Geometric Formulas for Computer Vision and Computer Graphics

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Sources include:
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/3_CSE4392-5369_IntroVision_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/6_CSE4392-5369_RigidBodyTransform_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/7_CSE4392-5369_CameraCalibrationResectioning_Mariottini.pdf
http://ranger.uta.edu/~gianluca/teaching/CSE4392-5369_F14/10_CSE4392-5369_EpipolarGeometry_Mariottini.pdf

Homogeneous ($\tilde{\mathbf{x}}$) to Non-Homogeneous ($\mathbf{x}$) - 2-d case:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}} \tag{1}$$

Homogeneous ($\tilde{\mathbf{x}}$) to Non-Homogeneous ($\mathbf{x}$) - 3-d case:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}} \tag{2}$$

Camera Resolution:

k_u and k_v give the *pixel resolution* of the image plane. (i.e *pixels per milli-meter*). u_0 and v_0 give the *optical center* of the image plane (in *pixels*).

$$\begin{gathered} \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \end{bmatrix} \tilde{\mathbf{x}} \\ \Downarrow \\ \tilde{\mathbf{u}} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} \end{gathered} \tag{3}$$

Camera Focal Length:

f givs the *focal length* along the optical axis (in *milli-meters*).

Using *Similarity of Triangles* we get $\dfrac{x}{f} = \dfrac{{}^C X}{{}^C Z}$:

$$\begin{gathered} s\tilde{\mathbf{x}} = s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \\ \Downarrow \\ \tilde{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda \end{gathered} \tag{4}$$

Where $s = {}^C Z = \dfrac{1}{\lambda}$.

Camera Calibration (K):

$$\begin{aligned}\tilde{\mathbf{u}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda \right) \\ = \left(\begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda = \begin{bmatrix} f k_u & 0 & u_0 \\ 0 & f k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda = \mathbf{K} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda \end{aligned} \tag{5}$$

Ideal Projection Matrix (Π_0):

$$\begin{aligned}\tilde{\mathbf{u}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) = \begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda \right) \\ = \left(\begin{bmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda = \begin{bmatrix} f k_u & 0 & u_0 \\ 0 & f k_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda = \mathbf{K} \begin{bmatrix} {}^C X \\ {}^C Y \\ {}^C Z \end{bmatrix} \lambda \end{aligned} \tag{6}$$

Geometric Tranformations:

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & \mathbf{1} \end{array} \right]$$

Cross Product (Skew-Symmetric Form):

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ [\mathbf{a}]_\times \mathbf{b} &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{aligned}$$

Change of Frame (of Reference):
Rearranging

$$\begin{aligned} {}^A\mathbf{x} &= {}^A_B\mathbf{R}\,{}^B\mathbf{x} + {}^A_B\mathbf{t} \\ &\Downarrow \\ {}^A\mathbf{x} - {}^A_B\mathbf{t} &= {}^A_B\mathbf{R}\,{}^B\mathbf{x} \\ &\Downarrow \\ {}^B\mathbf{x} &= {}^A_B\mathbf{R}^{-1} \left({}^A\mathbf{x} - {}^A_B\mathbf{t} \right) \\ &= {}^A_B\mathbf{R}^T \left({}^A\mathbf{x} - {}^A_B\mathbf{t} \right) \\ &= {}^A_B\mathbf{R}^T\,{}^A\mathbf{x} - {}^A_B\mathbf{R}^T\,{}^A_B\mathbf{t} \end{aligned}$$

Implies

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^T \qquad \text{and} \qquad {}^B_A\mathbf{t} = -{}^A_B\mathbf{R}^T\,{}^A_B\mathbf{t}$$

Where

$${}^B\mathbf{x} = {}^B_A\mathbf{R}\,{}^A\mathbf{x} + {}^B_A\mathbf{t}$$

Essential Matrix (Theoretical Calculation):

Relationship between ${}^{C}\mathbf{x}$ and ${}^{C'}\mathbf{x}$:

$${}^{C'}\mathbf{x} = {}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + {}^{C'}_{C}\mathbf{t}$$

Taking the *cross-product* with $[{}^{C'}_{C}\mathbf{t}]_{\times}$:

$$\begin{aligned}
[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} &= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{t} \\
&= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} + \mathbf{0} \\
&= [{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x}
\end{aligned}$$

Multiplying (on the left) by ${}^{C'}\mathbf{x}^{T}$:

$${}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} = {}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x}$$

Since ${}^{C'}\mathbf{x}^{T}$ is *orthogonal* to $[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x}$ the above is equal to

$$\begin{aligned}
{}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x} &= {}^{C'}\mathbf{x}^{T}[{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\,{}^{C}\mathbf{x} \\
{}^{C'}\mathbf{x}^{T}\left([{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}\mathbf{x}\right) &= {}^{C'}\mathbf{x}^{T}\left([{}^{C'}_{C}\mathbf{t}]_{\times}\,{}^{C'}_{C}\mathbf{R}\right){}^{C}\mathbf{x} \\
\mathbf{0} &= {}^{C'}\mathbf{x}^{T}\,{}^{C'}_{C}\mathbf{E}\,{}^{C}\mathbf{x}
\end{aligned}$$

Essential Matrix (Practical Calculation):