All Periodic Functions $f(t)$

$$a_0 = \frac{1}{P}\int_0^P f(t)dt$$
$$a_n = \frac{2}{P}\int_0^P f(t)\cdot\cos\left(\frac{2\pi}{P}n\cdot t\right)dt$$
$$b_n = \frac{2}{P}\int_0^P f(t)\cdot\sin\left(\frac{2\pi}{P}n\cdot t\right)dt$$

Complex Number Representation:

$$e^{i\theta} = \cos(\theta) + \sin(\theta)\cdot i$$
$$e^{-i\theta} = \cos(-\theta) + \sin(-\theta)\cdot = \cos(\theta) - \sin(\theta)\cdot i$$
$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Recovery of $f(t)$**:**

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} a_n\cdot\cos\left(\frac{2\pi}{P}n\cdot t\right) + \sum_{n=0}^{\infty} b_n\cdot\sin\left(\frac{2\pi}{P}n\cdot t\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cdot\cos\left(\frac{2\pi}{P}n\cdot t\right) + \sum_{n=1}^{\infty} b_n\cdot\sin\left(\frac{2\pi}{P}n\cdot t\right)\end{aligned}$$

`http://www.math.brown.edu/~pflueger/math19/`

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Important Note: *Since there are multiple ways to represent Markov Chains/Models, my solution may not match the notation or format given in Baron. It will be based off of my general knowledge, rather than any specific source.*

2.a. Show the 1-step transition matrix (5p)

The single-step probability of *Transitioning* from step i to step j is given by the variable ${}_i^jT$. A variable that has probability ${}_iS$ of *Starting* in state i, has probability of ${}^jF$ of *Finishing* in state j, which is given as:

$$\begin{bmatrix} {}^1F \\ {}^2F \\ {}^3F \\ {}^4F \\ {}^5F \\ {}^6F \end{bmatrix} = \begin{bmatrix} {}_1^1T & {}_2^1T & {}_3^1T & {}_4^1T & {}_5^1T & {}_6^1T \\ {}_1^2T & {}_2^2T & {}_3^2T & {}_4^2T & {}_5^2T & {}_6^2T \\ {}_1^3T & {}_2^3T & {}_3^3T & {}_4^3T & {}_5^3T & {}_6^3T \\ {}_1^4T & {}_2^4T & {}_3^4T & {}_4^4T & {}_5^4T & {}_6^4T \\ {}_1^5T & {}_2^5T & {}_3^5T & {}_4^5T & {}_5^5T & {}_6^5T \\ {}_1^6T & {}_2^6T & {}_3^6T & {}_4^6T & {}_5^6T & {}_6^6T \end{bmatrix} \times \begin{bmatrix} {}_1S \\ {}_2S \\ {}_3S \\ {}_4S \\ {}_5S \\ {}_6S \end{bmatrix}$$
$$\mathbf{F} = \mathbf{T}\times\mathbf{S}$$

Since for this example we are given the following transition probabilities

Begin in State 1:	${}_1^1T = 0.3$	${}_1^2T = 0.7$			$\sum = 1$
Begin in State 2:	${}_2^1T = 0.1$	${}_2^3T = 0.4$	${}_2^4T = 0.5$		$\sum = 1$
Begin in State 3:	${}_3^1T = 0.9$	${}_3^5T = 0.1$			$\sum = 1$
Begin in State 4:	${}_4^4T = 0.8$	${}_4^5T = 0.2$			$\sum = 1$
Begin in State 5:	${}_5^1T = 0.7$	${}_5^6T = 0.3$			$\sum = 1$
Begin in State 6:	${}_6^2T = 0.15$	${}_6^3T = 0.05$	${}_6^5T = 0.4$	${}_6^6T = 0.4$	$\sum = 1$

The above transition matrix becomes:

$$
\begin{bmatrix} {}^1F \\ {}^2F \\ {}^3F \\ {}^4F \\ {}^5F \\ {}^6F \end{bmatrix} = \boxed{\begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}} \times \begin{bmatrix} {}_1S \\ {}_2S \\ {}_3S \\ {}_4S \\ {}_5S \\ {}_6S \end{bmatrix}
$$

2.b. Derive the 3-step transition matrix (hint: matrix multtiplication) (5p)

The n-step transition matrix from time i to time j (where $n = j - i$) is given by ${}_i^j\mathbf{T}$. This means a variable with initial state given by ${}_i\mathbf{S}$, has final state ${}^j\mathbf{F}$

$$
\begin{aligned}
{}^j\mathbf{F} &= {}_i^j\mathbf{T} \times {}_i\mathbf{S} \\
{}^j\mathbf{F} &= \left({}_i^j\mathbf{T}\right) \times {}_i\mathbf{S} \\
{}^j\mathbf{F} &= \left({}_i^{i+1}\mathbf{T} \times {}_{i+1}^{i+2}\mathbf{T} \times \cdots \times {}_{j-2}^{j-1}\mathbf{T} \times {}_{j-1}^{j}\mathbf{T}\right) \times {}_i\mathbf{S}
\end{aligned}
$$

