Exercise n. 1

Consider the two following investments

$$Z_1 = \begin{cases} 20 & \text{with probability} \quad 1/4 \\ 10 & \text{with probability} \quad 1/2 \\ -10 & \text{with probability} \quad 1/4 \end{cases}$$

and Z_2 described by a continuous uniform distribution in the interval $(-10, 25)$. Determine which one of the two investments is preferred by an investor with an initial wealth $Y = 100$, according to the mean-variance principle and the expected utility criterion by considering the following Von Neumann-Morgestern utility functions:

a) $u(x) = x - 0.001x^2$;

b) $u(x) = 100(1 - e^{-x/100})$;

c) $u(x) = 10\log(x) + 5$.

1. **Mean-Variance principle**

   Define, at first, the random variables $X_1 = Y + Z_1$, and $X_2 = Y + Z_2$. We have

$$X_1 = \begin{cases} 120 & \text{with probability} \quad 1/4 \\ 110 & \text{with probability} \quad 1/2 \\ 90 & \text{with probability} \quad 1/4 \end{cases}$$

   while X_2 is a uniform random variable with support in the interval $(90, 125)$, i.e., $X_2 \sim U(90, 125)$.

$$E[X_1] = 120\frac{1}{4} + 110\frac{1}{2} + 90\frac{1}{4} = 107.5.$$

   Recalling that $\text{Var}[X_1] = E[X_1^2] - E^2[X_1]$,

$$E[X_1^2] = 120^2\frac{1}{4} + 110^2\frac{1}{2} + 90^2\frac{1}{4} = 11675,$$

   $\text{Var}[X_1] = 118.75$.

   We recall that the density function of a continuous uniform random variable, X, with support in the interval (a, b) is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in (a, b), \\ 0 & \text{elsewhere} \end{cases}$$

   Hence,

$$E[X] = \int_a^b x f_X(x)dx = \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\int_a^b xdx = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}.$$

$$E[X^2] = \int_a^b x^2 f_X(x)dx = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left|\frac{x^3}{3}\right|_a^b = \frac{b^3-a^3}{3(b-a)} =$$

$$= \frac{(b-a)(a^2+ab+b^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}.$$

Hence,

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12}.$$

In the considered case, $E[X_2] = \frac{90+125}{2} = 107.5$ and $\mathrm{Var}[X_2] = \frac{(125-90)^2}{12} = 102.0833$. To conclude, $E[X_1] = E[X_2]$, and $\mathrm{Var}[X_1] > \mathrm{Var}[X_2]$. Hence, $X_2 \succ X_1$ according to the mean-variance principle.

2. **Expected utility principle**

Let us consider, at first, the quadratic utility function. We have

$$E[u(X_1)] = u(120)\frac{1}{4} + u(110)\frac{1}{2} + u(90)\frac{1}{4} =$$

$$= (120 - 0.001 \cdot 120^2)\frac{1}{4} + (110 - 0.001 \cdot 110^2)\frac{1}{2} + (90 - 0.001 \cdot 90^2)\frac{1}{4} = 95.825.$$

$$E[u(X_2)] = \int_a^b u(x) f_{X_2}(x)dx = \frac{1}{35}\int_{90}^{125}(x - 0.001x^2)dx = \frac{1}{35}\left|\frac{x^2}{2} - 0.001\frac{x^3}{3}\right|_{90}^{125} =$$

$$= \frac{1}{35}\left[\frac{125^2}{2} - 0.001\frac{125^3}{3} - \frac{90^2}{2} + 0.001\frac{90^3}{3}\right] = 95.8417.$$

Hence, because $E[u(X_2)] > E[u(X_1)] \Rightarrow X_2 \succ X_1$.

