Eigenvalues, Eigenvectors, and Eigenspaces

DEFINITION: Let A be a square matrix of size n.

If a NONZERO vector $\vec{\mathbf{x}} \in \mathbb{R}^n$ and a scalar λ satisfy

$$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}, \qquad \text{or, equivalently, } (A - \lambda I_n)\vec{\mathbf{x}} = 0,$$

scalar λ is called an *eigenvalue* of A,
vector $\vec{\mathbf{x}} \neq 0$ is called an *eigenvector* of A associated with eigenvalue λ,
and the null space of $A - \lambda I_n$ is called the *eigenspace* of A associated with eigenvalue λ.

HOW TO COMPUTE?

The eigenvalues of A are given by the roots of the polynomial

$$\det(A - \lambda I_n) = 0.$$

The corresponding eigenvectors are the nonzero solutions of the linear system

$$(A - \lambda I_n)\vec{\mathbf{x}} = 0.$$

Collecting all solutions of this system, we get the corresponding eigenspace.

EXERCISES:

For each given matrix, find the eigenvalues, and for each eigenvalue give a basis of the corresponding eigenspace.

$$\text{(a) } \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad \text{(b) } \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad \text{(c) } \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix},$$

$$\text{(d) } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix}, \quad \text{(e) } \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, \quad \text{(f) } \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \text{(g) } \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix},$$

$$\text{(h) } \begin{bmatrix} -1 & 1 & 1 & -2 \\ -1 & 1 & 3 & 2 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

Turn over for the answers

Answers:

(a) Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$

The eigenspace associated to $\lambda_1 = 1$, which is $\mathrm{Ker}(A - I)$: $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_2 = 2$, which is $\mathrm{Ker}(A - 2I)$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives a basis.

(b) Eigenvalues: $\lambda_1 = \lambda_2 = 2$

$\mathrm{Ker}(A - 2I)$, the eigenspace associated to $\lambda_1 = \lambda_2 = 2$: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives a basis.

(c) Eigenvalues: $\lambda_1 = 2, \lambda_2 = 4$

$\mathrm{Ker}(A - 2I)$, the eigenspace associated to $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ gives a basis.

$\mathrm{Ker}(A - 4I)$, the eigenspace associated to $\lambda_2 = 4$: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives a basis.

(d) Eigenvalues: $\lambda_1 = 2, \lambda_2 = \lambda_3 = 3$

$\mathrm{Ker}(A - 2I)$, the eigenspace associated to $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gives a basis.

$\mathrm{Ker}(A - 3I)$, the eigenspace associated to $\lambda_2 = \lambda_3 = 3$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives a basis.

(e) Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

The eigenspace associated to $\lambda_1 = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_2 = 2$: $\mathbf{v}_2 = \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_3 = 3$: $\mathbf{v}_3 = \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$ gives a basis.

(f) Eigenvalues: $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$

The eigenspace associated to $\lambda_1 = 1$: $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_2 = \lambda_3 = 2$: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis.

(g) Eigenvalues: $\lambda_1 = 1, \lambda_2 = \lambda_3 = -2$

The eigenspace associated to $\lambda_1 = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_2 = \lambda_3 = -2$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ gives a basis.

(h) Eigenvalues: $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = -2, \lambda_4 = 2$

The eigenspace associated to $\lambda_1 = -1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_2 = 1$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_3 = -2$: $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ gives a basis.

The eigenspace associated to $\lambda_4 = 2$: $\mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ gives a basis.