buehler H., Volatility and dividends: Volatility modelling with cash dividends and simple credit risk. Working Paper, Institut fur Mathematik, MA 7-4, 10623 Berlin, Germany.

[2011] Coleman T.F., Li Y. and Wang c., Stable local volatility function calibration using spline kernel. Working Paper, University of Waterloo, May.

[2002] Cont R., da Fonseca J., Dynamics of implied volatility surfaces. *Quantitative Finance*, **2**, 45–60.

[2002] Cont R., Tankov P., Calibration of jump-diffusion option-pricing models : a robust non-parametric approach. CMAP, **42**, September.

[2005] Cont R., Model uncertainty and its impact on the pricing of derivative instruments. CMAP, Ecole Polytechnique, F-91128 Palaiseau, France.

[1996] Corrado C.J., Miller,Jr. T.W., A note on a simple, accurate formula to compute implied standard deviations. *Journal of Banking and Finance*, **20**, p 595–603.

[1976] Cox J.C. and Ross S.A., The valuation of options for alternative stochastic processes, *Journal of Financial Economics*, **3** (January-March), 145-166.

[2006] Daglish T., Hull J. and Suo W., Volatility surfaces: Theory, rules of thumb and empirical evidence. Working Paper, August.

[1998] Dempster A., Logicist statistics I. Models and modelling. *Statistical Science*, **13**, pp. 248–276.

[1999] Derman E., Regimes of volatility. *Risk*, **7** (2).

[1991] Detemple J., and Selden L., A general equilibrium analysis of option and stock market interactions. *International Economic Review*, **32**, pp 279–303.

[1994] Dupire B., Pricing with a smile. *Risk*, **7**, pp. 18–20.

[1997] El Karoui N., Modeles stochastiques de taux d'interet, Laboratoire de Probabilites, Universite de Paris 6.

[1998] ElKaroui N., Jeanblanc M. and Shreve S., Robustness of the BS formula, *Mathematical Finance*, **8**, 93–126.

[2005] Fengler M.R., Semiparametric modeling of implied volatility, *Springer Finance*, Springer-Verlag Berlin Heidelberg.

[2009] Fengler M.R., Arbitrage-free smoothing of the implied volatility surface. *Quantitative Finance*, **9** (4), pp. 417–428.

[2010] Fengler M.R., Option data and modeling BSM implied volatility. Working Paper, University of St. Gallen.

[1989] Figlewski S., What does an option pricing model tell us about option prices? *Financial Analysts Journal*, **45**: 12–15.

[1999] Gatheral J., The volatility skew: Arbitrage constraints and asymptotic behaviour. Merrill Lynch.

[2006] Gatheral J., The volatility surface: A practitioner's guide, John Wiley & Sons, Hoboken, New Jersey.

[2002] Hagan P.S., Kumar D., Lesniewski A.S. and Woodward D.E., Managing smile risk. *Wilmott Magazine*, September, 84–108.

[2003] Hentschel L., Errors in implied volatility estimation. Chapter 3 of PhD thesis "Caveat invertor: Errors in implied volatility estimation", Simon School, University of Rochester.

[1987] Hull J., and White A., The pricing of options on assets with stochastic volatilities. *Journal of Finance*, **42** (2), 281-300.

[2011] Hutchinson D., Relative value concepts in volatility. CBOE Risk Management Conference, Dana Point, CA.

[1998] Ingersoll J.E., Approximating american options and other financial contracts using barrier derivatives, *Journal of Computational Finance*, **2** (Fall), 85-112.

[1996] Jackwerth J.C., Rubinstein M., Recovering probability distributions from option prices. *Journal of Finance*, **51**, 1611–1631.

[2004] Kahalé N., An arbitrage-free interpolation of volatilities. *Risk*, **17** (5), pp. 102–106.

[2009] Kohler M., A review on regression-based Monte Carlo methods for pricing American options. Working Paper, Technical University of Darmstadt, Darmstadt.

[2009] Ladokhin S., Volatility modeling in financial markets. Master Thesis, VU University Amsterdam.

[2005] Lapeyre B., Sulem A. and Talay D., Simulation of financial models : mathematical foundation and applications, Working Paper, October.

[2003] Lettau M. and Ludvigson S., Measuring and modeling variation in the risk return tradeoff. *Handbook of Financial Econometrics*, forthcoming.

[1997] Malz A., Estimating the probability distribution of the future exchange rate from option prices. *Journal of Derivatives*, Winter, pp 18–36.

[1997] Melick W., Thomas C., Recovering an asset's implied pdf from option prices: An application to crude oil during the Gulf war. *Journal of Financial and Quantitative Analysis*, **32**, (1), 91–115.

[2003] Novales A. and Lafuente J.A., Optimal hedging under departures from cost-of-carry valuation : evidence from the spanish stock index futures market. *Journal of Banking and Finance*, **27**, 1053-1078.

[2002] Overhaus M., Ferraris A., Knudsen T., Milward R., Nguyen-Ngoc L., Schindlmayr G., Equity derivatives. Theory and Application John Wiley & Sons, New York.

[2004] Rebonato R., Cardoso M.T., Unconstrained fitting of implied volatility surfaces using a mixture of normals, Working Paper, QUARC and Oxford University, July 6.

[2004] Shreve S.E., Stochastic calculus for finance II, Continuous-time models. *Springer Finance*, Textbook, NY, USA.

[2003] Simpson J., Currency options and volatility. Goldman Sachs Working Paper, Chapter 11, Siegel M.J. at Stern NYU.

[2000] Tankov P., GRM smile model, Rapport de stage de fin d'etude de l'X.

[2005] Tankov P., Calibration de modeles et couverture de produits derives, Working Paper, Universite Paris VII.

[2004] Wang Y., Yin H. and Qi L., No-arbitrage interpolation of the option price function and its reformation, *Journal of Optimization Theory and Applications*, **120** (3), 627–649.