In the case of the negative exponential utility function,

$$E[u(X_1)] = 100(1 - e^{-\frac{120}{100}})\frac{1}{4} + 100(1 - e^{-\frac{110}{100}})\frac{1}{2} + 100(1 - e^{-\frac{90}{100}})\frac{1}{4} = 65.6623.$$

$$E[u(X_2)] = \int_a^b u(x) f_{X_2}(x)dx = \frac{1}{35}\int_{90}^{125} 100(1 - e^{-\frac{x}{100}})dx = \frac{100}{35}\left(\int_{90}^{125} dx - \int_{90}^{125} e^{-\frac{x}{100}}dx\right) =$$

$$= \frac{100}{35}\left(\left|x\right|_{90}^{125} + 100\left|e^{-\frac{x}{100}}\right|_{90}^{125}\right) = \frac{100}{35}\left(35 + 100(e^{-\frac{125}{100}} - e^{-\frac{90}{100}})\right) = 65.6958.$$

Hence, because $E[u(X_2)] > E[u(X_1)] \Rightarrow X_2 \succ X_1$.

In the case of the logarithmic utility function, we have

$$E[u(X_1)] = (10\log(120)+5)\frac{1}{4} + (10\log(110)+5)\frac{1}{2} + (10\log(90)+5)\frac{1}{4} = 51.7207.$$

$$E[u(X_2)] = \int_a^b u(x) f_{X_2}(x)dx = \frac{1}{35}\int_{90}^{125}(10\log(x)+5)dx = \frac{1}{35}\left(10\int_{90}^{125}\log(x)dx + 5\int_{90}^{125}dx\right) =$$

$$= \frac{1}{35}\left(10\big|x\log(x) - x\big|_{90}^{125} + 5\big|x\big|_{90}^{125}\right) =$$

$$= \frac{1}{35}\left(10\big(125\log(125) - 125 - 90\log(90) + 90\big) + 5\cdot 35\right) = 51.7304.$$

Hence, because $E[u(X_2)] > E[u(X_1)] \Rightarrow X_2 \succ X_1$.

Exercise n. 2

Consider the two following investments

$$Z_1 = \begin{cases} 0 & \text{with probability} \quad 1/12 \\ 30 & \text{with probability} \quad 1/3 \\ 70 & \text{with probability} \quad 7/12 \end{cases}$$

and Z_2 described by an exponential distribution with parameter $\lambda = 0.02$. Determine which of the two investments is preferred according to the expected utility principle by an investor with an initial wealth $Y = 100$, carachterized by a negative exponential utility function $u(x) = -e^{-bx}$, $b = 0.01$.

The first investment leads to the random position $X_1 = Y + Z_1$

$$X_1 = \begin{cases} 100 & \text{with probability} \quad 1/12 \\ 130 & \text{with probability} \quad 1/3 \\ 170 & \text{with probability} \quad 7/12 \end{cases}$$

The expected utility is

$$E[u(X_1)] = \frac{1}{12}(-e^{-0.01\cdot 100}) + \frac{1}{3}(-e^{-0.01\cdot 130}) + \frac{7}{12}(-e^{-0.01\cdot 130}) = -0.2281.$$

We recall that an exponential random variable is a continuous random variable with density function

$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0$$

and a distribution function

$$F_X(x) = \int_0^x \lambda e^{-\lambda u} du = 1 - e^{-\lambda x}, x \geq 0.$$

In order to determine the density function of $X_2 = Y + Z_2$, we observe that

$$F_{X_2}(x) = \text{Prob}(X_2 \leq x) = \text{Prob}(Y + Z_2 \leq x) = \text{Prob}(Z_2 \leq x - Y) =$$
$$= F_{Z_2}(x - Y) = 1 - e^{-\lambda(x-100)}, x \geq 100.$$

The density function of X_2 can be obtained by differentiating the distribution function

$$f_{X_2}(x) = \frac{dF_{X_2}(x)}{dx} = \lambda e^{-\lambda(x-100)}, \quad x \geq 100.$$

Hence, the expected utility of X_2 is

$$E[u(X_2)] = \int_{100}^{\infty} u(x) f_{X_2}(x) dx = \int_{100}^{\infty} -e^{-0.01x} 0.02 e^{-0.02(x-100)} dx = -0.02 \int_{100}^{\infty} e^{-0.03x+2} dx =$$
$$= -0.02 \lim_{t\to\infty} \int_{100}^{t} e^{-0.03x+2} dx = -0.02 \lim_{t\to\infty} \left| -\frac{e^{-0.03x+2}}{0.03} \right|_{100}^{t} = -0.02 \lim_{t\to\infty} \left(\frac{e^{-1}}{0.03} - \frac{e^{-0.03t+2}}{0.03} \right) =$$
$$= -0.02 \frac{e^{-1}}{0.03} = -0.2453.$$

Hence, because $E[u(X_1)] > E[u(X_2)] \Rightarrow X_1 \succ X_2$.

