Logistic Map, Euler & Runge-Kutta Method and Lotka-Volterra Equations

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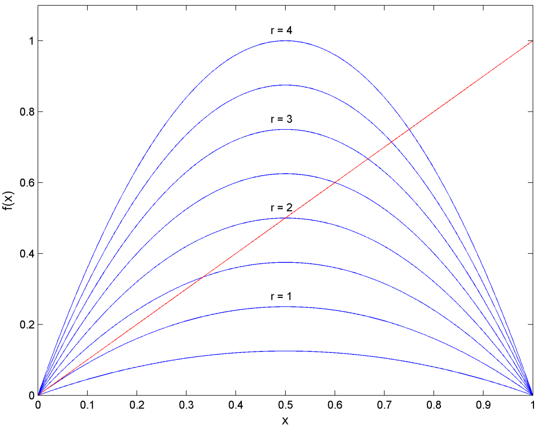
Contents

Logistic Map

Definition

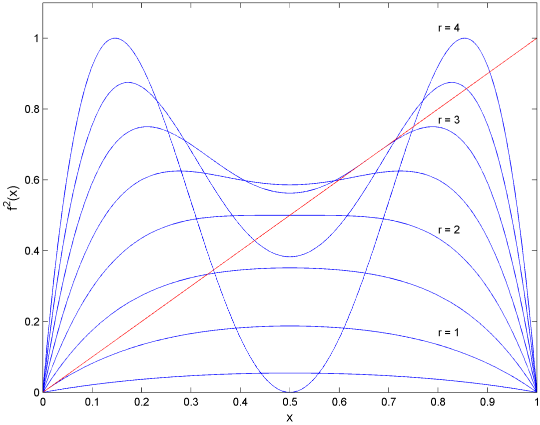
Let $f : [0, 1] \to [0, 1]$ such that $f(x) = rx(1 - x)$ where $r \in [0, 4]$. Then f is called a logistic map.

- Logistic Map $f(x) = rx(1-x)$



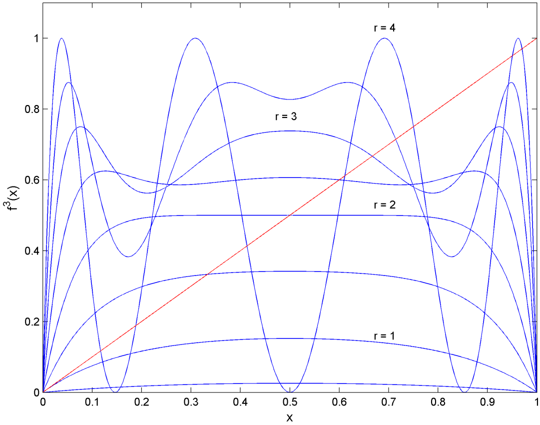
- Intersections are fixed points.

- Two iterations of logistic map $f^2(x)$



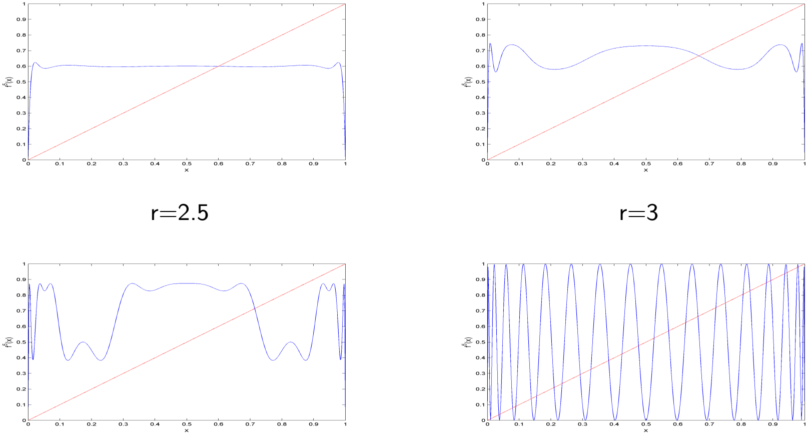
- Intersections are fixed points and period-2 points.

- Three iterations of logistic map $f^3(x)$



- Intersections are fixed points and period-3 points.

