1D Spherical Geometry

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Notation

Spacial cells are designated with indices in terms of i, with cell centers assigned whole integer indices (i) and cell edges assigned half-integer indices ($i + \frac{1}{2}$). The first spatial cell has index 1, the origin has index $\frac{1}{2}$, the last spatial cell has index I, and the external boundary has index $I + \frac{1}{2}$.

Angular cells are designated with indices in terms of m, with cell centers assigned whole integer indices (m) and cell edges assigned half-integer indices ($m + \frac{1}{2}$). Additionally for the angular finite element schemes, a superscript for quantities evaluated at a local Gauss S_2 point is included to indicate whether the point is to the left ($-$) or right ($+$) of the cell center. The first angular cell has index 1, the starting direction ($\mu = -1$) has index $\frac{1}{2}$, and the last angular cell has index N.

Angular flux values are assigned spatial and angular indices. The first index indicates direction and the second indicates position. For example, $\psi_{\frac{1}{2},2}$ is the starting direction angular flux in spatial cell 2, and $\psi_{2,4}^{(+)}$ is the angular flux for the local Gauss S_2 direction to the right of the center of angular cell 2 as evaluate in spatial cell 4.

1 Step-DD

The steady-state transport equation expressed in 1D spherical coordinates is given by:

$$\frac{\mu}{r^2}\frac{\partial\left[r^2\psi\right]}{\partial r}+\frac{1}{r}\frac{\partial\left[(1-\mu^2)\psi\right]}{\partial\mu}+\sigma_t\psi=Q\,. \tag{1}$$

When discretized in direction, this equation becomes:

$$\frac{\mu_m}{r^2}\frac{\partial\left[r^2\psi_m\right]}{\partial r}+\frac{1}{r}\frac{\alpha_{m+\frac{1}{2}}\psi_{m+\frac{1}{2}}-\alpha_{m-\frac{1}{2}}\psi_{m-\frac{1}{2}}}{w_m}+\sigma_t\psi_m=Q_m\,, \tag{2}$$

where

$$\begin{aligned}\mu_{m+\frac{1}{2}}&=\mu_{m-\frac{1}{2}}+w_m,\quad m=1,...,N,\quad \mu_{\frac{1}{2}}=-1\,,\\ \alpha_{m+\frac{1}{2}}&=\alpha_{m-\frac{1}{2}}-2\mu_m w_m,\quad m=1,...,N,\quad \alpha_{\frac{1}{2}}=0\,.\end{aligned}$$

Discretizing in space results in the following equation:

$$\begin{aligned}\mu_m(A_{i+\frac{1}{2}}\psi_{m,i+\frac{1}{2}}-A_{i-\frac{1}{2}}\psi_{m,i-\frac{1}{2}})+\frac{1}{2}(A_{i+\frac{1}{2}}-A_{i-\frac{1}{2}})\frac{\alpha_{m+\frac{1}{2}}\hat{\psi}_{m+\frac{1}{2},i}-\alpha_{m-\frac{1}{2},i}\hat{\psi}_{m-\frac{1}{2},i}}{w_m}&\\ +\sigma_{t,i}\overline{\psi}_{m,i}V_i=\overline{Q}_{m,i}V_i\,,\quad m=1,...,N,\quad i=1,...,I\,,&\end{aligned} \tag{3}$$

where

$$\begin{aligned}V_i&=\frac{4\pi}{3}(r^3_{i+\frac{1}{2}}-r^3_{i-\frac{1}{2}})\,,\\ A_{i\pm\frac{1}{2}}&=4\pi r^2_{i\pm\frac{1}{2}}\,,\end{aligned}$$

and

$$\begin{aligned}\hat{\psi}_{m\pm\frac{1}{2},i}&=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r\psi_{m\pm\frac{1}{2}}\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r\,dr}\,,\\ \overline{\psi}_{m,i}&=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\psi_m\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\,dr}\,,\\ \overline{Q}_{m,i}&=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2Q_m\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\,dr}\,.\end{aligned}$$

The following diamond difference relationship is used in space:

$$\psi_{m,i}=\frac{1}{2}(\psi_{m,i+\frac{1}{2}}+\psi_{m,i-\frac{1}{2}})\,, \tag{4}$$

and the following step difference relationship is used in angle:

$$\psi_{m+\frac{1}{2},i}=\psi_{m,i}\,. \tag{5}$$

The differencing scheme described results in the following equation when $\mu_m<0$:

$$\begin{aligned}\Big[-2\mu_m A_{i-\frac{1}{2}}+\alpha_{m+\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}}-A_{i-\frac{1}{2}}}{2w_m}\right)+\sigma_{t,i}V_i\Big]\psi_{m,i}=Q_{m,i}V_i-\mu_m(A_{i+\frac{1}{2}}+A_{i-\frac{1}{2}})\psi_{m,i+\frac{1}{2}}&\\ +\alpha_{m-\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}}-A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,,&\end{aligned} \tag{6}$$

and the following equation when $\mu_m > 0$:

$$\left[2\mu_m A_{i+\frac{1}{2}} + \alpha_{m+\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) + \sigma_{t,i}V_i\right]\psi_{m,i} = Q_{m,i}V_i + \mu_m(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}})\psi_{m,i-\frac{1}{2}} + \alpha_{m-\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,. \quad (7)$$

