Analysis of a Simple Pendulum's Amplitude in relation to the Period within Damped a Harmonic Oscillator System

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Abstract

Video editing & analysis applications are used to analyze the motion of a pendulum and organized into multiple sets of usable data in order to investigate whether the periodicity of the pendulum is dictated by the amplitude (also referred to as the angle) and whether periodicity is dictated by the pendulum length. Two pendulums are built and used for this study and the data collected from the multiple trials were transposed into multiple graphs with a regression line (hereinafter referred to as the "line of fit") being generated on a two dimensional plane. The relationships between the angle, periods of oscillation, elapsed times, lengths, and Q factor are analyzed and compared with each other. Furthermore, the present results are expected to provide information regarding the decay of the motion in addition to offering worthwhile data investigating the pendulum's model of a damped harmonic oscillator.

1 Introduction

The harmonic oscillator system has been an area of invaluable research in various fields with applications ranging from simple pendulums to more complex fields of research including RLC and radio circuits, as depicted in Figure 1. The damped harmonic oscillator is a concerning factor of the aforementioned system as it establishes itself as a more applicable model for physical systems. This is due to most models accounting for outside influences such as frictional forces and the dissipation of energies. Studies concerning the damped harmonic oscillator model can be found dating back to derivations of Newton's second law of motion [1].

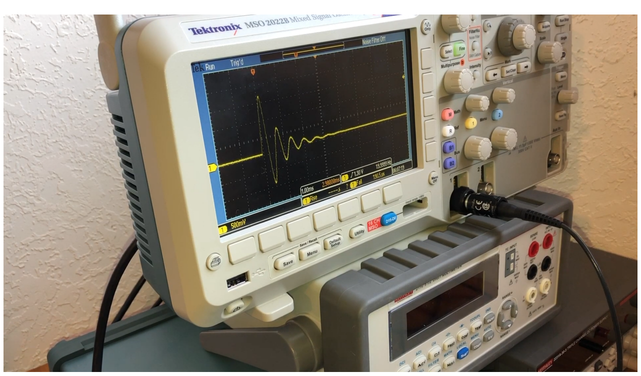

Figure 1: A decaying sine wave, also known as an under damped harmonic oscillator as seen on the Tektronix MSO2022B oscilloscope

In many of such studies, the author seeks to investigate the period of motion in relation to the amplitude in order to describe a generalized result relating to an oscillating system.

2 Methodology

2.1 Design Proposal and Model Scaling

A pendulum is built and used for this study due to the ability to adjust the length of string. A thin string is also used as to emulate an ideal string as much as possible with desired properties including near masslessness, minimal friction, and having uniform tension throughout its length. For weights a metal ball bearing is used, as referred to Figure 2.

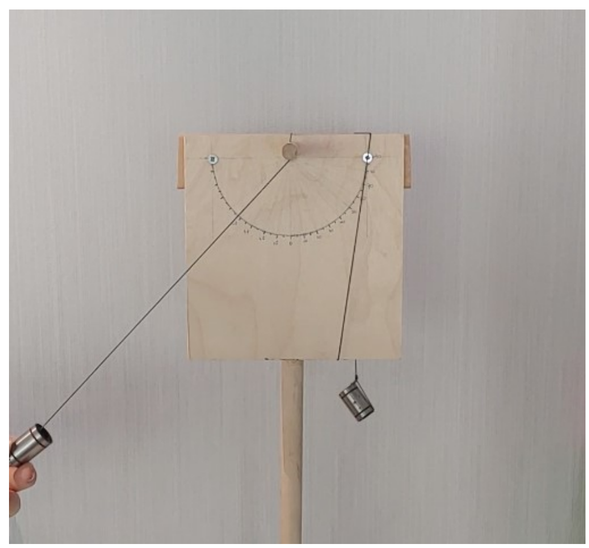

Figure 2: The first mock-up pendulum to be used for this report. The ball bearing mass is let go at a $\frac{\pi}{4}$ rad angle set at a string length of 0.236 m

The Angle v. Period graph is developed by timing each oscillation and taking the average of 4 oscillations per increment in angle. This is to be taken as the period. Doing so guarantees some baseline level of precision. As seen on Figure 3, the angle measurements' units are scaled to radians and times are scaled in seconds. The constructed graph along with its lines of fit are investigated and compared between the increments in radian measures.

