Introduction to Machine Learning
Homework 1

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1 [20pts] Basic review of probability

The probability distribution of random variable X follows:

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < x < 1; \\ \frac{1}{6} & 2 < x < 5; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

(1) [5pts] Please give the cumulative distribution function $F_X(x)$ for X;
解:

当 $x < 0$, $F_X(x) = 0$;
当$0 \le x < 1, F_X(x) = \int_{-\infty}^{x} f_X(x)dx = \int_0^x \frac{1}{2}dx = \frac{x}{2}$;
当$1 \le x < 2$, $F_X(x) = \frac{1}{2}$;
当$2 \le x < 5, F_X(x) = \frac{1}{2} + \int_2^x \frac{1}{6}dx = \frac{1}{2} + \frac{x}{6} - \frac{1}{3} = \frac{x+1}{6}$;
当$x \ge 5, F_X(x) = 1$;

因此,随机变量X的分布函数$F_X(x)$为

$$F_X(x) = \begin{cases} 0 & x < 0; \\ \frac{x}{2} & 0 \le x < 1; \\ \frac{1}{2} & 1 \le x < 2; \\ \frac{x+1}{6} & 2 \le x < 5; \\ 1 & x \ge 5. \end{cases} \tag{1.2}$$

(2) [5pts] Define random variable Y as $Y = 1/(X^2)$, please give the probability density function $f_Y(y)$ for Y;

解:

当$y \leq 0$, $F_Y(y) = 0$;

当$y > 0$, $F_Y(y) = P(Y \leq y) = P(\frac{1}{X^2} \leq y)$

$= P(X \geq \frac{1}{\sqrt{y}}) + P(X \leq -\frac{1}{\sqrt{y}}) = 1 - P(X < \frac{1}{\sqrt{y}}) + P(X \leq -\frac{1}{\sqrt{y}})$

$= 1 - F_X(\frac{1}{\sqrt{y}}) + F_X(-\frac{1}{\sqrt{y}}) = 1 - F_X(\frac{1}{\sqrt{y}})$;

进一步细分

当$y > 1$, $F_Y(y) = 1 - \frac{1}{2\sqrt{y}}$;

当$\frac{1}{4} < y \leq 1$, $F_Y(y) = 1 - \frac{1}{2} = \frac{1}{2}$;

当$\frac{1}{25} < y \leq \frac{1}{4}$, $F_Y(y) = 1 - \frac{1}{6}(\frac{1}{\sqrt{y}} + 1) = \frac{1}{6}(5 - \frac{1}{\sqrt{y}})$;

当$0 < y \leq \frac{1}{25}$, $F_Y(y) = 0$;

经过求导得, 随机变量y的概率密度函数$f_Y(y)$为

$$f_Y(y) = \begin{cases} \frac{1}{12\sqrt{y^3}} & \frac{1}{25} < y \leq \frac{1}{4}; \\ \frac{1}{4\sqrt{y^3}} & y > 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

(3) [10pts] For some random non-negative random variable Z, please prove the following two formulations are equivalent:

$$\mathbb{E}[Z] = \int_{z=0}^{\infty} z f(z) \mathrm{d}z, \tag{1.4}$$

$$\mathbb{E}[Z] = \int_{z=0}^{\infty} \Pr[Z \geq z] \mathrm{d}z, \tag{1.5}$$

Meantime, please calculate the expectation of random variable X and Y by these two expectation formulations to verify your proof.

解:

$\int_{z=0}^{\infty} \Pr[Z \geq z] \mathrm{d}z = \int_{z=0}^{\infty} \int_{z}^{\infty} f(t) dt dz = \int_{z=0}^{\infty} \int_{0}^{\infty} f(t) one(t) dt dz$

ps: $one(t) = 1$, 当$t \geq z$; $one(t) = 0$, 当$t < z$.

则由fubini定理, $\int_{z=0}^{\infty} \Pr[Z \geq z] \mathrm{d}z = \int_{z=0}^{\infty} f(t) \int_{0}^{\infty} one(z) dz dt = \int_{z=0}^{\infty} f(t) t dt$

ps: $one(z) = 1$, 当$z \leq t$; $one(z) = 0$, 当$z > t$.$thus$, $\int_{0}^{\infty} one(z) dz = t$

因此, 两种数学期望的表达形式是等价的.

接下来验证X和Y的分布满足这两种数学期望相等价的结论.