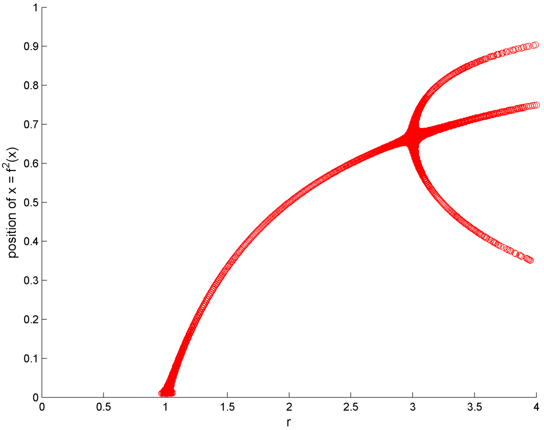
- Comparison of five iterations of logistic map $f^5(x)$ with repect to r



r=2.5 r=3

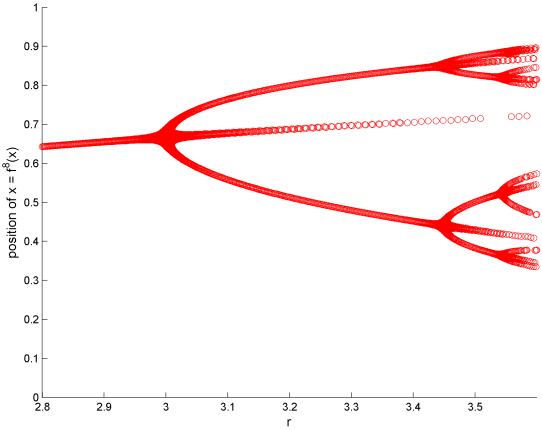
r=3.5 r=4

- Bifurcation of positions of fixed points and period-2 points



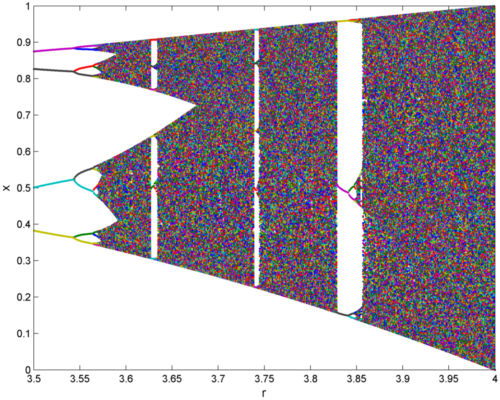
- The figure indicates positions of x satisfying $x = f^2(x)$ with respect to $0 \leq r \leq 4$.

- Doubling iterations



- The figure indicates positions of x satisfying $x = f^8(x)$ with respect to $2.8 \leq r \leq 3.6$.

- Let $x_0 = 0.5$ and $x_n = f^n(x_0)$. Plot positions of x_n while $n \in [500, 1500]$.



- The figure indicates positions that x_n wander with respect to $3.5 \le r \le 4$.

Euler and Runge-Kutta Method

- Let given ODE be $\dot{x} = f(x,t)$ and $\Delta t = h$.

$$\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{h}$$

- Explicit Euler Method

$$\begin{aligned} \frac{x_{n+1} - x_n}{h} &= f(x_n, t_n) \\ \Rightarrow \quad x_{n+1} &= x_n + hf(x_n, t_n), \quad t_{n+1} = t_n + h \end{aligned}$$

  h must be small.

- Implicit Euler Method

$$\begin{aligned} \frac{x_{n+1} - x_n}{h} &= f(x_{n+1}, t_{n+1}) \\ \Rightarrow \quad x_{n+1} &= g(x_n, t_n), \quad t_{n+1} = t_n + h \end{aligned}$$

  It is difficult to find out g.