Exercise n. 3

Compute the certainty equivalent and the insurance premiums of the risky investments described in Exercise n. 1.

Recalling that in the case of the quadratic utility function we have $E[u(X_1)] = 95.825$, the certainty equivalent $CE(Y, Z_1)$ is such that

$$u(Y + CE(Y, Z_1)) = E[u(X_1)].$$

Substituting, we obtain

$$Y + CE(Y, Z_1) - 0.001(Y + CE(Y, Z_1))^2 = E[u(X_1)],$$

$$100 + CE(Y, Z_1) - 0.001(10000 + 200CE(Y, Z_1) + CE(Y, Z_1)^2) = 95.825,$$

$$0.001CE(Y, Z_1)^2 - 0.8CE(Y, Z_1) + 5.825 = 0.$$

By solving the quadratic equation with respect to $CE(Y, Z_1)$, and discarding the greatest solution, $CE(Y, Z_1) = 7.3486$. In the same way, we obtain $CE(Y, Z_2) = 7.37$. The insurance premiums are

$$\Pi(Y, Z_1) = E[Z_1] - CE(Y, Z_1) = 0.1514; \quad \Pi(Y, Z_2) = E[Z_2] - CE(Y, Z_2) = 0.13.$$

For the negative exponential utility function,

$$100(1 - e^{-\frac{Y+CE(Y,Z_1)}{100}}) = 65.6623,$$

$$e^{-(1+\frac{CE(Y,Z_1)}{100})} = 0.3434; \quad CE(Y, Z_1) = 6.8859.$$

In the same way, $CE(Y, Z_2) = 7.0025$. The insurance premiums are

$$\Pi(Y, Z_1) = E[Z_1] - CE(Y, Z_1) = 0.6141; \quad \Pi(Y, Z_2) = E[Z_2] - CE(Y, Z_2) = 0.4975.$$

For the logarithmic utility function,

$$10 \log(Y + CE(Y, Z_1)) + 5 = 51.7207,$$

$$CE(Y, Z_1) = e^{4.6721} - 100 = 6.9188.$$

In the same way, $CE(Y, Z_2) = 7.0226$. The insurance premiums are

$$\Pi(Y, Z_1) = E[Z_1] - CE(Y, Z_1) = 0.5812; \quad \Pi(Y, Z_2) = E[Z_2] - CE(Y, Z_2) = 0.4774.$$

Exercise n. 4

Determine if the risky investments described in Exercise n. 1 and in Exercise n.2 can be ranked according to the first and the second-order stochastic dominance.

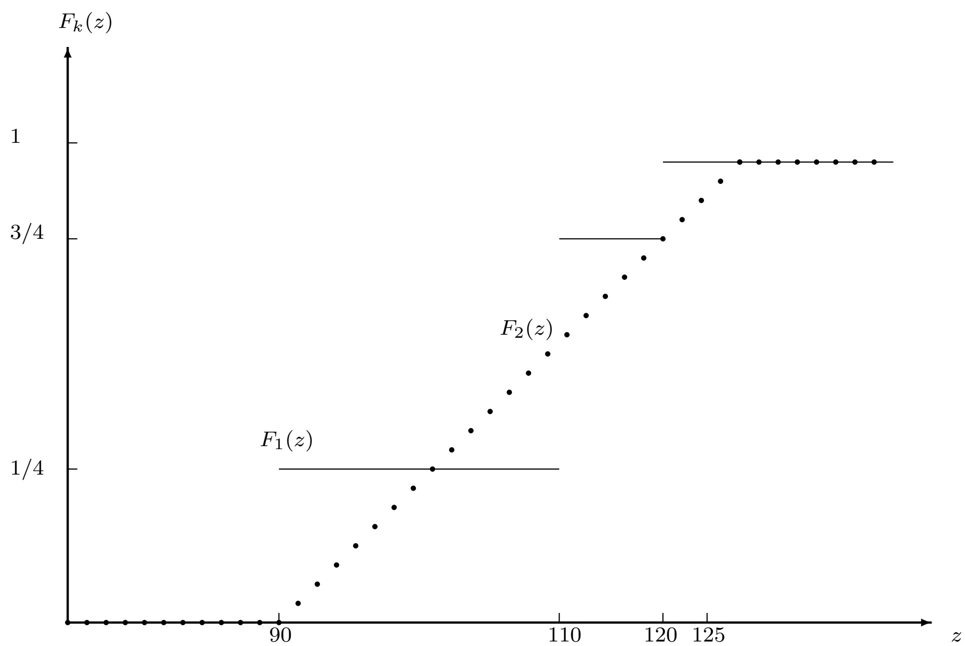
We have to define, at first, the distribution function $F_1(z) = \text{Prob}[X_1 \leq z]$ and $F_2(z) = \text{Prob}[X_2 \leq z]$.

