Orientation Control With Angle-Axis Representation

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Abstract: In this article I provide some basic definitions and proofs of identities for rotation matrices $\mathbf{R} \in \mathbb{SO}(3)$. I show that a rotation matrix can be represented as a matrix exponential. From this, Rodrigues' formula follows which expresses the matrix in terms of the angle and axis of rotation. I then show how to reverse this formula to obtain the angle and axis from an arbitrary rotation matrix. Then using the exponential form, and the angle-axis, I derive a control law for the angular velocity to perform feedback control on orientation error.

1. Euler's Rotation Theorem

Euler's rotation theorem states that any change in orientation of a rigid body can be described by:

- A single rotation α (rad),
- About an axis $\hat{\mathbf{a}} \in \mathbb{R}^3$

where $\hat{\mathbf{a}}$ is a unit vector such that $\|\hat{\mathbf{a}}\|^2 = \hat{\mathbf{a}}^{\mathrm{T}}\hat{\mathbf{a}} = 1$. So the 3 combined rotations in Fig. 1 can be to a single rotation about a single axis.

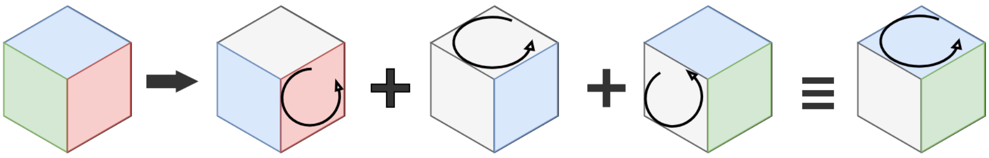

Figure 1: Any change in the orientation of a rigid body can be described by a single rotation about a single axis.

Any transformation of a vector $\mathbf{v} \in \mathbb{R}^n \rightarrow \mathbf{u} \in \mathbb{R}^n$ that preserves its length can be expressed with a product involving a rotation matrix:

$$\mathbf{u} = \mathbf{R}\mathbf{v}. \tag{1}$$

This matrix belongs to the Special Orthogonal group:

$$\mathbb{SO}(n) = \left\{ \mathbf{R} \in \mathbb{R}^{n \times n} \;\middle|\; \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}\,,\ \det(\mathbf{R}) = 1 \right\} \tag{2}$$

Given an arbitrary rotation matrix $\mathbf{R} \in \mathbb{SO}(3)$ we may be interested in finding the angle and axis of rotation. To do this, we need to define some other properties of $\mathbb{SO}(3)$ that we can exploit.

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2. Time Derivative & Exponential

If we take the time derivative of Eqn. (1), and assuming $\dot{\mathbf{v}} = \mathbf{0}$, then we arrive at:

$$\dot{\mathbf{u}} = \dot{\mathbf{R}}\mathbf{v}. \tag{3}$$

But in 3D, the time derivative of a vector is given by the cross product with the instantaneous angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ (rad/s):

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \times \mathbf{u} = \underbrace{S(\boldsymbol{\omega})}_{\dot{\mathbf{R}}} \overbrace{\mathbf{R}\mathbf{v}}^{\mathbf{u}} \tag{4}$$

where $S(\cdot)$ is the skew-symmetric matrix operator:

$$S(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_y & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \in \mathfrak{so}(3). \tag{5}$$

This is also the Lie algebra of $\mathbb{SO}(3)$. By equating Eqn. (3) with Eqn. (4), and substituting in Eqn. (1) we can see that the time derivative of the rotation matrix is "proportional" to itself:

$$\dot{\mathbf{R}} = S(\boldsymbol{\omega})\mathbf{R} \implies \mathbf{R}(t) = e^{S(\boldsymbol{\omega})t}\mathbf{R}(0) \tag{6}$$

This is a first-order differential equation whose solution is a (matrix) exponential. But the integral of the angular velocity is simply the angle-axis vector at any given point in time:

$$\int_0^t \boldsymbol{\omega}\ dt = \boldsymbol{\omega}t + \cancelto{0}{const.} = \alpha \cdot \hat{\mathbf{a}} = \mathbf{a}. \tag{7}$$

