Exotic Preferences in Heterogenous Agent Models

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1 Introduction

This document gives a brief explanation of Epstein-Zin preferences and Quasi-Hyperbolic discounting as well as how to implement them using the VFI Toolkit. I describe the preferences, give psuedocode to solve them, and provide examples for both OLG and infinite-horizon heterogeneous agent models that use them.

In standard economic models a single parameter determines both risk aversion and the intertemporal elasticity of substitution. Epstein-Zin preferences seperate the parameter that determines risk aversion from the parameter that determines the intertemporal elasticity of substitution. The usefulness of this in Economics seems self-evident.

Quasi-hyperbolic preferences allow us to model 'impatience', with the present-self taking decisions that are not in the interest of their own future-self. This is true of hyperbolic preferences in general, with Quasi-hyperbolic preferences simply being a tractable implementation. In standard exponential discounting we discount from any one period to the next at rate β; hence something two periods from now is discounted by β^2, and something three periods from now is discounted by β^3. With Quasi-hyperbolic discounting there are two parameters, β_0 which is the *additional* rate we discount between today and tomorrow, and β which is the rate we discount between any two consecutive periods. Hence something one period from now is discounted by $\beta_0\beta$, something two periods from now is discounted by $\beta_0\beta^2$, and something three periods from now is discounted by $\beta_0\beta^3$. Note that there are two ways solve Quasi-hyperbolic preferences, naive where the agent (incorrectly) assumes that in the future they will not be impatient, and sophisticated where the agent takes into account the fact that they will be 'impatient' in the future. Both are explained and implemented with naive being the default.

When implementing models using either of Epstein-Zin preferences or Quasi-hyperbolic discounting the value function problem is different from the standard case. There is however no change in terms of the algorithms used to simulate the models, calculate statistics of the agent distribution, nor in the algorithms used to find general equilibrium. Of course there is a partial exception in that any welfare calculations would need to be appropriately modified. With this in mind the VFI Toolkit simply needs to be told that the value function problem is going to be using these exotic preferences, which as is demonstated in the codes provided is very simple, typically requiring just a few lines of code.

For a more comprehensive discussion of exotic preferences see Backus, Routledge, and Zin (2004). Also of potential interest is a discussion of structural behavioural economics in DellaVigna (2018).

Note that it is not possible to use both Epstein-Zin preferences and Quasi-hyperbolic discounting at the same time in the VFI Toolkit. I am not aware of any article that shows that combining the two works in theory. I suspect they would, but don't want to implement them until I am sure. If you know of an article that shows how to combine them please contact me and let me know and I will try and implement it.

This pdf and all the relevant codes can be found at: https://github.com/vfitoolkit/VFItoolkit-matlab-examples/tree/master/Exotic Preferences

2 Epstein-Zin preferences

In standard economic models agents expectations about the future are based on 'von-Neumann-Morgenstern' expected utility. This involves two main aspects, preferences are time-seperable, and the future utilities matter in terms of their expectation. An unintentional side-effect of this is that both risk aversion and the intertemporal elasticity of consumption get determined by the same parameter. So for example in a standard two-period model of consumption-savings decision the 'value' in the first period is[1]

$$u(c_1) + \beta E[u(c_2)] \tag{1}$$

or if we were to use the constant-elasticity-of-substitution (CES) utility function,

$$\frac{c_1^{1-\gamma}}{1-\gamma} + \beta E\left[\frac{c_2^{1-\gamma}}{1-\gamma}\right] \tag{2}$$

where γ will determine both the risk aversion and the intertemporal elasticity of substitution. $\frac{-cu''(c)}{u'(c)} = \gamma$ is the relative risk aversion (RRA), the CES utility function displays constant RRA, called CRRA.[2] $ln\left(\frac{u'(c_{t+1})}{u'(c_t)}\right) = 1/\gamma$ is intertemporal elasticity of substitution. Notice how just one parameter γ determines both the risk aversion and the intertemporal elasticity of substitution.

Epstein-Zin preferences (Epstein and Zin, 1989) seperate the risk aversion from the intertemporal elasticity of substitution, using a different parameter to determine each of these. In our simple two-period model using Epstein-Zin preferences the 'value' in the first period becomes

$$\left[(1-\beta)c_1^{1-1/\psi} + \beta\big(E\big[((1-\beta)c_2^{1-1/\psi})^{1-\gamma}\big]\big)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}$$

where now γ is the CRRA and ψ is the intertemporal elasticity of substitution. It is important to note that we cannot write this in the form of von-Neumann-Morgenstern expected utility preferences as in (1). Except for the case when $1/\psi = \gamma$, in which case both play the role of γ in equation (2); note that with $1/\psi = \gamma$ the Epstein-Zin preferences will be vNM preferences, but that you still need to account for the use of $(1-\beta)$.

[1]To keep the notation easier I will only describe the utility functions, and skip over the budget constraints and the like.

[2]Actually this is not really the correct way to measure risk-aversion in models with many time periods and/or many goods, see Swanson (2012), and it becomes more subtle again when using Epstein-Zin preferences (Swanson, 2018).