对于X, $\mathbb{E}[X] = \int_{x=0}^{\infty} xf(x)dx = \int_0^1 \frac{x}{2}dx + \int_2^5 \frac{x}{6}dx = 2$;
同时$\mathbb{E}[X] = \int_{x=0}^{\infty} \Pr[X \geq x]\mathrm{d}x = \int_0^1 (1 - \frac{x}{2})dx + \int_1^2 \frac{1}{2}dx + \int_2^5 \frac{5-x}{6}dx$
$= \frac{3}{4} + \frac{1}{2} + \frac{3}{4} = 2$. 因此$X$的分布满足结论, 数学期望为 $\mathbb{E}[X] = 2$

对于Y, $\mathbb{E}[Y] = \int_{y=0}^{\infty} yf(y)dy = \int_{\frac{1}{25}}^{\frac{1}{4}} \frac{y}{12\sqrt{y^3}}dy + \int_1^{\infty} \frac{y}{4\sqrt{y^3}}dy = \infty$;
同时$\mathbb{E}[Y] = \int_{y=0}^{\infty} \Pr[Y \geq y]\mathrm{d}y = \int_0^{\frac{1}{25}} dy + \int_{\frac{1}{25}}^{\frac{1}{4}} \frac{1}{6}(1 + \frac{1}{\sqrt{y}})dy + \int_{\frac{1}{4}}^1 \frac{1}{2}dy$
$+ \int_1^{\infty} \frac{1}{2\sqrt{y}}dy = \infty$.
因此Y的分布满足结论, 数学期望为 $\mathbb{E}[Y] = \infty$.

2 [15pts] Probability Transition

(1) [5pts] Suppose P(rain today) = 0.30, P(rain tomorrow) = 0.60, P(rain today and tomorrow) = 0.25. Given that it rains today, what is the probability it will rain tomorrow?
解:

设A事件{今天下雨}; B事件{明天下雨}.
则由题干得, $P(A) = 0.3, P(B) = 0.6, P(AB) = 0.25$
则已知今天下雨的条件下，明天下雨的概率为$P(B|A) = \frac{P(AB)}{P(A)} = \frac{5}{6}$
因此，已知今天下雨的条件下，明天下雨的概率是$\frac{5}{6}$.

(2) [5pts] Give a formula for $P(G|\neg H)$ in terms of $P(G)$, $P(H)$ and $P(G \wedge H)$ only. Here H and G are boolean random variables.
解:

$P(G|\bar{H}) = \frac{P(G \wedge \bar{H})}{1-P(H)} = \frac{P(G)-P(G \wedge H)}{1-P(H)}$.

(3) [5pts] A box contains w white balls and b black balls. A ball is chosen at random. The ball is then replaced, along with d more balls of the same color (as the chosen ball). Then another ball is drawn at random from the box. Show that the probability that the second ball is white does not depend on d.

解:

设事件A_1{第一次拿白球},事件A_2{第一次拿黑球},

事件B{第二次拿白球}

则$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$

$= \frac{w}{w+b} \times \frac{w+d-1}{w+b+d-1} + \frac{b}{w+b} \times \frac{w}{w+b+d-1} = \frac{w}{w+b}$

显然结果和d无关

3 [20pts] Basic review of Linear Algebra

Let $x = (\sqrt{3}, 1)^\top$ and $y = (1, \sqrt{3})^\top$ be two vectors,

(1) [5pts] What is the value of $x_\perp$ where $x_\perp$ indicates the projection of x onto y.

解:

$x \cdot y = x^\top y = 2\sqrt{3}$, $|y| = \sqrt{1+3} = 2$

因此x在y上的投影大小为$||x_\perp|| = \frac{2\sqrt{3}}{2} = \sqrt{3}$, 投影为$x_\perp = (\frac{\sqrt{3}}{2}, \frac{3}{2})^\top$

(2) [5pts] Prove that $y \perp (x - x_\perp)$.

解:

$x - x_\perp = (\sqrt{3}, 1)^\top - (\frac{\sqrt{3}}{2}, \frac{3}{2})^\top = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top$

则$y^\top(x - x_\perp) = (1, \sqrt{3}) \cdot (\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top = 0$

因此 $y \perp (x - x_\perp)$.

