Geometric Formulas for Computer Vision and Computer Graphics

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Sources include:

http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/3_CSE4392-5369_IntroVision_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/6_CSE4392-5369_RigidBodyTransform_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/7_CSE4392-5369_CameraCalibrationResectioning_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/10_CSE4392-5369_EpipolarGeometry_Mariottini.pdf

Homogeneous ($\tilde{\mathbf{x}}$) to Non-Homogeneous ($\mathbf{x}$) - 2-d case:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}} \tag{1}$$

Homogeneous ($\tilde{\mathbf{x}}$) to Non-Homogeneous ($\mathbf{x}$) - 3-d case:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}} \tag{2}$$

Camera Resolution:

$$\begin{gathered} \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \end{bmatrix} \tilde{\mathbf{x}} \\ \Downarrow \\ \tilde{\mathbf{u}} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} \end{gathered} \tag{3}$$

Camera Focal Length:

Using *Similarity of Triangles* we get $\frac{x}{f} = \frac{^{C}X}{^{C}Z}$:

$$\begin{gathered} s\tilde{\mathbf{x}} = s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{C}X \\ ^{C}Y \\ ^{C}Z \end{bmatrix} \\ \Downarrow \\ \tilde{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{C}X \\ ^{C}Y \\ ^{C}Z \end{bmatrix} \lambda \end{gathered} \tag{4}$$

Where $s = {^{C}Z} = \frac{1}{\lambda}$.

Camera Calibration (K):

$$
\begin{aligned}
\tilde{\mathbf{u}} &= \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda \right) \\
&= \left(\begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda = \begin{bmatrix} fk_u & 0 & u_0 \\ 0 & fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda = \mathbf{K} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda
\end{aligned} \tag{5}
$$

Ideal Projection Matrix (Π_0):

$$
\begin{aligned}
\tilde{\mathbf{u}} &= \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda \right) \\
&= \left(\begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda = \begin{bmatrix} fk_u & 0 & u_0 \\ 0 & fk_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda = \mathbf{K} \begin{bmatrix} {}^{C}X \\ {}^{C}Y \\ {}^{C}Z \end{bmatrix} \lambda
\end{aligned} \tag{6}
$$

Geometric Tranformations:

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & \mathbf{1} \end{array} \right]$$

Cross Product (Skew-Symmetric Form):

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
[\mathbf{a}]_\times \mathbf{b} &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\end{aligned}
$$

Change of Frame (of Reference):
Rearranging

$$
\begin{aligned}
{}^{A}\mathbf{x} &= {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{x} + {}^{A}_{B}\mathbf{t} \\
&\Downarrow \\
{}^{A}\mathbf{x} - {}^{A}_{B}\mathbf{t} &= {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{x} \\
&\Downarrow \\
{}^{B}\mathbf{x} &= {}^{A}_{B}\mathbf{R}^{-1} \left({}^{A}\mathbf{x} - {}^{A}_{B}\mathbf{t} \right) \\
&= {}^{A}_{B}\mathbf{R}^{T} \left({}^{A}\mathbf{x} - {}^{A}_{B}\mathbf{t} \right) \\
&= {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}\mathbf{x} - {}^{A}_{B}\mathbf{R}^{T}\,{}^{A}_{B}\mathbf{t}
\end{aligned}
$$

Implies

$$ {}^{B}_{A}\mathbf{R} = {}^{A}_{B}\mathbf{R}^{T} \qquad \text{and} \qquad {}^{B}_{A}\mathbf{t} = -{}^{A}_{B}\mathbf{R}^{T}\,{}^{A}_{B}\mathbf{t} $$

Where

$$ {}^{B}\mathbf{x} = {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{x} + {}^{B}_{A}\mathbf{t} $$

Essential Matrix (Theoretical Calculation):

Relationship between ${}^{C}\mathbf{x}$ and ${}^{C'}\mathbf{x}$:

$$
{}^{C'}\mathbf{x} = {}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + {}^{C'}_{C}\mathbf{t}
$$

Taking the *cross-product* with $[{}^{C'}_{C}\mathbf{t}]_{\times}$:

$$
\begin{aligned}
[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} &= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{t} \\
&= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + \mathbf{0} \\
&= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x}
\end{aligned}
$$

Multiplying (on the left) by ${}^{C'}\mathbf{x}^{T}$:

$$
{}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} = {}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x}
$$

Since ${}^{C'}\mathbf{x}^{T}$ is *orthogonal* to $[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x}$ the above is equal to

$$
\begin{aligned}
{}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} &= {}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} \\
{}^{C'}\mathbf{x}^{T}\left([{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x}\right) &= {}^{C'}\mathbf{x}^{T}\left([{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\right){}^{C}\mathbf{x} \\
\mathbf{0} &= {}^{C'}\mathbf{x}^{T}\,{}^{C'}_{C}\mathbf{E}\,{}^{C}\mathbf{x}
\end{aligned}
$$

Essential Matrix (Practical Calculation):