Lie theory cheat sheet

| Lie group \mathcal{M}, \circ | | size | dim | $\Big \qquad \mathcal{X} \in \mathcal{M}$ | Constraint | $m{	au}^\wedge \in \mathfrak{m}$ | $m{	au} \in \mathbb{R}^m$ |
|--------------------------------|--------------------------|------|-----|--|---|--|--|
| Vector n-D | $ \mathbb{R}^n, +$ | n | n | $oldsymbol{\mathbf{v}} \in \mathbb{R}^n$ | $ \mathbf{v} - \mathbf{v} = 0$ | $\mathbf{v} \in \mathbb{R}^n$ | $\mathbf{v} \in \mathbb{R}^n$ |
| Complex number | $ S^1, \cdot $ | 2 | 1 | $ig \mathbf{z} \in \mathbb{C}$ | $\mathbf{z}^*\mathbf{z} = 1$ | $i\theta \in i\mathbb{R}$ | $\theta \in \mathbb{R}$ |
| 2D Rotation | $\mid SO(2), \cdot \mid$ | 4 | 1 | R | $\mathbf{R}^{	op}\mathbf{R} = \mathbf{I}$ | $\begin{bmatrix} \theta \end{bmatrix}_{\times} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \in \mathfrak{so}(2)$ | $\theta \in \mathbb{R}$ |
| 2D Rigid Motion | \mid SE(2), · | 9 | 3 | $ \mid \mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} $ | $\mid \mathbf{M}^{	op} \mathbf{M} = \mathbf{I}$ | $\begin{bmatrix} [\theta]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$ | $\left \begin{array}{c} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{array} \right \in \mathbb{R}^3 \ \left \begin{array}{c} \boldsymbol{\rho} \\ \end{array} \right $ |
| Quaternion | $ S^3, \cdot $ | 4 | 3 | $\big \qquad \mathbf{q} \in \mathbb{H}$ | $\mathbf{q}^*\mathbf{q} = 1$ | $\theta/2 \in \mathbb{H}_p$ | $oldsymbol{	heta} \in \mathbb{R}^3$ |
| 3D Rotation | $\mid SO(3), \cdot \mid$ | 9 | 3 | R | $\mathbf{R}^{	op}\mathbf{R} = \mathbf{I}$ | $ [\boldsymbol{\theta}]_{\times} = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \in \mathfrak{so}(3) $ | $oldsymbol{	heta} oldsymbol{	heta} \in \mathbb{R}^3$ |
| 3D Rigid Motion | \mid SE(3), · | 16 | 6 | $ \mid \mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} $ | $\mid \mathbf{M}^{	op} \mathbf{M} = \mathbf{I}$ | $\begin{bmatrix} [\boldsymbol{\theta}]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$ | $igg egin{bmatrix} oldsymbol{ ho} \ oldsymbol{	heta} \end{bmatrix} \in \mathbb{R}^6 \ igg $ |