Assuming we start from zero rotation ($\mathbf{R}(0) = \mathbf{I}$), then the rotation matrix is equivalent to a matrix exponential containing the angle-axis:

$$\mathbf{R} = e^{S(\mathbf{a})} \in \mathbb{SO}(3). \tag{8}$$

From the definition of the exponential:

$$e^{S(\mathbf{a})} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} S(\hat{\mathbf{a}})^k \tag{9}$$

we can reduce Eqn. (8) to Rodrigues' formula which features the angle and axis as separate parameters:

$$\mathbf{R}(\alpha, \hat{\mathbf{a}}) = \mathbf{I} + \sin(\alpha)S(\hat{\mathbf{a}}) + (1 - \cos(\alpha))S(\hat{\mathbf{a}})^2. \tag{10}$$

3. Angle & Axis from Rotation Matrix

Rodrigues' formula, Eqn. (10), contains 3 matrices with a particular structure to their respective diagonal elements. If we take the trace[1] we can see that:

- $\mathrm{trace}(\mathbf{I}) = 3$,
- $\mathrm{trace}\left(S(\hat{\mathbf{a}})\right) = 0$, and
- $\mathrm{trace}\left(S(\hat{\mathbf{a}})\right)^2 = -2$ since $\|\hat{\mathbf{a}}\| = 1$.

Hence the trace of a rotation matrix must be:

$$\mathrm{trace}(\mathbf{R}) = 3 - 2 \cdot (1 - \cos(\alpha)) \tag{11a}$$
$$= 1 + 2 \cdot \cos(\alpha). \tag{11b}$$

[1]Sum of diagonal elements

We can re-arrange this to solve for the angle of rotation:

$$\alpha = \cos^{-1}\left(\frac{\mathtt{trace}(\mathbf{R}) - 1}{2}\right). \tag{12}$$

If the angle of rotation is zero $\alpha = 0$, then the axis of rotation is arbitrary since $0 \cdot \hat{\mathbf{a}} = \mathbf{0}$.

The axis for a rotation matrix does not change $\mathbf{R}\hat{\mathbf{a}} = \hat{\mathbf{a}}$. This implies that it is an eigenvector whose corresponding eigenvalue $\lambda = 1$.[2] For any arbitrary eigenvector of $\mathbf{R}$ it must hold that:

$$\mathbf{R}\mathbf{v} = \mathbf{v}. \tag{13}$$

Multiplying this by the transpose of the rotation yields:

$$\overbrace{\mathbf{R}^\top\mathbf{R}}^{\mathbf{I}}\,\mathbf{v} = \mathbf{R}^\top\mathbf{v} \tag{14a}$$

$$\mathbf{v} = \mathbf{R}^\top\mathbf{v}. \tag{14b}$$

Equating Eqn. (13) and Eqn. (14) we obtain:

$$\mathbf{R}\mathbf{v} = \mathbf{R}^\top\mathbf{v} \tag{15a}$$

$$\underbrace{\left(\mathbf{R} - \mathbf{R}^\top\right)}_{S(\mathbf{v})}\mathbf{v} = \mathbf{0}. \tag{15b}$$

The matrix $\mathbf{R} - \mathbf{R}^\top$ must be skew-symmetric since $\mathbf{v} \times \mathbf{v} = S(\mathbf{v})\mathbf{v} = \mathbf{0}$. Expanding this we have:

$$\mathbf{R} - \mathbf{R}^\top = \begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix}. \tag{16}$$

Using what we know about the structure of skew-symmetric matrices, Eqn. (5), we can deduce that the eigenvector is:

$$\mathbf{v} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}. \tag{17}$$

We can then normalise this vector to obtain the axis of rotation $\hat{\mathbf{a}}$:

$$\hat{\mathbf{a}} = \begin{cases} \frac{\mathbf{v}}{\|\mathbf{v}\|} & \text{if } \alpha \neq 0 \\ \text{trivial} & \text{otherwise.} \end{cases} \tag{18}$$

Note that if $\mathbf{R} = \mathbf{I}$ (i.e. no rotation), then $\mathbf{v} = \mathbf{0}$ and hence $\nexists\|\mathbf{v}\|^{-1}$. In this case, we can assign any arbitrary value to the axis of rotation.