- 4th Order Runge-Kutta Method

$$\begin{aligned} \frac{x_{n+1} - x_n}{h} &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ \Rightarrow \quad x_{n+1} &= x_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

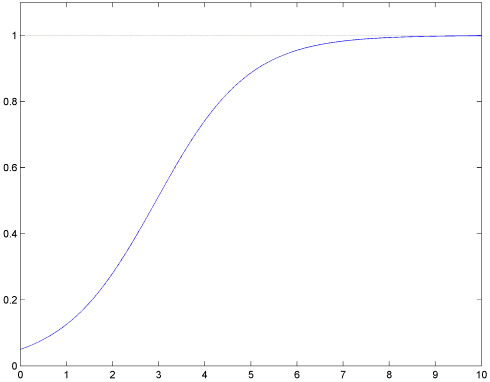
weighted average of k_1, k_2, k_3, k_4 where,

$$\begin{aligned} k1 &= f(x_n,\ t_n) \\ k2 &= f(x_n + \frac{1}{2}hk_1,\ t_n + \frac{1}{2}h) \\ k3 &= f(x_n + \frac{1}{2}hk_2,\ t_n + \frac{1}{2}h) \\ k4 &= f(x_n + hk_3,\ t_n + h) \\ t_{n+1} &= t_n + h \end{aligned}$$

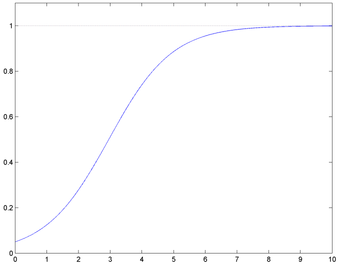
- Solve $\dot{x} = x(1-x)$.
- Exact solution is

$$x(t) = \frac{x_0 e^t}{1 + x_0(e^t - 1)}$$

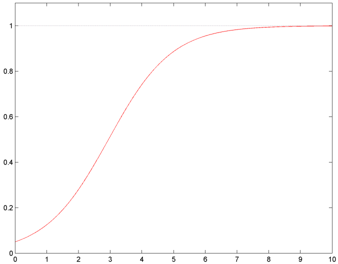
- Trajectory of $x(t)$ with $x_0 = 0.05$



- Let $x_0 = 0.05, h = 0.01.$
- Trajectories of $x(t)$ with Euler and Runge-Kutta methods

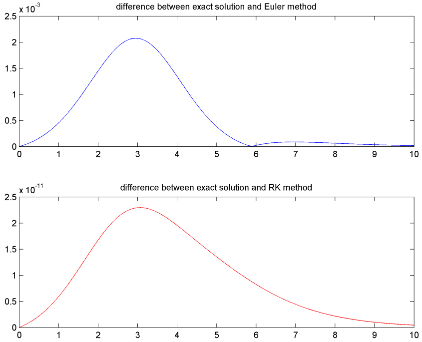


Euler method



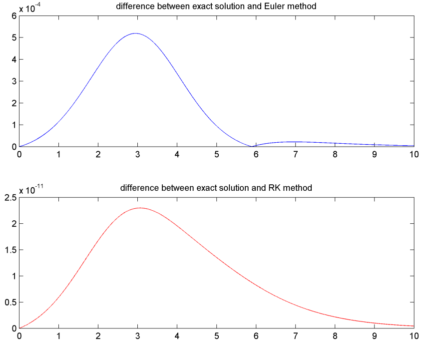
Runge-Kutta method

- Difference from exact solution



- Error of Euler method is less than 2.5×10^{-3}
- Error of Runge-Kutta method is less than 2.5×10^{-11}

- Using 4th order Runge-Kutta method needs 4 times more calculations than using Euler method.
- Let $h_1 = 0.01$ for Runge-Kutta method and $h_2 = 0.0025$ for Euler method in order to adjust the number of calculations.



- Error of Euler method is less than 6×10^{-4}
- Error of Runge-Kutta method is less than 2.5×10^{-11}

Runge-Kutta Method for System ODEs

- A system ODE

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \end{aligned}$$

- Runge-Kutta method for the system

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\ y_{n+1} &= y_n + \frac{1}{6}h(l_1 + 2l_2 + 2l_3 + l_4) \end{aligned}$$

where

$$
\begin{aligned}
k1 &= f(x_n,\ y_n,\ t_n), \\
l1 &= g(x_n,\ y_n,\ t_n), \\
k2 &= f(x_n + \frac{1}{2}hk_1,\ y_n + \frac{1}{2}hl_1,\ t_n + \frac{1}{2}h), \\
l2 &= g(x_n + \frac{1}{2}hk_1,\ y_n + \frac{1}{2}hl_1,\ t_n + \frac{1}{2}h), \\
k3 &= f(x_n + \frac{1}{2}hk_2,\ y_n + \frac{1}{2}hl_2,\ t_n + \frac{1}{2}h), \\
l3 &= g(x_n + \frac{1}{2}hk_2,\ y_n + \frac{1}{2}hl_2,\ t_n + \frac{1}{2}h), \\
k4 &= f(x_n + hk_3,\ y_n + hl_3,\ t_n + h), \\
l4 &= g(x_n + hk_3,\ y_n + hl_3,\ t_n + h), \\
t_{n+1} &= t_n + h.
\end{aligned}
$$