$$F_1(z) = \begin{cases} 0 & \text{for} \quad 0 \leq z < 90 \\ 1/4 & \text{for} \quad 90 \leq z < 110 \\ 3/4 & \text{for} \quad 110 \leq z < 120 \\ 1 & \text{for} \quad z \geq 120 \end{cases}$$

$$F_2(z) = \int_a^z f_{X_2}(z)dz = \int_a^z \frac{1}{b-a}du = \frac{1}{b-a}\int_a^z du = \frac{1}{b-a}|u|_a^z = \frac{z-a}{b-a}, \quad \text{for} \quad z \in (a,b)$$

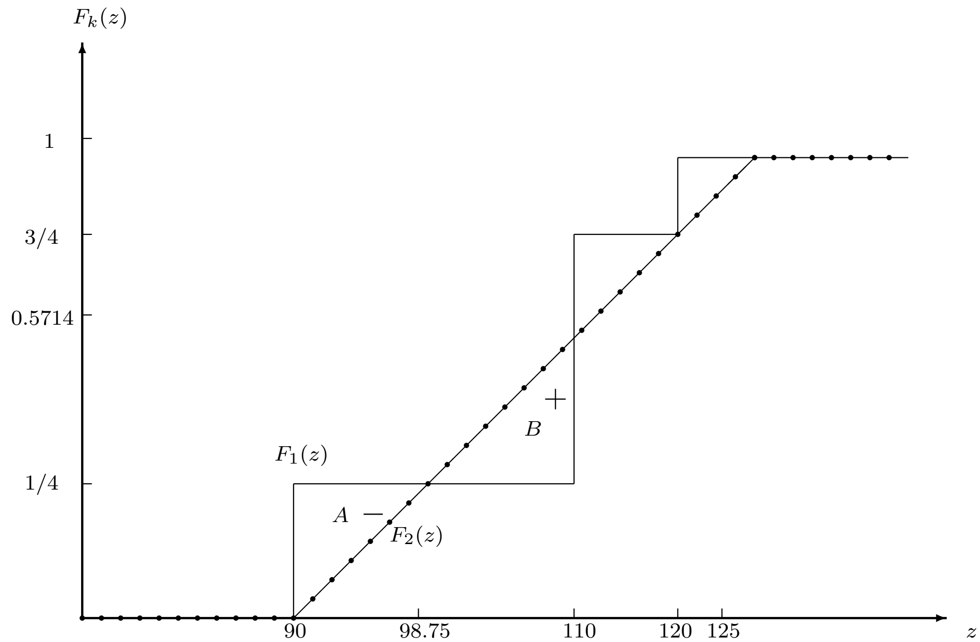
Hence,

$$F_2(z) = \begin{cases} 0 & \text{for} \quad z \leq 90 \\ \frac{z-90}{35} & \text{for} \quad 90 < z < 125 \\ 1 & \text{for} \quad z \geq 125 \end{cases}$$



For $z = 90$, we have $F_1(90) = 0.25 > F_2(90) = 0$. For $z = 109$, we have $F_1(109) = 0.25 < F_2(109) = 0.54$. Hence, the two investments Z_1 and Z_2 cannot be ranked according to the first-order stochastic dominance.