As seen in Figure 3, the graph shows as the radian measure increases so will the period. This is indicative of the positive parabolic relationship seen on the graph and the inference can

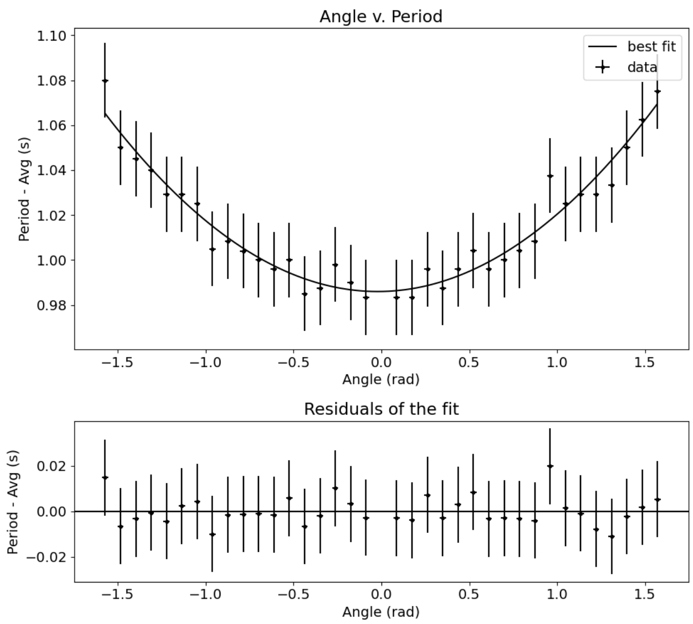

Figure 3: The radian measure versus the average of 4 oscillations is plotted into a Angle v. Period graph

be made that the amplitude contributes to the periodicity of 1 oscillation. However the presence of the restoring force (which is proportional to the acceleration) also contributes to the returning path of travel. This causes the mass to travel at a greater speed when travelling the path of oscillation when raised to greater amplitudes but it must be understood that the preceding attributes cancel each other out, causing the amplitude to have little to no effect on the periodicity of oscillation [2].

Further comparing the oscillations also presents the information of a strong-positive correlation as the r-value (correlation coefficient) is between the $\frac{2}{3}$ and $\frac{3}{3}$ range. This information is important as a progression of periodicity can be seen as the angle increases in either direction (which in turn increases the restoring force). Further investigation into the oscillations themselves reveals the tendencies of decay as detailed in the introduction of the Q Factor.

2.2 Evaluated Metrics and Implementation

The metrics under investigation for this study include the Q Factors, the Q Factor uncertainties, uncertainties in angular conversions, and finally, uncertainties in time measurement conversions in frames per second. To evaluate the Q Factor, Q, the following equation is used, which itself is derivative of Christiaan Huygens geometric proof of the Tautochrone curve [3],

$$Q = \pi \frac{\tau}{T} \tag{1}$$

where τ is the friction (or rather viscosity) and T is the period of oscillation. The Q Factor can alternatively be calculated by counting the number of oscillations until the amplitude reaches some value $e^{-4} \sim 4\%$ of the initial amplitude. However due to the total time being 25 seconds and the percentage amplitude falling short of the 4%, the increment of 35% is chosen instead, with the condition being, $Q/3$ ($e^{-\pi/3} \sim 35\%$).

The Q Factor uncertainty is being defined by the highest quotient between the variables τ and T, where the uncertainty is distinguished with subscript u,

$$\frac{\tau_u}{T} \tag{2}$$

$$\frac{T_u}{T} \tag{3}$$

The following uncertainty for $\Delta\theta$ is being defined as the conversion from degrees to radian measures,

$$1^\circ = 0.0174533 \rightarrow 0.02 \text{ rad} \tag{4}$$

and the uncertainty for time, s, is being defined as the conversion from the frame rate of the footage to seconds,

$$\frac{1}{60} = 0.016667 \rightarrow 0.017 \text{ sec} \tag{5}$$

Defining the following metrics allows insight to better understand the graphs. It also puts emphasis into how well the line of fit for each graphs fits within the data points, these are evident in Figures 3 & 5.