| Operation | Inverse | Compose | Exp | Log | \mid Right- \oplus | $\text{Right-}\ominus$ |
|-----------------|---|--|---|--|---|---|
| Right Jacobians | $oxed{\mathbf{J}_{oldsymbol{\chi}}^{oldsymbol{\chi}^{-1}} = -\mathbf{A}\mathbf{d}_{oldsymbol{\chi}}}$ | $ \begin{vmatrix} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi} \circ \boldsymbol{\gamma}} = \mathbf{A} \mathbf{d}_{\boldsymbol{\gamma}^{-1}} \\ \mathbf{J}_{\boldsymbol{\gamma}}^{\boldsymbol{\chi} \circ \boldsymbol{\gamma}} = \mathbf{I} \end{vmatrix} $ | $oxed{\mathbf{J}^{\mathrm{Exp}(oldsymbol{	au})}_{oldsymbol{	au}} = \mathbf{J}_r(oldsymbol{	au})}$ | $\mathbf{J}_{oldsymbol{\chi}}^{\mathrm{Log}(oldsymbol{\chi})} = \mathbf{J}_{r}^{-1}(oldsymbol{	au})$ | $ \begin{vmatrix} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi} \oplus \boldsymbol{\tau}} = (\mathbf{A} \mathbf{d}_{\mathrm{Exp}(\boldsymbol{\tau})})^{-1} \\ \mathbf{J}_{\boldsymbol{\tau}}^{\boldsymbol{\chi} \oplus \boldsymbol{\tau}} = \mathbf{J}_r(\boldsymbol{\tau}) \end{vmatrix} $ | $egin{aligned} \mathbf{J}_{oldsymbol{\chi}}^{oldsymbol{\gamma}\ominusoldsymbol{\chi}} &= -\mathbf{J}_{l}^{-1}(oldsymbol{	au}) \ \mathbf{J}_{oldsymbol{	au}}^{oldsymbol{\gamma}\ominusoldsymbol{\chi}} &= & \mathbf{J}_{r}^{-1}(oldsymbol{	au}) \end{aligned}$ |

Note: In accordance to manif implementation, all Jacobians in this document are **right Jacobians**, whose definition reads: $\frac{\delta f(X)}{\delta X} = \lim_{\varphi \to 0} \frac{f(X \oplus \varphi) \ominus f(X)}{\varphi}$. However, notice that one can relate the left- and right- Jacobians with the Adjoint, $\frac{\varepsilon_{\partial f(\mathcal{X})}}{\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{\kappa_{\partial f(\mathcal{X})}}{\partial \mathcal{X}}$, see [1] Eq. (46).

[1] J. Solà, J. Deray, and D. Atchuthan, "A micro Lie theory for state estimation in robotics," Tech. Rep. IRI-TR-18-01, Institut de Robòtica i Informàtica Industrial, Barcelona, 2018. Available at arxiv.org/abs/1812.01537.

| \mathcal{M}, \circ Op | Identity | Inverse | Compose | Act | Exp | Log |
|-------------------------|--|--|---|---|--|---|
| \mathbb{R}^n , + | $\mathbf{v} = [0]$ | $-\mathbf{v}$ | $ \mathbf{v}_1 + \mathbf{v}_2 $ | $ \mathbf{v} + \mathbf{p} $ | v | v |
| S^1, \cdot | z = 1 + i 0 | z^* | $ \hspace{.05cm} z_1 \hspace{.05cm} z_2 \hspace{.05cm}$ | z v | $z = \cos\theta + i\sin\theta$ | $\theta = \arctan2(\operatorname{Im}(z), \operatorname{Re}(z))$ |
| $SO(2), \cdot$ | $\mathbf{R} = \mathbf{I}$ | $\mathbf{R}^{-1} = \mathbf{R}^{\top}$ | ${f R}_1 \ {f R}_2$ | $\mathbf{R} \cdot \mathbf{v}$ | $\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ | $\theta = \arctan2(r_{21}, r_{11})$ |
| $\mathrm{SE}(2), \cdot$ | $\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ | $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^{\top} & -\mathbf{R}^{\top} \mathbf{t} \\ 0 & 1 \end{bmatrix}$ | $egin{aligned} \mathbf{M}_1 \mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \ 0 & 1 \end{bmatrix}$ | $\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R}\mathbf{p}$ | $\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\theta) & \mathbf{V}(\theta) \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix} {}^{(1)}$ | $egin{aligned} oldsymbol{	au} = egin{bmatrix} oldsymbol{ ho} \ 	heta \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}(heta) \ \mathbf{p} \ 	heta \end{bmatrix} egin{bmatrix} oldsymbol{1} \end{bmatrix} egin{bmatrix} oldsymbol{1} \end{bmatrix}$ |
| S^3 , · | $\mathbf{q} = 1 + i0 + j0 + k0$ | $\mathbf{q}^* = w - ix - jy - jz$ | $\mathbf{q}_1 \; \mathbf{q}_2$ | $\mathbf{q} \mathbf{v} \mathbf{q}^*$ | $\mathbf{q} = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$ | $oldsymbol{	heta} = 2 \mathbf{v} rac{rctan2(\ \mathbf{v}\ , w)}{\ \mathbf{v}\ }$ |
| $SO(3), \cdot$ | $\mathbf{R} = \mathbf{I}$ | $\mathbf{R}^{-1} = \mathbf{R}^{\top}$ | $\mathbf{R}_1 \; \mathbf{R}_2$ | $\mathbf{R} \cdot \mathbf{v}$ | $\mathbf{R} = \mathbf{I} + \sin \theta \left[\mathbf{u} \right]_{\times} + (1 - \cos \theta) \left[\mathbf{u} \right]_{\times}^{2}$ | $oldsymbol{	heta} = rac{	heta(\mathbf{R} - \mathbf{R}^T)^\wedge}{2\sin	heta}$ |
| $SE(3), \cdot$ | $\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$ | $ M^{-1} = \begin{bmatrix} \mathbf{R}^{\top} & -\mathbf{R}^{\top} \mathbf{t} \\ 0 & 1 \end{bmatrix} $ | $egin{aligned} \mathbf{M}_1 \mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \ 0 & 1 \end{bmatrix}$ | $\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R}\mathbf{p}$ | $\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta}) \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix} {}^{(2)}$ | $egin{aligned} oldsymbol{	au} = egin{bmatrix} oldsymbol{ ho} \ oldsymbol{	heta} \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}(oldsymbol{	heta}) \ \mathrm{Log}(\mathbf{R}) \end{bmatrix} \ \end{pmatrix}$ |