In order to determine the angular inflow for the first angular cell, $\psi_{\frac{1}{2},i}$, a starting direction sweep must be performed for $\mu_{\frac{1}{2}} = -1$. The equation for the starting direction flux is given by:

$$-\frac{\partial\psi_{\frac{1}{2}}}{\partial r} + \sigma_t\psi_{\frac{1}{2}} = Q_{\frac{1}{2}}\,, \quad (8)$$

which, when discritized in space becomes:

$$-(\psi_{\frac{1}{2},i+\frac{1}{2}} - \psi_{\frac{1}{2},i-\frac{1}{2}}) + \sigma_{t,i}\psi_{\frac{1}{2},i}\Delta r_i = Q_{\frac{1}{2},i}\Delta r_i\,. \quad (9)$$

A standard diamond difference relationship can be used to solve for the cell-averaged angular flux for the starting direction:

$$\psi_{\frac{1}{2},i} = \frac{\psi_{\frac{1}{2},i+\frac{1}{2}} + \frac{1}{2}Q_{\frac{1}{2},i}\Delta r_i}{1 + \frac{1}{2}\sigma_{t,i}\Delta r_i}\,. \quad (10)$$

The starting direction sweep also serves to determine the angular flux at the origin. The only particles at the origin will be ones travelling along $\mu = -1$, therefore the flux will be isotropic at the origin:

$$\psi_{m,\frac{1}{2}} = \psi_{\frac{1}{2},\frac{1}{2}}, \quad m = 1, ..., N\,.$$

The starting values for each sweep procedure are as follows:

- $\boldsymbol{\mu = -1}$: The incident flux at the boundary can be approximated by a linear extrapolation of the provided values:
$$\psi_{\frac{1}{2},I+\frac{1}{2}} = \psi_{1,I+\frac{1}{2}}\frac{\mu_2 + 1}{\mu_2 - \mu_1} - \psi_{2,I+\frac{1}{2}}\frac{\mu_1 + 1}{\mu_2 - \mu_1}$$
- $\boldsymbol{\mu < 0\ (\mu \neq -1)}$: The starting value is given by the initial conditions:
$$\psi_{m,I+\frac{1}{2}} = \psi_{\mathrm{bc},m}$$
- $\boldsymbol{\mu > 0}$: The starting value is given by the starting direction flux at the origin:
$$\psi_{m,\frac{1}{2}} = \psi_{\frac{1}{2},\frac{1}{2}}$$

This scheme is essentially useless unless used in preconditioning.

2 WD-DD

Beginning with (3) and using the same diamond difference relationship in space from (4) with the following weighted diamond difference relationship in angle:

$$\psi_{m,i} = \beta_m \psi_{m+\frac{1}{2},i} + (1-\beta_m)\psi_{m-\frac{1}{2},i}\,, \tag{11}$$

where

$$\beta_m = \frac{\mu_m - \mu_{m-\frac{1}{2}}}{\mu_{m+\frac{1}{2}} + \mu_{m-\frac{1}{2}}}\,,$$

results in the following difference equation when $\mu_m < 0$:

$$\begin{aligned}\Big[-2\mu_m A_{i-\frac{1}{2}} + \alpha_{m+\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2\beta_m w_m}\right) + \sigma_{t,i}V_i\Big]\psi_{m,i} &= Q_{m,i}V_i - \mu_m(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}})\psi_{m,i+\frac{1}{2}} \\ &+ \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\left[\alpha_{m+\frac{1}{2}}\left(\frac{1}{\beta_m} - 1\right) + \alpha_{m-\frac{1}{2}}\right]\psi_{m-\frac{1}{2},i}\,,\end{aligned} \tag{12}$$

and the following equation when $\mu_m > 0$:

$$\begin{aligned}\Big[2\mu_m A_{i+\frac{1}{2}} + \alpha_{m+\frac{1}{2}}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2\beta_m w_m}\right) + \sigma_{t,i}V_i\Big]\psi_{m,i} &= Q_{m,i}V_i + \mu_m(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}})\psi_{m,i-\frac{1}{2}} \\ &+ \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\left[\alpha_{m+\frac{1}{2}}\left(\frac{1}{\beta_m} - 1\right) + \alpha_{m-\frac{1}{2}}\right]\psi_{m-\frac{1}{2},i}\,.\end{aligned} \tag{13}$$

The starting direction sweep and incident fluxes at the boundaries are the same as they were for the scheme using weighted diamond difference in angle.

This is the scheme that is used in the python code.

3 LDPG-DD

Beginning with (1), the following linear discontinuous finite element representation can be used for the angular dependence:

$$\psi_m(\mu) = \begin{cases} \psi_{m-\frac{1}{2}}, & \mu = \mu_{m-\frac{1}{2}} \\ \psi_m^{(+)} B_m^{(+)}(\mu) + \psi_m^{(-)} B_m^{(-)}(\mu), & \mu \in (\mu_{m-\frac{1}{2}}, \mu_{m+\frac{1}{2}}] \,, \end{cases} \tag{14}$$

where $\mu_m^{(\pm)}$ are the local Gauss S_2 quadrature points on the interval $[\mu_{m-\frac{1}{2}}, \mu_{m+\frac{1}{2}}]$ with corresponding weights $w_m^{(\pm)}$ and

