QUANTUM OPERATIONS

REPORT

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ABSTRACT

The formalism offered by quantum operations allows easy handling of *open* quantum systems, which are those that interact with the environment. This formalism takes care of a variety of cases: coupled unitary evolution of the environment + system, interactions in which measurements are involved.

1 Basic cases

We denote the transformation of our open quantum system in terms of the density operator representation:

$$\rho' = \mathcal{E}(\rho)$$

For the simple case of unitary evolution, this map is simply: $\rho' = U\rho U^\dagger$.

Now, consider, $\{M_m\}$, a set of measurement operators ($\implies \sum_m M_m^\dagger M_m = I$). If a measurement of our system $\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$ results in the outcome m, the new state will be:

$$\begin{aligned}
\rho' &= \sum_i \frac{p(i|m) M_m \left|\psi_i\right\rangle \left\langle\psi_i\right| M_m^\dagger}{\left(\sqrt{\left\langle\psi_i\right| M_m^\dagger M_m \left|\psi_i\right\rangle}\right)^2} \\
&= \sum_i \frac{p(m|i)p(i) M_m \left|\psi_i\right\rangle \left\langle\psi_i\right| M_m^\dagger}{p(m) \left\langle\psi_i\right| M_m^\dagger M_m \left|\psi_i\right\rangle} \\
&\text{(since: } p(m|i) = \left\langle\psi_i\right| M_m^\dagger M_m \left|\psi_i\right\rangle) \\
&= \frac{M_m \rho M_m^\dagger}{p(m)}
\end{aligned}$$

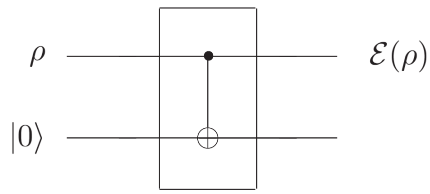
Moreover,

$$\begin{aligned}
p(m) &= \sum_i p(m|i)p(i) = \sum_i \left\langle\psi_i\right| M_m^\dagger M_m \left|\psi_i\right\rangle p_i \\
&= \sum_i \mathrm{tr}\big(M_m \left|\psi_i\right\rangle \left\langle\psi_i\right| M_m^\dagger\big) p_i = \mathrm{tr}\big(M_m \rho M_m^\dagger\big) \\
\implies \mathcal{E}(\rho) &= \frac{M_m \rho M_m^\dagger}{\mathrm{tr}\Big(M_m \rho M_m^\dagger\Big)}
\end{aligned}$$

Moving ahead, we introduce an environment, along with which the system forms a *closed* joint system. **An assumption** for now: Before the coupled evolution, the joint system is in a product state: $\rho \otimes \rho_{\text{env}}$. Hence:

$$\mathcal{E}(\rho) = \mathrm{tr}_{\text{env}}\left(U(\rho \otimes \rho_{\text{env}})U^\dagger\right)$$

Consider the example in the figure below. In this case, we begin with $\rho \otimes \left|0\right\rangle \left\langle 0\right|$. The controlled-NOT gate, serves as the unitary $U = \left|00\right\rangle \left\langle 00\right| + \left|01\right\rangle \left\langle 01\right| + \left|10\right\rangle \left\langle 11\right| + \left|11\right\rangle \left\langle 10\right| = P_0 \otimes I + P_1 \otimes X$,



where P_0, P_1 are the projection operators for the system qubit.

$$
\begin{aligned}
\implies U(\rho \otimes |0\rangle\langle 0|)U^\dagger &= P_0\rho P_0 \otimes |0\rangle\langle 0| + P_0\rho P_0 \otimes |1\rangle\langle 1| \\
\implies \rho' &= \mathrm{tr}_{\mathrm{env}}(P_0\rho P_0 \otimes |0\rangle\langle 0| + P_0\rho P_0 \otimes |1\rangle\langle 1|) \\
&= P_0\rho P_0 + P_1\rho P_1
\end{aligned}
$$

2 Operator-sum Representation

The transformations we have seen till now all have a common structure which arises because:

$$
\begin{aligned}
&\mathrm{tr}_{\mathrm{env}}(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger) \\
&= \sum_k (I \otimes \langle e_k|)(U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger)(I \otimes |e_k\rangle)
\end{aligned}
$$

Writing

$$
(I \otimes \langle e_k|)U|\psi e_0\rangle = E_k|\psi\rangle
$$

we end up getting:

$$
\begin{aligned}
\mathcal{E}(\rho) &= \sum_i p_i\, \mathrm{tr}_{\mathrm{env}}(U(|\psi_i e_0\rangle\langle\psi_i e_0|)U^\dagger) \\
&= \sum_i p_i \sum_k (I \otimes \langle e_k|)U(|\psi_i e_0\rangle\langle\psi_i e_0|)U^\dagger(I \otimes |e_k\rangle) \\
&= \sum_i p_i \sum_k E_k|\psi_i\rangle\langle\psi_i|E_k^\dagger = \sum_k E_k \sum_i (p_i|\psi_i\rangle\langle\psi_i|)E_k^\dagger \\
\mathcal{E}(\rho) &= \sum_k E_k\rho E_k^\dagger
\end{aligned}
$$

These operators $\{E_k\}$ which define $\mathcal{E}$ are known as its operation elements. Requiring $\mathcal{E}(\rho)$ to be a valid density operator for *all* $\rho \implies \mathrm{tr}\Big(\sum_k E_k^\dagger E_k\rho\Big) = 1\ \forall\rho$

Hence,

$$
\sum_k E_k^\dagger E_k = I
$$

Introducing measurements