MAT185 Linear Algebra
Assignment 1

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4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the aforementioned document. I have not violated these rules while completing and writing this assignment.
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Signatures: 1) Gordon Jeon

2) Harvey Zheng

1. Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Is V a real vector space with respect to the usual entry-wise vector addition and scalar multiplication? Why or why not?

No. Axioms, AIII and AIV (Medici, p104) fail to hold in V.

AIII states, *Zero: There exists a zero or null vector* $0 \in V$ *s.t.* $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$ (Medici pp104)

Therefore, by definition, there must be a zero or null vector, $0 \in V$ *s.t.* $(x_1, x_2) + 0 = (x_1, x_2)$.

Accordingly, for the previous statement to be true, the zero vector must have values $(0, 0)$:

$$(x_1, x_2) + 0 = (x_1, x_2)$$

$$(x_1, x_2) + (0, 0) = (x_1, x_2)$$

$$(x_1 + 0, x_2 + 0) = (x_1, x_2)$$

$$(x_1, x_2) = (x_1, x_2)$$

However, $x_1, x_2 > 0$ in V, therefore, the real vector space V does not contain a zero vector, proving AIII does not hold in V.

AIV states, *Negative: There exists a negative* $-u \in V$ *s.t* $\boldsymbol{u} + (\boldsymbol{-u}) = \boldsymbol{0}$ (Medeci pp104)

Therefore, by definition, for an inverse vector to exist, the sum of the vector and itself needs to be the zero vector. However, as seen above, V does not contain a zero vector. Therefore, V fails in AIV.

2 Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Define vector addition in V by

$$(x_1, x_2) + (y_1, y_2) = \frac{(x_1y_1, x_2y_2)}{x_1y_1 + x_2y_2}$$

and scalar multiplication in V by

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

Then V is a real vector space.

(a) Verify that axiom AIII. (Medici, pp104) holds in V.

To verify AIII, you need the zero vector. Therefore, consider the following:

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

In order for the expression to equal to 0, set $c = 0$, as values $x_1, x_2 > 0$ by definition:

$$\begin{aligned}
c(x_1, x_2) &= \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}, \quad c = 0 \\
&= \frac{(x_1^0, x_2^0)}{x_1^0 + x_2^0} \\
&= \frac{(1, 1)}{1 + 1} \\
&= (\frac{1}{2}, \frac{1}{2}) \\
\therefore (x_1, x_2) &= (\frac{1}{2}, \frac{1}{2}) \ s.t\ c = 0
\end{aligned}$$

As shown above $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ and as per definition $\frac{1}{2} > 0$, therefore AIII holds in V.

(b) Verify that axiom AIV. (Medici, pp104) holds in V.

To verify AIV, there must exist a negative $-u \in V\ s.t.\ u + (-u) = 0$. To find the zero vector, consider the following:

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

In order for the expression to become the inverse of (x_1, x_2), it must have values $(-x_1, -x_2)$ or $(-1)(x_1, x_2)$ (MIII, Medici, pp104) equivalently. Therefore, verify the scalar multiplication:

$$\begin{aligned}
c(x_1, x_2) &= -1(x_1, x_2) \\
&= \frac{(x_1^{-1}, x_2^{-1})}{x_1^{-1} + x_2^{-1}}, \ by\ definition \\
&= 1 \cdot \frac{(x_1^{-1}, x_2^{-1})}{x_1^{-1} + x_2^{-1}}, \ by\ MIV \\
&= \frac{x_1, x_2}{x_1 x_2} \cdot \frac{(\frac{1}{x_1}, \frac{1}{x_2})}{\frac{1}{x_1} + \frac{1}{x_2}}, \ by\ Q.E.D \\
&= \frac{(x_2, x_1)}{x_2 + x_1}, \ by\ MIII \\
&= \frac{(x_2, x_1)}{1}, \ by\ definition \\
&= (x_2, x_1)
\end{aligned}$$

2. Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, \text{ and } x_1 + x_2 = 1\}$. Define vector addition in V by

$$(x_1, x_2) + (y_1, y_2) = \frac{(x_1y_1, x_2y_2)}{x_1y_1 + x_2y_2}$$

and scalar multiplication in V by

$$c(x_1, x_2) = \frac{(x_1^c, x_2^c)}{x_1^c + x_2^c}$$

Then V is a real vector space.

(c) Verify that axiom MIII. (Medici, pp104) holds in V.
MIII states, *Distributivity: (a)* $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$ and (b) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (Medici pp104)