In order to determine if the two risky investments can be ranked according the second order stochastic dominance, we observe, at first, that $F_2(z)$ could dominate $F_1(z)$ but the converse is not true.



In order to have $F_2(z)$ SSD $F_1(z)$, we have to verify if

$$\int_{90}^{z} (F_2(u) - F_1(u))du \leq 0 \quad \forall z \in [90, 125].$$

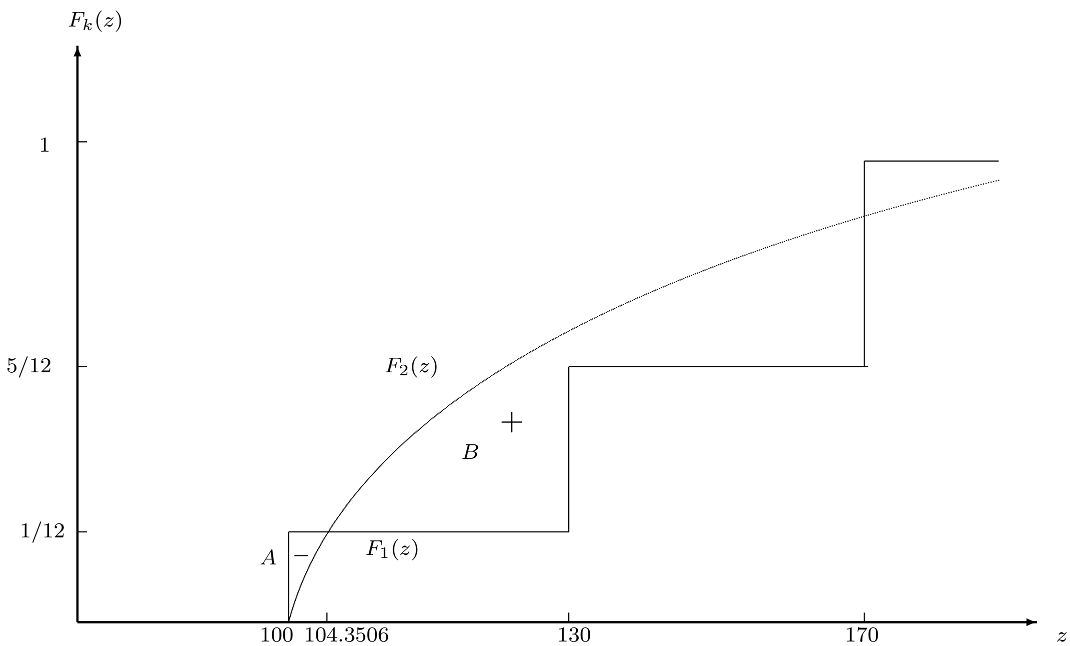
The first area to be considered as the difference between the areas below the two distribution functios is given by the right triangle A, and it has sign $-$. The length of one leg is $1/4$ while the length of the other leg can be obtained as the difference between the abscissa of the intersection point of the two distribution functions, and 90. Hence,

$$\frac{1}{4} = \frac{z - 90}{35} \rightarrow z = 98.75, \quad \text{and the length of the leg is} \quad 98.75 - 90 = 8.75,$$

and $A = (1/4 \times 8.75)/2 = 1.0938$.
The second area has sign $+$ and it is the right triangle B in the figure. The length of one leg is $110 - 98.75 = 11.25$ while the length of the other leg can be obtained as the difference between $F_2(110) = 0.5714$ and $1/4$. Hence, $(11.25 \times (0.5714 - 1/4))/2 = 1.8079$. Since

$$A + B = -1.0938 + 1.8079 = 0.7141 = \int_{90}^{110} (F_2(z) - F_1(z))dz > 0,$$

the two investments cannot be ranked according to the second order stochastic dominance.



