A1 - Vector Space p103-107

4.1 Vector Spaces

Def A vector space over a field, T, of elements $\{\alpha, \beta, \gamma, ...\}$ called scalars, is a set of elements $\{u, v, w\}$, called vectors, st the following are satisfied:

A1 Closure: $u+v \in V$

A2 Associative: $(u+v)+w = u+(v+w)$

A3 Zero: $u+0=u \mid 0 \in V$

A4 Negative: $u+(-u)=0 \mid -u \in V$

M1 Closure: $\alpha u \in V$

M2 Ass: $\alpha(\beta u) = (\alpha\beta)u$

M3 Distri: a) $(\alpha+\beta)u = \alpha u + \beta u$
b) $\alpha(u+v) = \alpha u + \alpha v$

M4 Unitary: $1u = u \mid 1 \in T$

Def Fields are sets of real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers

A2- Vector Space Properties p 109-112

4.3 More on vectors

Proposition 1

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B1 - Subspaces p 122 - 125

5.1 Subspaces

Subspace test:

1) Elements of W under Add & Mult closure are in $\in W$
2) Has 0-vector
3) Additive inverse is in W

B2 - Linear Combinations p126-127

5.3 Spanning Sets

Def A vector $v \in V$ is a linear combination of $\{v_1, v_2, ,, v_n\}$ $\subset V$ iff it can be written as:

$$v = \sum_{j=1}^{n} \lambda_j v_j = \lambda_1 v_1 + ,,, + \lambda_n v_n$$

for some $\lambda_j \in T$.

Ex $[5 \ 1] \in \mathbb{R}^2$ is combo of $\{[1 \ 1], [1 \ -1]\} \subset \mathbb{R}^2$
because $[5 \ 1] = 3[1 \ 1] + 2[1 \ -1]$

Ex $2+3x-2x^2+4x^3 \in \mathbb{P}_3$ is combo of $\{1-x^2, x, x+x^3\}$
because $2+3x-2x^2+4x^3 = 4(x+x^3) + 2(1-x^2) - 1(x)$

Def The span of $\{v_1, v_2, ,, v_n\} \subset V$, denoted span$\{v_1, v_2, ,, v_n\}$ is:

$$\text{span}\{v_1, v_2, ,, v_n\} = \left\{ v \,\middle|\, v = \sum_{j=1}^{n} \lambda_j v_j, \forall \lambda_j \in T \right.$$

span$\{v_1, v_2, ,, v_n\}$ is the set of all possible linear combinations of $\{v_1, v_2, ,, v_n\}$ ← this is the spanning set.
↳ The spanning set is a subset of V
└ Can be a subspace of V if passing test

Proposition 1

The span of $\{v_1, v_2, ..., v_n\} \subset V$ is a subspace of the vector space V.

↳ Apply subspace test 0, addition, and multiplication

Proposition 2

Let $U = \text{span}\{v_1, v_2, ..., v_n\} \subseteq V$. If W is a subspace of V containing the vectors $\{v_1, v_2, ..., v_n\}$, then $U \subseteq W$.

B3 - Linear Independence p132-135

6.1 Linear Independence

Def A set of vectors $\{v_1, v_2, ..., v_n\} \subset V$ is linearly independent iff:

$$\sum_{j=1}^{n} \lambda_j v_j = \lambda_1 v_1 + ... + \lambda_n v_n = 0$$

implies that all $\lambda_j = 0$.

- All coefs. $\lambda = 0$ then linearly independent, i.e λ is the 0-vector

But if linearly independent, then:

Proposition 1

If $\{v_1, v_2, ..., v_n\} \subset V$ is linearly independent and for all $v \in V$, then λ_j are uniquely undetermined.

Minimal Spanning sets:

Theorem 1

Let $\{v_1, v_2, ..., v_n\} \subset V$, a vector space. For every v_k $(k = 1, ..., n)$, $\text{span}\{v_1, ..., v_{k-1}, v_{k+1}, ..., v_n\} \subsetneq \text{span}\{v_1, ..., v_n\}$ iff $\{v_1, ..., v_n\}$ is linearly independent.

↳ If the spanning set of a vector space is linearly dependent, then we cannot remove a vector from the set without compromising the vector space.

Corollary (Negation)

Let $\{v_1, v_2, ..., v_n\} \subset V$, a vector space. For at least one v_k $(1 \leq k \leq n)$, $\text{span}\{v_1, ..., v_{k-1}, v_{k+1}, ..., v_n\} = \text{span}\{v_1, ..., v_n\}$ iff $\{v_1, ..., v_n\}$ is linearly dependent.