To verify MIII, both *(a)* and *(b)* need to pass concurrently, consider the following:

$$\begin{aligned}
(x_1, x_2) + (y_1, y_2) &= \frac{(x_1y_1, x_2y_2)}{x_1y_1 + x_2y_2},\ \textit{by definition} \\
&= (\frac{x_1y_1}{x_1y_1 + x_2y_2}, \frac{x_2y_2}{x_1y_1 + x_2y_2}),\ \textit{by definition} \\
a[(x_1, x_2), (y_1, y_2)] &= a(\frac{x_1y_1}{x_1y_1 + x_2y_2}, \frac{x_2y_2}{x_1y_1 + x_2y_2}),\ \textit{s.t. a is a scalar} \\
a(x_1, x_2) + a(y_1, y_2) &= (\frac{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}, \frac{(\frac{x_2y_2}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}),\ \textit{by definition} \\
(\frac{x_1^a, x_2^a}{x_1^a + x_2^a}) + (\frac{y_1^a, y_2^a}{y_1^a + y_2^a}) &= (\frac{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a, (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}),\ \textit{by definition} \\
(\frac{x_1^a}{x_1^a + x_2^a}, \frac{x_2^a}{x_1^a + x_2^a}) + (\frac{y_1^a}{y_1^a + y_2^a}, \frac{y_2^a}{y_1^a + y_2^a}) &= (\frac{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a, (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}),\textit{by definition} \\
(\frac{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}, \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)} + \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}) &= (\frac{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a, (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}),\textit{by definition}
\end{aligned}$$

Here, analyze the left-hand side (LHS) and right-hand side (RHS):

$$\begin{aligned}
(\frac{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}, \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)} + \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}) &= (\frac{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)} + \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}, \frac{\frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}{\frac{(x_1y_1)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)} + \frac{(x_2y_2)^a}{(x_1^a+x_2^a)(y_1^a+y_2^a)}}) \\
&= (\frac{\frac{(x_1y_1)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a)}}}{\frac{(x_1y_1)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a)}} + \frac{(x_2y_2)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a)}}}, \frac{\frac{(x_2y_2)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a)}}}{\frac{(x_1y_1)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a)}} + \frac{(x_2y_2)^a}{\cancel{(x_1^a+x_2^a)(y_1^a+y_2^a))}}}) \\
LHS &= (\frac{(x_1y_1)^a}{(x_1y_1)^a + (x_2y_2)^a}, \frac{(x_2y_2)^a}{(x_1y_1)^a + (x_2y_2)^a}) \\
(\frac{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a, (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}{(\frac{x_1y_1}{x_1y_1+x_2y_2})^a + (\frac{x_2y_2}{x_1y_1+x_2y_2})^a}) &= (\frac{(\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a}), (\frac{(x_2y_2)^a}{(x_1y_1+x_2y_2)^a})}{(\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a}) + (\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a})}) \\
&= (\frac{(\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a})}{(\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a}) + (\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a})}, \frac{(\frac{(x_2y_2)^a}{(x_1y_1+x_2y_2)^a})}{(\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a}) + (\frac{(x_1y_1)^a}{(x_1y_1+x_2y_2)^a})}) \\
&= (\frac{(\frac{(x_1y_1)^a}{\cancel{(x_1y_1+x_2y_2)^a}})}{(\frac{(x_1y_1)^a}{\cancel{(x_1y_1+x_2y_2)^a}}) + (\frac{(x_1y_1)^a}{\cancel{(x_1y_1+x_2y_2)^a}})}, \frac{(\frac{(x_2y_2)^a}{\cancel{(x_1y_1+x_2y_2)^a}})}{(\frac{(x_1y_1)^a}{\cancel{(x_1y_1+x_2y_2)^a}}) + (\frac{(x_1y_1)^a}{\cancel{(x_1y_1+x_2y_2)^a}})}) \\
RHS &= (\frac{(x_1y_1)^a}{(x_1y_1)^a + (x_2y_2)^a}, \frac{(x_2y_2)^a}{(x_1y_1)^a + (x_2y_2)^a})
\end{aligned}$$

As seen above the LHS and RHS are equivalent, proving MIII holds in V.

3. Recall that $P_3(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the set of polynomials of degree at most 3 with real coefficients, is a real vector space with respect to the usual polynomial addition and scalar multiplication.

(a) Give an example of a subset S of $P_3(\mathbb{R})$ that is closed under vector addition but not under scalar multiplication. You should both state clearly your subset S and demonstrate that S satisfies the requirement of the question.

Consider, $S = \{a_1x, a_2x^3 | \ a_1, a_2 \in R\}$

For closure in scalar multiplication, $a_1 \cdot a_2$ is always in the space of the subset S.

For closure in vector addition, the result is not always in the space of subset S.

Proof by counter-example:

$a_0x \cdot a_1x^3 = (a_0 \cdot a_1)x^4$, which is outside of the highest power of 3 for the variable x defined in subset S.

3. Recall that $P_3(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the set of polynomials of degree at most 3 with real coefficients, is a real vector space with respect to the usual polynomial addition and scalar multiplication.

(b) Give an example of a subset S of $P_3(\mathbb{R})$ that is closed under scalar multiplication but not under vector addition. You should both state clearly your subset S and demonstrate that S satisfies the requirement of the question.

Consider, $S = \{a_1, a_2 | \frac{\sqrt{2}}{2} < a_1 < 1, \frac{\sqrt{2}}{2} < a_2 < 1\}$

For closure in scalar multiplication, $a_1 \cdot a_2$ is always less than 1, which satisfies the definition of subset S.

For closure in vector addition, the result is always greater than one, outside of the definition of subset S.