Notes on Cubic Splines

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Suppose we have a series of points $x_1, \ldots, x_n$ that we want to reach at specific times $t_1, \ldots, t_n$. We could link them all as a continuous function of time using a polynomial. This requires at least $n-1$ degrees of freedom. Then if we wish to dictate velocity and acceleration at certain times each imposition requires an additional degree for the polynomial. As you can imagine this becomes increasingly complex to compute.

A compromise is to assign a single polynomial between any two points x_i and x_{i+1}. If this polynomial has sufficient degrees of freedom we can ensure continuity at the velocity and acceleration levels. In totality we would have $n-1$ polynomials linking n points which forms a spline. Each individual polynomial can then be solved algorithmically.

We can assign a cubic polynomial as:

$$x_i(t) = a_i(t-t_i)^3 + b(t-t_i)^2 + c_i(t-t_i) + d_i, \tag{1a}$$
$$\dot{x}_i(t) = 3a_i(t-t_i)^2 + 2b_i(t-t_i) + c_i, \tag{1b}$$
$$\ddot{x}_i(t) = 6a_i(t-t_i) + 2b_i. \tag{1c}$$

Equation (1a) is in fact the solution to a minimum acceleration trajectory across the given time interval:

$$\min\ \frac{1}{2}\int_{t_i}^{t_{i+1}} \ddot{x}_i(t)^2\ dt.$$

The cubic polynomial has four coefficients, therefore we can dictate the start positions, start velocity, end position, and end velocity for each section of the spline. We sacrifice control over the acceleration to gain some computational efficiency. At the very least we can ensure the start and end velocities of the spline are zero.

From Eqn. (1a) we can constrain the starting position as:

$$x_i(t_i) = d_i = x_i. \tag{2}$$

Next we can either constrain Eqn. (1b) or (1c). [Angeles, 2003] chooses to solve for starting accelerations along the spline since Eqn. (1c) has the fewest coefficients. However, a better choice is to solve for velocity. The return of investment for the additional computations will become apparent later. From Eqn. (1b) the starting velocity is given by:

$$\dot{x}_i(t_i) = c_i = \dot{x}_i. \tag{3}$$

We want to ensure continuity at the velocity level so there is a smooth transition between different sections of the spline. Therefore, from Eqn. (1b) it must hold that:

$$\dot{x}_i(t_{i+1}) = \dot{x}_{i+1}(t_{i+1}). \tag{4a}$$

Then using Eqn. (3) yields:

$$3a_i\Delta t_i^2 + 2b_i\Delta t_i + c_i = c_{i+1} \tag{4b}$$

$$3a_i\Delta t_i^2 + 2b_i t_i + \dot{x}_i = \dot{x}_{i+1} \tag{4c}$$

where $\Delta t_i = t_{i+1} - t_i$. Then solving for a_i:

$$a_i = \frac{\dot{x}_{i+1} - \dot{x}_i - 2b_i\Delta t_i}{3\Delta t_i^2}. \tag{4d}$$

Continuity at the position level is an obvious necessity:

$$x_i(t_{i+1}) = x_{i+1}(t_{i+1}). \tag{5a}$$

Substituting Eqns. (2) and (3) in to Eqn. (1a) leads to:

$$a_i\Delta t_i^3 + b_i\Delta t_i^2 + c_i\Delta t_i + d_i = d_{i+1} \tag{5b}$$

$$\left(\dot{x}_{i+1} - \dot{x}_i - 2b_i\Delta t_i\right)\Delta t_i + 3b_i\Delta t_i + 3\dot{x}_i\Delta t_i + 3x_i = 3x_{i+1} \tag{5c}$$

$$\left(\dot{x}_{i+1} + 2\dot{x}_i\right)\Delta t_i + b_i\Delta t_i^2 + 3x_i = 3x_{i+1}. \tag{5d}$$

Then solving for b_i:

$$b_i = \frac{3\Delta x_i}{\Delta t_i^2} - \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i} \tag{5e}$$

where $\Delta x_i = x_{i+1} - x_i$. Now we can substitute Eqn. (5e) back in to Eqn. (4d) to get the solution for a_i:

$$a_i = \frac{\dot{x}_{i+1} - \dot{x}_i}{3\Delta t_i^2} - \frac{2}{3\Delta t_i}\left(\frac{3\Delta x_i}{\Delta t_i^2} - \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i}\right) \tag{6a}$$

$$= \frac{\dot{x}_{i+1} - \dot{x}_i}{3\Delta t_i^2} - \frac{2\Delta x_i}{\Delta t_i^3} + \frac{2\dot{x}_{i+1} + 4\dot{x}_i}{3\Delta t_i^2} \tag{6b}$$

$$= \frac{\dot{x}_{i+1} + \dot{x}_i}{\Delta t_i^2} - \frac{2\Delta x_i}{\Delta t_i^3}. \tag{6c}$$