3 Setup and Analysis

3.1 Boundary Conditions

The use of boundary conditions based on real-world environments enhances the overall applicability of the results stemming from the experiments. It is therefore important to keep the proposed models appropriate and accurate as to keep the paper consistent.

3.2 Quantitative Analysis

In order to obtain the most accurate results as possible from the setups described, it is important to have the subject in question reflect real world scales properly. And through the process of using video analysis software and consulting precision measuring tools including calliper rulers, multiple scales of reference were set. The dimensions of the setup were measured to be 0.236 m for the string length and being let go at a starting angle of $+45^\circ = \frac{\pi}{4}$ rad, as shown in Figure 4.

Using the setups depicted, the process of recording 25 seconds at a frame rate of 30 frames per second is enacted to acquire the necessary data and with the use of Adobe's Premiere Pro to center a point of pivot and OSP's Tracker tool to acquire the data, a higher degree of precision is generated and used. Engaging in the following actions increases the level of accuracy and even bypasses the problem of having to organize and plot the data by hand. Through these processes, a set with over 750 data points is developed for the Time v. Amplitude graph for 25 seconds as depicted in Figure 5.

From the plot, the following values of note are found and using Formulas 1, 2, and 3, the Q factor and its uncertainty are defined,

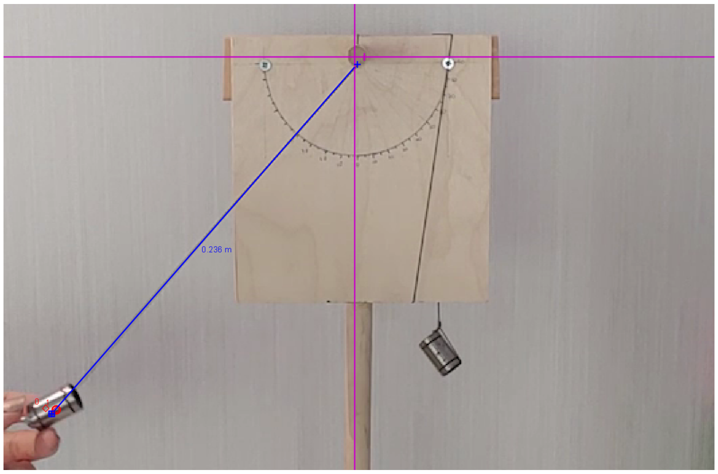

Figure 4: It should be noted that one end of the angular measure is positive and the other negative. The 90 degree mark is indicative of the center 0 degree

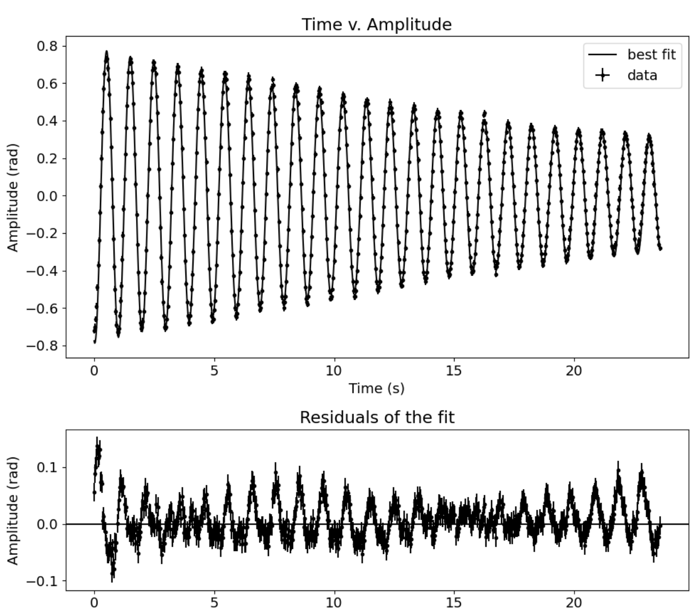

Figure 5: Time v. Amplitude graph with 750 data points in a 25 second span

$$T = 24.6 \pm 0.3$$
$$\tau = 0.98 \pm 0.0$$
$$\therefore Q = 78.4 \pm 0.01$$

Furthermore the Q factor can also be found using the alternative method,

$$25 \text{ oscillations} = Q/3 \ (e^{-\pi/3} \sim 35\%)$$
$$\therefore Q = 75$$

Comparing the two Q Factors, the Q Factor from Formula 1 is within the margin of uncertainty of 4.47% and 4.44%, which themselves are within a 0.74% margin, whereas the Q Factor from the alternative method defines itself with a factor of 75. This value falls within the margin of uncertainty of the Q Factor found with Formula 1, justifying the applicability of the alternative method.