We define, at first $F_1(z) = \text{Prob}[X_1 \le z]$ and $F_2(z) = \text{Prob}[X_2 \le z]$.

$$F_1(z) = \begin{cases} 0 & \text{for} \quad 0 \le z < 100 \\ 1/12 & \text{for} \quad 100 \le z < 130 \\ 5/12 & \text{for} \quad 130 \le z < 170 \\ 1 & \text{for} \quad z \ge 170 \end{cases}$$

while $F_2(z) = 1 - e^{-0.02(z-100)}$, for $z \ge 100$.
Since $F_1(100) = 1/12 > F_2(100) = 0$ and $F_1(129) = 1/12 < F_2(119) = 0.3161$, the two investments cannot be ranked according to the first order stochastic dominance.
In order to establish if the two investment can be ranked according to the second order stochastic dominance, since $F_1(z)$ can not be the dominating function we have to verify if

$$\int_{100}^{z} (F_2(u) - F_1(u))du \le 0 \quad \forall z \in [100, +\infty).$$

Since

$$\int_{100}^{130} (F_2(u) - F_1(u))du = \int_{100}^{130} 1 - e^{-0.02(u-100)} - \frac{1}{12}du =$$

$$= \frac{11}{12}|u|_{100}^{130} + \left| \frac{e^{-0.02(u-100)}}{0.02} \right|_{100}^{130} = \frac{330}{12} + \frac{e^{-0.6} - 1}{0.02} = 4.9406 > 0.$$

Hence, $F_1(z)$ and $F_2(z)$ cannot be ranked according to the second order stochastic dominance.

Exercise n. 5

Let us consider a market with a risk-free asset with rate of return $r_f = 2\%$, and the following three risky securities characterized by the linear correlation coefficients: $\rho_{12} = 0.2$, $\rho_{13} = -0.1$ and $\rho_{23} = 0.6$. Moreover,

Security	N. of shares	Price	Mean	St. deviation
1	100	4	4%	0.10
2	300	6	7%	0.14
3	200	5	6%	0.12

Under the assumptions of the C.A.P.M. model, determine the market portfolio, its expected rate of return and its standard deviation. Determine the CML, the SML, the betas of the risky assets, and the diversifiable and non-diversifiable risk for each of them.

The total market value of the securities is $100 \times 4 + 300 \times 6 + 200 \times 5 = 3200$. Hence,

$$\omega_{M1} = \frac{400}{3200} = 0.125, \quad \omega_{M2} = \frac{1800}{3200} = 0.5625, \quad \omega_{M3} = \frac{1000}{3200} = 0.3125.$$

The market portfolio is $\underline{\omega}_M^T = (0.125, 0.5625, 0.3125)$. The expected rate of return of the market portfolio is

$$e_M = \omega_{M1}e_1 + \omega_{M2}e_2 + \omega_{M3}e_3 = 0.125 \times 0.04 + 0.5625 \times 0.07 + 0.3125 \times 0.06 = 0.063125.$$

We have to determine the standard deviation of the market portfolio

$$\sigma_M^2 = \underline{\omega}_M^T V \underline{\omega}_M$$

$$V = \begin{bmatrix} 0.01 & 0.0028 & -0.0012 \\ 0.0028 & 0.0196 & 0.01008 \\ -0.0012 & 0.01008 & 0.0144 \end{bmatrix}$$

$$\sigma_M^2 = \underline{\omega}_M^T V \underline{\omega}_M = \begin{bmatrix} 0.125 & 0.5625 & 0.3125 \end{bmatrix} \begin{bmatrix} 0.01 & 0.0028 & -0.0012 \\ 0.0028 & 0.0196 & 0.01008 \\ -0.0012 & 0.01008 & 0.0144 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0.5625 \\ 0.3125 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.0024 & 0.0145 & 0.01 \end{bmatrix} \begin{bmatrix} 0.125 \\ 0.5625 \\ 0.3125 \end{bmatrix} = 0.011608, \quad \Rightarrow \sigma_M = \sqrt{0.011608} = 0.107740.$$

