CSE 3380 – Homework #5

Assigned:	Thursday, July 21, 2016
Due:	Thursday, July 28, 2016 at 5:20pm (the end of class)

Note the following about the homework:

1. You must show your work to receive credit.
2. If your submission has more than one page, staple the pages.

Assignment:
Process

1. Given vectors
$$\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$
what is the orthogonal projection of $\vec{y}$ onto $\vec{u}$?
2. Find the closest point to $\vec{y}$ in the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$.
$$\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
This is asking for the orthogonal projection of $\vec{y}$ onto the subspace spanned by $\vec{v}_i$.
3. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.
$$\left\{ \begin{bmatrix} 4 \\ -10 \\ -4 \\ -8 \end{bmatrix}, \begin{bmatrix} -6 \\ -14 \\ 4 \\ -12 \end{bmatrix} \right\}$$
4. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.
$$\left\{ \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 36 \\ 0 \\ 0 \end{bmatrix} \right\}$$
5. Orthogonalize the following set of vectors using the Gram-Schmidt procedure.
$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \\ -3 \end{bmatrix} \right\}$$

6. (hand solution) If

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$$

find the least squares solution $\vec{x}^*$.

7. (hand solution) If

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$$

find the least squares solution $\vec{x}^*$.

8. (hand solution) Use linear regression to determine the coefficients of $y_i = \beta_0 + \beta_1 x_i$ for the following set of x and y values.

x	1	2	4	7
y	-1	0	1	1

Theory

9. If we have an $m \times n$ real matrix A, show that $A^T A$ is symmetric. Do this for the general case, which means for a generic $m \times n$ matrix. Hint: this is easier using dot products.

10. In order to find the least squares solution of a system of equations using the normal equations, we must be able to produce $(A^T A)^{-1}$. This requires that A be of full column rank. If A is an $m \times n$ matrix and $m < n$ (i.e., less rows than columns), then it cannot be of full column rank. Show why $A^T A$ is noninvertible in this case. You can consider the case where A is a general 2×3 matrix instead of $m \times n$; this doesn't mean to use specific numbers.

    Hint: if A is wider than it is tall, then we know that at least one of the columns must be a linear combination of the other columns.

Applications

11. **(CS application: computer vision)** When we look at distant objects, the distance between them appears smaller than it really is. For example, when the distant object is the top of a tall building, the sides of the building appear to be moving toward each other. At times, we would like to transform a picture in which the relationship between points is skewed due to distance into a picture showing the correct relationships. For this, we can find a change-of-basis matrix.

    Our change-of-basis matrix will have the form

$$H = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

    Given four points $[(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)]$ in the skewed picture and four points $[(x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3), (x'_4, y'_4)]$ in the corrected picture, we can solve for the eight values

$a, b, \ldots, h$ using

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & -x_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & -y_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & -1 & 0 & 0 & 0 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -x_2' & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 & -y_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 0 & -1 & 0 & 0 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3' & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 & 0 & 0 & -y_3' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_3 & y_3 & 0 & 0 & -1 & 0 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_4' \\ 0 & 0 & 0 & x_4 & y_4 & 1 & 0 & 0 & 0 & 0 & 0 & -y_4' \\ 0 & 0 & 0 & 0 & 0 & 0 & x_4 & y_4 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

After constructing H from the values for $a, b, \ldots, h$ in the solution (note that the w_i values are not used), given a point (x, y) in the skewed picture we can produce

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}$$

Our corrected point (x', y') is found by dividing by w:

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

(a) On the course website is a program, `homography_main.m`, that contains the coordinate points for learning the mapping:

$$\text{points\_skewed} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}, \qquad \text{points\_corrected} = \begin{bmatrix} x_1' & x_2' & x_3' & x_4' \\ y_1' & y_2' & y_3' & y_4' \end{bmatrix}$$

The skewed (x, y) values are the corners of the red trapezoid in Figure 1. It also contains a matrix of the points to be mapped from the skewed coordinate system to the corrected coordinate system; these points represent the blue shape in Figure 1. Each column represents an (x, y) coordinate in homogeneous coordinates, so it's of the form

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(b) You will write the MATLAB function `homography_student.m`, which has the signature

```
[H, projected] = homography_student(points_skewed, points_corrected,
                                     test_points);
```

This function is given the skewed corner points, the corrected corner points, and the points to transform, and returns the transformation matrix H as well as a $3 \times n$ matrix of the n transformed points in homogeneous coordinates.