All of this can be more easily and generally done using recursive notation. We have that standard vNM preferences can be expressed as the value function

$$V(a,z) = max_{c,a' \in D(a,z)} u(c_t) + \beta E[V(a',z')]$$

where a is an endogenous state, and z is an exogenous stochastic state. c and a' are constrainted to be in some feasible decision space $D(a,z)$. In finite time $V(a',z')$ would be next period value function and so not the same as $V(a,z)$, in infinite time they are the same. Again, using the example of CES preferences we would get

$$V(a,z) = max_{c,a' \in D(a,z)} \frac{c^{1-\gamma}}{1-\gamma} + \beta E[V(a',z')]$$

We can express Epstein-Zin preferences as

$$V(a,z) = max_{c,a' \in D(a,z)} \left[(1-\beta)u_t^{1-1/\psi} + \beta(E[V(a',z')^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}$$

where u_t is the per-period utility function. In practice you are likely to want $u_t = u(c_t) = c_t$ in models with exogenous labour and $u_t = u(c_t, l_t) = (c_t^{\chi}(1-l_t)^{1-\chi})$ where l is labor supply and so $1-l$ is leisure.[3]

Solving the value function for Epstein-Zin preferences is concuptually simple. Consider the following standard value function iteration algorithm,

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_{n+1} - V_n|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for A **do**ll values of a and z
    Calculate $V_n(a,z) = \max_{d,a' \in D(a,z)} u_t(d,a',a,z) + \beta E[V_{old}(a',z')|z]$
    Calculate $g_n(a,z) = \arg\max_{d,a' \in D(a,z)} u_t(d,a',a,z) + \beta E[V_{old}(a',z')|z]$
  end for
end while
return V_n, g_n

The only change that needs to be made is to replace the two steps calculating of $V_n(a,z)$ and $g_n(a,z)$ with those that incorporate the Epstein-Zin preferences. Thus we have,

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_n - V_{n-1}|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for A **do**ll values of a and z

[3] In principle you could think of applying Epstein-Zin preferences to other agent problems with these as any 'two goods' that generate utility, rather than necessarily being consumption and leisure.

Calculate $V_n(a,z) = \max_{d,a' \in D(a,z)} (1-\beta)(u_t(d,a',a,z))^{1-1/\psi} + \beta(E[V_{old}(a',z')^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}}$

Calculate $g_n(a,z) = \arg\max_{d,a' \in D(a,z)} (1-\beta)(u_t(d,a',a,z))^{1-1/\psi} + \beta(E[V_{old}(a',z')^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}}$

end for

end while

return V_n, g_n

Note that this has two main practical applications for implementing the algorithm, the step calculating the expectation of next periods value function has to be appropriately modified, and the problem for which to compute the max (and/or argmax) has to be appropriately modified.[4] In a finite horizon problem the algorithm is modified in the obvious manner, namely that V_{old} will be the next period value function, and that solution will be iterating from the final period back to the first period.[5]

Notice that once you have solved the value function problem to get the optimal policy function everything else about a heterogeneous agent model depends only on the optimal policy and so we don't need to take account of Epstein-Zin preferences for things like simulating the model. The exception is any kind of welfare evaluation.

3 Quasi-Hyperbolic Discounting

Quasi-hyperbolic preferences are a tractable way to model 'impatience', with the present-self taking decisions that are not in the interest of their own future-self. The standard way to discount the future is exponential discounting, which uses the same discount factor β between any two consecutive periods. Quasi-hyperbolic discounting involves two discount factors, β_0 which is used as the additional discount factor between the current period and the next period, and β which is used as the discount factor between any two consecutive periods; so the discount factor between the current period and the next period is $\beta_0\beta$.[6] The concept of hyperbolic discounting has a long heritage, and the tractable case of quasi-hyperbolic was introduced in Laibson (1997). Let's introduce the idea using a four period model. I first will describe quasi-hyperbolic here in a deterministic setting (without uncertainty), and then switch to a stochastic setting when switching to recursive notation

[4] Whether this is solved with value function iteration, as described here, or policy function iteration, is not important in the sense that both will work. Similarly how the value function is approximated is left unspecificied.

[5] Note that the final period utility with Epstein-Zin preferences will then be $(1-\beta)(u_T(d,a',a,z))^{1-1/\psi}$, where T indicates the final/terminal period.

[6] We use $\beta_0\beta$ as the discount rate between this period and next period, rather than just some $\alpha \equiv \beta_0\beta$ because this notation becomes much more convenient when we want to look at, e.g., life-cycle models where there is a probability of dying; especially in terms of writing the codes. Note that papers that look at theory on Quasi-Hyperbolic discounting in simple models often just use one parameter to control this period to next period, e.g., $\alpha \equiv \beta_0\beta$.

In a four period model with standard exponential discounting

$$u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \beta^3 u(c_4) \tag{3}$$

Notice how β is used as the discount rate between periods one-and-two as well as between periods two-and-three and three-and-four. In contrast quasi-hyperbolic discounting uses $\beta_0\beta$ between periods one-and-two, but then uses β between periods two-and-three and three-and-four. Thus our four period model with quasi-hyperbolic discounting becomes

$$u(c_1) + \beta_0\beta u(c_2) + \beta_0\beta^2 u(c_3) + \beta_0\beta^3 u(c_4) \tag{4}$$

Because we have not fully specified the optimization problem (what is being choosen, just c_1? or also c_2, c_3, & c_4?) it is not so obvious in this formulation that there are two types of quasi-hyperbolic discounter: naive and sophisticated. A *naive* quasi-hyperbolic discounter assumes that their future self will not be impatient even though their present self is (their future self is using exponential discounting). A *sophisticated* quasi-hyperbolic discounter; one who realises that their future self will also behave impatiently by discounting in a quasi-hyperbolic manner. This will be much clearer when we now switch to recursive notation and write out the optimization problem in full.

We now express the same thing in recursive notation, but adding stochastics (z and the expectations). We have that standard exponential discounting can be expressed as the value function

$$V(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta E[V(a',z')] \tag{5}$$

where a is an endogenous state, and z is an exogenous stochastic state. c and a' are constrainted to be in some feasible decision space $D(a,z)$. In finite time $V(a',z')$ would be next period value function and so not the same as $V(a,z)$, in infinite time they are the same.

We can express naive quasi-hyperbolic discounting in terms of the exponential discounting solution. Let $\tilde{V}$ be the value function of the naive quasi-hyperbolic discounter, then

$$\tilde{V}(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta_0\beta E[V(a',z')] \tag{6}$$

notice how the expected next-period value function is that of the behavior of the exponential discounter, who the naive quasi-hyperbolic discounter (incorrectly) believes they will act like.