The capital market line is

$$e_p = rf + \frac{e_M - r_f}{\sigma_M}\sigma_p = 0.02 + 0.400406\sigma_p.$$

The security market line is

$$e_j = r_f + (e_M - r_f)\beta_{jM} = 0.02 + 0.043125\beta_{jM}.$$

$$\sigma_{1M} = \mathrm{COV}(\tilde{r}_1, \tilde{r}_M) = \mathrm{COV}(\tilde{r}_1, \omega_{M1}\tilde{r}_1 + \omega_{M2}\tilde{r}_2 + \omega_{M3}\tilde{r}_3) = \omega_{M1}\sigma_1^2 + \omega_{M2}\sigma_{12} + \omega_{M3}\sigma_{13} =$$

$$0.125 \times 0.1^2 + 0.5625 \times 0.0028 - 0.3125 \times 0.0012 = 0.00245$$

In the same way

$$\sigma_{2M} = \mathrm{COV}(\tilde{r}_2, \tilde{r}_M) = \mathrm{COV}(\tilde{r}_2, \omega_{M1}\tilde{r}_1 + \omega_{M2}\tilde{r}_2 + \omega_{M3}\tilde{r}_3) = \omega_{M1}\sigma_{12} + \omega_{M2}\sigma_2^2 + \omega_{M3}\sigma_{23} =$$

$$= 0.014525$$

and

$$\sigma_{3M} = \mathrm{COV}(\tilde{r}_3, \tilde{r}_M) = \mathrm{COV}(\tilde{r}_3, \omega_{M1}\tilde{r}_1 + \omega_{M2}\tilde{r}_2 + \omega_{M3}\tilde{r}_3) = \omega_{M1}\sigma_{13} + \omega_{M2}\sigma_{23} + \omega_{M3}\sigma_3^2 =$$

$$= 0.01002$$

$$\beta_{1M} = \frac{\sigma_{1M}}{\sigma_M^2} = \frac{0.00245}{0.011608} = 0.211061, \quad \beta_{2M} = \frac{\sigma_{2M}}{\sigma_M^2} = \frac{0.014525}{0.011608} = 1.251292,$$

$$\beta_{3M} = \frac{\sigma_{3M}}{\sigma_M^2} = \frac{0.01002}{0.011608} = 0.863198$$

The systematic and the non systematic risk, for each asset, are

$$\beta_{1M}\sigma_M = 0.211061 \times 0.107740 = 0.022740, \quad \sigma_1 - \beta_{1M}\sigma_M = 0.077260,$$

$$\beta_{2M}\sigma_M = 1.251292 \times 0.107740 = 0.134814, \quad \sigma_2 - \beta_{2M}\sigma_M = 0.005186,$$

$$\beta_{3M}\sigma_M = 0.863198 \times 0.107740 = 0.093001, \quad \sigma_3 - \beta_{3M}\sigma_M = 0.026999.$$

By applying the SML, it results

$$e_1 = 0.02 + 0.043125\beta_{1M} = 0.029102,$$

$$e_2 = 0.02 + 0.043125\beta_{2M} = 0.073962,$$

$$e_3 = 0.02 + 0.043125\beta_{3M} = 0.057225.$$

Securities 1 and 3 are overpriced, security 2 is underpriced.

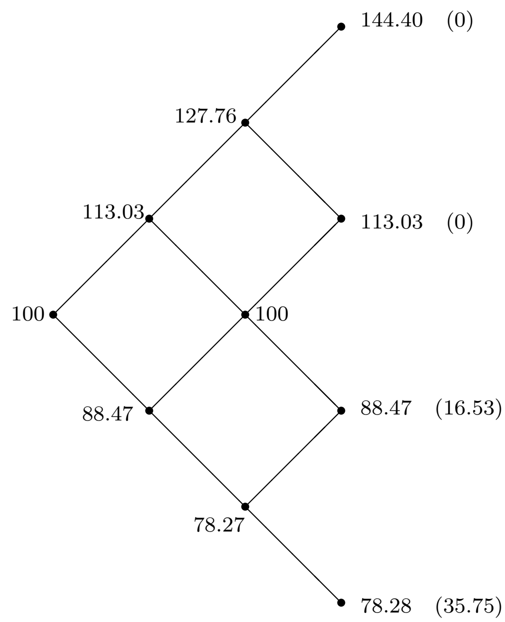
Exercise n. 6

Consider a European and an American put option written on an underlying security with initial price $S_0 = 100$, strike price $K = 105$, maturity $T = 6$ months. Compute the option prices with the Cox-Ross-Rubinstein model with $n = 3$ time steps when the volatility is $\sigma = 0.30$ and the continuously compounded rate of return is $r = 0.02$ on a yearly basis.