| \mathcal{M}, \circ Ad/Jac | Ad | \mathbf{J}_r | \mathbf{J}_l | $\begin{array}{c c} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi}\cdot\mathbf{p}} & \mathbf{J}_{\mathbf{p}}^{\boldsymbol{\chi}\cdot\mathbf{p}} \\ & (\mathrm{Act}) \end{array}$ | |
|-----------------------------|--|---|--|---|--------------------------------|
| \mathbb{R}^n , + | $\mathbf{I} \in \mathbb{R}^{n \times n}$ | I | I | I | I |
| S^1 , · | 1 1 | 1 | 1 | $\left \mathbf{R} \left[1 \right]_{\times} \mathbf{v} \right $ | \mathbf{R} |
| SO(2), · | 1 1 | 1 | 1 | $\mathbf{R}\left[1\right]_{\times}\mathbf{v}$ | \mathbf{R} |
| $\mathrm{SE}(2), \cdot$ | $\begin{bmatrix} \mathbf{R} & -[1]_{\times} \mathbf{t} \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \sin \theta/\theta & (1-\cos \theta)/\theta & (\theta\rho_1-\rho_2+\rho_2\cos \theta-\rho_1\sin \theta)/\theta^2\\ (\cos \theta-1)/\theta & \sin \theta/\theta & (\rho_1+\theta\rho_2-\rho_1\cos \theta-\rho_2\sin \theta)/\theta^2\\ 0 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} \sin \theta/\theta & (\cos \theta - 1)/\theta & (\theta \rho_1 + \rho_2 - \rho_2 \cos \theta - \rho_1 \sin \theta)/\theta^2 \\ (1 - \cos \theta)/\theta & \sin \theta/\theta & (-\rho_1 + \theta \rho_2 + \rho_1 \cos \theta - \rho_2 \sin \theta)/\theta^2 \\ 0 & 0 & 1 \end{bmatrix}$ | $\left \begin{array}{cc} \left[\mathbf{R} & \mathbf{R} \left[1 \right]_{\times} \mathbf{p} \right] \end{array} \right $ | R |
| S^3 , · | $\mathbf{R}(\mathbf{q})$ | $\mathbf{I} - rac{1-\cos	heta}{	heta^2}\left[oldsymbol{	heta} ight]_	imes + rac{	heta-\sin	heta}{	heta^3}\left[oldsymbol{	heta} ight]_	imes$ | $\mathbf{I} + rac{1-\cos	heta}{	heta^2}\left[oldsymbol{	heta} ight]_	imes + rac{	heta-\sin	heta}{	heta^3}\left[oldsymbol{	heta} ight]_	imes$ | $-\mathbf{R}(\mathbf{q}) \left[\mathbf{v}\right]_{\times} $ (3) | $\mathbf{R}(\mathbf{q})^{(3)}$ |
| SO(3), · | R | $\mathbf{I} - rac{1-\cos	heta}{	heta^2}\left[oldsymbol{	heta} ight]_	imes + rac{	heta-\sin	heta}{	heta^3}\left[oldsymbol{	heta} ight]_	imes$ | $\mathbf{I} + rac{1-\cos	heta}{	heta^2} \left[oldsymbol{	heta} ight]_	imes + rac{	heta-\sin	heta}{	heta^3} \left[oldsymbol{	heta} ight]_	imes$ | $\left -\mathbf{R}\left[\mathbf{v} ight]_{	imes} ight $ | \mathbf{R} |
| SE(3), · | $\left egin{array}{cc} \left[egin{array}{cc} \egin{array}{cc} egin{array}{cc} egin{array}{cc} \eq \egin{array}{cc} \egin{array} $ | $\begin{bmatrix} \mathbf{J}_r(\boldsymbol{\theta}) & \mathbf{Q}(-\boldsymbol{\rho}, -\boldsymbol{\theta}) \\ 0 & \mathbf{J}_r(\boldsymbol{\theta}) \end{bmatrix}^{(4)}$ | $\begin{bmatrix} \mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) \\ 0 & \mathbf{J}_l(\boldsymbol{\theta}) \end{bmatrix} {}^{(4)}$ | $\left \begin{array}{cc} \left[\mathbf{R} & -\mathbf{R} \left[\mathbf{p} \right]_{\times} \right] \end{array} \right $ | R |