(3) [10pts] Prove that for any $\lambda \in \mathbb{R}$, $||x - x_\perp|| \leq ||x - \lambda y||$

解:

由$||x - x_\perp|| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

又$||x - \lambda y|| = \sqrt{(\sqrt{3} - \lambda)^2 + (1 - \sqrt{3}\lambda)^2} = \sqrt{4\lambda^2 - 4\sqrt{3}\lambda + 4}$

$= 2\sqrt{(\lambda - \frac{\sqrt{3}}{2})^2 + \frac{1}{4}} \geq 2\sqrt{\frac{1}{4}} = 1$

故$||x - x_\perp|| \leq ||x - \lambda y||$, 当且仅当$\lambda = \frac{\sqrt{3}}{2}$, 取等号

4 [20pts] Hypothesis Testing

A coin was tossed for 50 times and it got 35 heads, please determine that *if the coin is biased for heads* with $\alpha = 0.05$.

解:

设p为硬币正面朝上的概率,则该问题可归结为如下假设检验问题:

$H_0 : E_X = 0.5$, $H_1 : E_X > 0.5$,

设x为正面朝上的次数，若原假设成立，则$x \sim (50, 0.5)$.

样本较大，由中心极限定理，近似为正态分布计算.

统计量取$u = \frac{x-50\times0.5}{\sqrt{50\times0.5\times(1-0.5)}}$,

即近似认为$u \sim N(0,1)$, 故检验拒绝域为$W = \{u > u_{1-\alpha}\}$

$= \{u > 1.645\}$

检验的p值近似计算为:

$p = P\{x \geq 35\} = P\{u \geq \frac{35-25}{\sqrt{\frac{25}{2}}}\} = P\{u \geq 2\sqrt{2}\} = 0.0023 < 0.5$

结果落在拒绝域内，因此拒绝原假设，认为硬币更倾向于正面朝上.

5 [25pts] Performance Measures

We have a set of samples that we wish to classify in one of two classes and a ground truth class of each sample (denoted as 0 and 1). For each example a classifier gives us a score (score closer to 0 means class 0, score closer to 1 means class 1). Below are the results of two classifiers (C_1 and C_2) for 8 samples,their ground truth values (y) and the score values for both classifiers (y_{C_1} and y_{C_2}).

y	1	0	1	1	1	0	0	0
y_{C_1}	0.6	0.31	0.58	0.22	0.4	0.51	0.2	0.33
y_{C_2}	0.04	0.1	0.68	0.24	0.32	0.12	0.8	0.51

(1) [10pts] For the example above calculate and draw the ROC curves for classifier C_1 and C_2. Also calculate the area under the curve (AUC) for both classifiers.

解:

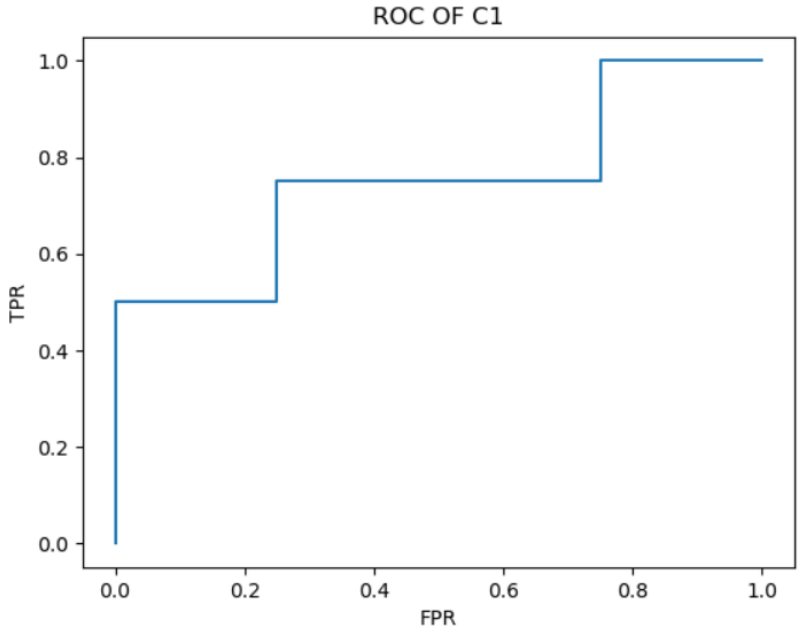
(a)先确定C1的ROC曲线和AUC大小

排序后C1表格为

y	0	1	0	0	1	0	1	1
y_{C_1}	0.2	0.22	0.31	0.33	0.4	0.51	0.58	0.6

由此得到ROC曲线上的点为$(0,0)$、$(0,\frac{1}{4})$、$(0,\frac{1}{2})$、$(\frac{1}{4},\frac{1}{2})$、$(\frac{1}{4},\frac{3}{4})$、$(\frac{1}{2},\frac{3}{4})$、$(\frac{3}{4},\frac{3}{4})$、$(\frac{3}{4},1)$、$(1,1)$