The Q Factor being greater than 1 is essential in justifying the relation between the amplitude and periodicity of oscillation. Up to this point the graphs have failed to explain why there was damping present within the sinusoid model. As mentioned earlier, the tautochrone problem was studied by Huygens when he realized, during his testing, his pendulum's movements were not isochronous, causing his models to keep differing times depending on how far the pendulums swung. Furthermore his testing beared fruitful results as the investigations of his clocks presented decisive results. With each pendulum he built, the string caused friction at the point of pivot. Additionally the circular error probable of his pendulums decreased as the arc length of the swing decreased, invoking an oversight in the quality of his setups themselves. These discrepancies align with the results found within this study and the line of fit as seen on Figure 5 and gives confirmation that the pendulum follows the model of a damped harmonic oscillator, as with each oscillation there was a loss in amplitude and as the amount of time increases, further amplitudes continue to exhibit a decay from some unknown source (now of which have been concluded to be from the force of friction, the opposing force, and the circular error probables)[4].

3.3 Numerical Uncertainties and Asymmetry

Asymmetry is to be identified when the mass on the pendulum starts swaying away from the path of travel on its axis. This causes the data to have a discrepancy on the damped sinusoidal graph, as seen on Figure 5. Despite this complication however, the line of fit still communicates the necessary data to make a conclusion for this experiment.

Moving forward, it would be in the best interest of the author to preserve accuracy and to ensure low experimental and computational costs. The importance of examining potential errors and uncertainties during experimentation have been demonstrated across numerous organizations and journals, and the necessity of quantifying these errors and uncertainties have led to well-established methods that attempt to describe effects of resolution to that of an educated conclusion.

Despite this, however, uncertainties are inevitable when testing and gathering data and to avoid these potential impediments the use of higher precision tools should be repeatedly consulted. On top of using better measuring devices including precision callipers, high frame rate cameras, and digital rulers, the setup of the pendulum itself should be improved. This can be done with a high quality bearing acting as the pivot with a lighter string in which length can be adjusted. Utilizing the bearing would not only mitigate the friction (or viscosity) between the string and the pivot point entirely, but it would also lessen the amount of sway towards another axis when testing the periodicity and amplitude during longer periods of time. This is to avoid the issue of having a linear decay rather than an exponential one. Following these steps curtails the level of uncertainty to the physical minimum and superior boundaries can be set when casting the data unto plots and in turn finding the metrics for further investigation.

4 New Setup Analysis

4.1 Design Proposal and Model Scaling

Because it was in the best interest of the experimenters to build a better pendulum, a new setup was made. This was in the expectations that the new set of generated data would be as accurate as possible for the following analysis. Therefore, in order to obtain the most accurate results possible, a lubricated bearing was chosen to act as the pivot rather than a direct tied connection between the string and point of pivot, as shown in Figure 6. This was on the basis that the bearing would curtail the friction as much as possible whereas the previous setup had demonstrated a clear deterioration of energy with each oscillation.

Figure 6: The second mock-up pendulum to be used for this study. The mass is let go at a $\frac{\pi}{2}$ rad angle set at a string length of 0.3 m

To keep the data uniform with the previous method, real world scales had to again be reflected. Multiple scales of reference were again set with the dimensions of the initial length being 0.3 m with 0.025 m decrements for each measure and the starting angle was set to $90° = \frac{\pi}{2}$ rad, as shown in Figure 7.