$$B_m^{(+)}(\mu) = \frac{\mu - \mu_m^{(-)}}{\mu_m^{(+)} - \mu_m^{(-)}} \,, \qquad B_m^{(-)}(\mu) = \frac{\mu_m^{(+)} - \mu}{\mu_m^{(+)} - \mu_m^{(-)}} \,,$$
$$\psi_{m-\frac{1}{2}} = \psi_{m-1}(\mu_{m-\frac{1}{2}}) \,,$$

which when combined with a Petrov-Galerkin weighted residual approach, with weight functions

$$W_m^{(-)}(\mu) = \begin{cases} 1, & \mu \in [\mu_{m-\frac{1}{2}}, \mu_m) \\ 0, & \text{otherwise} \end{cases}$$
$$W_m^{(+)}(\mu) = \begin{cases} 1, & \mu \in [\mu_m, \mu_{m+\frac{1}{2}}] \\ 0, & \text{otherwise} \end{cases}$$

results in the following equations:

$$\frac{\mu_m^{(-)}}{r^2} \frac{\partial \left[r^2 \psi_m^{(-)} \right]}{\partial r} + \frac{1}{r} \frac{\alpha_m \psi_m - \alpha_{m-\frac{1}{2}} \psi_{m-\frac{1}{2}}}{w_m^{(-)}} + \sigma_t \psi_m^{(-)} = Q_m^{(-)} \,, \tag{15}$$

$$\frac{\mu_m^{(+)}}{r^2} \frac{\partial \left[r^2 \psi_m^{(+)} \right]}{\partial r} + \frac{1}{r} \frac{\alpha_{m+\frac{1}{2}} \psi_{m+\frac{1}{2}} - \alpha_m \psi_m}{w_m^{(+)}} + \sigma_t \psi_m^{(+)} = Q_m^{(+)} \,, \tag{16}$$

where the α-coeffcients are defined with the following recursive formulas:

$$\alpha_m = \alpha_{m-\frac{1}{2}} - 2\mu_m^{(-)} w_m^{(-)} \,, \qquad \alpha_{m+\frac{1}{2}} = \alpha_m - 2\mu_m^{(+)} w_m^{(+)} \,, \qquad \alpha_{\frac{1}{2}} = 0$$

Discretizing these equations in space results in the following:

$$\mu_m^{(-)} \left[A_{i+\frac{1}{2}} \psi_{m,i+\frac{1}{2}}^{(-)} - A_{i-\frac{1}{2}} \psi_{m,i-\frac{1}{2}}^{(-)} \right] + \frac{1}{2} \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right) \frac{\alpha_m \hat{\psi}_{m,i} - \alpha_{m-\frac{1}{2}} \hat{\psi}_{m-\frac{1}{2},i}}{w_m^{(-)}} + \sigma_{t,i} \overline{\psi}_{m,i}^{(-)} V_i = \overline{Q}_{m,i}^{(-)} V_i \,, \tag{17}$$

$$\mu_m^{(+)} \left[A_{i+\frac{1}{2}} \psi_{m,i+\frac{1}{2}}^{(+)} - A_{i-\frac{1}{2}} \psi_{m,i-\frac{1}{2}}^{(+)} \right] + \frac{1}{2} \left(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}} \right) \frac{\alpha_{m+\frac{1}{2}} \hat{\psi}_{m+\frac{1}{2},i} - \alpha_m \hat{\psi}_{m,i}}{w_m^{(+)}} + \sigma_{t,i} \overline{\psi}_{m,i}^{(+)} V_i = \overline{Q}_{m,i}^{(+)} V_i \,, \tag{18}$$

where the same definitions for $V, A, \hat{\psi}, \overline{\psi}$, and $\overline{Q}$ have been used here. Using the same spatial diamond difference relationship from (4) as well as that

$$\psi_{m,i} = \frac{1}{2} \left(\psi_{m,i}^{(+)} + \psi_{m,i}^{(-)} \right) \,, \tag{19}$$

and utilizing the fact that

$$\psi_{m+\frac{1}{2},i} = \psi_m^{(+)} B_m^{(+)}(\mu_{m+\frac{1}{2}}) + \psi_m^{(-)} B_m^{(-)}(\mu_{m+\frac{1}{2}}) \,,$$

results in the following 2×2 system when $\mu_m < 0$:

$$\left[-2\mu_m^{(-)} A_{i-\frac{1}{2}} + \alpha_m \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(-)} + \alpha_m \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) \psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(-)} V_i - \mu_m^{(-)} \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \psi_{m,i+\frac{1}{2}}^{(-)} + \alpha_{m-\frac{1}{2}} \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(-)}}\right) \psi_{m-\frac{1}{2},i}\,, \tag{20}$$

$$\left[\alpha_{m+\frac{1}{2}} B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}\alpha_m\right] \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) \psi_{m,i}^{(-)}$$
$$+ \left[-2\mu_m^{(+)} A_{i-\frac{1}{2}} + \left\{\alpha_{m+\frac{1}{2}} B_m^{(+)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}\alpha_m\right\} \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(+)} V_i - \mu_m^{(+)} \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \psi_{m,i+\frac{1}{2}}^{(+)}\,, \tag{21}$$

and the following 2×2 system when $\mu_m > 0$:

$$\left[2\mu_m^{(-)} A_{i+\frac{1}{2}} + \alpha_m \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(-)} + \alpha_m \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) \psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(-)} V_i + \mu_m^{(-)} \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \psi_{m,i-\frac{1}{2}}^{(-)} + \alpha_{m-\frac{1}{2}} \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(-)}}\right) \psi_{m-\frac{1}{2},i}\,, \tag{22}$$

$$\left[\alpha_{m+\frac{1}{2}} B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}\alpha_m\right] \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) \psi_{m,i}^{(-)}$$
$$+ \left[2\mu_m^{(+)} A_{i+\frac{1}{2}} + \left\{\alpha_{m+\frac{1}{2}} B_m^{(+)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}\alpha_m\right\} \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(+)} V_i + \mu_m^{(+)} \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \psi_{m,i-\frac{1}{2}}^{(+)}\,, \tag{23}$$

For the first angular cell, the following quadratic continuous approximation is made in order to more strongly incorporate the starting direction flux:

$$\psi_1(\mu) = \psi_1^{(+)} B_1^{(+)}(\mu) + \psi_1^{(-)} B_1^{(-)}(\mu) + \psi_{\frac{1}{2}} B_{\frac{1}{2}}(\mu)\,, \tag{24}$$

where

$$B_1^{(+)}(\mu) = \frac{(\mu - \mu_1^{(-)})(\mu + 1)}{(\mu_1^{(+)} - \mu_1^{(-)})(\mu_1^{(+)} + 1)}\,, \qquad B_1^{(-)}(\mu) = \frac{(\mu - \mu_1^{(+)})(\mu + 1)}{(\mu_1^{(-)} - \mu_1^{(+)})(\mu_1^{(-)} + 1)}$$
$$B_{\frac{1}{2}}(\mu) = \frac{(\mu - \mu_1^{(+)})(\mu - \mu_1^{(+)})}{(-1 - \mu_1^{(+)})(-1 - \mu_1^{(+)})}$$

are the Lagrangian basis functions and $\psi_{\frac{1}{2}}$ is the starting direction flux. This results in the following 2×2 system for the first angular cell:

$$\left[-2\mu_1^{(-)} A_{i-\frac{1}{2}} + \alpha_1 \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right) B_1^{(-)}(\mu_1) + \sigma_{t,i} V_i\right] \psi_{1,i}^{(-)} + \left[\alpha_1 \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right) B_1^{(+)}(\mu_1)\right] \psi_{1,i}^{(+)}$$
$$= Q_{m,i}^{(-)} V_i - \mu_1^{(-)} \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right) \psi_{1,i+\frac{1}{2}}^{(-)} - \alpha_1 \left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right) B_{\frac{1}{2}}(\mu_1)\, \psi_{\frac{1}{2},i} \tag{25}$$

$$
\begin{aligned}
&\left[\left(\alpha_{\frac{3}{2}} B_1^{(-)}(\mu_{\frac{3}{2}}) - \alpha_1 B_1^{(-)}(\mu_1)\right)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right)\right]\psi_{1,i}^{(-)} \\
&\qquad + \left[-2\mu_1^{(+)} A_{i-\frac{1}{2}} + \left(\alpha_{\frac{3}{2}} B_1^{(+)}(\mu_{\frac{3}{2}}) - \alpha_1 B_1^{(+)}(\mu_1)\right)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right) + \sigma_{t,i} V_i\right]\psi_{1,i}^{(+)} \\
&= Q_{1,i}^{(+)} V_i - \mu_1^{(+)}\left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\psi_{1,i+\frac{1}{2}}^{(+)} - \left(\alpha_{\frac{3}{2}} B_{\frac{1}{2}}(\mu_{\frac{3}{2}}) - \alpha_1 B_{\frac{1}{2}}(\mu_1)\right)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right)\psi_{\frac{1}{2},i}
\end{aligned}
\tag{26}
$$

4 LDG-DD

Beginning with (1), the same linear discontinuous finite element representation in (14) can be used for the angular dependence, which when combined with a Galerkin weighted residual approach, with weight functions

$$W_m^0(\mu) = \begin{cases} 1, & \mu \in [\mu_{m-\frac{1}{2}}, \mu_{m+\frac{1}{2}}] \\ 0, & \text{otherwise} \end{cases}$$
$$W_m^1(\mu) = \begin{cases} \mu, & \mu \in [\mu_{m-\frac{1}{2}}, \mu_{m+\frac{1}{2}}] \\ 0, & \text{otherwise} \end{cases}$$

results in the following equations:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\mu_m^{(-)}\psi_m^{(-)} + \mu_m^{(+)}\psi_m^{(+)})\right] + \frac{1}{r}\frac{(1-\mu_{m+\frac{1}{2}}^2)\psi_{m+\frac{1}{2}} - (1-\mu_{m-\frac{1}{2}}^2)\psi_{m-\frac{1}{2}}}{w_m} + \sigma_t\left[\psi_m^{(-)} + \psi_m^{(+)}\right] = Q_m^{(-)} + Q_m^{(+)}\,, \tag{27}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\mu_m^{(-)2}\psi_m^{(-)} + \mu_m^{(+)2}\psi_m^{(+)})\right] + \frac{1}{r}\frac{\mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\psi_{m+\frac{1}{2}} - \mu_{m-\frac{1}{2}}(1-\mu_{m-\frac{1}{2}}^2)\psi_{m-\frac{1}{2}}}{w_m} - \frac{1}{r}\left[(1-\mu_m^{(-)2})\psi_m^{(-)} + (1-\mu_m^{(+)2})\psi_m^{(+)}\right] + \sigma_t\left[\mu_m^{(-)}\psi_m^{(-)} + \mu_m^{(+)}\psi_m^{(+)}\right] = \mu_m^{(-)}Q_m^{(-)} + \mu_m^{(+)}Q_m^{(+)}\,, \tag{28}$$