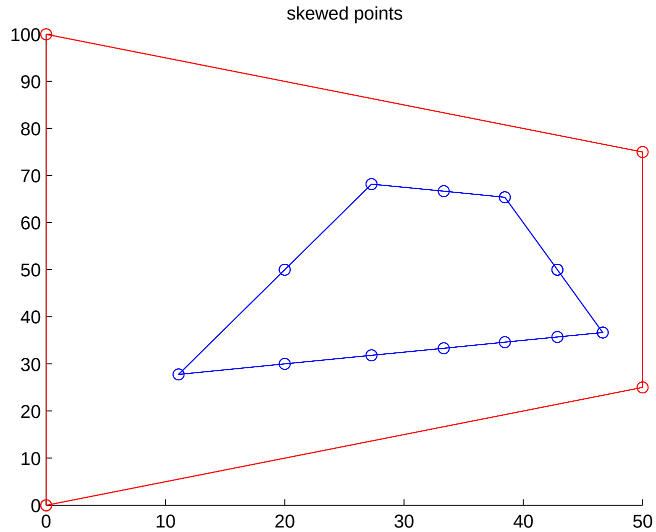

Figure 1: Data points for computer vision problem.

(c) Your function should also plot the original points to be transformed (the blue trapezoid) as well as the corrected shape. Note that to plot a set of points with circles for the points and lines connecting the points, in MATLAB you can use `plot(X, Y, '-o')` where `X` is a vector of the x-coordinates of all of the points and `Y` is a vector of the y-coordinates of all of the points. To completely enclose a shape, the first column of the matrix is repeated at the end of the matrix.

12. **(CS application: data modeling)** (MATLAB solution) One application of least squares is to find a function that best fits a set of data. For example, given a set of t values and the corresponding $f(t)$ values, we might want to find the coefficients of $f(t) = a_0 + a_1 t + \cdots + a_n t^n$. However, we might not know in advance the degree of the polynomial that best fits the data. We can try polynomials of different degrees and compare the residuals to see which polynomial best fits the data.

(a) On the course website is a file, `least_squares_main.m`, that contains two sets of data. Each set contains a set of t values and the corresponding $f(t)$ values. This function will give each set of values to a function, `least_squares_student.m`, with the signature

```
[x_lin, norm_l, x_q, norm_q] = least_squares_student(t, ft);
```

(b) Write the MATLAB function `least_squares_student.m` such that it

i. uses least squares to fit it to a linear model, $f(t) = a_0 + a_1 t$. These values will be returned as the vector

$$\text{x\_lin} = \left[\begin{array}{cc} a_0 & a_1 \end{array}\right]$$

ii. calculates the norm of the residual, $||\vec{b} - A\vec{x}||_2$, for the linear model; this value is returned as `norm_l`.

iii. uses least squares to fit it to a quadratic model, $f(t) = a_0 + a_1 t + a_2 t^2$. These values will be returned as the vector

$$\text{x\_q} = \left[\begin{array}{ccc} a_0 & a_1 & a_2 \end{array}\right]$$

iv. calculates the norm of the residual, $||\vec{b} - A\vec{x}||_2$, for the quadratic model; this value is returned as `norm_q`.

v. plots the original points, that is, t versus $f(t)$. To plot individual points, you can use `plot(t, f, '.')`.

(c) As an exercise for yourself, use the norms and the plots to determine which model you think best fits the data.

(d) When fitting the models, set up the appropriate A matrix and b vector and solve using $A\backslash b$. Do not use MATLAB's `polyfit()` function.

General requirements about the MATLAB problems:

a As a comment in each file, include your name.

b The MATLAB program should do the work. Don't perform the calculations and then hard-code the values in the code or look at the data and hard-code to this data unless instructed to do so.

c Your function should use the data passed to it and should work if I were to change the data.

d The program should not prompt the user for values or read from files unless instructed to do so.

e Don't use the `Symbolic` Toolbox or other special libraries.

To submit the MATLAB portion, do the following:

a Create a directory with a name of the form `lastname_hwXX`, where `lastname` is your actual last name and `XX` is a two-digit number representing the homework number.

b Place your .m files in this directory.

c Zip the directory. You must use the zip format and the extension must be `.zip`.

d Upload the zip'd file to Blackboard.

Review Questions

These are not for credit, but instead are intended to test your understanding of the concepts. You should be able to answer these without simply regurgitating equations.

1. What does it mean for two vectors to be orthogonal?
2. What is the difference between orthogonal and orthonormal sets?
3. What is the orthogonal projection of $\vec{x}$ onto $\vec{v}$?
4. What makes a matrix an 'orthogonal matrix'?
5. What problem is least squares intended to solve? That is, why did we need to learn it?
6. What is the relationship between least squares and Col(A)?
7. What is the relationship between $\vec{b}$ and Col(A) when $A\vec{x} \neq \vec{b}$.