The sophisticated quasi-hyperbolic discounter understands that their future self will behave as a quasi-hyperbolic discounter. Let $\hat{V}$ be the value function of the sophisticated quasi-hyperbolic discounter, then

$$\hat{V}(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta_0\beta E[\underline{\mathrm{V}}(a',z')] \tag{7}$$

and the argmax of this same expression gives the (optimal) policies $\hat{c}$ and $\hat{a}'$. Using $\hat{c}$ and $\hat{a}'$ we get the definition of $\underline{\mathrm{V}}(a,z)$ as

$$\underline{\mathrm{V}}(a,z) = u(\hat{c}) + \beta E[\underline{\mathrm{V}}(\hat{a}',z')] \tag{8}$$

note that to compute this we will need, given next period $\underline{\mathrm{V}}$, to first get current period $\hat{V}$ and $\hat{c}$ and $\hat{a}'$ from equation (7), and then use these in equation (8) to calculate the current period $\underline{\mathrm{V}}$.

Notice that standard exponential discounting is nested as the case $\beta_0 = 1$, and quasi-hyperbolic discounting involves $0 < \beta_0 < 1$. For infinite horizon models $0 < \beta < 1$, but for finite horizon models with a probability of dying it can sometimes be greater than 1.

Let's now look at psuedocode for solving these. Solving the naive quasi-hyperbolic discounting thus still involves solving the exponential discounting case (to get V using equation (5)) with the added step of then using equation (6). In the infinite horizon this means

Declare initial value V_0.
Declare iteration count $n = 0$.
while $||V_n - V_{n-1}|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for A **do**ll values of a and z
    Calculate $V_n(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta E[V(a',z')]$
    Calculate $g_n(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta E[V(a',z')]$
  end for
end while
Calculate $\tilde{V}_n = \max_{d,a' \in D(a,z)} F(d,a',a,z) + \beta_0 \beta V_n$
Calculate $\tilde{p}_n = \arg\max_{d,a' \in D(a,z)} F(d,a',a,z) + \beta_0 \beta V_n$
return $\tilde{V}_n$, $\tilde{g}_n$

Notice that the finite horizon version of this would involve calculating $\tilde{V}_n$ and $\tilde{g}_n$ inside the for (and while) loops, immediately after calculating each V_n and g_n. Finite horizon would start in the final period, and then use a for loop backwards over the time periods in place of the while loop. Note that while $\tilde{V}$ depends on V, V does not depend on $\tilde{V}$.

Solving for the sophisticated quasi-hyperbolic discounting is slightly more complicated as $\hat{V}$ and $\underline{\mathrm{V}}$ each depend on the other. In the infinite horizon the psuedocode for the algorithm is

Declare initial value $\hat{V}_0$ and $\underline{\mathrm{V}}_0$.
Declare iteration count $n = 0$.
while $||\hat{V}_n - \hat{V}_{n-1}|| >$ 'tolerance constant' **do**
  Increment n. Let $V_{old} = V_{n-1}$.
  for A **do**ll values of a and z
    Calculate $\hat{V}_n(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta_0 \beta E[\underline{\mathrm{V}}(a',z')]$
    Calculate $\hat{g}_n(a,z) = max_{c,a' \in D(a,z)} u(c) + \beta_0 \beta E[\underline{\mathrm{V}}(a',z')]$

Calculate $\underline{\mathrm{V}}_n = F(\hat{g}_n^d(a,z), \hat{g}_n^{a'}(a,z), a, z) + \beta E[\underline{\mathrm{V}}_n(\hat{g}_n^{a'}(a,z), z')]$

end for

end while

return $\hat{V}_n$, $\hat{g}_n$

The finite-horizon version of this is obvious, simply starting from the final time period and doing backward induction in time in place of the while loop. Note that $\hat{g}_n(a,z)$ is a choice of both d and a', I denote $\hat{g}_n^d(a,z)$ as the choice for d, and analagously for $\hat{g}_n^{a'}(a,z)$ and a'.

4 Examples in an Infinite-Horizon model

In this section I explain the Aiyagari model, as well as a version of the model with Epstein-Zin preferences, a version of the model with Quasi-Hyperbolic discounting, a version of the model with endogenous labor, and a version of the model with both endogenous labor and Epstein-Zin preferences.

4.1 Aiyagari model

I assume the reader is already familiar with the model of Aiyagari (1994), an infinite-horizon model with incomplete markets, borrowing-constraints, and a representative firm. I will therefore proceed directly to give the equations. I will use z to denote the household-level exogenous labor, and L the aggregate (rather than the more obvious l and L) because this will make it much easier to later introduce an extension with endogenous labor.

There is a household problem, in which a household faces an exogenous income process and makes consumption-savings decisions. It can be represented by the following value function problem:

$$
\begin{aligned}
V(k,z) = &\max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta E[V(k',z')|z] \\
&\text{s.t. } c + k' = (1+r)k + wz \\
&\qquad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2)
\end{aligned}
\tag{9}
$$

where k is household capital (assets), c is consumption, r is the interest rate (net of depreciation), w is the wage, and z is exogenous labor supply and follows an AR(1) process.[7] Denote the optimal policy function that solves this value function problem by $k' = g(k,z)$. Denote the (implied) Markov transition function for z as $Q(z',z)$.

There is a representative firm with Cobb-Douglas production function, $Y = K^\alpha L^{1-\alpha}$, and there is perfect competition in the factor markets (capital and labor) so the firm's profit maximization

[7] σ is the standard deviation of l, σ_ϵ is the standard deviation of the innovations to the AR(1) process; a standard time series result is that $\sigma_\epsilon^2 = \sigma^2(1-\rho^2)$).

problem give us that,

$$w = (1-\alpha)K^{\alpha}L^{-\alpha} \tag{10}$$
$$r = \alpha K^{\alpha-1}L^{1-\alpha} - \delta \tag{11}$$

note that r is the interest rate net of depreciation, hence the addition of the $-\delta$ term.