Discretizing these equations in space results in the following:

$$A_{i+\frac{1}{2}}\left[\mu_m^{(-)}\psi_{m,i+\frac{1}{2}}^{(-)} + \mu_m^{(+)}\psi_{m,i+\frac{1}{2}}^{(+)}\right] - A_{i-\frac{1}{2}}\left[\mu_m^{(-)}\psi_{m,i-\frac{1}{2}}^{(-)} + \mu_m^{(+)}\psi_{m,i-\frac{1}{2}}^{(+)}\right] + \frac{1}{2}(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}})\frac{(1-\mu_{m+\frac{1}{2}}^2)\hat{\psi}_{m+\frac{1}{2},i} - (1-\mu_{m-\frac{1}{2}}^2)\hat{\psi}_{m-\frac{1}{2},i}}{w_m} + \sigma_{t,i}\left[\overline{\psi}_{m,i}^{(-)} + \overline{\psi}_{m,i}^{(+)}\right]V_i = \left[\overline{Q}_{m,i}^{(-)} + \overline{Q}_{m,i}^{(+)}\right]V_i\,, \tag{29}$$

$$A_{i+\frac{1}{2}}\left[\mu_m^{(-)2}\psi_{m,i+\frac{1}{2}}^{(-)} + \mu_m^{(+)2}\psi_{m,i+\frac{1}{2}}^{(+)}\right] - A_{i-\frac{1}{2}}\left[\mu_m^{(-)2}\psi_{m,i-\frac{1}{2}}^{(-)} + \mu_m^{(+)2}\psi_{m,i-\frac{1}{2}}^{(+)}\right] + \frac{1}{2}(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}})\frac{\mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\hat{\psi}_{m+\frac{1}{2},i} - \mu_{m-\frac{1}{2}}(1-\mu_{m-\frac{1}{2}}^2)\hat{\psi}_{m-\frac{1}{2},i}}{w_m} - \frac{1}{2}(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}})\left[(1-\mu_m^{(-)2})\hat{\psi}_{m,i}^{(-)} + (1-\mu_m^{(+)2})\hat{\psi}_{m,i}^{(+)}\right] + \sigma_{t,i}\left[\mu_m^{(-)}\overline{\psi}_{m,i}^{(-)} + \mu_m^{(+)}\overline{\psi}_{m,i}^{(+)}\right]V_i = \left[\mu_m^{(-)}\overline{Q}_{m,i}^{(-)} + \mu_m^{(+)}\overline{Q}_{m,i}^{(+)}\right]V_i\,, \tag{30}$$

where the same definitions for $V, A, \hat{\psi}, \overline{\psi}$, and $\overline{Q}$ have been used here. Using the same spatial diamond difference relationship from (4) as well as that

$$\psi_{m,i} = \frac{1}{2}\left(\psi_{m,i}^{(+)} + \psi_{m,i}^{(-)}\right)\,, \tag{31}$$

and utilizing the fact that

$$\psi_{m+\frac{1}{2}} = \psi_m^{(+)}B_m^{(+)}(\mu_{m+\frac{1}{2}}) + \psi_m^{(-)}B_m^{(-)}(\mu_{m+\frac{1}{2}})\,,$$

results in the following 2×2 system when $\mu_m < 0$:

$$\begin{aligned}
&\left[-2\mu_m^{(-)} A_{i-\frac{1}{2}} + (1-\mu_m^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(-)}(\mu_{m+\frac{1}{2}}) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(-)} \\
&\qquad + \left[-2\mu_m^{(+)} A_{i-\frac{1}{2}} + (1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(+)}(\mu_{m+\frac{1}{2}}) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(+)} \\
&\qquad = \left[Q_{m,i}^{(-)} + Q_{m,i}^{(+)}\right] V_i - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_m^{(-)} \psi_{m,i+\frac{1}{2}}^{(-)} + \mu_m^{(+)} \psi_{m,i+\frac{1}{2}}^{(+)}\right) \\
&\qquad\qquad + (1-\mu_{m-\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\Bigg[-2{\mu_m^{(-)}}^2 A_{i-\frac{1}{2}} + \mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-{\mu_m^{(-)}}^2)(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}) \\
&\quad + \mu_m^{(-)}\sigma_{t,i} V_i\Bigg]\psi_{m,i}^{(-)} + \Bigg[-2{\mu_m^{(+)}}^2 A_{i-\frac{1}{2}} + \mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(+)}(\mu_{m+\frac{1}{2}}) \\
&\quad - \frac{1}{2}(1-{\mu_m^{(+)}}^2)(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}) + \mu_m^{(+)}\sigma_{t,i} V_i\Bigg]\psi_{m,i}^{(+)} = \left[\mu_m^{(-)} Q_{m,i}^{(-)} + \mu_m^{(+)} Q_{m,i}^{(+)}\right] V_i \\
&- \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left({\mu_m^{(-)}}^2 \psi_{m,i+\frac{1}{2}}^{(-)} + {\mu_m^{(+)}}^2 \psi_{m,i+\frac{1}{2}}^{(+)}\right) + \mu_{m-\frac{1}{2}}(1-\mu_{m-\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,,
\end{aligned} \tag{33}$$

