MAT185 Linear Algebra
Assignment 4

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: JEON, Woohyun (Gordon)

Student number: 1008990451

Full Name: BOURDÉ, Vincent

Student number: 1009014291

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Gordon Jeon

2) Vincent Bourdé

1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(a) Find a matrix A such that $A\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}$.

Such that $A\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}$, A must be a 2×2 matrix. Consider the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_n + 1 \\ b_n + 1 \end{bmatrix}, \textbf{ s.t. } \begin{Bmatrix} a \cdot a_n + b \cdot b_n & = a_n + 1 \\ c \cdot a_n + d \cdot b_n & = b_n + 1 \end{Bmatrix}$$
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

Finding the eigenvalues:

$$\textbf{s.t. } A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$
$$C_A(\lambda) = \textbf{det}(\lambda I - A) = \textbf{det}(\begin{bmatrix} \lambda - 1 & -2 \\ -1 & \lambda - 1) \end{bmatrix})$$
$$= \lambda^2 - 2\lambda - 1$$
$$\lambda = 1 \pm \sqrt{2}$$
$$\therefore D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$$

Finding the eigenvectors:

$$\textbf{for } \lambda = 1 + \sqrt{2} \Rightarrow E_{1+\sqrt{2}}(A) = \textbf{null}\begin{bmatrix} \sqrt{2} & -2 \\ -1 & \sqrt{2} \end{bmatrix} = \textbf{span}\{\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}\}$$
$$\textbf{for } \lambda = 1 - \sqrt{2} \Rightarrow E_{1-\sqrt{2}}(A) = \textbf{null}\begin{bmatrix} -\sqrt{2} & -2 \\ -1 & -\sqrt{2} \end{bmatrix} = \textbf{span}\{\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}\}$$
$$\textbf{and s.t. } D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix},$$
$$\textbf{for } E_{1+\sqrt{2}}(A), v_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$
$$\textbf{for } E_{1-\sqrt{2}}(A), v_2 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$
$$\therefore S = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}$$

1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(c) Use your answer from part (b) to find explicit formulas for a_n and b_n, and then show that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$.

Consider $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$. If $\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$
\begin{aligned}
\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 7 \\ 5 \end{bmatrix} \Rightarrow AA \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\vdots \\
\begin{bmatrix} a_n \\ b_n \end{bmatrix} &= A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= (SDS^{-1})^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= SD^nS^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= (\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\begin{bmatrix} a_n \\ b_n \end{bmatrix} &= \begin{bmatrix} \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n + \sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n)}{2} \\ \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n + \sqrt{2}((1+\sqrt{2})^n + (1-\sqrt{2})^n)}{2\sqrt{2}} \end{bmatrix} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{(\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n}{(\sqrt{2}+1)(1+\sqrt{2})^n + (\sqrt{2}-1)(1-\sqrt{2})^n} \\
&= \lim_{n\to\infty} \frac{(\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n}{(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)(1-\sqrt{2})^n} \\
&= \lim_{n\to\infty} \frac{(\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n}{(\sqrt{2}+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{(\sqrt{2}+2)(1+\sqrt{2})^n}{(\sqrt{2}+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2}+2}{\sqrt{2}+1} \\
&= \lim_{n\to\infty} \frac{\sqrt{2}(1+\sqrt{2})}{\sqrt{2}+1} \\
&= \lim_{n\to\infty} \sqrt{2} \\
&= \sqrt{2}
\end{aligned}
$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$$

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(a) Prove that $\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank} A$.

Theorem IV of *10.4 Conditions of Diagonalizability* (Medici p. 256) states that for m_α, the geometric multiplicity of the eigenvalues of $A \in {}^n\mathbb{R}^n$ (i.e. $m_\alpha = \mathbf{dim}\ E_\alpha(A)$) we nave $m_1 + m_2 + ... + m_r \leq n$. Here,

$$\begin{aligned}\mathbf{dim}\ E_1(A) &= m_1\\ \mathbf{dim}\ E_2(A) &= m_2\\ \mathbf{dim}\ E_0(A) &= m_0\end{aligned}$$

By definition: $E_0(A) = \mathbf{null}(0I - A) = \mathbf{null}(-A)$. Furthermore by of the Rank-Nullity Theorem:

$$\begin{aligned}\mathbf{dim}\ \mathbf{null}(-A) &= n - \mathbf{rank}(-A)\\ \mathbf{dim}\ E_0(A) &= n - \mathbf{rank}(A)\\ \mathbf{rank}(A) &= n - \mathbf{dim}\ E_0(A)\\ \mathbf{rank}(A) &= n - m_0\end{aligned}$$

$$\begin{aligned}\mathbf{dim}\ \mathbf{null}(-A) &= \mathbf{dim}\ E_0(A),\\ \mathbf{rank}(-A) &= \mathbf{rank}(A)\\ \mathbf{dim}\ E_0(A) &= m_0\end{aligned}$$

from Theorem IV, $n \geq m_1 + m_2 + m_0 \Leftrightarrow n - m_0 \geq m_1 + m_2$.

$$\begin{aligned}\therefore \mathbf{rank}(A) &= n - m_0 \geq m_1 + m_2\\ \mathbf{rank}(A) &\geqslant \mathbf{dim}\ E_1(A) + \mathbf{dim}\ E_2(A)\end{aligned}$$

Therefore $\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank} A$ is proven.

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(b) Prove that if $\dim E_1(A) + \dim E_2(A) = \operatorname{rank} A$, then A is diagonalizable.

From (a), the corollary of Theorem IV states:

"If $A \in {}^n\mathbb{R}^n$ has distinct eigenvalues λ_α, $\alpha = 1, 2, ... \, r$ and $m_\alpha = \dim \, E_\alpha(A)$ (the geometric multiplicity), $\lambda = 1, 2, ... \, r$ such that $m_1 + m_2 + \ ... \ + m_r = n$, then A is diagonalizable.

Furthermore from (a), $\operatorname{rank}(A) = n - m_0$. And if $\operatorname{rank}(A) = \dim \, E_1(A) + \dim \, E_2(A) = m_1 + m_2$, then $n - m_0 = m_1 + m_2$.

$\Rightarrow n = m_1 + m_2 + m_0$

$\therefore m_1 + m_2 + m_0 = n$ for $A \in {}^n\mathbb{R}^n$ with eigenvalues, 0, 1, 2. Therefore by Theorem IV, A is diagonalizable.