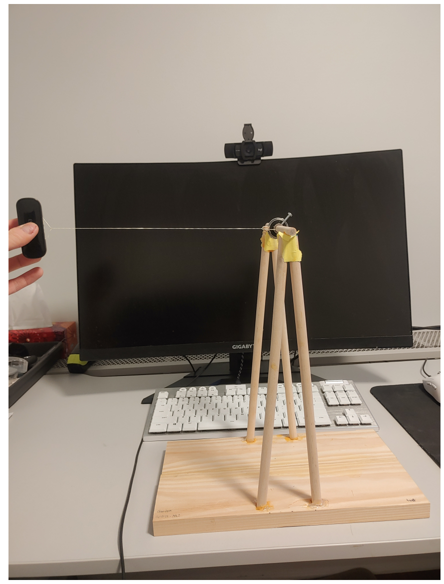

Figure 7: Similar to Figure 4, one end of the angular measure is positive and the other negative. The absolute minimum point of travel is indicative of the center 0 degree

4.2 Quantitative Analysis

With the new setup and its scales established, the process of recording 30 seconds at a frame rate of 30 frames per second for each pendulum length is conducted to acquire the necessary data and through this process a set with 12 data points is developed for the Length v. Period graph for 12 lengths as depicted in Figure 8. This data is to be interpreted as the experimental data.

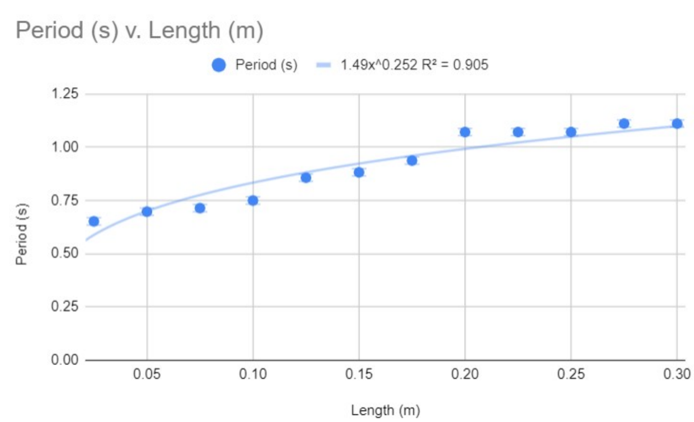

Figure 8: Length v. Period experimental graph with 12 periods over 12 varying lengths

Additionally another set is developed using the predicted power law function. This data is to be interpreted as the theoretical data,

$$T = kL^n \tag{6}$$

Where it is predicted that $k = 2$ and $n = 0.5$ within uncertainties, which in turn generates the following Length v. Period graph, as depicted in Figure 9.

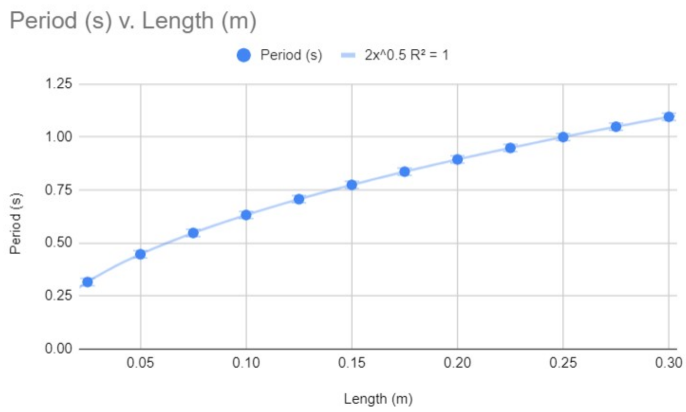

Figure 9: Length v. Period theoretical graph with 12 periods over 12 varying lengths

Analyzing the line of fit between the two graphs shows promising coefficients of determination (R^2 value) of 0.905 and 1 which are within a margin of difference of 9.97%. And while the line of fit was generated in accordance to Formula 6, the data generated a function that veered far from the predicted values of $k = 2$ and $n = 0.5$. Additionally these discrepancies are to be attributed to the new setup of the pendulum, specifically at the point of the pivot which is a metal ball bearing.

It should be noted however that discretion is advised at this point of the study as the data analysis process produces a plot with points that vary far from the expected points produced with Formula 6. Therefore such circumstance warrants further investigation into a possible case of a better model of fit.