因此ROC曲线图为



则$AUC_{C1} = \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \times \frac{3}{4} + \frac{1}{4} \times 1 = \frac{3}{4}$

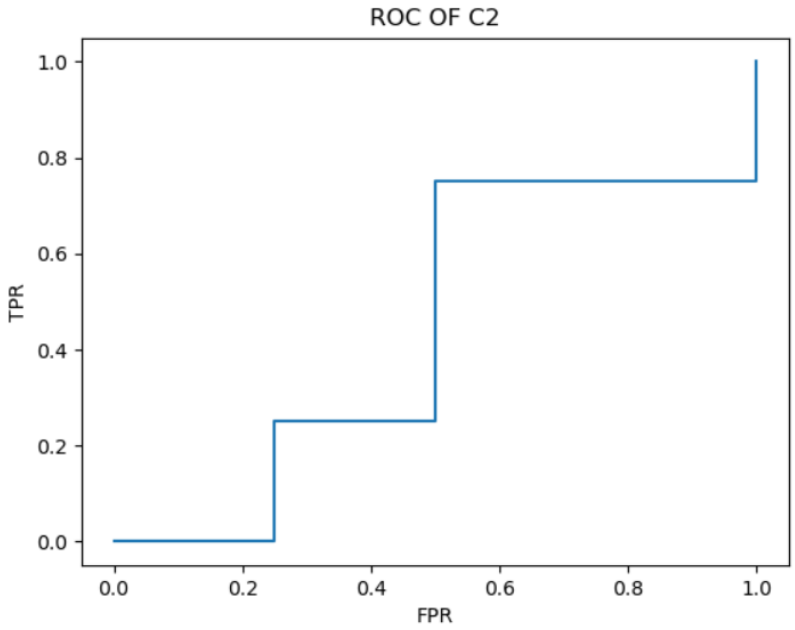
(b)再确定C2的ROC曲线和AUC大小

排序后C2表格为

由此得到ROC曲线上的点为$(0,0)$、$(\frac{1}{4},0)$、$(\frac{1}{4},\frac{1}{4})$、$(\frac{1}{2},\frac{1}{4})$、$(\frac{1}{2},\frac{1}{2})$、

y	1	0	0	1	1	0	1	0
y_{C_2}	0.04	0.1	0.12	0.24	0.32	0.51	0.68	0.8

$(\frac{1}{2}, \frac{3}{4})$、$(\frac{3}{4}, \frac{3}{4})$、$(1, \frac{3}{4})$、$(1, 1)$
因此ROC曲线图为



$AUC_{C2} = \frac{1}{4} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{4} = \frac{7}{16}$
明显看出C1比C2效果要好得多

(2) [15pts] For the classifier C_1 select a decision threshold $th_1 = 0.33$ which means that C_1 classifies a sample as class 1, if its score $y_{C_1} > th_1$, otherwise it classifies it as class 0. Use it to calculate the confusion matrix and the F_1 score. Do the same thing for the classifier C_2 using a threshold value $th_2 = 0.1$.

解:

(a)依然是先确定C1的混淆矩阵和F1度量

		prediction outcome		
		p	**n**	**total**
actual value	**p′**	TP=3	FN=1	P′
	n′	FP=1	TN=3	N′
	total	P	N	

$F1_{C1} = \frac{2TP}{N+TP-TN} = \frac{6}{8+3-3} = \frac{3}{4}$

(b)然后确定C2的混淆矩阵和F1度量

		prediction outcome		
		p	**n**	**total**
actual value	**p′**	TP=3	FN=1	P′
	n′	FP=3	TN=1	N′
	total	P	N	

$F1_{C2} = \frac{2TP}{N+TP-TN} = \frac{6}{8+3-1} = \frac{3}{5}$

从F1-Score来看依然是C1效果要好