Chapter 9: Elastic Deflection of Beams

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Coverage: Direct integration, Area-Moment method, Macaulay's method, Singularity functions, Beam Analysis

1 Assumptions

Before we get into the content, we need to go over some assumptions. For the following applications we are **assuming** that the material is homogeneous and isotropic. We are **assuming** that the beam will only deform elastically. We are **assuming** that the deflection is a fraction of the overall thickness. We are **assuming** the beam will deform in the manner of circular curves. We are **assuming** that the deflection of the neutral axis is identical to that of the centroidal axis. As you can guess, analyzing deflection is more nuanced than that, but we need to create a jumping off point from somewhere.

2 Curvature caused by a Bending Moment

Refer to Figure 1, an exaggerated graph illustrating the curvature of a beam's deflection.

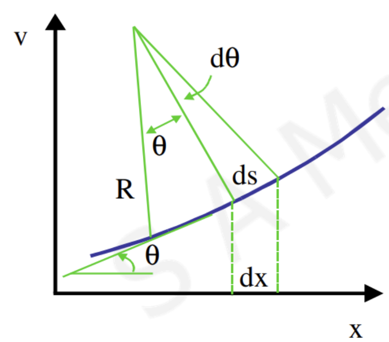

Figure 1: v-x graph representing the curvature of beam deflection

People smarter than I, derived and defined the bending moment as:

$$M = \frac{EI}{R} = EI\frac{d^2v}{dx^2} \tag{1}$$

where,

E	Young's modulus
I	second moment of inertia
R	radius of curvature

This relationship, in general, will get you most of the way through understanding the deflection mechanics.

3 Boundary Conditions

However, as displayed in Figure 2, it goes without saying that some boundary conditions exist.

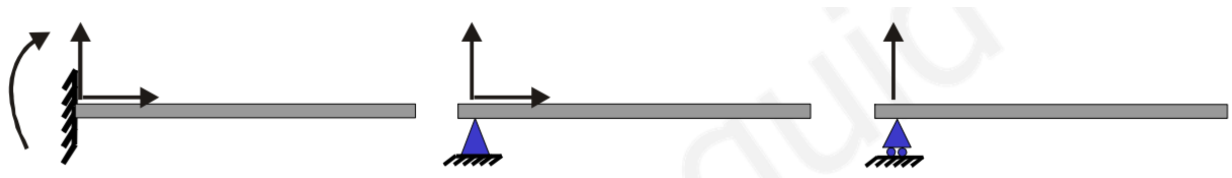

Figure 2: Boundary conditions including a cantilever beam, a pin connection, and a roller connection respectively

For such boundary conditions we make some the following assumptions:

1. Deflection at the supports, v or $y = 0$, unless stated otherwise
2. The slope at built-in supports, $\frac{dy}{dx} = 0$
3. The slope at the center of a symmetrical load is zero
4. The bending moment at pin connections, $M = 0$

4 Method of Direct/Double integration

In the event we're solving a beam problem, we simply need to draw the respective FBD, draw our reactionary forces and bending moments and solve for following integrations.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{2}$$

$$\frac{dy}{dx} = \int \frac{M}{EI} dx \tag{3}$$

$$y = \iint \frac{M}{EI} dxdx \tag{4}$$

Well this seems easy enough right? These formulas will get us most of the way to a solution for simple beam designs. But for some beams where the calculations get difficult, we need another tool.

5 Macaulay's Method

Say we have a beam profile with multiple point loads and even UDLs. How on earth do we tackle such a problem? We consult the use of singularity functions, $\langle x - a \rangle$. These functions are akin to the piecewise functions we may be familiar with but with the bending moments of the beam.

$$\langle x - a \rangle = \begin{cases} 0, & x \leq a \\ x - a, & x > a \end{cases} \tag{5}$$

Be careful here, Macaulay brackets don't follow the standard rules of integration. For singularity functions we follow a different approach and recall concepts such as the Dirac delta and unit step functions.

$$\int P\langle x - a \rangle dx = P\frac{\langle x - a \rangle^2}{2} + C_m \tag{6}$$

6 Mohr's/Moment Area Method

In Chapter 5, our friend Mohr and his circle helped us find the relevant information to a stress element. He will now help us in solving beam problems using his Area-Moment method. This method is based off of finding the area under the bending moment diagram and consists of two theorems.

6.1 Theorem 1

The first theorem revolves finding the difference of angles between two points on a beam. The left-hand side is the difference in angular deflection, while the right-hand side is the area under the BMD between the two points divided by the flexural rigidity, EI.

$$\theta_2 - \theta_1 = \frac{A}{EI} \tag{7}$$

6.2 Theorem 2

The second theorem revolves around finding the tangential deviation at a point. This may not seem useful at first but using trigonometry we are able to easily find the angle of deflection by just scaling the triangle.

$$t_{2/1} = \frac{A\bar{x}}{EI} \tag{8}$$