1D Spherical Geometry

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1 Weighted Diamond-Diamond Difference

The steady-state transport equation expressed in 1D spherical coordinates is given by:

$$\frac{\mu}{r^2}\frac{\partial\left[r^2\psi\right]}{\partial r}+\frac{1}{r}\frac{\partial\left[(1-\mu^2)\psi\right]}{\partial\mu}+\sigma_t\psi=Q\,. \tag{1}$$

When discretized in direction, this equation becomes:

$$\frac{\mu_m}{r^2}\frac{\partial\left[r^2\psi_m\right]}{\partial r}+\frac{1}{r}\frac{\alpha_{m+\frac{1}{2}}\psi_{m+\frac{1}{2}}-\alpha_{m-\frac{1}{2}}\psi_{m-\frac{1}{2}}}{w_m}+\sigma_t\psi_m=Q_m\,, \tag{2}$$

where

$$\mu_{m+\frac{1}{2}}=\mu_{m-\frac{1}{2}}+w_m,\quad m=1,...,N,\quad \mu_{\frac{1}{2}}=-1\,, \tag{3}$$

$$\alpha_{m+\frac{1}{2}}=\alpha_{m-\frac{1}{2}}-2\mu_m w_m,\quad m=1,...,N,\quad \alpha_{\frac{1}{2}}=0\,. \tag{4}$$

Discretizing in space results in the following equation:

$$\begin{aligned}\mu_m(A_{i+\frac{1}{2}}\psi_{m,i+\frac{1}{2}}-A_{i-\frac{1}{2}}\psi_{m,i-\frac{1}{2}})+\frac{1}{2}(A_{i+\frac{1}{2}}-A_{i-\frac{1}{2}})\frac{\alpha_{m+\frac{1}{2}}\hat{\psi}_{m+\frac{1}{2},i}-\alpha_{m-\frac{1}{2},i}\hat{\psi}_{m-\frac{1}{2},i}}{w_m}&\\ +\sigma_{t,i}\overline{\psi}_{m,i}V_i=Q_{m,i}V_i,\quad m=1,...,N,\quad i=1,...,I&\end{aligned} \tag{5}$$

where

$$V_i=\frac{4\pi}{3}(r^3_{i+\frac{1}{2}}-r^3_{i-\frac{1}{2}})\,,$$

$$A_{i\pm\frac{1}{2}}=4\pi r^2_{i\pm\frac{1}{2}}\,,$$

and

$$\hat{\psi}_{m\pm\frac{1}{2},i}=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r\psi_{m\pm\frac{1}{2}}\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r\,dr}\,,$$

$$\overline{\psi}_{m,i}=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\psi_m\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\,dr}\,,$$

$$Q_{m,i}=\frac{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2 Q_m\,dr}{\int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}}r^2\,dr}\,.$$

The following diamond difference relationship is used in space:

$$\psi_{m,i}=\frac{1}{2}(\psi_{m,i+\frac{1}{2}}+\psi_{m,i-\frac{1}{2}})\,, \tag{6}$$

and the following weighted diamond difference relationship is used in angle:

$$\psi_{m,i}=\beta_m\psi_{m+\frac{1}{2},i}+(1-\beta_m)\psi_{m-\frac{1}{2},i}\,, \tag{7}$$

where

$$\beta_m = \frac{\mu_m - \mu_{m-\frac{1}{2}}}{\mu_{m+\frac{1}{2}} + \mu_{m-\frac{1}{2}}} \,. \tag{8}$$

The differencing scheme described results in the following equation when $\mu_m < 0$:

$$\begin{aligned}\Big[-2\mu_m A_{i-\frac{1}{2}} + \frac{\alpha_{m+\frac{1}{2}}(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}})}{2\beta_m w_m} + \sigma_{t,i}V_i\Big]\psi_{m,i} &= Q_{m,i}V_i - 2\mu_m(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}})\psi_{m,i+\frac{1}{2}} \\ &\quad + \frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\Big[\alpha_{m+\frac{1}{2}}\left(\frac{1}{\beta_m} - 1\right) + \alpha_{m-\frac{1}{2}}\Big]\psi_{m-\frac{1}{2},i} \,,\end{aligned} \tag{9}$$

and the following equation when $\mu_m > 0$:

$$\begin{aligned}\Big[2\mu_m A_{i+\frac{1}{2}} + \frac{\alpha_{m+\frac{1}{2}}(A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}})}{2\beta_m w_m} + \sigma_{t,i}V_i\Big]\psi_{m,i} &= Q_{m,i}V_i + 2\mu_m(A_{i+\frac{1}{2}} + A_{i-\frac{1}{2}})\psi_{m,i+\frac{1}{2}} \\ &\quad + \frac{A_{i+\frac{1}{2}} - A_{i-\frac{1}{2}}}{2w_m}\Big[\alpha_{m+\frac{1}{2}}\left(\frac{1}{\beta_m} - 1\right) + \alpha_{m-\frac{1}{2}}\Big]\psi_{m-\frac{1}{2},i} \,.\end{aligned} \tag{10}$$

In order to determine the angular inflow for the first angular cell, $\psi_{\frac{1}{2},i}$, a starting direction sweep must be performed for $\mu_{\frac{1}{2}} = -1$. The equation for the starting direction flux is given by:

$$-\frac{\partial \psi_{\frac{1}{2}}}{\partial r} + \sigma_t \psi_{\frac{1}{2}} = Q_{\frac{1}{2}} \,, \tag{11}$$

which, when discritized in space becomes:

$$-(\psi_{\frac{1}{2},i+\frac{1}{2}} - \psi_{\frac{1}{2},i-\frac{1}{2}}) + \sigma_{t,i}\psi_{\frac{1}{2},i}\Delta r_i = Q_{\frac{1}{2},i}\Delta r_i \,. \tag{12}$$

A standard diamond difference relationship can be used to solve for the cell-averaged angular flux for the starting direction:

$$\psi_{\frac{1}{2},i} = \frac{\psi_{\frac{1}{2},i+\frac{1}{2}} + \frac{1}{2}Q_{\frac{1}{2},i}\Delta r_i}{1 + \frac{1}{2}\sigma_{t,i}\Delta r_i} \,. \tag{13}$$

The starting direction sweep also serves to determine the angular flux at the origin. The only particles at the origin will be ones travelling along $\mu = -1$, therefore the flux will be isotropic at the origin:

$$\psi_{m,\frac{1}{2}} = \psi_{\frac{1}{2},\frac{1}{2}}, \quad m = 1, ..., N \,.$$

The starting values for each sweep procedure are as follows:

- $\boldsymbol{\mu = -1}$: The incident flux at the boundary can be approximated by a linear extrapolation of the provided values:

$$\psi_{\frac{1}{2},I+\frac{1}{2}} = \psi_{1,I+\frac{1}{2}}\frac{\mu_2 + 1}{\mu_2 - \mu_1} - \psi_{2,I+\frac{1}{2}}\frac{\mu_1 + 1}{\mu_2 - \mu_1}$$

- $\boldsymbol{\mu < 0}$ **(**$\boldsymbol{\mu \neq -1}$**)**: The starting value is given by the initial conditions:

$$\psi_{m,I+\frac{1}{2}} = \psi_{\mathrm{bc},m}$$

- $\boldsymbol{\mu > 0}$: The starting value is given by the starting direction flux at the origin:

$$\psi_{m,\frac{1}{2}} = \psi_{\frac{1}{2},\frac{1}{2}}$$