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B4 - Fundamental theorem p135-136

Fundamental Theorem:

Theorem 2:

Let V be a vector space spanned by n vectors. If a set of m vectors from V is linearly independent, then $m \leq n$.

↳ If a set of vectors is linearly independent, then there are no more than n vectors

↳ If more than n then linearly dependent

B5 - Bases   p 137-138

Def A set of vectors $\{e_1, e_2, ..., e_n\} \in V$ is a basis for the vector space V iff:

1. $\{e_1, ..., e_n\}$ is lin. independent
2. $\{e_1, ..., e_n\}$ spans V
   ↳ If one basis has n vectors, so does all its bases

Theorem 3:
Every basis for a given vector space contains the same number of vectors.

Def The dimensions of a vector space V, denoted by $\dim V$, is the number of vectors in any of its bases.

- $V = \{0\}$ has no basis, $\therefore \dim \{0\} = 0$

Proposition 2:
Let V be a finite-dimensional vector space with $\dim V = n$. Then,

1. A linearly indepedent set of vectors in V can at most contain n vectors
2. A spanning set for V must at least contain n vectors

- Basis can be called a minimal spanning set or a maximal independent set.

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C1 - Row & Column Spaces   pg 154-156

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = [c_1, c_2, \ldots, c_n]$$

Def The row space of $A \in {}^m\mathbb{R}^n$, denoted row A, is

$$\text{row } A \triangleq \text{span}\{r_1, \ldots, r_m\}$$

where $r_i \in \mathbb{R}^n$ are the rows of A. The column space denoted col A is

$$\text{col } A \triangleq \text{span}\{c_1, \ldots, c_n\}$$

where $c_i \in {}^m\mathbb{R}$ are the columns of A.

↳ row $A \leq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ ⇒ row $A \leq n$

↳ col $A \leq {}^m\mathbb{R}$ is a subspace of ${}^m\mathbb{R}$ ⇒ col $A \leq m$

↳ (dim row A & dim col A) $\leq \min\{m, n\}$

↳ Column space of A is the image space or range of A

↳ col $A = \{y \mid y = Ax, x \in {}^m\mathbb{R}\}$

↳ dim row A = dim col A

Dimensions of Row & Column spaces:

Proposition 1 Let $A \in {}^m\mathbb{R}^n$, $U \in {}^m\mathbb{R}^m$, and $V \in {}^n\mathbb{R}^n$. Then row $UA \subseteq$ row A with equality holding if U is invertible. Furthermore, col $AV \subseteq$ col A with equality holding if V is invertible.

Proposition 2 Let $\{x_1, \ldots, x_r\} \subset {}^m\mathbb{R}$ and let $U \in {}^m\mathbb{R}^m$ be invertible. Then $\{x_1, \ldots, x_r\}$ is lin. ind. iff $\{Ux_1, \ldots, Ux_r\}$ is lin. ind.

C2 - Row & Column space dimension pg 158-161

Lemma 1 Let $A \in {}^m\mathbb{R}^n$. Then row $\tilde{A}$ = row A, where $\tilde{A}$ is the RREF of A, hence dim row $\tilde{A}$ = dim row A. The nonzero rows of $\tilde{A}$ constitute a basis for row A.
↳ dim row $A = r$ = dim row $\tilde{A}$

Lemma 2 Let $A \in {}^m\mathbb{R}^n$. Then,
1) The set of columns of leading ones of $\tilde{A}$, constitutes a basis for col $\tilde{A}$
2) The set of corresponding columns $\{c_{j_1}, \ldots, c_{j_r}\}$ of A constitutes a basis for col A.
∴ dim col $\tilde{A}$ = dim col A

Theorem Let $A \in {}^m\mathbb{R}^n$. Dim row A = dim col A

row A = row $\tilde{A}$
dim row A = dim row $\tilde{A}$
dim col A = dim col $\tilde{A}$

Proof
→ dim row A = dim row $\tilde{A}$
$= r$ [L1]
= dim col $\tilde{A}$
= dim col A [L2]

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

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C3-Finding the basis pg 161-163

Ex Basis found by:

1) Find spanning set
2) Do RREF
3) Basis is the non-zero rows = also equals dim of spanning set, S

C4- Rank and Dimension formula pg 164-165

Rank is the common dimensions of the row A and col A.