Combining the policy function (which maps from (k,z) into k) with the exogenous (labor) shock process (which maps from z into z) we get a law-of-motion for the agent distribution (a mapping from (k,z) into (k,z)).[8] We will denote the law-of-motion for the agent distribution as $P = p \circ Q$. The stationary distribution of agents, Λ, satifies $\Lambda = P\Lambda$.

A stationary equilibrium consists of the following.

i The household value fn and policy fn solve the household value fn problem, (9).

ii The interest rate r and wage w satisfy the firms profit maximization problem, (10) & (11).

iii The aggregate variables satify:
$K = \int k d\Lambda$
$L = \int z d\Lambda$, note that this equals one by definition of the process on z.

iv The agent distribution is stationary $\Lambda = P\Lambda$.

Finding the stationary equilbrium can be posed as finding the (equilibrium) interest rate at which this holds (this is part of what I assume you already know about this model).

Solving this problem consists of four steps.

Start with a guess of r.

1. Solve agents value function problem (to get the policy).
2. Find stationary distribution of agents.
3. Calculate the aggregate variables.
4. Check the general equilibrium conditions, namely whether the implied r from the aggregate variables is the same as our guess of r.
5. If the general equilibrium condition is satisfied (implied r from firm optimization conditions is the 'same' the guess then we have found the general equilibrium value for r) then we are finished and the current r is the general equilibrium value, otherwise update our guess for r and return to the first step.

[8] I have been lazy with notation, the mappings are from the spaces, not the elements of the spaces, but you get the idea.

When we compute exotic preferences the only change is to step 1. All of the other steps remain unchanged.

When we endogenize the labor decision we need to include a guess for w alongside that of r, and update the implied w as well as r in Steps 4 & 5. With exogenous labor we only need to evaluate one of r and w because exogenous labor meant the wage is just a function of r,[9] while with endogenous labor w and r become genuinely different and so we need to use both the firm optimization conditions.[10]

Codes implementing the Aiyagari model can be found at this link. Specifically, run the main code *Aiyagari1994.m*.

4.2 Aiyagari with Epstein-Zin preferences

The only change is to the household problem, equation (9), which now becomes,

$$
\begin{aligned}
V(k,z) = &\max_{c,k'} (1-\beta)c_t^{1-1/\psi} + \beta(E[V(k',z')^{1-\gamma}|z])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}} \\
&\text{s.t. } c + k' = (1+r)k + wz \\
&\qquad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2)
\end{aligned}
$$

All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged. Note that when implementing this with the VFI Toolkit this involves both setting $vfoptions$ and also requires changing the return function (to return c_t rather than $\frac{c_t^{1-\gamma}}{1-\gamma}$ which is what it returns for standard Aiyagari model).

Codes implementing the Aiyagari model with Epstein-Zin preferences can be found at this link. Specifically, run the main code *Aiyagari1994_EpsteinZin.m*.

4.3 Aiyagari with Quasi-Hyperbolic discounting

I describe first the case of naive quasi-hyperbolic discounting, and then the case of sophisticated quasi-hyperbolic discounting.

For *naive* quasi-hyperbolic discounting the only change is to the household problem, equation

[9] You can use the equations (10) and (11) from the optimization problem of the firm to show this. Rewrite the second in terms of K/L, then substitute into the first to get $w = (1-\alpha)((r+\delta)/\alpha)^{\frac{\alpha}{\alpha-1}}$; note that this is useful as we don't need to check that the labor market clears as it trivally does with exogenous labor (we know $L=1$, so we can just use this directly).

[10] Whether you prefer to think of general eqm as checking market clearance in both the labor and capital markets, or checking that the prices are 'correct', does not matter as the two are equivalent interpretations.

(9), which now becomes,

$$\begin{aligned} V(k,z) = &\max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta E[V(k',z')|z] \\ &\text{s.t. } c + k' = (1+r)k + wz \\ &\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2) \end{aligned} \tag{12}$$

together with

$$\begin{aligned} \tilde{V}(k,z) = &\max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta_0 \beta E[V(k',z')|z] \\ &\text{s.t. } c + k' = (1+r)k + wz \\ &\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2) \end{aligned} \tag{13}$$

All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged.

For *sophisticated* quasi-hyperbolic discounting the only change is to the household problem, equation (9), which now becomes,

$$\begin{aligned} \hat{V}(k,z) = &\max_{\hat{c},\hat{k}'} \frac{\hat{c}^{1-\gamma}}{1-\gamma} + \beta_0 \beta E[\underline{\mathrm{V}}(\hat{k}',z')|z] \\ &\text{s.t. } \hat{c} + \hat{k}' = (1+r)\hat{k} + wz \\ &\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2) \end{aligned} \tag{14}$$

together with

$$\begin{aligned} \underline{\mathrm{V}}(k,z) = &\frac{\hat{c}^{1-\gamma}}{1-\gamma} + \beta E[\hat{V}(\underline{\mathrm{k}}',z')|z] \\ ln(z)' = &\rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2) \end{aligned} \tag{15}$$

All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged.

Codes implementing the Aiyagari model with Quasi-Hyperbolic discounting can be found at this link. Specifically, run the main code *Aiyagari1994_QuasiHyperbolic.m*. Note that the same code is used for both the naive and sophisticated quasi-hyperbolic preferences, and switching between the two just involves one line of code setting $vfoptions.quasi\_hyperbolic =' Naive'$; or $vfoptions.quasi\_hyperbolic =' Sophisticated'$;.