Since ${}_i^{i+1}\mathbf{T} = {}_{i+1}^{i+2}\mathbf{T} = \cdots = {}_{j-2}^{j-1}\mathbf{T} = {}_{j-1}^{j}\mathbf{T} = \mathbf{T}$, this becomes

$$
\begin{aligned}
{}^j\mathbf{F} &= {}_i^j\mathbf{T} \times {}_i\mathbf{S} \\
{}^j\mathbf{F} &= \left({}_i^{i+1}\mathbf{T} \times {}_{i+1}^{i+2}\mathbf{T} \times \cdots \times {}_{j-2}^{j-1}\mathbf{T} \times {}_{j-1}^{j}\mathbf{T}\right) \times {}_i\mathbf{S} \\
{}^j\mathbf{F} &= (\mathbf{T} \times \mathbf{T} \times \cdots \times \mathbf{T} \times \mathbf{T}) \times {}_i\mathbf{S} \\
{}^j\mathbf{F} &= \left(\mathbf{T}^{j-i}\right) \times {}_i\mathbf{S}
\end{aligned}
$$

So the the *3*-step transition matrix (say from time $i = 0$ to time $j = 3$) is given by

$$
\begin{aligned}
{}_0^3\mathbf{T} &= \mathbf{T}^3 \\
&= \begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}^3 \\
&= \boxed{\begin{bmatrix} 0.3210 & 0.2220 & 0.1650 & 0.1540 & 0.2140 & 0.2955 \\ 0.1120 & 0.2730 & 0.2425 & 0.1070 & 0.1650 & 0.2800 \\ 0.0840 & 0.0280 & 0.2535 & 0.0030 & 0.2200 & 0.0380 \\ 0.3850 & 0.3550 & 0.3150 & 0.5120 & 0.2675 & 0.0900 \\ 0.0980 & 0.0800 & 0.0120 & 0.1520 & 0.0495 & 0.1350 \\ 0.0000 & 0.0420 & 0.0120 & 0.0720 & 0.0840 & 0.1615 \end{bmatrix}}
\end{aligned}
$$

2.c. Derive the 10-step transition matrix (hint: matrix multiplication) (5p)

Similaryly, the *10*-step transition matrix from time $i = 0$ to time $j = 10$ is given by

$$
\begin{aligned}
{}_{0}^{10}\mathbf{T} = \mathbf{T}^{10} \\
&= \begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}^{10} \\
&= \boxed{\begin{bmatrix} 0.2158 & 0.2141 & 0.2141 & 0.2156 & 0.2170 & 0.2191 \\ 0.1576 & 0.1597 & 0.1563 & 0.1594 & 0.1587 & 0.1643 \\ 0.0648 & 0.0661 & 0.0676 & 0.0661 & 0.0683 & 0.0691 \\ 0.4004 & 0.3985 & 0.4025 & 0.3952 & 0.3984 & 0.3928 \\ 0.1083 & 0.1076 & 0.1074 & 0.1083 & 0.1062 & 0.1041 \\ 0.0532 & 0.0540 & 0.0521 & 0.0555 & 0.0513 & 0.0505 \end{bmatrix}}
\end{aligned}
$$

2.d. If the intitial state distribution is $P_0(X) = (0.5, 0.25, 0.15, 0.05, 0.05, 0)$, what is state distribution after 10 steps $P_{10}(X)$ (hint: vector-matrix multiplication) (10p)

Setting ${}_0\mathbf{S} = P_0(X) = \begin{bmatrix} 0.50 & 0.25 & 0.15 & 0.05 & 0.05 & 0.00 \end{bmatrix}^T$, the state distribution at *time* $= 10$ (${}^{10}\mathbf{F} = P_{10}(X)$), is given by

$$\begin{aligned}
{}^{10}\mathbf{F} &= {}^{10}_{0}\mathbf{T} \times {}_0\mathbf{S} \\
&= (\mathbf{T})^{10} \times {}_0\mathbf{S} \\
&= \begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix}^{10} \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.15 \\ 0.05 \\ 0.05 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.2158 & 0.2141 & 0.2141 & 0.2156 & 0.2170 & 0.2191 \\ 0.1576 & 0.1597 & 0.1563 & 0.1594 & 0.1587 & 0.1643 \\ 0.0648 & 0.0661 & 0.0676 & 0.0661 & 0.0683 & 0.0691 \\ 0.4004 & 0.3985 & 0.4025 & 0.3952 & 0.3984 & 0.3928 \\ 0.1083 & 0.1076 & 0.1074 & 0.1083 & 0.1062 & 0.1041 \\ 0.0532 & 0.0540 & 0.0521 & 0.0555 & 0.0513 & 0.0505 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.15 \\ 0.05 \\ 0.05 \\ 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0.2152 \\ 0.1580 \\ 0.0658 \\ 0.3999 \\ 0.1079 \\ 0.0533 \end{bmatrix}}
\end{aligned}$$