4. Orientation Feedback

We can use the angle-axis vector to perform feedback on the orientation of an automated system. Suppose $\mathbf{R}_d \in \mathbb{SO}(3)$ is the desired orientation, and $\mathbf{R} \in \mathbb{SO}(3)$ is our actual orientation. We can define our orientation error as:

$$\mathbf{E} \triangleq \mathbf{R}_d\mathbf{R}^\top = e^{S(\boldsymbol{\epsilon})}. \tag{19}$$

If $\mathbf{R} = \mathbf{R}_d$ then $\mathbf{E} = \mathbf{I}$, implying no difference between orientations. From Eqn. (6) the time derivative of our rotation error is:

$$\dot{\mathbf{E}} = S(\dot{\boldsymbol{\epsilon}})\mathbf{E}\,,\ \ \dot{\boldsymbol{\epsilon}} = \boldsymbol{\omega}_d - \boldsymbol{\omega}. \tag{20}$$

where:

- $\boldsymbol{\omega}_d \in \mathbb{R}^3$ is the desired angular velocity (rad/s),and

[2] For any arbitrary matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ the eigenvector $\mathbf{v} \in \mathbb{C}^m$ and eigenvalue $\lambda \in \mathbb{C}$ obey the identity $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

- $\boldsymbol{\omega} \in \mathbb{R}^3$ is the actual angular velocity (rad/s).

Assuming $\boldsymbol{\omega}$ is our control input, we can define the control law:

$$\boldsymbol{\omega} \triangleq \boldsymbol{\omega}_d + \mathbf{K}\boldsymbol{\epsilon} \tag{21}$$

where $\mathbf{K} \in \mathbb{R}^{3\times 3}$ is a positive-definite gain matrix.[3] The desired angular velocity $\boldsymbol{\omega}_d$ becomes a feed-forward term, whereas $\mathbf{K}\boldsymbol{\epsilon}$ is a proportional feedback on the orientation error.[4]

If we substitute Eqn. (21) in to Eqn. (20) we obtain:

$$\dot{\boldsymbol{\epsilon}} = -\mathbf{K}\boldsymbol{\epsilon} \implies \boldsymbol{\epsilon}(t) = e^{-\mathbf{K}t}\boldsymbol{\epsilon}(0). \tag{22}$$

This form implies exponential decay. As the error angle approaches zero $\boldsymbol{\epsilon} \to \mathbf{0}$ then the orientation error will approach the identity $\mathbf{E} \to \mathbf{I}$ such that $\mathbf{R} \to \mathbf{R}_d$.[5]

Figure 2 shows the ergoCub rotating a box using this method of orientation control.

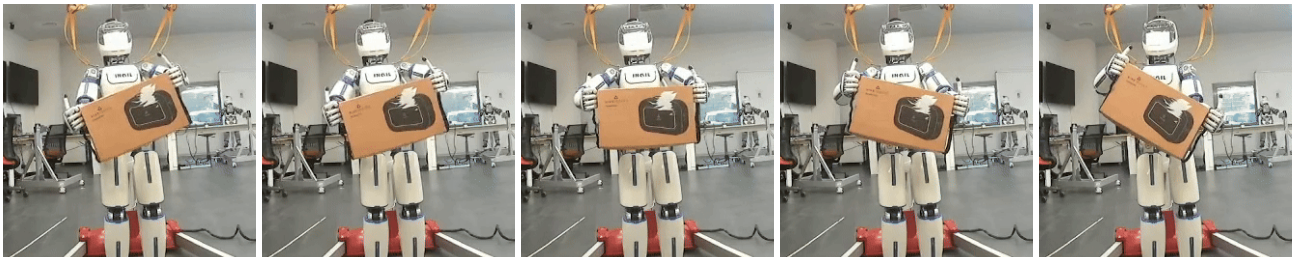

Figure 2: Orientation feedback control with angle-axis representation was used to control an object being held by the ergoCub robot.

[3] An easy choice here is a diagonal matrix with positive values.
[4] In such cases where $\boldsymbol{\omega}_d$ is unavailable, then $\boldsymbol{\omega} = \mathbf{K}\boldsymbol{\epsilon}$ is sufficient.
[5] This follows from the fact that $e^0 = 1$.