4.4 Aiyagari with Endogenous Labor

The households problem now includes the choice of how much to work; l becomes an decision variable. To include labor supply l in the utility function we think of the household as getting utility from leisure, which we set as $1-l$. The household value fn problem becomes,

$$\begin{aligned} V(k,z) =& \max_{c,k',l} \frac{c^{1-\gamma}}{1-\gamma} + \chi\frac{(1-l)^{1-\gamma_l}}{1-\gamma_l} + \beta E[V(k',z')|z] \\ &\text{s.t. } c+k' = (1+r)k + wlz - \theta\mathbb{I}_{l\{>0\}} \\ &\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma^2_\epsilon) \end{aligned} \tag{16}$$

where z is now an idiosyncratic labor productivity shock. To keep some features of the model we will make one small change. To ensure that some people choose $l=0$ and are thus unemployed we add a fixed cost of working (which is paid when $l>0$; imagine commuting or similar). You can see this as the $-\theta\mathbb{I}_{l\{>0\}}$ term in the budgent constraint. Without this fixed cost of working noone would choose to work zero, and so we would have no unemployed. Notice that the pension was always calculated on 'potential life-time earnings', not actual life-time earnings, and this remains the case.[11]

The other change to the model equations that matter for equilbrium is that now $L = \inf lzd\Lambda$.[12] Note also that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization and as a result we need to find both equilibrium w and r —equilibrium in both the labor and capital markets— so we do need to change the general equilibrium equations in the codes to add the one for w. Note also that the VFI Toolkit needs you to redefine the $FnsToEvaluate$, such as K and L since these functions are evaluated on the decision varibles and state variables, and we have added a decision variable.

Note that in this version we have used seperable preferences over consumption and leisure, $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} + \chi\frac{(1-l)^{1-\gamma_l}}{1-\gamma_l}$.[13] We could alternatively use non-seperable preferences over consumption and leisure. This can be done using the preferences $u(c,l) = \frac{(c^{1-\chi}(1-l)^{\chi}))^{1-\gamma}}{1-\gamma}$.[14]

The households problem now includes the choice of how much to work, or equivalently how much leisure (leisure is $1-l$),

$$\begin{aligned} V(k,z) =& \max_{c,k',l} \frac{(c^{1-\chi}(1-l)^{\chi}))^{1-\gamma}}{1-\gamma} + \beta E[V(k',z')|z] \\ &\text{s.t. } c+k' = (1+r)k + wlz - \theta\mathbb{I}_{l\{>0\}} \\ &\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma^2_\epsilon) \end{aligned} \tag{17}$$

[11]To allow pensions to depend on actual life-time earnings we would have to add a state variable that kept track of these.

[12]L is now 'effective units of labor', think of being like hours worked times productivity per hour adding up to the effective units. To be precise in this model l is not hours worked but fraction of total time worked.

[13]For $u(c,l)$ seperable preferences mean that $u_c(c,l)$ does not depend on l and that $u_l(c,l)$ does not depend on c. That is, the marginal utility of each of c and l is 'seperate' from the other. Time seperable preferences play an important role in models with exogenous growth as non-seperable preferences are typically not compatible with the existence of a balanced-growth path since non-seperable preferences mean as people get richer they will gradually work less and less, but we won't go into that here.

[14]More general form of non-seperable preferences is $u(c,l) = \frac{([(1-\chi)c^{1-\upsilon} + \chi(1-l)^{1-\upsilon})]^{\frac{1}{1-\upsilon}})^{1-\gamma}}{1-\gamma}$ for $\upsilon \neq 1$, for the case of $\upsilon = 1$ it simplifies to $u(c,l) = \frac{(c^{1-\chi}(1-l)^{\chi}))^{1-\gamma}}{1-\gamma}$ which is what we use here. υ can be used to give different weights to c and l, whereas χ is determining the elasticity of substitution between the two.

The other change to the model equations that matter for equilbrium is that now $L = \inf lzd\Lambda$.[15] Note also that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization and as a result we need to find both equilibrium w and r —equilibrium in both the labor and capital markets— so we do need to change the general equilibrium equations in the codes to add the one for w. Note also that the VFI Toolkit needs you to redefine the $FnsToEvaluate$, such as K and L since these functions are evaluated on the decision varibles and state variables, and we have added a decision variable.

Codes implementing the Aiyagari model with with endogenous labour can be found at this link. Specifically, run the main code *Aiyagari1994_EndoLabor.m*.

4.5 Aiyagari with Endogenous Labor and Epstein-Zin preferences

The households problem now includes the choice of how much to work, or more precisely I will write it as how much leisure (work will then be 1-leisure). This means that we now have Epstein-Zin preferences over two goods (consumption and leisure).

$$
\begin{aligned}
V(k,z) = &\max_{c,k',l}(1-\beta)(c^{1-\chi}(1-l)^{\chi}))^{1-1/\psi} + \beta(E[V(k',z')^{1-\gamma}|z])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}} \\
&\text{s.t. } c + k' = (1+r)k + wlz - \theta \mathbb{I}_{l\{>0\}} \\
&\quad ln(z)' = \rho ln(z) + \epsilon, \quad \epsilon \sim N(0,\sigma_\epsilon^2)
\end{aligned}
$$

Notice that the consumption and leisure use non-seperable preferences.[16] There is no change to any of the other equations relating to equilibrium. Note however that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization.

The other change to the model equations that matter for equilbrium is that now $L = \inf lzd\Lambda$.[17] Note also that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization and as a result we need to find both equilibrium w and r —equilibrium in both the labor and capital markets— so we do need to change the general equilibrium equations in the codes to add the one for w. Note also that the VFI Toolkit needs you to redefine the $FnsToEvaluate$, such as K and L since these functions are evaluated on the decision varibles and state variables, and we have added a decision variable.