In summary, the coefficients for the i^{th} cubic spline are:

$$\begin{aligned} a_i &= -\frac{2\Delta x_i}{\Delta t_i^3} + \frac{\dot{x}_{i+1} + \dot{x}_i}{\Delta t_i^2} \\ b_i &= \frac{3\Delta x_i}{\Delta t_i^2} - \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i} \\ c_i &= \dot{x}_i \\ d_i &= x_i. \end{aligned}$$

Notice that each of the coefficients is a function of position and velocity. Given $x_1, \ldots, x_n$ we want to solve for the corresponding velocities $\dot{x}_1, \ldots, \dot{x}_n$. We need to somehow eliminate the coefficients from Eqns. (1a) - (1c). To do this we can equate the accelerations between sections:

$$\ddot{x}_{i-1}(t_i) = \ddot{x}_i(t_i). \tag{7a}$$

Then using Eqns. (5e) and (6c) we have:

$$3a_{i-1}\Delta t_{i-1} + b_{i-1} = b_i \tag{7b}$$

$$\frac{3\left(\dot{x}_i + \dot{x}_{i-1}\right)}{\Delta t_{i-1}} - \frac{6\Delta x_{i-1}}{\Delta t_{i-1}^2} + \frac{3\Delta x_{i-1}}{\Delta t_{i-1}^2} - \frac{\dot{x}_i + 2\dot{x}_{i-1}}{\Delta t_{i-1}} = \frac{3\Delta x_i}{\Delta t_i^2} - \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i} \tag{7c}$$

$$\frac{2\dot{x}_i + \dot{x}_{i-1}}{\Delta t_{i-1}} - \frac{3\Delta x_{i-1}}{\Delta t_{i-1}^2} = \frac{3\Delta x_i}{\Delta t_i^2} - \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i}. \tag{7d}$$

Now we place the velocities on the left hand side and positions on the right hand side:

$$\frac{2\dot{x}_i + \dot{x}_{i-1}}{\Delta t_{i-1}} + \frac{\dot{x}_{i+1} + 2\dot{x}_i}{\Delta t_i} = \frac{3\Delta x_{i-1}}{\Delta t_{i-1}^2} + \frac{3\Delta x_i}{\Delta t_i^2} \tag{7e}$$

which we can arrange in to matrix/vector form:

$$\begin{bmatrix} \Delta t_{i-1}^{-1} & 2\left(\Delta t_{i-1}^{-1} + \Delta t_i^{-1}\right) & \Delta t_i^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}_{i-1} \\ \dot{x}_i \\ \dot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} 3\Delta t_{i-1}^{-2} & 3\Delta t_i^{-2} \end{bmatrix} \begin{bmatrix} \Delta x_{i-1} \\ \Delta x_i \end{bmatrix}. \tag{7f}$$

We could have expressed the velocities in terms of x instead of Δx, but this choice will make it easier to solve orientation-based trajectories later. There is also problem with Eqn. (7f). We have 1 equation and 3 unknowns. Or, in terms of our complete spline, $n - 2$ equations and n unknowns. This is indeterminable. We require 2 additional constraint equations. Because [Angeles, 2003] solves for acceleration, he chooses to set the start and end accelerations as zero. The implication is the start and end velocities are non-zero. This is a problem since if we wish to stop and start at rest (a reasonable requirement), then any object tracking the trajectory has to accelerate and brake infinitely hard. Fortunately with our approach we can set the start and end velocities to zero:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_{n-2} \\ \Delta x_{n-1} \end{bmatrix} \tag{8}$$

We can now solve all the velocities along the spline using linear algebra:

$$\mathbf{A}\begin{bmatrix}\dot{\mathrm{x}}_1\\ \vdots\\ \dot{\mathrm{x}}_\mathrm{n}\end{bmatrix} = \mathbf{B}\begin{bmatrix}\Delta\mathrm{x}_1\\ \vdots\\ \Delta\mathrm{x}_{\mathrm{n}-1}\end{bmatrix} \tag{9a}$$

$$\begin{bmatrix}\dot{\mathrm{x}}_1\\ \vdots\\ \dot{\mathrm{x}}_\mathrm{n}\end{bmatrix} = \mathbf{A}^{-1}\mathbf{B}\begin{bmatrix}\Delta\mathrm{x}_1\\ \vdots\\ \Delta\mathrm{x}_{\mathrm{n}-1}\end{bmatrix}. \tag{9b}$$

Then we can solve the coefficients for each section of the spline. Note that by using Eqns. (7f) and (8) that **B** is both rectangular and singular. However, it's not an issue since we don't need to invert it.

References

[Angeles, 2003] Angeles, J. (2003). *Fundamentals of robotic mechanical systems: theory, methods, and algorithms.* Springer.