$$^{(1)}\mathbf{V}(\theta) = \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1-\cos\theta}{\theta}\left[1\right]_{\times}$$

$$^{(2)}\mathbf{V}(\theta) = \mathbf{I} + \frac{1-\cos\theta}{\theta} \left[\mathbf{u}\right]_{\times} + \frac{\theta-\sin\theta}{\theta} \left[\mathbf{u}\right]_{\times}^{2}$$

$${}^{(2)}\mathbf{V}(\theta) = \mathbf{I} + \frac{1-\cos\theta}{\theta} \left[\mathbf{u} \right]_{\times} + \frac{\theta-\sin\theta}{\theta} \left[\mathbf{u} \right]_{\times}^{2}$$

$${}^{(3)}\mathbf{R}(\mathbf{q}) = \begin{bmatrix} w^{2}+x^{2}-y^{2}-z^{2} & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & w^{2}-x^{2}+y^{2}-z^{2} & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wz) & w^{2}-x^{2}-y^{2}+z^{2} \end{bmatrix}$$

$$^{(4)}\mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) = 1/2\left[\boldsymbol{\rho}\right]_{\times} + \frac{\theta - \sin\theta}{\theta^{3}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times} + \left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1 - \frac{\theta^{2}}{2} - \cos\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}^{2}\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1}{2}(\frac{1 - \frac{\theta^{2}}{2} - \cos\theta}{\theta^{4}} - 3\frac{\theta - \sin\theta - \frac{\theta^{3}}{6}}{\theta^{5}})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \cos\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \theta^{2} - \cos\theta}{\theta^{4}} - \frac{\theta^{2} - \sin\theta}{\theta^{4}}) + \frac{\theta^{2} - \sin\theta}{\theta^{4}}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol$$