Codes implementing the Aiyagari model with with endogenous labour can be found at this link. Specifically, run the main code *Aiyagari1994_EndoLabor_EpsteinZin.m*.

[15] L is now 'effective units of labor', think of being like hours worked times productivity per hour adding up to the effective units. To be precise in this model l is not hours worked but fraction of total time worked.

[16] I have not seen any paper using Epstein-Zin together with seperable preferences, if you know that it is possible, or know that it is impossible, please let me know so I can update this.

[17] L is now 'effective units of labor', think of being like hours worked times productivity per hour adding up to the effective units. To be precise in this model l is not hours worked but fraction of total time worked.

5 Examples in an OLG model

In this section I explain the model of Imrohoroglu, Imrohoroglu, and Joines (1995). I then present a version of the model with Epstein-Zin preferences, and a version of the model with Quasi-Hyperbolic discounting. I further provide an extension to endogenous labor, and then a version with both endogenous labor and Epstein-Zin prefereces.

5.1 OLG model of IIJ1995

We will look at a brief description of the model of of Imrohoroglu, Imrohoroglu, and Joines (1995). I provide only the equations, not the economic motivation, and make numerous notational changes; for a full explanation of the model you should consult the original paper.

The model is a general equilibrium OLG model. The finite-horizon value function problem has one exogenous state (which takes two possible values, employed and unemployed), one endogenous state (assets), and 65 periods. The household value function problem is given by

$$\begin{aligned} V(k,z,j) = &\max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta s_j E_j[V(k',z',j+1)|z] \\ &\text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j z \mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \cdots \\ &\qquad\qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\ &\qquad k' \geq 0 \end{aligned} \tag{18}$$

There are $J = 65$ periods and $V(k,z,J+1) = 0$ for all k, & z. So household faces employment-status shocks (z) and solve a consumption-savings problem of choosing consumption c and next period assets k'. There are some basic taxes which are used to fund pensions SS that are received once retirement age, $J_r = 45$, is reached. When people die their assets are redistributed lump-sum accoss the living as Tr_{beq}. Households discount the future by pure discount factor β and conditional probability of survival s_j.

The earnings process z consists of two states, 'employed' which is when $z = 1$ and 'unemployed' which is when $z = 0$. While in principle it could be markov (and IIJ1995 describes it as such) the markov transition matrix is defined so that the actual shock is iid. (The rows of the markov transition matrix are all the same.) ϵ_j is a deterministic spline of earnings in terms of age and is used to generate the age profile of earnings. When unemployed agents receive unemployment benefits u.

The initial distribution of agents at birth is for them to have zero assets and the stationary distribution of shocks.

The government budget constraint consists of the following two (seperate) parts: the unemployment benefits tax, τ_u, pays for unemployment benefits u. The social security payroll tax, τ_s, pays for social security (pension) benefits SS.

The model has five general equilibrium constraints, the first is that the interest rate r equals the marginal product of capital minus the depreciation rate δ. The next two are fiscal: that the unemployment benefits tax balances unemployment benefits, and that the social security tax pays for social security pensions. The fifth is that the the (total across the population of the) lump-sum transfer of accidental bequests Tr_{beq} much equal the assets left behind by people on dying.

Notational differences from Imrohoroglu, Imrohoroglu, and Joines (1995), originals in parentheses: I refer to the replacement rate for social security (pension) benefits as b (θ) and the benefits themselves as SS (b). Age of retirement is Jr (j^*). Population growth rate n (ρ). Age-conditional survival probability is s_j (ψ_j). Exogenous shock –employment or unemployment– is z (s). In the Cobb-Douglas production function I use α as share of capital ($1-\alpha$) and A as the total factor productivity (B). Lump-sum transfers due to accidental bequests are Tr_beq (T).

Note that I use $V(k,z,j)$ and $V(k,z,j+1)$. The alternative notation of $V_j(k,z)$ and $V_{j+1}(k,z)$ is often used.

For more details on the model see Imrohoroglu, Imrohoroglu, and Joines (1995).

Codes implementing the IIJ1995 model can be found at this link. Specifically, run the main code *ImrohorogluImrohorogluJoines1995_Example.m*.

The actual paper of IIJ1995 contains a number of extensions of the model (to exogenous technology growth, medical shocks when old, welare evaluation, etc.) and the *ImrohorogluImrohoroglu-Joines1995_ExampleFull.m* code implementing this can be found at the same place, it is just the standard Example code with a few things added.

5.2 OLG model of IIJ1995 with Epstein-Zin preferences

The only change is to the household problem, equation (18). Only the first line changes, all of the constraints remain the same.

$$
\begin{aligned}
V(k,z,j) = & \max_{c,k'} (1-\beta s_j) c_t^{1-1/\psi} + \beta s_j (E[V(k',z',j+1)^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}} \\
\text{subject to } & c + k' \leq (1 + r(1-\tau_k))k + (1-\tau_s-\tau_u) wh\epsilon_j z \mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \dots \\
& \quad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\
& k' \geq 0
\end{aligned}
$$

All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged. Note that when implementing this with the VFI Toolkit this involves both setting $vfoptions$ and also requires changing the return function (to return c_t rather than $\frac{c_t^{1-\gamma}}{1-\gamma}$ which is what it returns for standard IIJ1995 model).

Codes implementing the IIJ1995 model with Epstein-Zin preferences can be found at this link. Specifically, run the main code *ImrohorogluImrohorogluJoines1995_EpsteinZin.m*.