Properties:

1. rank A = rank $\tilde{A}$
   ↳ row A = row $\tilde{A}$

2. rank A = rank A^T
   ↳ dim col A = dim row A^T

Rank nullity theorem:
dim null $A = n -$ rank A , n: no. of columns of A

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D1 - Linear Transformations   pg 180-182

Def The mapping $\ell : V \to W$ is a linear transformation (operator) iff:

L1 Distribution: $\ell(u+v) = \ell(u) + \ell(v)$ for all $u, v \in V$

L2 Homogeneity: $\ell(\lambda v) = \lambda \ell(v)$ for all $\lambda \in \Gamma$, $v \in V$

Axiom L1 & L2 are preserved by ℓ. Can be combined.
addition   multiplication

$$\ell(\lambda u + \mu v) = \lambda \ell(u) + \mu \ell(v)$$

Note V is the domain of ℓ
and W is the codomain/target space of ℓ

Consider v lives in V but $\ell(v)$ lives in W. $u+v$ uses vector addition of V but $\ell(u) + \ell(v)$ uses vector addition of W.

Operators

Trace Sum of the diagonal entries in a square matrix
$tr\ A$

Differential Differential operation $\to$ $T : P_n \to P_n$, $T(p) = p'$ $\Rightarrow$ $T(p+q)' = T(p)' + T(q)'$, $T(cp)' = cT(p)'$

Identity Leaves vector unchanged ie $i(v) = v$ for any $v \in V$
$i : V \to V$

Zero Returns zero, $z(0) = 0$

D2 - Image & Kernals   pg 184-186

Def The image of a linear transformation, $l: V \to W$, is denoted as:

$$\text{im } l = \{ w \mid w = l(v), \forall v \in V \}$$

- Image can also be called the range
- Image is not the codomain
  ↳ $\text{im } l = l(V) \subseteq W$

If $\text{im } l = W$, l maps V "onto" W instead of "into". ie the transformation is surjective.

If l produces for each $v \in V$ a different $w \in W$, ie $l(u) = l(v)$
$u = v$, l is one-to-one or injective.

If a transformation is both surjective & injective, the transformation is byjective.

Whereas $Ax = 0$ forms a nullspace, for transformations it is called a kernel.

Def The kernel of a linear transformation, $l: V \to W$, is denoted as:

$$\ker l = \{ v \in V \mid l(v) = 0 \}$$

Note null space → null A
kernel → ker A

Formula Dimension:

$$\dim \ker l + \dim \text{im } l = \dim V$$

Note image is the span of vectors of the linear transformation

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E1 - Determinants pg 209-210

Some determinants:

Ex $\det A = \begin{bmatrix} 4 & 8 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & 3 \end{bmatrix} = 0$

$\times 2$

Ex $\det A = \begin{bmatrix} -1 & 2 & 4 \\ 4 & -8 & -16 \\ 3 & 0 & 5 \end{bmatrix} = 0$ $\times 2$

Ex $\det A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix} = 0$

Ex $\det A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = 0$

Ex $\det A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

Equal

Ex $\det A = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = (a+c)d - (b+d)c$

Ex $\det A = \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - da \rightarrow -1$

Ex $\det I = 1 \rightarrow$ Identity matrix

E4 - Adjoints pg 231-233

For a 2x2, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Def The (i,j)-cofactor of matrix $A \in {}^n\mathbb{R}^n$ is:

$$C_{ij}(A) = (-1)^{i+j} \det M_{ij}(A)$$

Def Laplace expansion for the determinant:

$$\det A = \sum_{j=1}^{n} a_{kj} C_{kj}$$

↳ Assumes expansion across a row

Def The adjoint of matrix $A \in {}^n\mathbb{R}^n$ is:

$$\operatorname{adj} A = C^T \in {}^n\mathbb{R}^n$$

where $C = [C_{ij}] \in {}^n\mathbb{R}^n$ and $C_{ij}(A)$ is the (i,j)-cofactor of A.

Theorem Let $A \in {}^n\mathbb{R}^n$. Then:

$$A \operatorname{adj} A = (\operatorname{adj} A) A = (\det A) I$$

Furthermore:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

F1 - Eigen problem   pg 238-242

Def Eigenproblem:

$Ap = \lambda p$ or $(\lambda I - A)p = 0$, $p \neq 0$

For nontrivial sol: $\det(\lambda I - A) = 0$

For characteristic polynomial/eigenpolynomial of A:

$c_A(\lambda) = \det(\lambda I - A)$

The eigenspace of λ:

$E_\lambda = \text{null}(\lambda I - A) = \{p \in {}^n\mathbb{R} \mid Ap = \lambda p\}$

F4 - Diagonalization Test pg 254-256

Ex Can you diagonilize $A = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Characteristic eq'n:

$C_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & 4 & 0 \\ -1 & -\lambda-4 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix}$

Laplace expansion:

$C_A(\lambda) = (\lambda-2)[\lambda(\lambda-4)+4] = (\lambda-2)(\lambda^2-4\lambda+4)$

$= (\lambda-2)^3$

$= 0$

↳ Eigen value $\lambda = 2$

↳ Eigenspace $\mathcal{E}_{\lambda=2} = \text{null}(2I-A)$

↳ $2I - A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rank=1

$(\text{rank}(2I-A)=1)$

Note $\text{rank}(2I-A)=1 \rightarrow \therefore \dim \mathcal{E}_{\lambda=2} = 3 - \text{rank}(2I-A)$

$= 3 - \overbrace{\text{rank}(2I-A)}$

$= 2$

$2 \neq 3 \quad \therefore$ Not diagonalizable

Theorem Let $A \in {}^n\mathbb{R}^n$ with distinct $\lambda_1, \dots, \lambda_r$, $r \leq n$.
If $x_\alpha \in \mathcal{E}_{\lambda_\alpha} \setminus \{0\}$, $\alpha = 1, \dots, r$, then $\{x_1, \dots, x_r\}$
is lin. ind.

Corollary If $A \in {}^n\mathbb{R}^n$ has n distinct λ, A is diagonalizable

Lemma Let $A \in {}^n\mathbb{R}^n$ with distinct $\lambda_1, \ldots, \lambda_r$, $r \leq n$. If $x_\alpha \in \mathcal{E}_{\lambda_\alpha}$ and $x_1 + \ldots x_r = 0$, then $x_\alpha = 0$, $\alpha = 1, \ldots, r$.

Theorem Let $A \in {}^n\mathbb{R}^n$ with distinct $\lambda_1, \ldots, \lambda_r$, $r \leq n$. If H_{λ_α} is a linearly ind. set in $\mathcal{E}_{\lambda_\alpha}$, then

$$H = H_{\lambda_1} \cup H_{\lambda_2} \cup \ldots \cup H_{\lambda_r} \equiv \bigcup_{\alpha=1}^{r} H_{\lambda_\alpha}$$

is lin. ind. and

$$m_1 + \ldots m_r \leq n$$

where $m_\alpha = \dim \mathcal{E}_{\lambda_\alpha}$

F5 - Geometric & Algebraic Multiplicities pg 257-260

Def Let $A \in {}^n\mathbb{R}^n$ with eigenvalues λ_α. The highest power n_α of $\lambda - \lambda_\alpha$ that divides the characteristic polynomial $c_A(\lambda)$ st $c_A(\lambda) = (\lambda - \lambda_\alpha)^{n_\alpha} g(\lambda)$ is the algebraic multiplicity of λ_α.

The dimension m_α of E_{λ_α} is the geometric multiplicity of λ_α.

Note If $n_\alpha > 1$ then repeated / multiple eigenvalue.

Proposition Let $A = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \in {}^n\mathbb{R}^n$ where $B \in {}^r\mathbb{R}^{n-r}$, $C \in {}^{n-r}\mathbb{R}^{n-r}$, is $I \in {}^r\mathbb{R}^r$ the $r \times r$ identity matrix. Then $\det A = \det C$.

Theorem Multiplicity theorem:
Let λ_α be an eiganvalue of $A \in {}^n\mathbb{R}^n$. Then $1 \leq m_\alpha \leq n_\alpha$, where m_α and n_α are the geometric and algebraic multiplicities of λ_α. Note if $n_\alpha = 1$ then $m_\alpha = n_\alpha = 1$

Theorem Diagonalization test:
Let $A \in {}^n\mathbb{R}^n$ with distinct eigenvalues λ_α, $\alpha = 1, \ldots, r$
Then, A is diagonalizable iff $m_\alpha = n_\alpha$, $\alpha = 1, \ldots, r$.
↳ If $m_\alpha = n_\alpha$

F6 - Differential Eq'ns pg 264-266

Let $\dot{x} = Ax$ where $x(t) \in {}^n F[\mathbb{R}]$ and $A \in {}^n\mathbb{R}^n$, and $x_0 = x(0)$ are the initial cond, Assume a is diagonalized by $P = [\, p_1 \,,,, \, p_n \,]$, therefore

$$P^{-1}AP = \Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_r \end{bmatrix} \quad \text{and} \quad Ap_\alpha = \lambda_\alpha p_\alpha$$

where each λ_α, $\alpha = 1,,, r$ is repeated $m_\alpha = n_\alpha$ times