and the following 2×2 system when $\mu_m > 0$:

$$\begin{aligned}
&\left[2\mu_m^{(-)} A_{i+\frac{1}{2}} + (1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(-)}(\mu_{m+\frac{1}{2}}) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(-)} \\
&\qquad + \left[2\mu_m^{(+)} A_{i+\frac{1}{2}} + (1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(+)}(\mu_{m+\frac{1}{2}}) + \sigma_{t,i} V_i\right] \psi_{m,i}^{(+)} \\
&\qquad = \left[Q_{m,i}^{(-)} + Q_{m,i}^{(+)}\right] V_i + \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_m^{(-)} \psi_{m,i-\frac{1}{2}}^{(-)} + \mu_m^{(+)} \psi_{m,i-\frac{1}{2}}^{(+)}\right) \\
&\qquad\qquad + (1-\mu_{m-\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\Bigg[2{\mu_m^{(-)}}^2 A_{i+\frac{1}{2}} + \mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-{\mu_m^{(-)}}^2)(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}) \\
&\quad + \mu_m^{(-)}\sigma_{t,i} V_i\Bigg]\psi_{m,i}^{(-)} + \Bigg[2{\mu_m^{(+)}}^2 A_{i+\frac{1}{2}} + \mu_{m+\frac{1}{2}}(1-\mu_{m+\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right) B_m^{(+)}(\mu_{m+\frac{1}{2}}) \\
&\quad - \frac{1}{2}(1-{\mu_m^{(+)}}^2)(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}) + \mu_m^{(+)}\sigma_{t,i} V_i\Bigg]\psi_{m,i}^{(+)} = \left[\mu_m^{(-)} Q_{m,i}^{(-)} + \mu_m^{(+)} Q_{m,i}^{(+)}\right] V_i \\
&+ \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left({\mu_m^{(-)}}^2 \psi_{m,i-\frac{1}{2}}^{(-)} + {\mu_m^{(+)}}^2 \psi_{m,i-\frac{1}{2}}^{(+)}\right) + \mu_{m-\frac{1}{2}}(1-\mu_{m-\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\right)\psi_{m-\frac{1}{2},i}\,.
\end{aligned} \tag{35}$$

The same quadratic continuous representation in (24) can be used in the first angular cell to more strongly incorporate the starting direction flux, resulting in the following 2×2 system:

$$
\begin{aligned}
&\left[-2\mu_1^{(-)} A_{i-\frac{1}{2}} + (1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_1^{(-)}(\mu_{\frac{3}{2}}) + \sigma_{t,i} V_i\right] \psi_{1,i}^{(-)} \\
&\quad + \left[-2\mu_1^{(+)} A_{i-\frac{1}{2}} + (1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_1^{(+)}(\mu_{\frac{3}{2}}) + \sigma_{t,i} V_i\right] \psi_{1,i}^{(+)} = \left[Q_{1,i}^{(-)} + Q_{1,i}^{(+)}\right] V_i \\
&\qquad - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_1^{(-)}\psi_{1,i}^{(-)} + \mu_1^{(+)}\psi_{1,i}^{(+)}\right) - (1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_{\frac{1}{2}}(\mu_{\frac{3}{2}})\, \psi_{\frac{1}{2},i}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&\left[-2\mu_1^{(-)^2} A_{i-\frac{1}{2}} + \mu_{\frac{3}{2}}(1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_1^{(-)}(\mu_{\frac{3}{2}}) - (1-\mu_1^{(-)^2})\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2}\right) + \mu_1^{(-)}\sigma_{t,i} V_i\right] \psi_{m,i}^{(-)} \\
&+\left[-2\mu_1^{(+)^2} A_{i-\frac{1}{2}} + \mu_{\frac{3}{2}}(1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_1^{(+)}(\mu_{\frac{3}{2}}) - (1-\mu_1^{(+)^2})\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2}\right) + \mu_1^{(+)}\sigma_{t,i} V_i\right] \psi_{m,i}^{(+)} \\
&\qquad = \left[\mu_1^{(-)} Q_{1,i}^{(-)} + \mu_1^{(+)} Q_{1,i}^{(+)}\right] V_i - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_1^{(-)^2}\psi_{1,i}^{(-)} + \mu_1^{(+)^2}\psi_{1,i}^{(+)}\right) \\
&\qquad\qquad - \mu_{\frac{3}{2}}(1-\mu_{\frac{3}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1}\right) B_{\frac{1}{2}}(\mu_{\frac{3}{2}})\, \psi_{\frac{1}{2},i}
\end{aligned} \tag{37}
$$