2.e. Derive the steady state probabilities. (hint: eigenvalue problem) (10p)

For the markov chain ${}^{i+1}\mathbf{F} = {}^{i+1}_{i}\mathbf{T} \times {}_i\mathbf{S}$ we wish to find the vectors of ${}_i\mathbf{S}$ and ${}^{i+1}\mathbf{F}$ such that ${}_i\mathbf{S} = {}^{i+1}\mathbf{F}$. As an *eigen-value* problem this is equivalent to finding the *eigen-vector* ($\mathbf{v}$) corresponding to the *eigen-value* of $\lambda = 1$, such that $\mathbf{T} \times \mathbf{p} = \lambda \cdot \mathbf{p}$.

MatLab gives the following vector for the *eigen-value* of $\lambda = 1$

$$\begin{bmatrix} -0.4324 \\ -0.3189 \\ -0.1329 \\ -0.7971 \\ -0.2159 \\ -0.1080 \end{bmatrix} = -2.0052 \cdot \boxed{\begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix}}$$

of which the vector on the right-hand side (which sums to 1 and is therefore the steady-state probability) was obtained by weighting the vector on the left hand side.

We see that

$$\begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix} = \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix}$$

or

$$\begin{bmatrix} 0.3 & 0.1 & 0.9 & 0 & 0.7 & 0 \\ 0.7 & 0 & 0 & 0 & 0 & 0.15 \\ 0 & 0.4 & 0 & 0 & 0 & 0.05 \\ 0 & 0.5 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.1 & 0.2 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.3 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix} - \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix} = \mathbf{0}$$

2.f. What is the expected value of random variable X (hint: if you did 2.e, this should be trivial) (5p)

Using the vector $\mathbf{X} = \begin{bmatrix} 5 & 15 & 3 & 5 & 20 & 0 \end{bmatrix}$, multiplying by the steady-state probability vector gives the expected value of

$$\begin{aligned} E[\mathbf{X}] &= \mathbf{X} \times P_{SS}(\mathbf{X}) \\ &= \begin{bmatrix} 5 & 15 & 3 & 5 & 20 & 0 \end{bmatrix} \times \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix} \\ &= \boxed{7.8035} \end{aligned}$$

2.g. If we want to make sure that every state probability is within 0.01 of its steady state value, how many steps do we need to take from the initial state? (hint: successive matrix multiplications and comparisons) (5p)

Using the initial probabilities (${}_0\mathbf{S}$) from **2.d.**, and applying multiple matrix multiplications of $\mathbf{T}$ we get the following vector for ${}^6\mathbf{F}$

$$\begin{aligned} {}^6\mathbf{F} &= {}_0^6\mathbf{T} \times {}_0\mathbf{S} \\ &= \begin{bmatrix} 0.2179 \\ 0.1587 \\ 0.0712 \\ 0.4041 \\ 0.1031 \\ 0.0450 \end{bmatrix} \end{aligned}$$

The difference with the steady state vector is given by

$$P_{SS}(\mathbf{X}) - {}^6\mathbf{F} = \begin{bmatrix} 0.2179 \\ 0.1587 \\ 0.0712 \\ 0.4041 \\ 0.1031 \\ 0.0450 \end{bmatrix} - \begin{bmatrix} 0.2156 \\ 0.1590 \\ 0.0663 \\ 0.3975 \\ 0.1077 \\ 0.0538 \end{bmatrix} = \begin{bmatrix} 0.0023 \\ -0.0003 \\ 0.0049 \\ 0.0066 \\ -0.0046 \\ -0.0088 \end{bmatrix}$$

The maximum difference is given by

$$\begin{aligned} MAX\left(ABS\left(\begin{bmatrix} 0.0023 \\ -0.0003 \\ 0.0049 \\ 0.0066 \\ -0.0046 \\ -0.0088 \end{bmatrix} \right) \right) &= MAX\left(\begin{bmatrix} 0.0023 \\ 0.0003 \\ 0.0049 \\ 0.0066 \\ 0.0046 \\ 0.0088 \end{bmatrix} \right) \\ &= 0.0088 \end{aligned}$$

since $0.0088 < 0.01$ this is the first vector with a difference of less than 0.01 occurring at $\boxed{time = 6}$.