MAT185 Linear Algebra
Assignment 3

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Gordon Jeon

2) Vincent Bourdé

1. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Suppose A_1 and A_2 are subspaces of V.

Read and then write a critique the following "proof" that $T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$. The proof consists of five lines, not including assumptions. Your critique should identify, at a minimum, which lines are not correct; where exactly the proof breaks down; and what exactly are the incorrect statements or deductions.

"Proof":

Suppose that $\mathbf{x} \in T(A_1 \cap A_2)$.

Line 1: Then there exists a vector $\mathbf{y} \in A_1 \cap A_2$ such that $T\mathbf{y} = \mathbf{x}$.

Line 2: Since $\mathbf{y} \in A_1$ and $\mathbf{y} \in A_2$, we have $T\mathbf{y} \in T(A_1)$ and $T\mathbf{y} \in T(A_2)$, so that $\mathbf{x} \in T(A_1) \cap T(A_2)$.In other words, we have shown that $T(A_1 \cap A_2) \subseteq T(A_1) \cap T(A_2)$.

Now suppose that $\mathbf{x} \in T(A_1) \cap T(A_2)$.

Line 3: Then there exists a vector $\mathbf{y} \in A_1$ such that $T\mathbf{y} = \mathbf{x}$ and there exists a $\mathbf{y} \in A_2$ such that $T\mathbf{y} = \mathbf{x}$.

Line 4: But, $\mathbf{y} \in A_1 \cap A_2$ so that $\mathbf{x} \in T(A_1 \cap A_2)$. In other words, we have shown that $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$.

Line 5: Since we have shown both $T(A_1 \cap A_2) \subseteq T(A_1) \cap T(A_2)$, and $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$ we have $T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$.

- Line 3 is incorrect because it uses the same notation for the input $\mathbf{y}$ in A_1 and A_2 of which the transformation will output $\mathbf{x}$. Furthermore it isn't explicitly declared that the input $\mathbf{y} \in A_2$ is a vector.
- Line 4 is incorrect because carrying on from Line 3, Line 4 makes the incorrect deduction that $\mathbf{y}$ is the equivalent input for both subspaces, A_1 and A_2. Therefore it can not be assumed that the same $\mathbf{y}$ is in the intersection of A_1 and A_2. Furthermore if the aforementioned is incorrect, $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$ cannot hold.
- Line 5 is incorrect because it carries on the false deduction from Line 4. $T(A_1) \cap T(A_2) \subseteq T(A_1 \cap A_2)$ has not been proven. Therefore $T(A_1 \cap A_2) = T(A_1) \cap T(A_2)$ cannot be guaranteed to hold.

2. Let $c \in \mathbb{R}$, and let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the linear transformation defined by $T(p(x)) = cp(x) - xp'(x)$.

Determine all values of c such that T is bijective?

For T to be bijective, it must pass the check for both injectivity and surjectivity. Therefore consider the following:

Such that $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$, T can equivalently be interpreted as $T : V \to V$. Furthermore by the Dimension Theorem, dim im T + dim ker T = dim V (*MAT185 Lec. 16, p.5*). Therefore if the dimension of the kernal equals 0, dim im T = dim im V.

Now consider the injectivity of T. For the linear transformation to be injective, dim ker T must equal 0. Therefore the following must hold true:

$$\begin{aligned} \dim \operatorname{im} T + \dim \ker T &= \dim V \\ \dim \operatorname{im} T + 0 &= \dim \operatorname{im} V \\ \dim \operatorname{im} T &= \dim \operatorname{im} V \\ \operatorname{im} T &= \dim V \end{aligned}$$

As shown above, im T = dim V. This passes the check for surjectivity. From here only injectivity needs to be confirmed (i.e. ker $T = 0$), therefore consider the transformation and how it's a transformation on a first order differential equation. Solving for $p(x)$:

$$\begin{aligned} T(p(x)) = cp(x) - xp'(x) &= 0 \\ cp(x) - xp'(x) &= 0 \\ p(x) &= cx^c \end{aligned}$$

As mentioned above, however, for the transformation to be injective, ker $T = \{0\}$. This means ker $T \neq \{0\}$ iff $cp(x) - xp'(x) = 0$ has a non-zero solution. Therefore if $c \leq n$, there exists some $p(x) = cx^c \in P_n(\mathbb{R})$ for which $T(p(x)) = 0$ which in turn concludes ker $T = \{0\}$ iff $c > n$.

$\therefore$ Any value of c s.t. $c > n$, makes the linear transformation T bijective as the only solution to $T(p(x)) = 0$ is $p(x) = 0$.

3. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v_3}$ be a basis for V.

(a) Prove that if T is bijective, then $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis for W.

If T is bijective then:

- dim ker $T = 0$, by injectivity
- dim im $T =$ dim W, by surjectivity

And by the Dimension Theorem:

$$\begin{aligned} \text{dim } V &= \text{dim ker } T + \text{dim im } T \\ 3 &= 0 + \text{dim } V \\ \therefore \text{dim } W &= 3 \end{aligned}$$

the basis of W will contain 3 vectors. Furthermore, by Lemma 16.1 (*MAT185 Lec. 16, p.1*):

$$\begin{aligned} \text{im } T = T(V) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\ &= c_1T\mathbf{v}_1 + c_2T\mathbf{v}_2 + c_3T\mathbf{v}_3 \\ &= \text{span}\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\} \end{aligned}$$

As mentioned before, we know im $T \subseteq W$, and dim im $T =$ dim $W = 3$. Therefore im $T = W =$ span$\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\}$

$\therefore$ Knowing dim $W = 3$, and the list $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ has three vector that span W, then $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ must be a basis of W.

3. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v_3}$ be a basis for V.

(b) Prove that if $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis for W, then T is bijective.

Consider the following:

- If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of V, then $\dim V = 3$

- If $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis of W, then $W = \operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\}$ and $\dim W = 3$

Then by Lemma 16.1,

$$\operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\} = \operatorname{im} T$$
$$\operatorname{span}\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\} = W = \operatorname{im} T$$

It can be determined that $\dim \operatorname{im} T = 3$. Moreover, by the Dimension Theorem,

$$\begin{aligned} \dim V &= \dim \ker T + \dim \operatorname{im} T \\ 3 &= \dim \ker T + 3 \\ \therefore \dim \ker T &= 0 \end{aligned}$$

$\therefore$ If $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ is a basis of W, T is bijective by:

- Surjectivity: $\dim \operatorname{im} T = \dim W$
- Injectivity: $\dim \ker T = 0$