5.3 OLG model of IIJ1995 with Quasi-Hyperbolic discounting

The only change is to the household problem. For *naive* quasi-hyperbolic discounting we continue to have the same equation (18),

$$\begin{aligned} V(k,z,j) = & \max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta s_j E_j[V(k',z',j+1)|z] \\ & \text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j z \mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \dots \\ & \qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\ & \qquad k' \geq 0 \end{aligned}$$

but now with the addition of an expression for $\tilde{V}$,

$$\begin{aligned} \tilde{V}(k,z,j) = & \max_{c,k'} \frac{c^{1-\gamma}}{1-\gamma} + \beta_0 \beta s_j E_j[V(k',z',j+1)|z] \\ & \text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j z \mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \dots \\ & \qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\ & \qquad k' \geq 0 \end{aligned}$$

Note that in the expression for $\tilde{V}$ the discount is based on $\beta_0\beta$. All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged.

For *sophisticated* quasi-hyperbolic discounting we instead relace equation (18) with expressions for $\hat{V}$ and $\underline{\text{V}}$,

$$\begin{aligned} \hat{V}(k,z,j) = & \max_{\hat{c},\hat{k}'} \frac{\hat{c}^{1-\gamma}}{1-\gamma} + \beta_0 \beta s_j E_j[\underline{\text{V}}(\hat{k}',z',j+1)|z] \\ & \text{subject to } \hat{c} + \hat{k}' \leq (1+r(1-\tau_k))\hat{k} + (1-\tau_s-\tau_u)wh\epsilon_j z \mathbb{I}_{(j<J_r)} + u(1-z)\mathbb{I}_{(j<J_r)} \dots \\ & \qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\ & \qquad k' \geq 0 \end{aligned}$$

together with

$$\underline{\text{V}}(k,z,j) = \frac{\hat{c}^{1-\gamma}}{1-\gamma} + \beta s_j E_j[\underline{\text{V}}(\hat{k}',z',j+1)|z]$$

Note that the expression for $\underline{\text{V}}$ involves evaluating a function using the optimal polices calculted in the expression for $\hat{V}$ but based on exponential discounting via β and depending on next period $\underline{\text{V}}$. All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equilbrium remain unchanged.

Codes implementing the IIJ1995 model with Quasi-Hyperbolic discounting can be found at this link. Specifically, run the main code *ImrohorogluImrohorogluJoines1995_QuasiHyperbolic.m*. Note that the same code is used for both the naive and sophisticated quasi-hyperbolic discounting, and switching between the two just involves one line of code setting $vfoptions.quasi\_hyperbolic =' Naive';$ or $vfoptions.quasi\_hyperbolic =' Sophisticated';$.

5.4 OLG model of IIJ1995 with Endogenous Labor

The households problem now includes the choice of how much to work, or more precisely I will write it as how much leisure (work will then be 1-leisure),

$$
\begin{aligned}
V(k,z,j) = &\max_{c,k',l} \frac{c^{1-\gamma_c}}{1-\gamma_c} + \chi\frac{(1-l)^{1-\gamma_l}}{1-\gamma_l} + \beta s_j E_j[V(k',z',j+1)|z] \\
&\text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j(1-l)z\mathbb{I}_{(j<J_r)} + u\mathbb{I}_{\{l=1\}}\mathbb{I}_{(j<J_r)} \cdots \\
&\qquad\qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\
&\qquad k' \geq 0
\end{aligned}
$$

Notice that z has been changed substantially, and that unemployment benefits are now given when leisure takes it's maximum value (of 1) which implies that household is not working. There is no change to any of the other equations relating to equilibrium. Note however that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization.[18]

Note that in this version we have used seperable preferences over consumption and leisure, $u(c,l) = \frac{c^{1-\gamma_c}}{1-\gamma_c} + \chi\frac{(1-l)^{1-\gamma_l}}{1-\gamma_l}$.[19] We could alternatively use non-seperable preferences over consumption and leisure. This can be done using the preferences $u(c,l) = \frac{(c^{1-\chi}(1-l)^{\chi})^{1-\gamma}}{1-\gamma}$.[20]

The households problem now includes the choice of how much to work, or more precisely I will

[18] Actually when implementing the codes I just calculated L for each parameterization anyway as it was fast and easy to do, even though unnecessary.

[19] For $u(c,l)$ seperable preferences mean that $u_c(c,l)$ does not depend on l and that $u_l(c,l)$ does not depend on c. That is, the marginal utility of each of c and l is 'seperate' from the other. Time seperable preferences play an important role in models with exogenous growth as non-seperable preferences are typically not compatible with the existence of a balanced-growth path since non-seperable preferences mean as people get richer they will gradually work less and less, but we won't go into that here.

[20] More general form of non-seperable preferences is $u(c,l) = \frac{([(1-\chi)c^{1-\upsilon}+\chi(1-l)^{1-\upsilon})]^{\frac{1}{1-\upsilon}})^{1-\gamma}}{1-\gamma}$ for $\upsilon \neq 1$, for the case of $\upsilon = 1$ it simplifies to $u(c,l) = \frac{(c^{1-\chi}(1-l)^{\chi}))^{1-\gamma}}{1-\gamma}$ which is what we use here. υ can be used to give different weights to c and l, whereas χ is determining the elasticity of substitution between the two.

write it as how much leisure (work will then be 1-leisure),

$$
\begin{aligned}
V(k,z,j) = &\max_{c,k',l} \frac{(c^{1-\chi}(1-l)^{\chi})^{1-\gamma}}{1-\gamma} + \beta s_j E_j[V(k',z',j+1)|z] \\
&\text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j(1-l)z\mathbb{I}_{(j<J_r)} + u\mathbb{1}_{\{l=1\}}\mathbb{I}_{(j<J_r)} \cdots \\
&\qquad\qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\
&\qquad k' \geq 0
\end{aligned}
$$

There is no change to any of the other equations relating to equilibrium. Note however that when labor was exogenous we could simply precompute L as it was exogenous, while now we need to calculate L for each parameterization.

While the other equations relating to equilibrium are in principle unchanged the VFI Toolkit needs you to redefine the $FnsToEvaluate$, such as K and L since these functions are evaluated on the decision varibles and state variables, and we have added a decision variable.