To evaluate the European put option we compute at first $u = \exp(\sigma\sqrt{\Delta t}) = \exp(0.3\sqrt{1/6}) = 1.1303$ and d=$1/u = 0.8847$. The risk-neutral probability of an up-step is

$$q = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.02\times(1/6)} - 0.8847}{1.1303 - 0.8847} = 0.4830$$

while the risk neutral probability of a down-step is $1 - q = 0.5170$. The binomial tree describing the evolution of the underlying asset price is the following:



Note that associated with each final node, in brackets, is reported the corresponding payoff of the European option, so that $\max(105-127.76, 0) = 0$, $\max(105-108.51, 0) = 0$, $\max(105-92.16, 0) = 12.84$, and $\max(105-78.28, 0) = 26.72$. The risk-neutral probability of reaching the asset price 88.47 is

$$\binom{3}{1} 0.4830 \times 0.5170^2 = 0.3873,$$

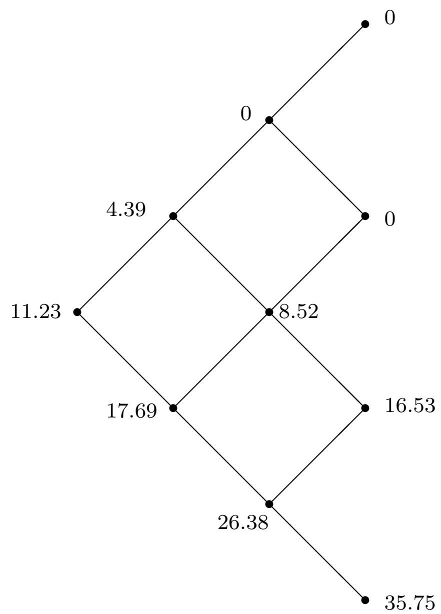
the risk-neutral probability of reaching asset price 78.28 is

$$\binom{3}{0} 0.5170^3 = 0.1382.$$

Hence the price at time $t = 0$ of the European put option is

$$P_0 = e^{-0.02\times 0.5}(0.3873 \times 16.53 + 0.1382 \times 35.75) = 11.23.$$

The same price can be obtained by computing backward the option price at each node of the binomial tree



For example, the option price corresponding to the underlying asset price 78.27 can be computed as

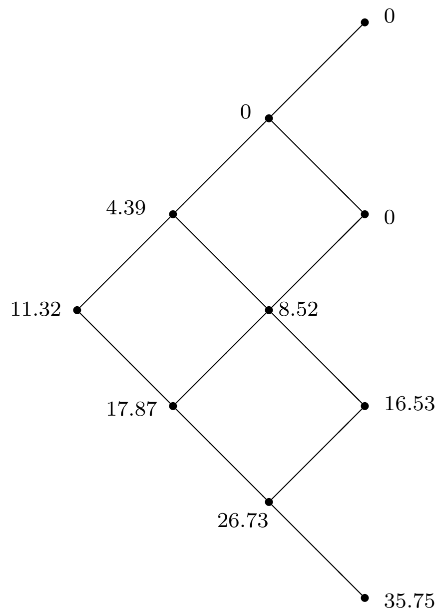
$$e^{-0.02\times\frac{0.5}{3}}(0.4830 \times 16.53 + 0.517 \times 35.75) = 26.38.$$

In the same way, it is possible to compute the option prices associated with all the nodes in the binomial tree.

If the put option is of American type, working backward along the tree we must set the option price equal to the maximum between the option price if the option is not exercised and the option price if the option is exercised. For example, the option price corresponding to the underlying asset price 78.27 is obtained as

$$\max(105 - 78.27, e^{-0.02\times\frac{0.5}{3}}(0.4830 \times 16.53 + 0.517 \times 35.75)) = 105 - 78.27 = 26.73.$$

In the same way we can compute the option prices associated with the ramaining nodes in the binomial tree and we arrive at the initial option price equal to 11.32.


0
0
4.39
0
11.32
8.52
17.87
16.53
26.73
35.75