5 "True" LDPG

Beginning with (1), the same linear discontinuous finite element representation from (14) can be used for the angular dependence with the same basis and weight functions used in the original Petrov-Galerkin. However, this time the integrals of angular flux over direction will be evaluated exactly to obtain the following spatially analytic equations:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\mu_m^{(-,-)}\psi_m^{(-)}+\mu_m^{(-,+)}\psi_m^{(+)}\right]+\frac{1}{r}\frac{(1-\mu_m^2)\psi_m-(1-\mu_{m-\frac12}^2)\psi_{m-\frac12}}{w_m^{(-)}} \\ +\sigma_t\left[\beta_m^{(-,-)}\psi_m^{(-)}+\beta_m^{(-,+)}\psi_m^{(+)}\right]=Q_m^{(-)}\,, \quad (38)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\mu_m^{(+,-)}\psi_m^{(-)}+\mu_m^{(+,+)}\psi_m^{(+)}\right]+\frac{1}{r}\frac{(1-\mu_{m+\frac12}^2)\psi_{m+\frac12}-(1-\mu_m^2)\psi_m}{w_m^{(+)}} \\ +\sigma_t\left[\beta_m^{(+,-)}\psi_m^{(-)}+\beta_m^{(+,+)}\psi_m^{(+)}\right]=Q_m^{(+)}\,, \quad (39)$$

with the following definitions

$$\beta_m^{(-,\pm)}=\frac{1}{w_m^{(-)}}\int_{\mu_{m-\frac12}}^{\mu_m}B_m^{(\pm)}(\mu)\,d\mu\,,\qquad \beta_m^{(+,\pm)}=\frac{1}{w_m^{(+)}}\int_{\mu_m}^{\mu_{m+\frac12}}B_m^{(\pm)}(\mu)\,d\mu\,,$$

$$\mu_m^{(-,\pm)}=\frac{1}{w_m^{(-)}}\int_{\mu_{m-\frac12}}^{\mu_m}\mu B_m^{(\pm)}(\mu)\,d\mu\,,\qquad \mu_m^{(+,\pm)}=\frac{1}{w_m^{(+)}}\int_{\mu_m}^{\mu_{m+\frac12}}\mu B_m^{(\pm)}(\mu)\,d\mu\,,$$

$$Q_m^{(-)}=\frac{1}{w_m^{(-)}}\int_{\mu_{m-\frac12}}^{\mu_m}Q(\mu)\,d\mu\,,\qquad Q_m^{(+)}=\frac{1}{w_m^{(+)}}\int_{\mu_m}^{\mu_{m+\frac12}}Q(\mu)\,d\mu\,.$$

Discretizing the above equations in space results in the following fully discrete equations:

$$A_{i+\frac12}\left[\mu_m^{(-,-)}\psi_{m,i+\frac12}^{(-)}+\mu_m^{(-,+)}\psi_{m,i+\frac12}^{(+)}\right]-A_{i-\frac12}\left[\mu_m^{(-,-)}\psi_{m,i-\frac12}^{(-)}+\mu_m^{(-,+)}\psi_{m,i-\frac12}^{(+)}\right] \\ +\left(\frac{A_{i+\frac12}-A_{i-\frac12}}{2w_m^{(-)}}\right)\left[(1-\mu_m^2)\psi_{m,i}-(1-\mu_{m-\frac12}^2)\psi_{m,i-\frac12}\right]+\sigma_{t,i}\left[\beta_m^{(-,-)}\psi_{m,i}^{(-)}+\beta_m^{(-,+)}\psi_{m,i}^{(+)}\right]V_i \\ =Q_{m,i}^{(-)}V_i\,, \quad (40)$$

$$A_{i+\frac12}\left[\mu_m^{(+,-)}\psi_{m,i+\frac12}^{(-)}+\mu_m^{(+,+)}\psi_{m,i+\frac12}^{(+)}\right]-A_{i-\frac12}\left[\mu_m^{(+,-)}\psi_{m,i-\frac12}^{(-)}+\mu_m^{(+,+)}\psi_{m,i-\frac12}^{(+)}\right] \\ +\left(\frac{A_{i+\frac12}-A_{i-\frac12}}{2w_m^{(+)}}\right)\left[(1-\mu_{m+\frac12}^2)\psi_{m+\frac12,i}-(1-\mu_m^2)\psi_{m,i}\right]+\sigma_{t,i}\left[\beta_m^{(+,-)}\psi_{m,i}^{(-)}+\beta_m^{(+,+)}\psi_{m,i}^{(+)}\right]V_i \\ =Q_{m,i}^{(+)}V_i\,. \quad (41)$$

These produce the following 2×2 system for $\mu_m<0$:

$$\left[-2\mu_m^{(-,-)}A_{i-\frac12}+(1-\mu_m^2)\left(\frac{A_{i+\frac12}-A_{i-\frac12}}{4w_m^{(-)}}\right)+\beta_m^{(-,-)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(-)} \\ +\left[-2\mu_m^{(-,+)}A_{i-\frac12}+(1-\mu_m^2)\left(\frac{A_{i+\frac12}-A_{i-\frac12}}{4w_m^{(-)}}\right)+\beta_m^{(-,+)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(+)} \\ =Q_{m,i}^{(-)}V_i-\left(A_{i+\frac12}+A_{i-\frac12}\right)\left(\mu_m^{(-,-)}\psi_{m,i+\frac12}^{(-)}+\mu_m^{(-,+)}\psi_{m,i+\frac12}^{(+)}\right)+(1-\mu_{m-\frac12}^2)\left(\frac{A_{i+\frac12}-A_{i-\frac12}}{2w_m^{(-)}}\right)\psi_{m-\frac12,i}\,, \quad (42)$$

