MAT185 Linear Algebra
Assignment 2

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Gordon Jeon

2) Vincent Bourdé

1. Let W be the subspace of ${}^n\mathbb{R}^n$ defined by

$$W = \{A = [a_{ij}] \in {}^n\mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} = 0, \text{ for every } i, \text{ and } \sum_{i=1}^{n} a_{ij} = 0, \text{ for every } j\}$$

What is $\dim W$?

By the Fundamental Theorem of Linear Algebra (*Medici, p.136*) , dim W is the maximum number of linearly independent vector in W.

Consider the smallest $n \times n$ matrix (1), in W:

$$\begin{bmatrix} a & -a \\ -a & a \end{bmatrix}$$

Therefore any square $n \times n$ matrix containing only matrix (1) must be in W as it follows the condition that the rows and colums add up to 0.

Now consider the set of linearly independent vectors containing exclusively the entries in matrix (1):

$$\{w_1, w_2, w_3, \dots\ w_k\}$$

For this set to be linearly independent, the entries must not overlap, therefore matrix (1) needs to be in a different position in W.

Stating this however, the question may be raised: *"How many different positions of matrix (1) is an $n \times n$ square matrix?"*

Remembering that matrix (1) is a 2×2 matrix, matrix (1) can fit $n-1$ times in each row and $n-1$ times in each column. And because the matrix is a square $n \times n$ matrix, the total number of unique positions of matrix (1) is:

$$(n-1) \times (n-1) = (n-1)^2$$

Therefore, by definition, the maximum number of elements in the above set is $k = (n-1)^2$ which therefore proves dim $W = (n-1)^2$.

$$\therefore \text{ dim } W = (n-1)^2$$

2. Let X be any non-empty set and define $F(X) = \{f \mid f : X \to \mathbb{R}\}$. Define vector addition and scalar multiplication in $F(X)$ in the usual way:

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x), \quad c \in \mathbb{R}$$

for all $x \in X$. Then $F(X)$ is a vector space (cf. Medici, Section 4.2, page 105).

Let n be a positive integer. If $X = \{1, 2, 3, \ldots, n\}$, what is $\dim F(X)$?

Consider the definition for a vector space and how its defined by closure under vector addition and scalar multiplication. It's impossible to compare functions when evaluated at differing values of $x \in X$.

$\Rightarrow$ For all values of $x \in X$, there exists a subspace W in $F(X)$ such that:

$$W_i = \{f \mid f : x_i \to \mathbb{R}\}$$

Furthermore, if X has n elements, $F(X)$ must have $n \times W_i$ subspaces and the vector sum:

$$\sum_{i=1}^{n} W_i = F(X)$$

because each W_i has a corresponding x value. Therefore, all W_i are independent sets from each other.

$\Rightarrow$ Knowing $f : x_i \to \mathbb{R}$ and $g : x_i \to \mathbb{R}$, there must exist a scalar $c \in \mathbb{R}$ such that $g(x_i) = cf(x_i)$, as any real number is a scalar multiple of another.

$$\therefore \text{ For span}\{f\} = W_i \text{ and } f \neq \hat{0}, \ \dim W_i = 1.$$

$\Rightarrow$ Because $\dim W_i = 1$ for each W_i, there must exist 1 independent vector to span it. To span $F(X)$, the spanning set of W_i needs to exist. Therefore n independent vectors must also exist.

$$\therefore \dim F(X) = n \text{ , i.e. the number of integers in } X$$

3. Let U and W be subspaces of a vector space V. If $\dim U = k$, and $\dim W = l$, prove that $\dim (U + W) \leq k + l$. Under what condition would $\dim (U + W) = k + l$?

Consider the following:

$$\begin{aligned} \text{If dim } u &= k \\ U &= \text{span}\{u_1, u_2, ...u_k\} \\ \text{and if dim } w &= l \\ W &= \text{span}\{w_1, w_2, ...w_l\} \end{aligned}$$

Therefore $U + W$ must be defined as:

$$U + W = \{u + w \mid u \in U, w \in W\}$$

This also must mean $U + W$ can be defined as:

$$\text{span}\{u_1, u_2, ...\ u_k, w_1, w_2, ...\ w_l\}$$

This proves the following:

1) If fully linearly independent:
   $k + l$: independent vectors that span $\dim(u + k) = k + l$

2) If linearly dependent:
   Vectors can be removed and still span $\dim(u + w) < k + l$

$$\therefore\ \text{dim } (u + w) < k + l$$

4. Show that if

$$A = \sum_{i=1}^{k} \mathbf{x}_i \mathbf{y}_i^{\mathrm{T}}$$

for some $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in {}^m\mathbb{R}$, and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k \in {}^n\mathbb{R}$, then rank $A \leq k$.

By definition,

$$\begin{aligned} A &= x_1y_1^T + x_2y_2^T + x_3y_3^T + ... + x_ky_k^T \\ &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_2 & ... & y_k \end{bmatrix} \end{aligned}$$

The resulting matrix has dimension $m \times n$ and each column is a scalar multiple of x_i.

$$\therefore \ \text{rank}(x_iy_i^T) = 1$$

Furthermore, $x_iy_i^T$ is linearly independent from $x_{i+1}y_{i+1}^T$ if and only if $x_i \neq cx_{i+1}$.

Finding col A:

$$\begin{aligned} \text{col } A = Ax &= (x_1y_1^T + x_2y_2^T + x_3y_3^T + ... + x_ky_k^T)x \\ &= x_1y_1^T \cdot x + x_2y_2^T \cdot x + x_3y_3^T \cdot x + ... + x_ky_k^T \cdot x && \text{by matrix distribution} \\ &= \text{col } x_1y_1^T + \text{col } x_2y_2^T + \text{col } x_3y_3^T + ... + \text{col } x_ky_k^T && \text{by definition} \end{aligned}$$

$\therefore$ Col A is the set of the column spaces of the intermediate matrices. Knowing rank$(x_iy_i^T) = 1$ by the rank theorem, dim $cdx_iy_i^T = 1$. Therefore, because dim $x_iy_i^T = 1$, all the column spaces of the matrices $x_iy_i^T$, can be spanned by a single vector, v_i.

$$\begin{aligned} \text{col } A &= \text{span}\{v_1\} + \text{span}\{v_2\} + ... + \text{span}\{v_k\} \\ &= \text{span}\{v_1, v_2, v_2, ...\ v_k\} \end{aligned}$$

Consider the following:

$\Rightarrow$ If the set of vectors $v_1, v_2, ...\ v_k$ are linearly independent, then col A is spanned by k linearly independent vectors, so dim col $A = k$, and by the rank theorem, rank $A = k$

$\Rightarrow$ If the set of vectors $x_1,\ x_2,\ x_3, ...\ x_k$ are linearly dependent, then the vectors can be removed until the set is linearly independent but still span col A. The resulting set, has less than k vectors, therefore dim col $A < k$, and by the Rank Theorem, rank $A < k$

$$\therefore \ \text{rank } A \leq k$$