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1.A: $\mu 1 = 9.2$, $\sigma 1 = 1.6$ grams.
B: $\mu 2 = 9.6$, $\sigma 2 = 1.2$ grams.

1/a.

Let's define probability they're from A P(A), probability they're from B P(B). And probability that the rocks are from Location B is four times the probability that the rocks are from Location A,
P(B)= 4P(A) …(1)
Since they come from the same location,
P(B)+P(A) = 1…(2)
From (1) and (2), We get
5P(A) = 1
P(A)=0.2, P(B)=0.8

1/b.

According to Bayes Rule: $P(A|x) = \frac{P(x|A) \cdot P(A)}{P(x)}$
We got $P(A) = 0.2$ from 1/a
We also have $x_1 = 9.3,\ x_2 = 8.8,\ x_3 = 9.8$

$$p(x_i|A) = \frac{1}{\sqrt{2\pi}\sigma_1} exp(-\frac{x_i - \mu_1}{2\sigma_1^2}) = \frac{1}{\sqrt{2\pi}1.6} exp(-\frac{x_i - 9.2}{2 \cdot 1.6^2}), p(x|A) = \prod_{3}^{i=1} p(x_i|A);$$

$$p(x|A) = (\frac{1}{\sqrt{2\pi}1.6})^3 exp(-\frac{9.3 - 9.2}{2 \cdot 1.6^2}) \cdot exp(-\frac{8.8 - 9.2}{2 \cdot 1.6^2}) \cdot exp(-\frac{9.8 - 9.2}{2 \cdot 1.6^2}) = 0.014$$

$$p(x_i|B) = \frac{1}{\sqrt{2\pi}\sigma_2} exp(-\frac{x_i - \mu_2}{2\sigma_2^2}) = \frac{1}{\sqrt{2\pi}1.2} exp(-\frac{x_i - 9.6}{2 \cdot 1.2^2}), p(x|B) = \prod_{3}^{i=1} P(x_i|B);$$

$$p(x|B) = (\frac{1}{\sqrt{2\pi}1.2})^3 exp(-\frac{9.3 - 9.6}{2 \cdot 1.2^2}) \cdot exp(-\frac{8.8 - 9.6}{2 \cdot 1.2^2}) \cdot exp(-\frac{9.8 - 9.6}{2 \cdot 1.2^2}) = 0.028,$$

And $P(x) = P(x|A)P(A) + P(x|B)P(B) = 0.014 \cdot 0.2 + 0.028 \cdot 0.8 = 0.0252$

As a result, $P(A|x) = \frac{P(x|A) \cdot P(A)}{P(x)} = \frac{1.4 \cdot 10^{-2} \cdot 0.2}{4.2 \cdot 10^{-2}} = 0.111$

1/c.
$p(x|A) = 0.014,\ p(x|B) = 0.028$
Since $p(x|A) < p(x|B)$, the rocks are more likely from B.

2. method 1: fp=0.15, fn=0.1;
method 2: fp= 0.05, fn=0.03
0.02% of the population has this disease

random person from the population is screened using Screening method 1, and it returns a positive result
P(Y) as possibility that a random person actually has the disease. P(!Y) as possibility that a random person does not have the disease. Define C as all kinds of incidence including{!Y, Y},
$P(x_i)$ as the possibility of being diagnosed as having the disease with method i.
$P(x_1 | Y) = tp_1 = 0.9,\ P(x_2 | Y) = tp_2 = 0.97$
$P(x_1 | !Y) = fp_1 = 0.15,\ P(x_2 | !Y) = fp_2 = 0.05$
$P(!x_1 | !Y) = tn_1 = 0.85,\ P(!x_2 | !Y) = tn_2 = 0.95$
$P(!x_1 | Y) = fn_1 = 0.1,\ P(!x_2 | Y) = fn_2 = 0.03$

2/a

For MAP estimation, $P(Y) = 0.0002,\ P(!Y) = 0.9998,$
$C = argmax_C P(C | x) = argmax_C\{\frac{P(x | Y) \cdot P(Y)}{P(x)}, \frac{P(x | !Y) \cdot P(!Y)}{P(x)}\}$
$= argmax_C\{P(x | Y)P(Y), P(x | !Y)P(!Y)\} = argmax_C\{tp \cdot P(Y)\ , fp \cdot P(!Y)\}$
$= argmax_C\{0.9 \cdot 0.0002, 0.15 \cdot 0.9998\} = argmax_C\{0.00018,\ 0.14997\}$
Since $P(x | !Y)P(!Y) > P(x | Y)P(Y)$
As we can see, it's more likely the person does not have the disease.

2/b.

For ML, $C = argmax_C P(C | x) = argmax_C\{P(x | Y)/P(x), P(x | !Y)/P(x)\}$
$= argmax_C\{P(x | Y), P(x | !Y)\} = argmax_C\{tp, fp\} = argmax_C\{0.9,\ 0.15\}$
Since $P(x | Y) > P(x | !Y)$
As we can see, it's more likely the person has the disease.