Knowing that sin $\theta = \theta$ (as per the first term of the Taylor series for sin x at small angles), θ can be used to model the torque of the system,

$$\sum \tau = I \times \alpha \tag{7}$$

$$-mgLsin\ \theta = I\alpha \tag{8}$$

$$-mgL\theta = I\alpha \tag{9}$$

where $I = mL^2$, $\alpha = -\omega^2\theta$, m is the mass of the point mass, g is the gravitational acceleration constant, and L is the string length,

$$mgL\theta = mL^2\ddot{\theta} \tag{10}$$

$$\text{Note: } mL^2 = mL^2 + \frac{1}{2}MR^2$$

where $\ddot{\theta}$ is the second derivative of the angle with respect to time, which is angular acceleration,

$$\ddot{\theta} = -\frac{g}{L}\theta \tag{11}$$

which can be used to further derive for the angular velocity,

$$\omega = 2\pi\sqrt{\frac{L}{g}} \tag{12}$$

From here consult Formula 6 and reproduce a function for the period,

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{13}$$

$$\text{Note: } \frac{2\pi}{\sqrt{g}} \approx 2$$

Consult Formula 10 and derive for $\ddot{\theta}$ but with the inclusion of the two systems,

$$\ddot{\theta} = \frac{mgL}{mL^2 + \frac{1}{2}MR^2}\theta \tag{14}$$

where M is the mass of the bearing and R is the radius of the bearing. From here consult Formula 13 and rederive the period, T,

$$T = 2\pi\sqrt{\frac{mL^2 + \frac{1}{2}MR^2}{mgL}} \tag{15}$$

And despite this function generating a root plot in the format of $T = a\sqrt{bL + \frac{c}{L}}$), it can be further derived into a power law function,

$$T = a\sqrt{bL + \frac{c}{L}} = 2\pi\sqrt{\frac{mL^2 + \frac{1}{2}MR^2}{mgL}} \tag{16}$$

$$T^2 = \frac{2\pi^2(2mL^2 + MR^2)}{mgL} \tag{17}$$

$$T^2L = \frac{2\pi^2(2mL^2 + MR^2)}{mg} \tag{18}$$

With Formula 15 and Formula 18 derived, new boundary conditions and uncertainties were defined,

$$m = 0.02\ kg \pm 0.001$$
$$M = 0.01\ kg \pm 0.001$$
$$R = 0.01\ m \pm 0.01$$

Which generates a more fitting Length v. Period graph, as depicted in Figure 10.

Defining the following models allows insight to not only better understand the graphs but also to investigate a better model compared to Formula 6 which did not agree with the experimental data. Therefore for the case of the new setup where there were two systems including the bearing and the string, Formula 15 and Formula 18 satisfy the power and log-log function stipulation.

4.3 Relationship and Q Factor Uncertainty

The relationship between the Q factor and the pendulum length was investigated and the alternative method was used. This decision was made due to its applicability shown when

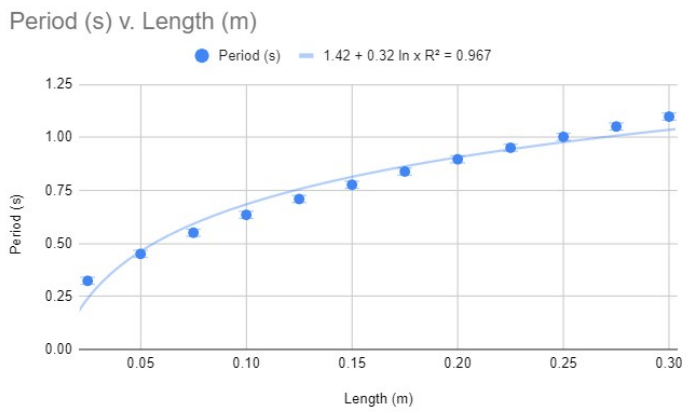

Figure 10: Length v. Period experimental graph with 12 periods over 12 varying lengths using Formula 15

comparing both methods in Section 3.2 and a following graph was generated. Furthermore using the alternative method completely removes the concern for a counting uncertainty as counting is the only method of measurement free from uncertainty, such that the count is an exact number.

As seen on Figure 11, the graph displays the relationship that as the length of the pendulum increases, so will the Q factor. This is indicative by the positive linear relationship seen on the graph and the inference can be made that the pendulum does have an impact on the Q factor.