We still need to (re)define the markov process for the labor productivity shocks z. The codes use z in an AR(1) process: $z' = \rho z + \epsilon_z$, $\epsilon \sim N(0, \sigma_{\epsilon_z}$. Parameters are $\rho = 0.6$ and $\sigma_z = 0.2$. Note that the standard deviation of z is σ_z, and this is related to the standard deviation of the innovations ϵ_z by $\sigma^2_{\epsilon_z} = \sigma^2_z(1-\rho)^2$.

Codes implementing the IIJ1995 model with endogenous labor can be found at this link. Specifically, run the main code *ImrohorogluImrohorogluJoines1995_EndoLabor.m*.

5.5 OLG model of IIJ1995 with Endogenous Labor and Epstein-Zin preferences

The only change is to the household problem, equation (18). Only the first line changes, all of the constraints remain the same.

$$
\begin{aligned}
V(k,z,j) = &\max_{c,k',l}(1-\beta s_j)\left(c^{1-\chi}(1-l)^{\chi}\right)^{1-1/\psi} + \beta s_j (E[V(k',z',j+1)^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}}]^{\frac{1}{1-1/\psi}} \\
&\text{subject to } c + k' \leq (1+r(1-\tau_k))k + (1-\tau_s-\tau_u)wh\epsilon_j(1-l)z\mathbb{I}_{(j<J_r)} + u\mathbb{1}_{\{l=1\}}\mathbb{I}_{(j<J_r)} \cdots \\
&\qquad\qquad + SS\mathbb{I}_{(j\geq J_r)} + Tr_{beq} \\
&\qquad k' \geq 0
\end{aligned}
$$

All of the other equations, the definition of a stationary equilibrium, and the steps involved in computing the stationary general equlibrium remain unchanged. Note that when implementing this with the VFI Toolkit this involves both setting $vfoptions$ and also requires changing the return function (to return $c^{1-\chi}(1-l)^{\chi}$ rather than which is what it returns for standard IIJ1995 model).

We still need to (re)define the markov process for the labor productivity shocks z. The codes use z in an AR(1) process: $z' = \rho z + \epsilon_z$, $\epsilon \sim N(0, \sigma_{\epsilon_z}$. Parameters are $\rho = 0.6$ and $\sigma_z = 0.2$.

Note that the standard deviation of z is σ_z, and this is related to the standard deviation of the innovations ϵ_z by $\sigma^2_{\epsilon_z} = \sigma^2_z(1-\rho)^2$.

Codes implementing the IIJ1995 model with endogenous labor and Epstein-Zin preferences can be found at this link. Specifically, run the main code *ImrohorogluImrohorogluJoines1995_EndoLabor_EpsteinZin.m*.

6 Brief Note on Other Exotic Preferences

The following describes a short list of other exotic preferences and their uses:

1. **Recursive time preferences (not time-seperable and/or not time-additive)**: in a standard model the utility received in one period does not depend on actions/utility in the periods before and after. A generalized form for recursive preferences is Koopman's time aggregator. Other than the Epstein-Zin preferences discussed above and implemented in VFI Toolkit, the main other form of these in practice tends to mean setting the discount factor to depend either on todays consumption ($\beta(c)$) or on today's utility ($\beta(u)$); this is little used as far as I can tell (except in international trade models with free capital mobility where for purely mathematical reasons it means an equilibrium exists where a constant β would lead to no equilibrium). VFI Toolkit cannot currently do this, although it can allow β to depend on next periods exogenous state.

2. **Habit Formation**: One trivial way to make preferences in one time period depend on actions in other time periods is 'habit formation', which typically involves setting $u(c_t, c_{t-1})$ so the utility from consumption today depends on the 'habit' consumption from last period; utility is increasing in consumption today, and decreasing in the difference between consumption today and the habit consumption from last period. This is easy to implement as a standard value function iteration problem as it simply requires adding an endogenous state capturing last periods consumption, so it is something the VFI Toolkit can already solve in a standard manner. Note that in principle the habit formation could be in anything, but in practice it is most often consumption.[21]

3. **Prospect theory, a.k.a. loss aversion (Kahneman & Tversky)**: This can be modelled using standard value fn iteration, by adding the lag of consumption (or whatever other variable is considered the reference point) as an endogenous state. A short example with VFI Toolkit can be found at: http://discourse.vfitoolkit.com/t/prospect-theory/149

4. Risk-sensitive preferences/Robust control: conceptually these are different concepts, but mathematically they are the same. Robust control can be interpreted as trying to solve

[21] If you instead make the 'habit' the certainty equivalent of the future then you get 'Disappointment aversion' preferences; disappointment aversion and related weighted utility increase risk aversion.

an optimization problem when there is uncertainty about the underlying model. VFI Toolkit does not presently solve these.

5. Ambiguity and max-min preferences: these abandon rational expectations and are premised on the idea that the probability distribution of future outcomes is unknown. VFI Toolkit does not presently solve these.

6. Time inconsistent preferences: we saw the case of quasi-hyperbolic discounting. One aspect which we did not consider is committment; that an agent in period t might commit to a plan that applies in future periods whether or not the same agent wants to stick to that plan next period (or some other future period).

7. Temptation preferences: another form of time-inconsistent preferences. Gul-Pesendorfer provide a formalization. VFI Toolkit does not presently solve these.[22]

Some final comments. When using preferences that depend on a reference point, like habit formation, disappointment aversion, or loss aversion it is empicially challenging to determine what the appropriate reference point should be (and if it evolves over time, then how exactly does it do so). Obviously this list of exotic preferences is not complete, but it does cover a solid range. One substantial area left unaddressed is that these do not cover any kinds of 'social' preferences (the agents in these are always solely self-interested) such as keeping-up-with-the-joneses or social norms.

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