$$\left[-2\mu_m^{(+,-)}A_{i-\frac{1}{2}} + \left\{(1-\mu_{m+\frac{1}{2}}^2)B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-\mu_m^2)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \beta_m^{(+,-)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(-)}$$
$$+\left[-2\mu_m^{(+,+)}A_{i-\frac{1}{2}} + \left\{(1-\mu_{m+\frac{1}{2}}^2)B_m^{(+)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-\mu_m^2)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \beta_m^{(+,+)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(+)}V_i - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_m^{(+,-)}\psi_{m,i+\frac{1}{2}}^{(-)} + \mu_m^{(+,+)}\psi_{m,i+\frac{1}{2}}^{(+)}\right), \quad (43)$$

and the following 2×2 system for $\mu_m > 0$:

$$\left[2\mu_m^{(-,-)}A_{i+\frac{1}{2}} + (1-\mu_m^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) + \beta_m^{(-,-)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(-)}$$
$$+\left[2\mu_m^{(-,+)}A_{i+\frac{1}{2}} + (1-\mu_m^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{4w_m^{(-)}}\right) + \beta_m^{(-,+)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(-)}V_i + \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_m^{(-,-)}\psi_{m,i-\frac{1}{2}}^{(-)} + \mu_m^{(-,+)}\psi_{m,i-\frac{1}{2}}^{(+)}\right) + (1-\mu_{m-\frac{1}{2}}^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(-)}}\right)\psi_{m-\frac{1}{2},i}, \quad (44)$$

$$\left[2\mu_m^{(+,-)}A_{i+\frac{1}{2}} + \left\{(1-\mu_{m+\frac{1}{2}}^2)B_m^{(-)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-\mu_m^2)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \beta_m^{(+,-)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(-)}$$
$$+\left[2\mu_m^{(+,+)}A_{i+\frac{1}{2}} + \left\{(1-\mu_{m+\frac{1}{2}}^2)B_m^{(+)}(\mu_{m+\frac{1}{2}}) - \frac{1}{2}(1-\mu_m^2)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m^{(+)}}\right) + \beta_m^{(+,+)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(+)}$$
$$= Q_{m,i}^{(+)}V_i + \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_m^{(+,-)}\psi_{m,i-\frac{1}{2}}^{(-)} + \mu_m^{(+,+)}\psi_{m,i-\frac{1}{2}}^{(+)}\right). \quad (45)$$

The same quadratic continuous approximation as described for previous schemes is used in the first angular cell in order to more strongly incorporate the starting direction angular flux. However, it should be noted that here the quadratic continuous approximation is only made in the angular derivative term, and the remaining terms continue to utilize the linear discontinuous approximation. This results in the following 2×2 system in the first angular cell:

$$\left[-2\mu_1^{(-,-)}A_{i-\frac{1}{2}} + (1-\mu_1^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right)B_1^{(-)}(\mu_1) + \beta_1^{(-,-)}\sigma_{t,i}V_i\right]\psi_{1,i}^{(-)}$$
$$+\left[-2\mu_1^{(-,+)}A_{i-\frac{1}{2}} + (1-\mu_1^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right)B_1^{(+)}(\mu_1) + \beta_1^{(-,+)}\sigma_{t,i}V_i\right]\psi_{m,i}^{(+)}$$
$$= Q_{1,i}^{(-)}V_i - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_1^{(-,-)}\psi_{1,i+\frac{1}{2}}^{(-)} + \mu_1^{(-,+)}\psi_{1,i+\frac{1}{2}}^{(+)}\right) - (1-\mu_1^2)\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(-)}}\right)B_{\frac{1}{2}}(\mu_1)\psi_{\frac{1}{2},i}, \quad (46)$$

$$\left[-2\mu_1^{(+,-)}A_{i-\frac{1}{2}} + \left\{(1-\mu_{\frac{3}{2}}^2)B_1^{(-)}(\mu_{\frac{3}{2}}) - (1-\mu_1^2)B_1^{(-)}(\mu_1)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right) + \beta_1^{(+,-)}\sigma_{t,i}V_i\right]\psi_{1,i}^{(-)}$$
$$+\left[-2\mu_1^{(+,+)}A_{i-\frac{1}{2}} + \left\{(1-\mu_{\frac{3}{2}}^2)B_1^{(+)}(\mu_{\frac{3}{2}}) - (1-\mu_1^2)B_1^{(+)}(\mu_1)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right) + \beta_1^{(+,+)}\sigma_{t,i}V_i\right]\psi_{1,i}^{(+)}$$
$$= Q_{1,i}^{(+)}V_i - \left(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}}\right)\left(\mu_1^{(+,-)}\psi_{1,i+\frac{1}{2}}^{(-)} + \mu_1^{(+,+)}\psi_{1,i+\frac{1}{2}}^{(+)}\right)$$
$$-\left\{(1-\mu_{\frac{3}{2}}^2)B_{\frac{1}{2}}(\mu_{\frac{3}{2}}) - (1-\mu_1^2)B_{\frac{1}{2}}(\mu_1)\right\}\left(\frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_1^{(+)}}\right)\psi_{\frac{1}{2},i}, \quad (47)$$