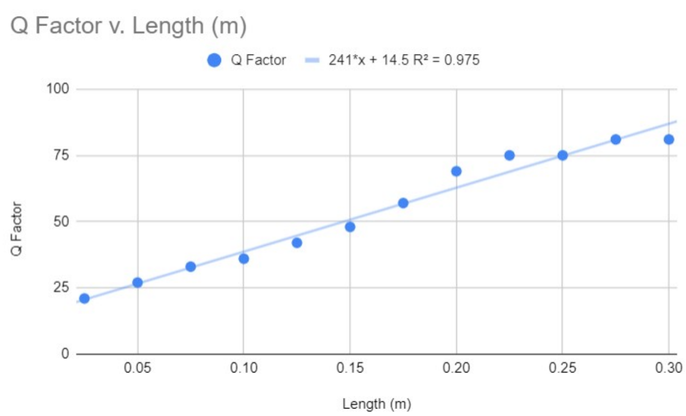

Figure 11: Length v. Q factor graph with 12 Q factors over 12 varying lengths

Additionally, while the inference is made that the pendulum length has an impact on the Q factor, consulting more length measurements may produce a differing graph, leading to the conclusion that while the length has an impact on the Q factor, the results are arbitrary, with Q factors having an initial linear increasing trend in some beginning interval but then generating arbitrary Q factors after aforementioned interval that will make the graph decrease. In the future, an investigation with more lengths would be required in order to plot out the full relationship between the length and the Q factor. Furthermore making a comparison with both methods of finding the Q factor would further solidify the applicability of the alternative method.

5 Summaries and Conclusions

This paper has offered a unique compendium of data providing insight into the characteristics of the damped harmonic oscillator model. As such, the results have indicated that the periodicity of the pendulum is not dependent on the amplitude. However it must not be understated that the range in amplitudes with each oscillation still have a delta in which independence is present, although minuscule (referring to the differences in peaks/valleys per oscillation as seen on Figure 5 which have a delta of $< 1\%$). Furthermore the Q Factors calculated by the two methods incurred a uncertainty margin of $< 1\%$ as well with the calculated Q Factors of 78.4 and 75. The Q Factors themselves being > 1 also demonstrated that there is viscosity within the system, with the most amount of friction being likely located at the point of pivot.

A major shortcoming to this study was the decision to not build a better pendulum setup. As noted earlier, a significant source of potential inaccuracy was when calculating for the Q Factors, and this affected the final investigation on whether the periodicity of the pendulum was affected by the amplitude. This oversight therefore blunts the overall applicability of the results found in this study to actual pendulum models. Furthermore the new setup generated its own set of complications with the requirement to account for the two-system setup. It is important to note, however, that the Q Factor being greater than 1 proves that there is a presence of decay (as emulated with Huygens' geometric proofs), as reflected on both the graphs and the relatively low Q Factor itself. Additionally, even though the Q Factor was as low as it was, comparisons done with other experimenters' results show that the outcomes from this study were strongly promising.

For future studies, it would be in the best interest of the author to, in the near-term, re-evaluate the current work without the hindrance of friction at the pivot and larger sample size in terms of the time. Work done using a computational approach should also be repeatedly completed to further gauge the effects and inaccuracies of finding the metrics in relation to time.

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Appendix A: Supplementary Material

Reading material on the Tautochrone curve:
`https://en.wikipedia.org/wiki/Tautochrone_curve`
Ecapements and how their mechanisms work:
`https://en.wikipedia.org/wiki/Escapement`
Damping and its effects on the damped harmonic oscillator:
`https://en.wikipedia.org/wiki/Damping`
Circular error probable and its appliance and relation from military ballistics to the pendulum:
`https://en.wikipedia.org/wiki/Circular_error_probable`
Harmonic oscillators and the properties of their motions within

a pendulum model:
https://en.wikipedia.org/wiki/Harmonic_oscillator#
Damped_harmonic_oscillator
Multicollinearity and group effects:
https://en.wikipedia.org/wiki/Multicollinearity#
Multicollinearity_and_group_effects
Log-log plot for Length v. Period
https://drive.google.com/file/d/
18lCWTgKv3M6iT9BYsnl8R2--Hy6J0imX/view?usp=sharing

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