Notes on Quintic Polynomials

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We want to generate a nice, smooth trajectory x(t) for a robot or vehicle to follow. In doing so we can reduce any hard acceleration and braking, thus reducing wear on the motors and joints. To do this we can optimize the jerk (derivative of acceleration) from the start to end of our trajectory:

$$\min \frac{1}{2}\int_{t_0}^{t_f} \dddot{x}^2 \, dt. \tag{1}$$

If the jerk is small, the acceleration will be smooth, as will the velocity and displacement.

To solve Eqn. (1) we can turn to the Euler-Poisson equation which gives the solution:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial \dddot{x}}\right) = 0. \tag{2}$$

where L is the Lagrangian. In our case we have $L = \frac{1}{2}\dddot{x}$ hence Eqn. (2) reduces to:

$$\frac{d^3}{dt^3}\left(\frac{\partial L}{\partial \dddot{x}}\right) = 0 \tag{3a}$$

$$\frac{d^6 x}{dt^6} = 0. \tag{3b}$$

That is, the 6^{th} time derivative of the trajectory must be zero. This means the trajectory takes the form of a 5^{t} order (quintic) polynomial:

$$x(t) = a(t - t_0)^5 + b(t - t_0)^4 + c(t - t_0)^3 + d(t - t_0)^2 + e(t - t_0) + f \tag{4a}$$

$$\dot{x}(t) = 5a(t - t_0)^4 + 4b(t - t_0)^3 + 3c(t - t_0)^2 + 2d(t - t_0) + e \tag{4b}$$

$$\ddot{x}(t) = 20a(t - t_0)^3 + 12b(t - t_0)^2 + 6c(t - t_0) + 2d. \tag{4c}$$

This polynomial has 6 coefficients and therefore we must impose 6 constraint equations to solve it.

At the beginning of the trajectory $t = t_0$ we know the starting position hence we can immediately solve:

$$x(t_0) = f = x_0. \tag{5a}$$

Likewise, for the velocity we have:

$$\dot{x}(t_0) = e = \dot{x}_0. \tag{5b}$$

We can similarly define the starting acceleration. For convenience we can choose it to be zero:

$$\ddot{x}(t_0) = 2d = \ddot{x}_0 \Longrightarrow d = \tfrac{1}{2}\ddot{x}_0 = 0. \tag{5c}$$

The final conditions require a bit more work. For $t_f - t_0 = \Delta t$ we can designate the position as:

$$x(t_f) = a\Delta t^5 + b\Delta t^4 + c\Delta t^3 + d\Delta t^2 + e\Delta t + f = x_f \tag{6a}$$

$$a\Delta t^5 + b\Delta t^4 + c\Delta t^3 + \dot{x}_0\Delta t + x_0 = x_f \tag{6b}$$

$$a\Delta t^2 + b\Delta t + c = \frac{x_0 - x_f}{\Delta t^3} - \frac{\dot{x}_0}{\Delta t^2} \tag{6c}$$

where the values for Eqns. (5a), (5b), and (5c) have been substituted in. Likewise, given the final velocity we can substitute and rearrange to produce:

$$\dot{x}(t_f) = 5a\Delta t^4 + 4b\Delta t^3 + 3c\Delta t^2 + 2d\Delta t + e = \dot{x}_f \tag{7a}$$

$$5a\Delta t^4 + 4b\Delta t^3 + 3c\Delta t^2 + \dot{x}_0 = \dot{x}_f \tag{7b}$$

$$5a\Delta t^2 + 4b\Delta t + 3c = \frac{\dot{x}_f - \dot{x}_0}{\Delta t^2}. \tag{7c}$$

And lastly, mostly for convenience, we can set the final acceleration as zero:

$$\ddot{x}(t_f) = 20a\Delta t^3 + 12b\Delta t^2 + 6c\Delta t + 2d = 0 \tag{8a}$$

$$20a\Delta t^2 + 12\Delta t + 6c = 0. \tag{8b}$$

Putting Eqns. (6c), (7c), and (8b) together in matrix form we have:

$$\begin{bmatrix} \Delta t^2 & \Delta t & 1 \\ 5\Delta t^2 & 4\Delta t & 3 \\ 20\Delta t^2 & 12\Delta t & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \Delta t^{-3}\left(x_f - x_0\right) - \Delta t^{-2}\dot{x}_0 \\ \Delta t^{-2}\left(\dot{x}_f - \dot{x}_0\right) \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} \tag{9a}$$

where the substitutions y_1 and y_2 are denoted for future ease of use. This system of equations is small enough to apply Gaussian elimination to solve by hand.

To start, the last row minus 4 times the second row yields:

$$\begin{bmatrix} \Delta t^2 & \Delta t & 1 \\ 5\Delta t^2 & 4\Delta t & 3 \\ 0 & -4\Delta t & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ -4y_2 \end{bmatrix}. \tag{9b}$$

Then the second row minus 5 times the first row produces:

$$\begin{bmatrix} \Delta t^2 & \Delta t & 1 \\ 0 & -\Delta t & -2 \\ 0 & -4\Delta t & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - 5y_1 \\ -4y_2 \end{bmatrix}. \tag{9c}$$

And finally, we can take the last row and subtract 4 times the second row:

$$\begin{bmatrix} \Delta t^2 & \Delta t & 1 \\ 0 & -\Delta t & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - 5y_1 \\ -4y_2 - 4(y_2 - 5y_1) \end{bmatrix}. \tag{9d}$$

The matrix on the left-hand side is now upper-triangular, therefore we can solve the remaining coefficients recursively like so:

$$c = 10y_1 - 4y_2 \tag{10a}$$

$$b = \frac{7y_2 - 15y_1}{\Delta t} \tag{10b}$$

$$a = \frac{y_1 - \Delta t b - c}{\Delta t^2} = \frac{6y_1 - 3y_2}{\Delta t^2}. \tag{10c}$$

It is sufficient to use the substitutions of y_1 and y_2 to make the calculations easier:

$$y_1 = \frac{x_f - x_0}{\Delta t^3} - \frac{\dot{x}_0}{\Delta t^2},$$

$$y_2 = \frac{\dot{x}_f - \dot{x}_0}{\Delta t^2}.$$

But by substituting these back in to Eqns. (10a), (10b), and (10c), a complete solution can be derived as:

$$a = \frac{6\,(x_f - x_0)}{\Delta t^5} - \frac{3\,(\dot{x}_f - \dot{x}_0)}{\Delta t^4}$$

$$b = -\frac{15\,(x_f - x_0)}{\Delta t^4} + \frac{4\dot{x}_f + 8\dot{x}_0}{\Delta t^3}$$

$$c = \frac{10\,(x_f - x_0)}{\Delta t^3} - \frac{4\dot{x}_f + 6\dot{x}_0}{\Delta t^2}$$

$$d = 0$$

$$e = \dot{x}_0$$

$$f = x_0.$$

Thus we have a solution for the trajectory coefficients in terms of $x_0,\ \dot{x}_0,\ x_f,\ \dot{x}_f,\ t_0$ and t_f.

Note that if $\dot{x}_0 = \dot{x}_f = 0$ then the coefficients a, b and c reduce to the same values deduced by [Angeles, 2003]. This is ideal if the object tracking the trajectory starts and ends at rest. However, if we wish to pass through multiple waypoints, we can equate the non-zero velocities between the end of one trajectory and the start of the next. This ensures continuity of motion and a smooth transition between them.

References

[Angeles, 2003] Angeles, J. (2003). *Fundamentals of robotic mechanical systems: theory, methods, and algorithms.* Springer.