- Higher order ODE can be transformed into a system ODE

$$\ddot{x} + a\dot{x} + b = 0$$

Let $x_1 := x$, $x_2 := \dot{x}$ then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1 - b \end{aligned}$$

Lotka-Volterra Equations

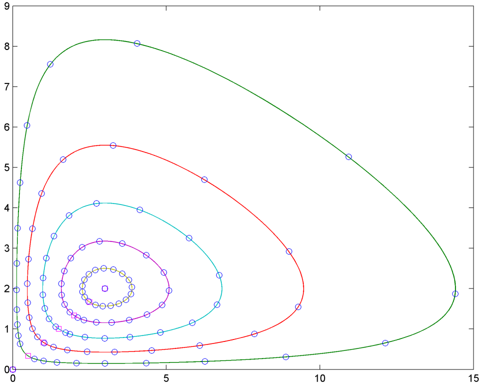
- Lotka-Volterra equation

$$\begin{aligned} \dot{x} &= x(a - by) \\ \dot{y} &= y(-c + dx) \end{aligned}$$

with positive a, b, c, d.

- We use 4th order Runge-Kutta method for calculations.

- Set $a = 4, b = 2, c = 3, d = 1$ and $h = 0.01$.

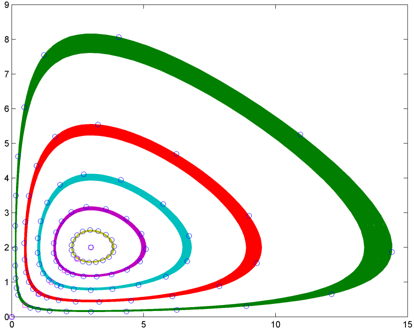


- We plot 10000 iterations starting at square shaped points.
- We plot 'o' shape at every 10 iterations.
- In this system, the phase follow the orbits counterclockwise.
- If the phase is far from the point $(3, 2)$, the phase moves fast.

- Lotka-Volterra equation with small perturbations.

$$\begin{aligned}\dot{x} &= x(a - \varepsilon_1 x - by)\\ \dot{y} &= y(-c + dx - \varepsilon_2 y)\end{aligned}$$

- Set $a = 4, b = 2, c = 3, d = 1, \varepsilon_1 = 0.0001, \varepsilon_2 = 0.0001$.

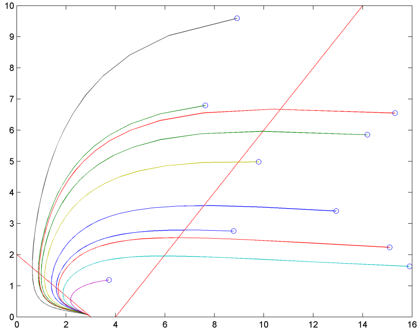


- In this system, the phase moves to $(3, 2)$ slowly.

- Lotka-Volterra equation with intraspecific competition.

$$\begin{aligned}\dot{x} &= x(a - ex - by) \\ \dot{y} &= y(-c + dx - fy)\end{aligned}$$

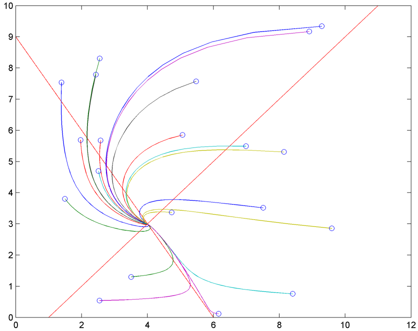
- Set $a = 6, b = 3, c = 4, d = 1, e = 2, f = 1$.



- Lotka-Volterra equation with intraspecific competition.

$$\begin{aligned} \dot{x} &= x(a - ex - by) \\ \dot{y} &= y(-c + dx - fy) \end{aligned}$$

- Set $a = 18, b = 2, c = 1, d = 1, e = 3, f = 1$.



Thank You