2/c

The person have been received positive results from both methods.
$P(x_1, x_2 | Y) = P(x_1 | Y)P(x_2 | Y) = 0.97 \cdot 0.9 = 0.873,\ P(Y) = 0.0002,$
$P(x_1, x_2) = P(x_1, x_2 | Y)P(Y) + P(x_1, x_2 | !Y)P(!Y) = P(x_1 | Y)P(x_2 | Y)P(Y) + P(x_1 | !Y)P(x_2 | !Y)P(!Y) =$
$0.97 \cdot 0.9 \cdot 0.0002 + 0.15 \cdot 0.05 \cdot 0.9998 = 1.74 \cdot 10^{-4} + 7.5 \cdot 10^{-3} = 7.67 \cdot 10^{-3}$
$P(Y | x_1, x_2) = \frac{P(x_1, x_2 | Y)P(Y)}{p(x1,x2)} =\ 0.0228$

3. Tails $1-\theta$, heads θ

3/a

P(T T HT T H|θ) = $(1-\theta)^4\theta^2$

3/b

$argmax_{\theta} logP(TTHTTH|\theta) = log(1-\theta)^4\theta^2 = 4log(1-\theta) + 2log\theta$

3/c

$argmax_{\theta} P(TTHTTH|\theta) = argmax_{\theta} logP(TTHTTH|\theta) = argmax_{\theta}[4log(1-\theta) + 2log\theta]$

For $y = 4log(1-\theta) + 2log\theta$, We take its derivative, $\frac{dy}{d\theta} = -\frac{4}{1-\theta} + \frac{2}{\theta}$, $0 < \theta < 1$

To get its maximum , we first need all local maxima/minima. $\frac{dy}{d\theta} = -\frac{4}{1-\theta} + \frac{2}{\theta} = 0$

$\rightarrow -4\theta + 2(1-\theta) = 0 \rightarrow \theta = 1/3$

$y|_{\theta=1/3} = 4log(2/3) + 2log(1/3) = 4log2 - 6log3$ is indeed larger than $y|_{\theta'<1/3}$ and $y|_{\theta'>1/3}$

4.

4/a

If s = 3, given X = {3,1,1,2,3}

$P(\theta_1) = t/N = 2/5$

4/b

If s = 3, what is the estimate for θ1 using add-1 smoothing, given X = {3, 1, 1, 2, 3}

$P(\theta_1) = \dfrac{t+1}{N+s} = (2+1)/(3+5) = 3/8$

4/c

Suppose s = 2

pdf $p(\theta_1) = 6(1-\theta_1)\theta_1$ when θ1 ∈ [0,1], else p(θ1) = 0

MAP estimate of θ1, for a generic sample X of size N with t 1's:

$argmax_{\theta_1} P(N, t \,|\, \theta_1) \cdot P(\theta_1) = (1-\theta_1)^{(N-t)}\theta_1^t \cdot 6\theta_1(1-\theta_1) = 6\theta_1^{t+1}(1-\theta_1)^{N-t+1}$

$(6\theta^{t+1}(1-\theta)^{N-t+1})' = 6\theta_1^t(1-\theta_1)^{(N-t)}(t+1-(2+N)\theta_1)$=0

possible maxima: $\theta_1 = 0$, $\theta_1 = 1$, $\theta_1 = (t+1)/(N+2)$

After inspecting the pdf, 0 and 1 gives us the extreme points which are minima.

As we can see, $\theta_1 = \dfrac{t+1}{N+2}$

5.

5/a

m = 0.3, $P(x) = \dfrac{t + m}{N + sm}$

P (x_1 = Low|+) = (1+0.3)/(2+3*0.3) = 1.3/2.9 = 0.448

P (x_2 = Yes|+) = (0+0.3)/(2+2*0.3) = 0.3/2.6 = 0.115

P (x_3 = Green|+) = (1+0.3)/(2+2*0.3) = 1.3/2.6 = 0.5

P (x_1 = Low|−) = (1+0.3)/(3+3*0.3) = 1.3/3.9 = 0.333

P (x_2 = Yes|−) = (2+0.3)/(3+2*0.3) = 2.3/3.6 = 0.639

P (x_3 = Green|−) = (2+0.3)/(3+2*0.3) = 2.3/3.6 = 0.639

5/b

P (x|+) and P (x|−), for x = [Low, Yes, Green]

P (x|+) = P (x_1 = Low|+) P (x_2 = Yes|+) P (x_3 = Green|+) = 0.026

P (x|-) = P (x_1 = Low|−) P (x_2 = Yes|−) P (x_3 = Green|−) = 0.136

5/c

ML label for x = [Low, Yes, Green] is "–"

5/d

Estimate of the priors would be P (+) = 2/5 and P (−) = 3/5. Using these priors, and the values you calculated for x = [Low, Yes, Green], what is the MAP label for the example x = [Low, Yes, Green]?

$P(Low, Yes, Green) = P(Low) \cdot P(Yes) \cdot P(Green) = \dfrac{12}{125}$

$P(Low, Yes, Green \mid +) = 0.026$, $P(Low, Yes, Green \mid -) = 0.136$

$P(+) = \dfrac{2}{5}, P(-) = \dfrac{3}{5}$

$$P(+ \mid Low, Yes, Green) = \frac{P(Low, Yes, Green \mid +)P(+)}{P(Low, Yes, Green)} = 2/5 \cdot 0.026/12 \cdot 125 = 0.108$$

$$P(- \mid Low, Yes, Green) = \frac{P(Low, Yes, Green \mid -)P(-)}{P(Low, Yes, Green)} = 3/5 \cdot 0.136/12 \cdot 125 = 0.567$$

Since 0.567> 0.108, for MAP estimation, - is more likely the label than +.